Attitude determination for three-vehicle heterogeneous formations

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Resumo

A determinação da atitude em formações de veículos tem um papel fundamental em muitas missões. Esta tese propõe uma formação heterogénea com três veículos, na qual existem restrições dos vectores em linha de vista, que são apenas medidos entre um veículo chefe e todos os outros. Cada veículo tem também uma medição inercial para completar a informação acerca da sua própria atitude. A solução determinística para as diferentes matrizes de atitude do problema é dividida em vários passos. Primeiro, a relação entre o chefe e cada um dos outros veículos é determinada resultando, em geral, em duas soluções. No entanto, existem casos degenerados com infinitas soluções e também casos coplanares com uma solução única. Seguidamente, a atitude inercial do chefe é calculada a partir de cada candidato da fase anterior. Esta nova fase resulta, em geral, numa solução única para cada candidato ou soluções infinitas nos casos degenerados. Posteriormente, uma comparação conduz aos candidatos correctos e todas as restantes matrizes de atitude podem ser obtidas através da combinação das matrizes já conhecidas. Em geral, o problema é resolvido univocamente. No entanto, os casos degenerados em qualquer das fases podem levar a que seja impossível encontrar uma solução única. Além disso, existem casos especiais onde a solução correcta não pode ser distinguida de um segundo candidato. A solução proposta foi simulada introduzindo ruído de acordo com um modelo de sensores conhecido e os resultados são coerentes com o que foi obtido noutras soluções determinísticas para problemas de atitude.

Palavras Chave: Determinação de atitude, formações heterogéneas, navegação através de visão, matriz de rotação.

Abstract

Attitude determination in formations of vehicles plays a critical role in many missions. This thesis proposes a representative heterogeneous formation with three vehicles, in which the line-of-sight vector measurements are constrained to be between a chief vehicle and the other two. Moreover, every vehicle has access to an inertial measurement, possibly distinct for each vehicle. The deterministic solution for the different attitude matrices of the problem is developed employing a multistage method. Firstly, the relation between the chief and each of the other vehicles is considered, which gives, in general, two solutions. Nonetheless, there are degenerate cases that have infinite solutions and specific configurations that give a unique solution. Secondly, the inertial attitude of the chief is found from each of the candidates from the first stage. This stage has, in general, a unique solution for each candidate or infinite solutions in the degenerate cases. Afterwards, a comparison leads to the selection of the correct candidate and all the other attitude matrices are, finally, computed from the ones already determined. In general, there is a unique solution for all the attitude matrices, both relative and inertial. Besides the degenerate cases that have infinite solutions, there are special cases where the correct solution cannot be distinguished from two final candidates. This situation is fully characterized. The proposed solution is simulated using a convenient sensor noise model and the results are coherent with state-of-the-art approaches to similar problems.

Keywords: Attitude determination, heterogeneous formation, vision-based navigation, rotation matrix.

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Acronyms

ESOQ Estimator of the Optimal Quaternion. 3

ESOQ2 Second Estimator of the Optimal Quaternion. 3

FOAM Fast Optimal Attitude Matrix. 3

FOV Field-of-View. 4

FPD Focal Plane Detector. 4, 43, 49

LOS line-of-sight. 5, 8, 9, 39, 40, 44, 45, 49

PSD Positive Sensitive Diode. 4

QUEST Quaternion Estimator. 3

SVD Singular Value Decomposition. 3, 42

TRIAD Tri-Axial Attitude Determination. 3, 23, 24

UAV Unmanned Aerial Vehicle. 1

VISNAV Vision Based Navigation. xi, 4

VLA Very Large Array. 1

Chapter 1

Introduction

Attitude determination is a dynamic field of study and it is connected to the astronautic sciences, since the field saw many advancements to answer problems that rose from the development of satellites and space missions. In reality, its applications go beyond this field and are seen also in aerial, ground and underwater vehicles. It is embedded into multiple systems that go from the civil to the military spectrum, from the simple commercial Unmanned Aerial Vehicle (UAV) to the complex interplanetary probe and in all of them attitude determination is usually critical to accomplish the system mission.

In space design, however, there are multiple constraints and trade-offs to be considered and as a general principle less is better. As a consequence, there is a tendency towards designing distributed heterogeneous platforms that cooperate to achieve a common objective. One such case is to distribute attitude sensors across a number of vehicles that share this information, in order to calculate their relative and inertial attitude, which allows each platform to know how it is oriented relative to each of the elements of the formation and also to the inertial reference.

A heterogeneous formation means that not all sensors must be placed in every vehicle, or not as many, or even not as accurate, relying on the higher accuracy of some of them. This can impact not only cost, but also weight, which are the main drivers of the space industry. These benefits are followed by an increase of complexity, since the characteristics of the measurements are different from an homogeneous formation. Certain specific situations that do not allow every sensor to work properly, such as malfunctions or an off design formation geometries, may also lead to a heterogeneous situation.

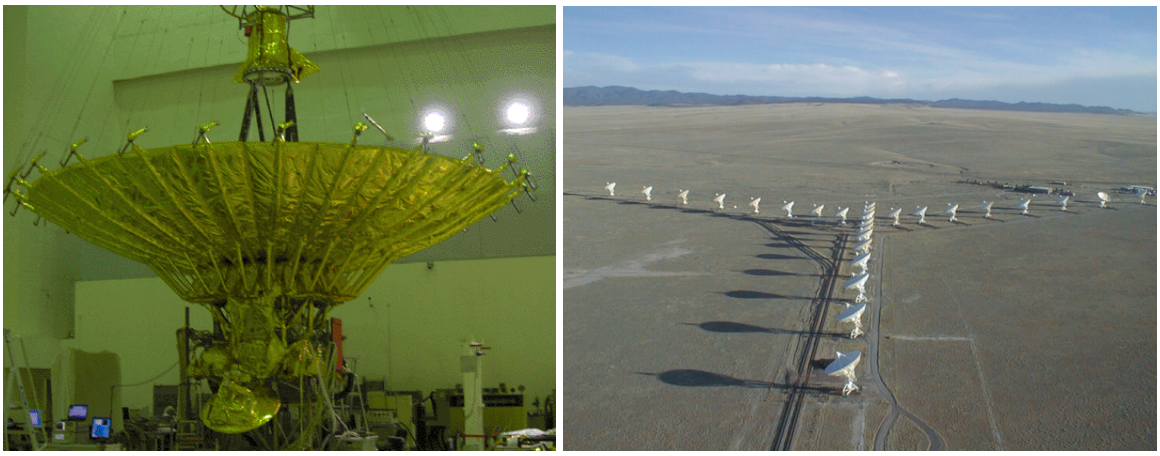
1.1 Motivation and relevance

As described before, the attitude determination of formations brings up the problem of having an heterogeneous formation in terms of sensor measurements, due to limited resources imposed by design or temporary constraints. The solutions that exist for the homogeneous cases are the foundation for the development of methods to solve the problem in constrained situations. However, they cannot address this kind of problems, because it is out of their scope.

Therefore, a representative heterogeneous formation was designed to emulate a practical distributed system where the attitude determination is critical. In such a system, not all vehicles would be able to make the same measurements, thus being a heterogeneous formation.

An example of such system is a distributed space radio telescope. Radio telescopes sense radio waves emitted from celestial bodies, which allow to answer some questions about the universe, relevant for science, moreover radio telescopes are also used in interplanetary communication systems, since they can detect the signals sent from distant probes. Radio waves have a large wavelength which means that large telescopes are required for a good resolution. [2] However, very large telescope dishes are impractical, so

an useful architecture that is already in use on ground infrastructures is the distributed array, composed by several smaller antenna dishes working together. This architecture is very practical because it allows to use cheaper telescopes and increase the overall sensitivity just by increasing their number. The Allen Telescope Array is an example of that using 350 small dishes that have the combined resolving power of a dish with several kilometres. [2] Radio Telescopes have also been sent to space as the Spektr-R with a 10 metre dish, seen in Fig. 1.1a, [3], which has the advantage of having less interferences from the atmosphere or human activities, even though the size of the disc is severely limited by the launch vehicles capabilities. An idea is to send a formation of several small radio telescopes to space that work as an array, taking advantage of the space environment to obtain better results. Moreover, it would also be possible to configure such formation in any shape desired and change it with time, since this can be advantageous. For example the Very Large Array Very Large Array (VLA) has its dishes in a disposition shaped like an Y, as is seen in Fig. 1.1b. [4] In such a system, attitude determination would be critical, because the aperture synthesis techniques require that the direction the antenna is facing is known. [2] The viability of such a project would be dependent on the overall cost, thus distributing sensors unevenly is helpful. Furthermore, this distributed telescope could take advantage of the space environment if it spread through a large area, which would make it harder, for every vehicle to detect every other vehicle. Therefore, an heterogeneous formation is indeed a good solution for this case and it is worth studying.



(a) Spektr-R Radio Telescope: ground testing [5]

(b) Very Large Array: aerial view [6]

Figure 1.1: Radio telescopes examples

Static attitude determination is an important kind of attitude determination, because it does not require the knowledge of any prior estimate of the attitude. This kind of methods just need the current measurements to give the attitude at any given moment. Nonetheless, it requires that all measurements are close enough in time for the vehicle motion not to affect the measurement itself. The other important category is dynamic attitude determination that typically includes filters that need some kind of knowledge of the vehicle motion. [7] The problem studied here is a static attitude determination, furthermore it is a deterministic approach, since it is the simpler way to study and characterize a new problem, since optimal solutions can be derived later, based on the study of the problem made with this approach.

Hence, a deterministic solution was derived, characterized and simulated. This solution is important because it is a system architecture that brings valuable benefits, such as the possibility for wider and denser formations, cost optimization, better redundancy, higher reliability and other advantages. The deterministic solution and characterization of the framework allow to develop more complex methods to address the problem later on.

1.2 State of the art

Before addressing the new problem that is proposed, the most recent methods to determine the static attitude were researched, focusing on problems where there are limited resources or constraints to the general homogeneous formation. Also, the most common sensor models were investigated, to be used in the simulation validation.

Static Attitude Determination

Shuster et al. [8] devised a deterministic algorithm called Tri-Axial Attitude Determination (TRIAD) that finds the rotation between two frames from two vector measurements. It uses the two vectors represented in two different frames to build two right-handed orthogonal triads. A right-handed orthogonal triad is a set of three orthogonal unit vectors, that can represent a right-handed coordinate frame. Then, using these triads it finds the rotation that takes a vector from one frame to the other, since this rotation is given simply by the sum of the outer products between each of the vectors from both orthogonal triads. This solution is not symmetric, as in the construction of the triads some information is discarded, which means that precision is more influenced by one of the vectors. The TRIAD is limited to cases where there are only two different measurements, but is applicable to many situations, given that the noise is not too important.

Allowing more than two measurements is important because of the way many attitude determination devices, like star trackers, work, moreover weighing conveniently each measurement can also prove to be useful, since some measurements are better than other, as seen in the TRIAD. These aspects are considered by the optimization problem proposed by Wahba [9]. The problem is to find the rotation matrix $\mathbf{A}$ that minimizes the loss function

$$L(\mathbf{A}) \equiv \frac{1}{2}\sum_{i=1}^{N} a_i \|\mathbf{b}_i - \mathbf{A}\mathbf{r}_i\|^2 \ ,$$

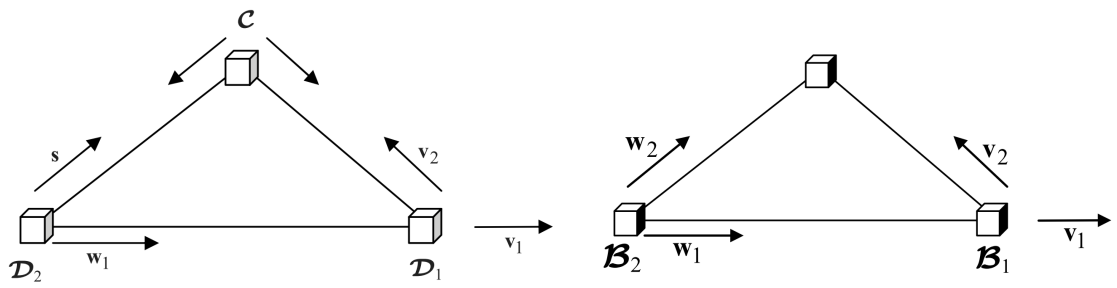
where $\{\mathbf{b}_i\}$ is a set of N unit vectors measured in the body frame, $\{\mathbf{r}_i\}$ is a set of N unit vectors measured in a reference frame and $\{a_i\}$ is a set non-negative weights. This problem is equivalent to the orthogonal Procrustes problem with the determinant of the orthogonal matrix necessarily equal to $+1$.

There are several methods to solve the Wahba's problem, some that use the attitude matrix directly, such as the Singular Value Decomposition (SVD) Method [10] and the Fast Optimal Attitude Matrix (FOAM) [11] and others that use the quaternion representation of the attitude matrix, as the Davenport's q Method [12], the Quaternion Estimator (QUEST) method [8], the Method of Sequential Rotations [13], the Estimator of the Optimal Quaternion (ESOQ) [14] and the Second Estimator of the Optimal Quaternion (ESOQ2) [15].

Constrained Attitude Determination

New problems that introduced constraints to the general problem have been approached by the scientific community as a natural response to new situations, technologies and requirements. A constrained deterministic attitude problem has been considered by Linares et al. [16] and by Andrle et al. [17]. Both give the solution to a similar problem, where observation constraints with the second measurement, as seen in Fig. 1.2, makes it impossible to apply any of the methods enumerated before, because there is only one vector that is represented in both frames. The second measurement of each vehicle is relative to a third object or vehicle, meaning that a new relation has to be considered to relate these two vectors, and then find the attitude matrix.

The constraint in both cases is the lack of the second vector measurement relative to each vehicle.



(a) (from [17]). (b) (from [16]).

Figure 1.2: Geometry of the constrained problems

The solutions proposed use the trigonometric relations between the vectors relative to a common object or vehicle and use them to build the relative rotation matrix between both body-frames.

Sensors and Noise

VISNAV has been used in space vehicles for some time now, since these vehicles often rely on sensors like star trackers, sun sensors, horizon sensors and other similar devices. [18] Therefore, the sensors considered by many in the scientific community are often based on the Focal Plane Detector (FPD). This device detects the light sources and gives the measurements as vectors that are computed by the photogrammetry collinearity equations, [19] that relate the focal plane frame and the object frame, thus allowing to know the measurements in both frames given the knowledge of the sensor boresight. Real sensors are subject to systematic and random errors, the first kind can be eliminated by means of calibration and other techniques and the second kind are always present and thus have to be modelled. Shuster [20] has modelled the FPD as a zero mean Gaussian random noise with a covariance that is influenced by the distance of the measurement vector from the boresight axis. Hence, a measurement is less accurate when the vector is further from the boresight of the respective sensor. Cheng et al. [1] devised a model of a wide Field-of-View (FOV) sensor, the Positive Sensitive Diode (PSD), based on the FPD model. Fig. 1.3 shows a scheme of the image space and object space relations considering a PSD which relies on the photogrammetry collinearity equations to relate both frames, similarly to the FPD.

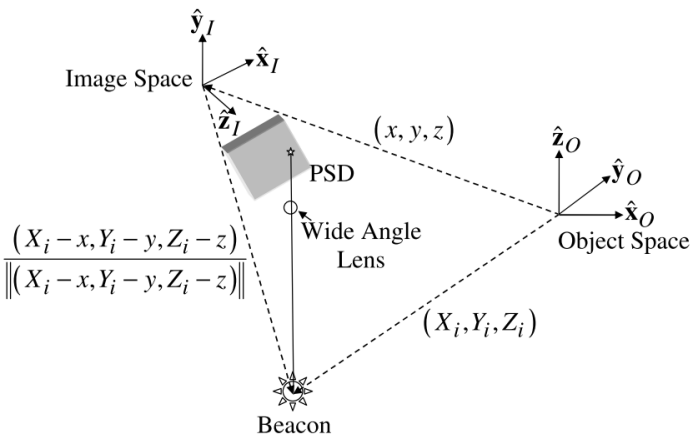


Figure 1.3: VISNAV: object-space and image-space scheme [1]

Considerations

The constrained problem is interesting and allows to reduce the physical link between two vehicles by decreasing the number of vector measurements between the vehicles that are related by the attitude matrix. However, every vehicle still has to have at least one measurement vector relative to every other vehicle, to determine the different attitude relations. The constraint, indeed, improves the flexibility of the formation, since the vehicles are not dependant on measuring the same inertial reference but can measure a common target. But the flexibility is still limited, since the distances must allow the measurements between all vehicles to be made and the swarm cannot grow too much because of line of sight obstructions. Therefore, a new possibility is to distribute the formation and its measurements in such a way that it allows the vehicles not to detect every vehicle in order to relate to it, which means that a very large and spread out formation is possible. This can be accomplished by distributing heterogeneously the sensors, which means that there must be a way to relate the attitudes of vehicles that cannot detect each other.

1.3 Contributions

This work contributes with the proposal of an attitude determination system for a heterogeneous three-vehicle formation, where all the inertial measurements are equally distributed, since each vehicle measures one inertial reference, and only one of the vehicles is in the line of sight of the other two, thus the vehicles can measure line-of-sight (LOS) vectors only relative to this vehicle. In particular, the deterministic solution is derived considering the geometric relations of the problem. Additionally, the full characterization of the solution is carried out, considering a formal setting, and conditions with physical insight are established. Finally, simulations are shown, in the presence of realistic noise, to assess the performance of the proposed solution.

Chapter 2

Problem statement

2.1 Notation and definitions

Throughout this document scalars are represented in regular typeface, whereas vectors and matrices are represented in bold. Reference frames are represented in calligraphic typeface and between brackets, such as $\{\mathcal{I}\}$. Body-fixed frames are numbered and represented by the letter $\mathcal{B}$, for example, $\{\mathcal{B}_1\}$. The symbol $\mathbf{0}$ represents the null vector or matrix, according to the situation. The symbol $\mathbf{I}$ represents the identity matrix with the appropriate dimensions, and whenever it is convenient to explicitly state the dimension, this dimension is given as a subscript, as in $\mathbf{I}_{\mathbf{3}\times\mathbf{3}}$. The angle between two given vectors, $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^3$, is denoted by $(\hat{\mathbf{xy}})$.

The set of unit vectors in $\mathbb{R}^3$ is denoted by

$$S(2) := \left\{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1\right\} \ .$$

The special orthogonal group of dimension 3 is denoted by

$$SO(3) := \left\{\mathbf{X} \in \mathbb{R}^{3\times 3} : \mathbf{X}\mathbf{X}^T = \mathbf{X}^T\mathbf{X} = \mathbf{I} \ \wedge \ \det(\mathbf{X}) = 1\right\} \ .$$

The skew-symmetric matrix parametrized by $\mathbf{x} \in \mathbb{R}^3$, which encodes the cross product between $\mathbf{x}$ and another vector, is denoted by

$$\mathbf{S}(\mathbf{x}) := \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}, \ \mathbf{x} = [x_1, x_2, x_3] \, .$$

The rotation matrix in $SO(3)$ that transforms a given vector in $\mathbb{R}^3$ expressed in the reference frame $\{\mathcal{A}\}$ to the reference frame $\{\mathcal{C}\}$ is denoted by $\mathbf{R}_A^C$. If any of the reference frames is a body-fixed frame, the number that identifies the body is used instead, *i.e.*, the rotation from $\{\mathcal{B}_i\}$ to $\{\mathcal{B}_j\}$, $i, j \in \mathbb{N}$, is denoted by $\mathbf{R}_i^j$. [1] Moreover, the rotation matrix through an angle $\theta \in \mathbb{R}$ about the unit vector $\mathbf{x} \in \mathrm{S}(2)$ is denoted by $\mathbf{R}(\theta, \mathbf{x})$, which is expressed as [7]

$$\mathbf{R}(\theta, \mathbf{x}) := \cos(\theta)\,\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + (1 - \cos(\theta))\mathbf{x}\,\mathbf{x}^T - \sin(\theta)\,\mathbf{S}(x) \ . \tag{2.1}$$

The four-quadrant inverse tangent function is denoted by $\operatorname{atan2}(b, a)$.

[1] Note that the inverse of a rotation matrix is given by its transpose, because it is an orthogonal matrix. So, whenever a rotation is transposed, its indices switch position, *i.e.*, $\left(\mathbf{R}_i^j\right)^T = \left(\mathbf{R}_i^j\right)^{-1} = \mathbf{R}_j^i$.

2.2 Problem definition

Consider a scenario where three vehicles operate in a formation, in which each of the vehicles has its own body-fixed frame, $\{\mathcal{B}_1\}$, $\{\mathcal{B}_2\}$, and $\{\mathcal{B}_3\}$, accordingly. An inertial reference frame, $\{\mathcal{I}\}$, is also considered.

In the proposed framework, there are two kinds of measurements: one is a LOS vector that points to the position of another vehicle and the other is an inertial vector, for example, a known physical field or inertial direction. The LOS vector is only known in the body-fixed frame of the vehicle that measures it, while the inertial vector is available both in the body-fixed frame, where it is measured, and the inertial frame, since the sensor has knowledge about the inertial reference that it is measuring. It is worth mentioning that all measurement vectors are considered as unit vectors, so they only give information about the direction that the sensor detects, as it is common in attitude determination problems.

In the formation, there are some constraints which result in different vehicles getting different kinds of measurements from each other. The main constraint is that only one of the vehicles can measure LOS vectors to both the other vehicles, whereas the other two can only measure a single LOS vector relative to this first vehicle. Meaning, for example, that these two are too far from each other for the sensors to measure the LOS or that their sensors are not oriented in such a way that they can get both LOS. Furthermore, every vehicle has a sensor that allows to measure one inertial vector. The vehicle that can measure LOS to the other two is identified as vehicle 1 and will sometimes be referred as chief, since it connects the formation in terms of measurements. The other vehicles are often called deputies and are identified as vehicles 2 and 3, respectively. The geometry of the scenario is represented in Fig. 2.1.

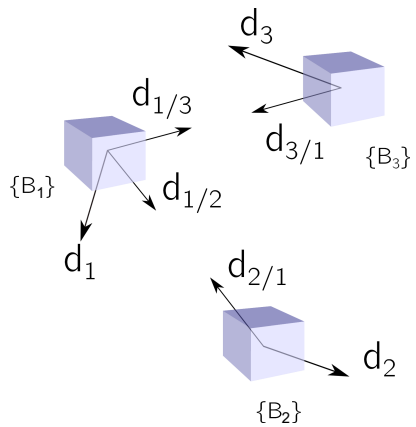


Figure 2.1: Three-vehicle heterogeneous formation

In the figure and throughout this document, the LOS measurements are denoted by $\mathbf{d}_{i/j}$, $i, j = 1, 2, 3$, $i \neq j$, which represents the unit vector from the i-th to the j-th vehicles expressed in the i-th vehicle body-fixed frame, $\{\mathcal{B}_i\}$. The inertial vector measurements are denoted by $\mathbf{d}_i$, $i = 1, 2, 3$, which represents the inertial vector measured by the i-th platform and expressed in its own body-fixed frame.

If a vector is expressed in coordinates other than its own, a superscript is used, for both the LOS vectors and the inertial vectors, to specify the frame where the vector is described. For example, ${}^j\mathbf{d}_{i/j}$, $i, j = 1, 2, 3$, $i \neq j$, is the LOS from the i-th to the j-th vehicle expressed in $\{\mathcal{B}_j\}$ and ${}^I\mathbf{d}_i$, $i = 1, 2, 3$, is the inertial vector from the i-th vehicle, expressed in the inertial reference frame, $\{\mathcal{I}\}$.

The problem that is here considered is that of determining all the rotation matrices, both relative ($\mathbf{R}_2^1$, $\mathbf{R}_3^1$, $\mathbf{R}_3^2$) and inertial ($\mathbf{R}_1^I$, $\mathbf{R}_2^I$, $\mathbf{R}_3^I$), using the measurement vectors that were described. The knowledge of these matrices fully determines the attitude of the formation.

Chapter 3

Attitude determination

The proposed solution can be divided in four stages. In the first stage, the candidates for $\mathbf{R}_2^1$ and for $\mathbf{R}_3^1$ are found. In general there are two candidates for each of these rotations. Then, a stage where each of these candidates is used to find candidates for $\mathbf{R}_1^I$. Afterwards, these candidates are compared according to their origin, in order to disambiguate between the several possibilities for the inertial attitude of the chief vehicle. This third stage finds which candidates are identical while having different origins, because the real $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$ give the same candidates for $\mathbf{R}_1^I$. Finally, the inertial and relative attitudes that are still unknown, that is, $\mathbf{R}_3^2$, $\mathbf{R}_2^I$ and $\mathbf{R}_3^I$, are determined using the product between the rotations already calculated.

3.1 Solution and characterization: partial solution

In order to determine a solution to this problem, the formation is first considered as two branches, each corresponding to two vehicles. The chief and each of the deputies form each branch, from which each the candidates for the relative rotation matrices are obtained.

3.1.1 Partial solution for $\mathbf{R}_2^1$

In this section the goal is to find $\mathbf{R}_2^1$, consequently, the relation between vehicle 1, the chief, and the deputy vehicle 2, is considered.

In this problem, two constraints are taken into account to determine the relative rotation matrix $\mathbf{R}_2^1$. The first relation is associated with the anti-parallel association between the LOS vectors of the two vehicles, because they are both measuring opposing directions, considering a common frame, as given by

$$-\mathbf{d}_{1/2} = \mathbf{R}_2^1 \mathbf{d}_{2/1} \ . \tag{3.1}$$

The second relation is associated with the conservation of the angle between two vectors, regardless of the coordinate frame, provided that it is the same for both. Note that the vectors considered are unitary, so the inner product also represents the cosine of the angle between these vectors. Consequently,

$${}^I\mathbf{d}_1^T \, {}^I\mathbf{d}_2 = \mathbf{d}_1^T \mathbf{R}_2^1 \mathbf{d}_2 \ . \tag{3.2}$$

The problem of finding $\mathbf{R}_2^1$ is addressed by decomposing this rotation matrix into a product of two rotations, which can be found using (3.1) and (3.2). This approach is possible because the product of two proper rotation matrices is still a proper rotation matrix, as shown in Appendix A.1. This rotation decomposition is defined as

$$\mathbf{R}_2^1 := \mathbf{R}\left(\theta_2, \mathbf{n}_2\right) \mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \ . \tag{3.3}$$

A possible solution is to consider $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$ as any rotation that verifies (3.1). There are infinite rotation matrices that can be used, because the relation leaves one degree of freedom of the rotation undetermined. Therefore, $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$ must be such that both (3.1) and (3.2) are verified by the product $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right) \mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$. As a consequence, the choice of $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$ influences the determination of $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$.

Before describing the proposed solution, it is noticed that there are two degenerate cases in which it is not possible to uniquely determine $\mathbf{R}_2^1$. It is useful to define these cases before the actual characterization, because it gives some hints about the solution. The following lemma addresses these cases.

Lemma 1. *If*

$$\mathbf{d}_1 = \pm \mathbf{d}_{1/2} \tag{3.4a}$$

or

$$\mathbf{d}_2 = \pm \mathbf{d}_{2/1} \ , \tag{3.4b}$$

then $\mathbf{R}_2^1$ *has an infinite number of solutions.*

Proof. Fix $\mathbf{R}_2^1$, such that both (3.1) and (3.2) are verified, and define

$$\mathbf{A}\left(\theta\right) := \mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right) \mathbf{R}_2^1 \ ,$$

where $\theta \in \mathbb{R}$ is arbitrary. It will be shown by direct computation that this arbitrary rotation satisfies both (3.1) and (3.2), if either (3.4a) or (3.4b) hold. Indeed,

$$\mathbf{A}\left(\theta\right) \mathbf{d}_{2/1} = \mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right) \mathbf{R}_2^1 \mathbf{d}_{2/1} \ ,$$

which, using (3.1), simplifies to

$$\mathbf{A}\left(\theta\right) \mathbf{d}_{2/1} = -\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right) \mathbf{d}_{1/2} \ . \tag{3.5}$$

Substituting (2.1) in (3.5) gives

$$\mathbf{A}\left(\theta\right) \mathbf{d}_{2/1} = -\left[\cos\left(\theta\right) \mathbf{I}_{\mathbf{3}\times\mathbf{3}} + \left(1 - \cos\left(\theta\right)\right) \mathbf{d}_{1/2} \left(\mathbf{d}_{1/2}\right)^T - \sin\left(\theta\right) \mathbf{S}\left(-\mathbf{d}_{1/2}\right)\right] \mathbf{d}_{1/2} \ .$$

Using the distributive property, it follows that

$$\mathbf{A}\left(\theta\right) \mathbf{d}_{2/1} = -\cos\left(\theta\right) \mathbf{d}_{1/2} - \left(1 - \cos\left(\theta\right)\right) \mathbf{d}_{1/2} \left(\mathbf{d}_{1/2}\right)^T \ \mathbf{d}_{1/2} + \sin\left(\theta\right) \mathbf{S}\left(-\mathbf{d}_{1/2}\right) \mathbf{d}_{1/2} \ . \tag{3.6}$$

Next, notice that

$$\left(\mathbf{d}_{1/2}\right)^T \mathbf{d}_{1/2} = 1 \tag{3.7}$$

because the inner product between two identical unit vectors is equal to one. Furthermore, notice that

$$\mathbf{S}\left(-\mathbf{d}_{1/2}\right) \mathbf{d}_{1/2} = 0 \ , \tag{3.8}$$

because the cross product of two parallel vectors is equal to zero. Therefore, (3.6) is given by

$$\mathbf{A}\left(\theta\right) \mathbf{d}_{2/1} = -\cos\left(\theta\right) \mathbf{d}_{1/2} - \left(1 - \cos(\theta)\right) \mathbf{d}_{1/2} \ ,$$

or equivalently,

$$\mathbf{A}\left(\theta\right)\mathbf{d}_{2/1} = -\mathbf{d}_{1/2} \ . \tag{3.9}$$

Thus, there is a rotation with an arbitrary angle that keeps the result of (3.1) unchanged. Up until this point none of the assumptions of this lemma were used, which means that (3.1) can be satisfied by an infinite number of rotations, regardless of the situation. Next, take the right expression of (3.2) and compute

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left(\mathbf{d}_1\right)^T\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{R}_2^1\mathbf{d}_2 \ . \tag{3.10}$$

Without loss of generality, the first assumption is used with the positive sign, *i.e.*, $\mathbf{d}_1 = \mathbf{d}_{1/2}$, which applied to (3.10) yields

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left(\mathbf{d}_{1/2}\right)^T\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{R}_2^1\mathbf{d}_2 \ . \tag{3.11}$$

Using (2.1) in (3.11) gives

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left(\mathbf{d}_{1/2}\right)^T\left[\cos(\theta)\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + (1-\cos(\theta))\mathbf{d}_{1/2}\left(\mathbf{d}_{1/2}\right)^T - \sin(\theta)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\right]\mathbf{R}_2^1\mathbf{d}_2 \ ,$$

which, distributing $\left(\mathbf{d}_{1/2}\right)^T$, is rewritten as

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left[\cos(\theta)\left(\mathbf{d}_{1/2}\right)^T + (1-\cos(\theta))\left(\mathbf{d}_{1/2}\right)^T\mathbf{d}_{1/2}\left(\mathbf{d}_{1/2}\right)^T - \sin(\theta)\left(\mathbf{d}_{1/2}\right)^T\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\right]\mathbf{R}_2^1\mathbf{d}_2 \ . \tag{3.12}$$

Next, the properties described in (3.7) and (3.8) allow to simplify (3.12), as given by

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left[\cos(\theta)\mathbf{d}_{1/2}^T + (1-\cos(\theta))\left(\mathbf{d}_{1/2}\right)^T\right]\mathbf{R}_2^1\mathbf{d}_2 \ .$$

or equivalently,

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left(\mathbf{d}_{1/2}\right)^T\mathbf{R}_2^1\mathbf{d}_2 \ ,$$

which, using $\mathbf{d}_1 = \mathbf{d}_{1/2}$ again, can be rewritten as

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left(\mathbf{d}_1\right)^T\mathbf{R}_2^1\mathbf{d}_2 \ . \tag{3.13}$$

Substituting (3.2) in (3.13) yields

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left({}^I\mathbf{d}_1\right)^{T_I}\mathbf{d}_2 \ . \tag{3.14}$$

This concludes the first part of the proof. Indeed, (3.9) and (3.14) are verified for all θ, which means that there are infinite rotations $\mathbf{R}_2^1$ that satisfy both (3.9) and (3.14). Taking, instead, the assumption $\mathbf{d}_2 = \mathbf{d}_{2/1}$, without loss of generality, and substituting in (3.10) gives

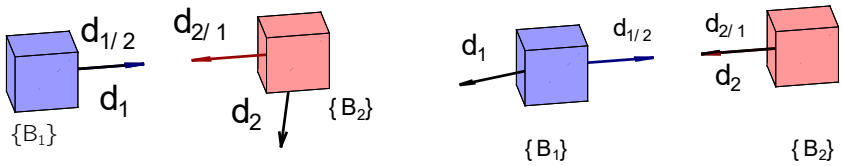
$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = \left(\mathbf{d}_1\right)^T\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{R}_2^1\mathbf{d}_{2/1} \ . \tag{3.15}$$

Using (3.1) in (3.15) allows to write

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = -\left(\mathbf{d}_1\right)^T\mathbf{R}\left(\theta, -\mathbf{d}_{1/2}\right)\mathbf{d}_{1/2} \ ,$$

which using (2.1) gives

$$\left(\mathbf{d}_1\right)^T\mathbf{A}\left(\theta\right)\mathbf{d}_2 = -\left(\mathbf{d}_1\right)^T\left[\cos(\theta)\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + (1-\cos(\theta))\mathbf{d}_{1/2}\left(\mathbf{d}_{1/2}\right)^T - \sin(\theta)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\right]\mathbf{d}_{1/2} \ .$$



(a) $\mathbf{d}_{1/2} = \mathbf{d}_1$ (b) $\mathbf{d}_{2/1} = \mathbf{d}_2$

Figure 3.1: Degenerate cases of the partial problem

Then, distributing $\mathbf{d}_{1/2}$ yields

$$(\mathbf{d}_1)^T \mathbf{A}(\theta)\,\mathbf{d}_2 = -(\mathbf{d}_1)^T \left[\cos(\theta)\mathbf{d}_{1/2} + (1-\cos(\theta))\mathbf{d}_{1/2}\left(\mathbf{d}_{1/2}\right)^T \mathbf{d}_{1/2} - \sin(\theta)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\mathbf{d}_{1/2}\right] \ ,$$

which, using the properties (3.7) and (3.8), is rewritten as

$$(\mathbf{d}_1)^T \mathbf{A}(\theta)\,\mathbf{d}_2 = -(\mathbf{d}_1)^T \left[\cos(\theta)\mathbf{d}_{1/2} + (1-\cos(\theta))\mathbf{d}_{1/2}\right] \ ,$$

or equivalently,

$$(\mathbf{d}_1)^T \mathbf{A}(\theta)\,\mathbf{d}_2 = -(\mathbf{d}_1)^T \mathbf{d}_{1/2} \ .$$

Next, using (3.1) again, yields

$$(\mathbf{d}_1)^T \mathbf{A}(\theta)\,\mathbf{d}_2 = (\mathbf{d}_1)^T \mathbf{R}_2^1 \mathbf{d}_{2/1} \ ,$$

and substituting (3.4b) again, gives

$$(\mathbf{d}_1)^T \mathbf{A}(\theta)\,\mathbf{d}_2 = (\mathbf{d}_1)^T \mathbf{R}_2^1 \mathbf{d}_2 \ . \tag{3.16}$$

Finally, substituting (3.2) in (3.16)

$$(\mathbf{d}_1)^T \mathbf{A}(\theta)\,\mathbf{d}_2 = \left({}^I\mathbf{d}_1\right)^T {}^I\mathbf{d}_2 \ .$$

Hence, there are infinite solutions for $\mathbf{A}(\theta)$ that satisfy both (3.1) and (3.2) when $\mathbf{d}_2 = \pm\mathbf{d}_{2/1}$. □

Remark. The conditions that lead to infinite solutions are configurations of the measurement vectors in which the problem is degenerate. That is a consequence of a degree of freedom that is undetermined. Fig. 3.1 shows the representation of both the degenerate cases considered in Lemma 1.

Determination of $\mathbf{R}(\theta_1, \mathbf{n}_1)$

As shown in the previous proof, there are infinite rotations that can be used as $\mathbf{R}(\theta_1, \mathbf{n}_1)$, which means that a choice must be made. However, this choice affects $\mathbf{R}(\theta_2, \mathbf{n}_2)$, since the product between both rotations must be conserved, recalling that it satisfies both (3.1) and (3.2). The following lemma addresses the choice for $\mathbf{R}(\theta_1, \mathbf{n}_1)$.

Lemma 2. *Suppose that*

$$\mathbf{d}_1 \neq \pm\mathbf{d}_{1/2} \ .$$

Then, the transformation $\mathbf{R}(\theta_1, \mathbf{n}_1)$ *that verifies*

$$\mathbf{R}(\theta_1, \mathbf{n}_1)\,\mathbf{d}_{2/1} = -\mathbf{d}_{1/2} \tag{3.17}$$

is described by

$$\theta_1 = \pi \tag{3.18}$$

and

$$\begin{cases} \mathbf{n}_1 = \frac{\mathbf{d}_{2/1}-\mathbf{d}_{1/2}}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|} & , \text{ for } \mathbf{d}_{2/1} \neq \mathbf{d}_{1/2} \\ \\ \mathbf{n}_1 = \frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|} & , \text{ for } \mathbf{d}_{2/1} = \mathbf{d}_{1/2} \end{cases} . \tag{3.19}$$

Proof. The proof follows by direct computation for both cases. Firstly, suppose that $\mathbf{d}_{2/1} \neq \mathbf{d}_{1/2}$ and take the left expression of (3.17), considering the parameters from (3.18) and (3.19) which is expressed as

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_{2/1} = \mathbf{R}\left(\pi, \frac{\mathbf{d}_{2/1}-\mathbf{d}_{1/2}}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|}\right)\mathbf{d}_{2/1} \ .$$

Then, expand it using (2.1), which gives

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_{2/1} = \Bigg[\cos(\pi)\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + (1-\cos(\pi))\frac{\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)^T}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2} - \sin(\pi)\mathbf{S}\left(\frac{\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|}\right)\Bigg]\mathbf{d}_{2/1} \ ,$$

or equivalently,

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_{2/1} = \left[-\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + 2\frac{\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)^T}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\right]\mathbf{d}_{2/1} \ .$$

Using the distributive property, it follows that

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_{2/1} = -\mathbf{d}_{2/1} + 2\frac{\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)^T}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\mathbf{d}_{2/1} \ .$$

The product $\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)^T\mathbf{d}_{2/1}$ is scalar, since it is an inner product, thus it can be commutated, yielding

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_{2/1} = -\mathbf{d}_{2/1} + 2\frac{\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)^T\mathbf{d}_{2/1}}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)$$

or equivalently,

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_{2/1} = -\mathbf{d}_{2/1} + 2\frac{\left(\mathbf{d}_{2/1}^T\mathbf{d}_{2/1} - \mathbf{d}_{1/2}^T\mathbf{d}_{2/1}\right)}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right) \ . \tag{3.20}$$

Noticing that

$$\frac{\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)^T\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2} = 1$$

and due to $\mathbf{d}_{2/1}\mathbf{d}_{2/1}^T = 1$, (3.20) is rewritten as

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_{2/1} = -\frac{\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)^T\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\mathbf{d}_{2/1} + 2\frac{\left(1-\mathbf{d}_{1/2}^T\mathbf{d}_{2/1}\right)}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right) \ . \tag{3.21}$$

As

$$\mathbf{d}_{2/1}^T\mathbf{d}_{1/2} = \mathbf{d}_{1/2}^T\mathbf{d}_{2/1} \ , \tag{3.22}$$

it follows that

$$\begin{aligned}\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right)^T\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right) &= \|\mathbf{d}_{1/2}\|^2-\mathbf{d}_{2/1}^T\mathbf{d}_{1/2}-\mathbf{d}_{1/2}^T\mathbf{d}_{2/1}+\|\mathbf{d}_{2/1}\|^2 \\ &= 2-2\mathbf{d}_{2/1}^T\mathbf{d}_{1/2}\end{aligned} \quad , \tag{3.23}$$

so, substituting (3.23) in (3.21) it follows that

$$\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1}=-\frac{2-2\mathbf{d}_{1/2}^T\mathbf{d}_{2/1}}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\mathbf{d}_{2/1}+\frac{\left(2-2\mathbf{d}_{1/2}^T\mathbf{d}_{2/1}\right)}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\left(\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\right) \ ,$$

which, by summing symmetric terms, is simplified to

$$\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1}=\frac{\left(2-2\mathbf{d}_{1/2}^T\mathbf{d}_{2/1}\right)}{\|\mathbf{d}_{2/1}-\mathbf{d}_{1/2}\|^2}\left(-\mathbf{d}_{1/2}\right) \ . \tag{3.24}$$

By the definition of norm, $\|\mathbf{x}\|=\sqrt{\mathbf{x}^T\mathbf{x}}$, the denominator in (3.24) is given by (3.23), hence, using (3.22) to show that the numerator equals the denominator, finally leads to

$$\begin{aligned}\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1} &= \frac{\left(2-2\mathbf{d}_{1/2}^T\mathbf{d}_{2/1}\right)}{\left(2-2\mathbf{d}_{1/2}^T\mathbf{d}_{2/1}\right)}\left(-\mathbf{d}_{1/2}\right) \\ &= -\mathbf{d}_{1/2} \ ,\end{aligned}$$

which concludes the first part of the lemma. Now, suppose instead that $\mathbf{d}_{2/1}=\mathbf{d}_{1/2}$, which means that

$$\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1}=\mathbf{R}\left(\pi,\frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|}\right)\mathbf{d}_{2/1} \ .$$

Then, using (2.1) in the left expression of (3.17) gives

$$\begin{aligned}\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1}=\Bigg[&\cos\left(\pi\right)\mathbf{I}+\left(1-\cos\left(\pi\right)\right)\left(\frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|}\right)\left(\frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|}\right)^T \\ &-\sin\left(\pi\right)\mathbf{S}\left(\frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|}\right)\Bigg]\mathbf{d}_{2/1} \ ,\end{aligned}$$

or equivalently,

$$\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1}=\left(-\mathbf{I}+2\frac{\left(\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\right)\left(\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\right)^T}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|^2}\right)\mathbf{d}_{2/1} \ . \tag{3.25}$$

Next, using $\mathbf{d}_{2/1}=\mathbf{d}_{1/2}$, it follows that (3.25) is rewritten as

$$\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1}=\left(-\mathbf{I}+2\frac{\left(\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\right)\left(\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\right)^T}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|^2}\right)\mathbf{d}_{1/2} \ ,$$

which, using the distributivity, is expressed as

$$\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1}=-\mathbf{d}_{1/2}-2\frac{\left(\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\right)\left(\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\right)^T\mathbf{d}_{1/2}}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|^2} \ ,$$

or equivalently,

$$\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\mathbf{d}_{2/1}=-\mathbf{d}_{1/2}-2\frac{\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\mathbf{d}_1^T\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_{1/2}}{\|\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_1\|^2} \ . \tag{3.26}$$

Finally, noticing that $\mathbf{S}\left(\mathbf{d}_{1/2}\right)\mathbf{d}_{1/2} = \mathbf{0}$, (3.26) is given as

$$\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_{2/1} = -\mathbf{d}_{1/2} \ ,$$

thus concluding the proof for the second condition of $\mathbf{n}_1$. □

When $\mathbf{d}_{1/2} \neq \mathbf{d}_{2/1}$, Lemma 2 has a clear geometrical interpretation, see Fig. 3.2, where it is clear that a rotation by 180° of two vectors around an axis that is the result of their sum, transforms one vector into the other, if both have the same length, which in this case is true. Indeed, a rotation of 180° generates two other vectors in the same plane, which creates a diamond shape. So, the generated vectors have the orientation of the opposing edge of the diamond. These also have the same length, since both original vectors have equal length. Therefore, the vectors generated by the rotation are equal to the respective vector that makes the opposing edge of the diamond.

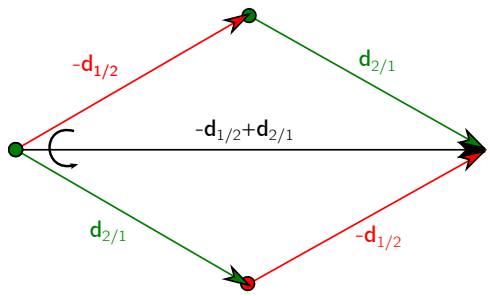


Figure 3.2: Geometry of the rotation of π about the axis $\left(-\mathbf{d}_{1/2} + \mathbf{d}_{2/1}\right)$

When $\mathbf{d}_{1/2} = \mathbf{d}_{2/1}$, any vector that is not parallel to $\mathbf{d}_{1/2}$ could have been used in the place of $\mathbf{d}_1$, since the cross product with $\mathbf{d}_{1/2}$ would still be perpendicular to $\mathbf{d}_{1/2}$.

Determination of $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$

The determination of $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$ must be in accordance with the choice for $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$, that is expressed in Lemma 2. The axis of $\mathbf{n}_2$ must be one that does not change (3.1), because this relation is already verified using the rotation parameters θ_1 and $\mathbf{n}_1$ already defined in Lemma 2. Furthermore, θ_2 is also conditioned by $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$, because (3.1) and (3.2) must both be verified by the product $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$. The next lemma gives the parameters for $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$.

Lemma 3. *Consider the decomposition* (3.3) *with* $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$ *parameterized by* (3.18) *and* (3.19). *Assume also that*

$$\mathbf{d}_1 \neq \pm\mathbf{d}_{1/2}$$

and

$$\mathbf{d}_2 \neq \pm\mathbf{d}_{2/1} \ .$$

Then, the angle and axis of $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$ *are given by*

$$\theta_2 = \operatorname{atan2}\left(a_{s_{12}}, a_{c_{12}}\right) \pm \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) \tag{3.27}$$

and

$$\mathbf{n}_2 = -\mathbf{d}_{1/2} \ , \tag{3.28}$$

with

$$\begin{cases} a_{p_{12}} & := \mathbf{d}_1^T(-\mathbf{d}_{1/2})(-\mathbf{d}_{1/2})^T\mathbf{d}_2^* - {}^I\mathbf{d}_1^T\,{}^I\mathbf{d}_2 \\ a_{c_{12}} & := \mathbf{d}_1^T(\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^2)\mathbf{d}_2^* \\ a_{s_{12}} & := \mathbf{d}_1^T(\mathbf{S}\left(-\mathbf{d}_{1/2}\right))\mathbf{d}_2^* \end{cases} , \tag{3.29}$$

where

$$\mathbf{d}_2^* = \mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_2 \ . \tag{3.30}$$

Proof. For any rotation angle, the rotation of a vector about that very same vector does not change this vector, as shown in Appendix A.2, and therefore, $\mathbf{n}_2$ is limited to $\left(\pm\mathbf{d}_{1/2}\right)$, since (3.1) and (3.17) are both verified. The signal choice of the axis only affects the signal of the resulting angle of the rotation, so the negative signal, *i.e.*, $\mathbf{n}_2 = -\mathbf{d}_{1/2}$, was chosen for coherence with the signal of (3.1). Now, the proof for θ_2 is made by direct computation. Applying Lemma 24, shown in Appendix B.1, to (2.1) results in an alternative rotation representation given by

$$\mathbf{R}\left(\theta, \mathbf{x}\right) = \mathbf{x}\mathbf{x}^T - \cos(\theta)\mathbf{S}\left(x\right)^2 - \sin(\theta)\mathbf{S}\left(x\right) \ , \tag{3.31}$$

which, applied to $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$ with $\mathbf{n}_2 = -\mathbf{d}_{1/2}$, gives

$$\mathbf{R}(\theta_2, -\mathbf{d}_{1/2}) = \left(-\mathbf{d}_{1/2}\right)\left(-\mathbf{d}_{1/2}\right)^T - \cos(\theta_2)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^2 - \sin(\theta_2)\mathbf{S}\left(-\mathbf{d}_{1/2}\right) \ . \tag{3.32}$$

Then, substituting $\mathbf{R}_2^1 = \mathbf{R}\left(\theta_2, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$, where $\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)$ is as given in Lemma 2, in (3.2) gives

$$\left(\mathbf{d}_1\right)^T\mathbf{R}\left(\theta_2, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right)\mathbf{d}_2 = \left({}^I\mathbf{d}_1\right)^T\,{}^I\mathbf{d}_2 \ ,$$

which, using (3.32) and also the notation from (3.30), is given as

$$\left(\mathbf{d}_1\right)^T\left[(-\mathbf{d}_{1/2})(-\mathbf{d}_{1/2})^T - \cos(\theta_2)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^2 - \sin(\theta_2)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\right]\mathbf{d}_2^* = \left({}^I\mathbf{d}_1\right)^T\,{}^I\mathbf{d}_2 \ . \tag{3.33}$$

Making use of the coefficients defined in (3.29), it follows that (3.33) is expressed as a trigonometric equation, which is written as

$$a_{p_{12}} = a_{c_{12}}\ \cos(\theta_2) + a_{s_{12}}\ \sin(\theta_2) \ .$$

The solution for this equation is given in Lemma 29, Appendix C, and leads to the possible angles for this rotation, which are given by (3.27). □

Relative Rotation Matrix

The combination of Lemmas 2 and 3 to produce the rotation $\mathbf{R}_2^1$ concludes the partial solution, which is stated in the following corollary.

Corollary 1. *Assume that*

$$\mathbf{d}_1 \neq \pm\mathbf{d}_{1/2}$$

and

$$\mathbf{d}_2 \neq \pm\mathbf{d}_{2/1} \ .$$

Then $\mathbf{R}_2^1$ *is given by*

$$\mathbf{R}_2^1 = \begin{cases} \mathbf{R}\left(\operatorname{atan2}\left(a_{s_{12}}, a_{c_{12}}\right) \pm \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right), -\mathbf{d}_{1/2}\right) \mathbf{R}\left(\pi, \frac{\mathbf{d}_{2/1} - \mathbf{d}_{1/2}}{\|\mathbf{d}_{2/1} - \mathbf{d}_{1/2}\|}\right) & , \text{ if } \mathbf{d}_{2/1} \neq \mathbf{d}_{1/2} \\ \mathbf{R}\left(\operatorname{atan2}\left(a_{s_{12}}, a_{c_{12}}\right) \pm \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right), -\mathbf{d}_{1/2}\right) \mathbf{R}\left(\pi, \frac{\mathbf{d}_{1/2} \times \mathbf{d}_1}{\|\mathbf{d}_{1/2} \times \mathbf{d}_1\|}\right) & , \text{ if } \mathbf{d}_{2/1} = \mathbf{d}_{1/2} \end{cases} . \tag{3.34}$$

where $a_{p_{12}}$, $a_{s_{12}}$ and $a_{c_{12}}$ are as defined in (3.29).

Proof. Given by the construction of the partial solution in Lemmas 1, 2, and 3.

Remark. Notice that there are, in general, two solutions for $\mathbf{R}(\theta_2, \mathbf{n}_2)$, which result in two candidates for $\mathbf{R}_2^1$ that satisfy both (3.1) and (3.2), in the cases when $\mathbf{d}_1 \neq \pm\mathbf{d}_{1/2}$ and $\mathbf{d}_2 \neq \pm\mathbf{d}_{2/1}$.

Interestingly enough, both possible solutions are related to each other and it is convenient to write this result explicitly, since it will be useful later in the characterization. That relation is addressed in the next corollary.

Corollary 2. *In the conditions of Corollary 1, let $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$ represent both solutions for $\mathbf{R}_2^1$. Recall the angle θ_2 defined by* (3.27) *and, without loss of generality, consider that $(\theta_2)_A$ is the angle associated with $\left(\mathbf{R}_2^1\right)_A$ and $(\theta_2)_B$ is the angle associated with $\left(\mathbf{R}_2^1\right)_B$, given by*

$$(\theta_2)_A := \operatorname{atan2}\left(a_{s_{12}}, a_{c_{12}}\right) + \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) \tag{3.35}$$

and

$$(\theta_2)_B := \operatorname{atan2}\left(a_{s_{12}}, a_{c_{12}}\right) - \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) . \tag{3.36}$$

Then, the relation between $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$ is expressed as

$$\left(\mathbf{R}_2^1\right)_A = \mathbf{R}\left(2\arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right), -\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B . \tag{3.37}$$

Proof. Comparing (3.35) and (3.36) allows to write

$$(\theta_2)_A = (\theta_2)_B + 2\arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) . \tag{3.38}$$

Taking into consideration the decomposition (3.3) and the solution (3.34), then $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$ are expressed as

$$\left(\mathbf{R}_2^1\right)_A = \mathbf{R}\left((\theta_2)_A, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \tag{3.39}$$

and

$$\left(\mathbf{R}_2^1\right)_B = \mathbf{R}\left((\theta_2)_B, -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) . \tag{3.40}$$

Substituting (3.38) in (3.39) gives

$$\left(\mathbf{R}_2^1\right)_A = \mathbf{R}\left((\theta_2)_B + 2\arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right), -\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) . \tag{3.41}$$

As shown in Appendix A.3,

$$\mathbf{R}\left(\phi+\lambda,\mathbf{n}\right)=\mathbf{R}\left(\phi,\mathbf{n}\right)\mathbf{R}\left(\lambda,\mathbf{n}\right)=\mathbf{R}\left(\lambda,\mathbf{n}\right)\mathbf{R}\left(\phi,\mathbf{n}\right)\ ,$$

hence (3.41) can be rewritten as

$$\left(\mathbf{R}_2^1\right)_A=\mathbf{R}\left(2\arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2+a_{c_{12}}^2}}\right),-\mathbf{d}_{1/2}\right)\mathbf{R}\left(\left(\theta_2\right)_B,-\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_1,\mathbf{n}_1\right)\ . \tag{3.42}$$

Substituting (3.40) in (3.42) readily yields (3.37). □

3.1.2 Partial solution for $\mathbf{R}_3^1$

The other branch of the problem includes the chief vehicle and vehicle 3. To have both branches assessed, it is also necessary to determine the relative rotation matrix $\mathbf{R}_3^1$. However, the solution for $\mathbf{R}_3^1$ is analogous to the one of $\mathbf{R}_2^1$, since both branches have equivalent information. Therefore, the cases where $\mathbf{R}_3^1$ has an infinite number of solutions, the solution for $\mathbf{R}_3^1$ and the relation between the candidates of $\mathbf{R}_3^1$ are respectively addressed in the next three corollaries.

Corollary 3. *If*

$$\mathbf{d}_1=\pm\mathbf{d}_{1/3}$$

or

$$\mathbf{d}_3=\pm\mathbf{d}_{3/1}\ ,$$

then $\mathbf{R}_3^1$ *has an infinite number of solutions.*

Proof. This proof follows analogously to the proof of Lemma 1 and thus it is omitted.

Corollary 4. *Assume that*

$$\mathbf{d}_1\neq\pm\mathbf{d}_{1/3}$$

and

$$\mathbf{d}_3\neq\pm\mathbf{d}_{3/1}\ .$$

Then $\mathbf{R}_3^1$ *is given by the decomposition*

$$\mathbf{R}_3^1=\mathbf{R}\left(\theta_4,\mathbf{n}_4\right)\mathbf{R}\left(\theta_3,\mathbf{n}_3\right)\ ,$$

which is parameterized by

$$\theta_3=\pi\ ,$$

$$\mathbf{n}_3=\begin{cases}\frac{\mathbf{d}_{3/1}-\mathbf{d}_{1/3}}{\|\mathbf{d}_{3/1}-\mathbf{d}_{1/3}\|} & ,\text{ if }\mathbf{d}_{3/1}\neq\mathbf{d}_{1/3}\\ \frac{\mathbf{S}\left(\mathbf{d}_{1/3}\right)\mathbf{d}_1}{\|\mathbf{S}\left(\mathbf{d}_{1/3}\right)\mathbf{d}_1\|} & ,\text{ if }\mathbf{d}_{3/1}=\mathbf{d}_{1/3}\end{cases}\ ,$$

$$\theta_4=\operatorname{atan2}(a_{s_{13}},a_{c_{13}})\pm\arccos\left(\frac{a_{p_{13}}}{\sqrt{a_{s_{13}}^2+a_{c_{13}}^2}}\right)\ , \tag{3.43}$$

and

$$\mathbf{n}_4=-\mathbf{d}_{1/3}\ ,$$

where

$$\begin{cases} a_{p_{13}} & := \mathbf{d}_1^T(-\mathbf{d}_{1/3})(-\mathbf{d}_{1/3})^T\mathbf{d}_3^* - {}^I\mathbf{d}_1^T\,{}^I\mathbf{d}_3 \\ a_{c_{13}} & := \mathbf{d}_1^T(\mathbf{S}\left(-\mathbf{d}_{1/3}\right)^2)\mathbf{d}_3^* \\ a_{s_{13}} & := \mathbf{d}_1^T(\mathbf{S}\left(-\mathbf{d}_{1/3}\right))\mathbf{d}_3^* \ , \end{cases}$$

with

$$\mathbf{d}_3^* = \mathbf{R}\left(\theta_3, \mathbf{n}_3\right)\mathbf{d}_3 \ .$$

Proof. This result is analogous to Corollary 1 and thus it is omitted.

Corollary 5. *In the conditions of Corollary 4, let* $\left(\mathbf{R}_3^1\right)_C$ *and* $\left(\mathbf{R}_3^1\right)_D$ *represent both solutions for* $\mathbf{R}_3^1$*. Recall the angle* θ_4 *defined by* (3.43) *and, without loss of generality, consider that* $(\theta_4)_C$ *is the angle associated with* $\left(\mathbf{R}_3^1\right)_C$ *and* $(\theta_4)_D$ *is the angle associated with* $\left(\mathbf{R}_3^1\right)_D$*, given by*

$$(\theta_4)_C := \operatorname{atan2}\left(a_{s_{13}}, a_{c_{13}}\right) + \arccos\left(\frac{a_{p_{13}}}{\sqrt{a_{s_{13}}^2 + a_{c_{13}}^2}}\right)$$

and

$$(\theta_4)_D := \operatorname{atan2}\left(a_{s_{13}}, a_{c_{13}}\right) - \arccos\left(\frac{a_{p_{13}}}{\sqrt{a_{s_{13}}^2 + a_{c_{13}}^2}}\right) \ .$$

Then, the relation between $\left(\mathbf{R}_3^1\right)_C$ *and* $\left(\mathbf{R}_3^1\right)_D$ *is expressed as*

$$\left(\mathbf{R}_3^1\right)_C = \mathbf{R}\left(2\arccos\left(\frac{a_{p_{13}}}{\sqrt{a_{s_{13}}^2 + a_{c_{13}}^2}}\right), -\mathbf{d}_{1/3}\right)\left(\mathbf{R}_3^1\right)_D \ . \tag{3.44}$$

Proof. This proof is analogous to the one of Corollary 2 and therefore it is omitted.

3.1.3 Characterization of the partial solution

The solution from Corollary 1 has different outcomes for different situations, which influence the uniqueness of $\mathbf{R}_2^1$. This characterization seeks to describe all these situations and their results. [1]

In the general case, there are two solutions for $\mathbf{R}_2^1$, as it is seen by the duality of θ_2 in (3.27). In the degenerate case, there are infinite solutions, as shown in Lemma 1. However, there are some situations where the partial problem immediately results in a unique solution. Analysing (3.37), it is possible to conclude that the solution is unique when

$$\arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) = \pi\ \mathrm{k}\ ,\ \mathrm{k} \in \mathbb{Z}\ , \tag{3.45}$$

as this condition allows to rewrite (3.37) as

$$\left(\mathbf{R}_2^1\right)_A = \mathbf{R}\left(2\pi\mathrm{k}, -\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\ ,\ k \in \mathbb{Z}\ ,$$

or equivalently,

$$\left(\mathbf{R}_2^1\right)_A = \left(\mathbf{R}_2^1\right)_B \ .$$

[1] This characterization refers to the determination of $\mathbf{R}_2^1$. However, all the conclusions can be taken, by analogy, to the determination of $\mathbf{R}_3^1$

Moreover, notice that (3.45) is verified when

$$\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}} = \pm 1 \ .$$

This situation is discussed in the following lemma.

Lemma 4. *Consider the partial solution given by Corollary 1. A unique solution exists when all the vectors from this problem, expressed in the same frame, (for example* $\mathbf{d}_{1/2}$, $\mathbf{d}_1$, $\mathbf{R}_2^1\mathbf{d}_2$ *and* $\mathbf{R}_2^1\mathbf{d}_{2/1}$*), are coplanar.*

Proof. Notice from (3.1) that $-\mathbf{d}_{1/2} = \mathbf{R}_2^1\mathbf{d}_{2/1}$, so there are only 3 independent vectors to consider and hence, the coplanarity of the framework can be translated algebraically by

$$\mathbf{d}_1^T \ \mathbf{S}\left(-\mathbf{d}_{1/2}\right)\mathbf{R}_2^1\mathbf{d}_2 = 0 \ ,$$

which, using (3.3), becomes

$$\mathbf{d}_1^T \ \mathbf{S}\left(-\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)\mathbf{R}\left(\theta_1, \mathbf{n}_1\right) \ \mathbf{d}_2 = 0 \ . \tag{3.46}$$

Then, substituting (3.30), rewrite (3.46) as

$$\mathbf{d}_1^T \ \mathbf{S}\left(-\mathbf{d}_{1/2}\right)\mathbf{R}\left(\theta_2, \mathbf{n}_2\right) \ \mathbf{d}_2^* = 0 \ .$$

Expand $\mathbf{R}\left(\theta_2, \mathbf{n}_2\right)$ using (3.28) and (3.31), which gives

$$\mathbf{d}_1^T \ \mathbf{S}\left(-\mathbf{d}_{1/2}\right)\left[(-\mathbf{d}_{1/2})(-\mathbf{d}_{1/2})^T - \cos(\theta_2)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^2 - \sin(\theta_2)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\right] \ \mathbf{d}_2^* = 0 \ . \tag{3.47}$$

Observing that

$$\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\left(-\mathbf{d}_{1/2}\right) = \mathbf{0} \ ,$$

and distributing, (3.47) is given as

$$-\cos(\theta_2)\mathbf{d}_1^T \ \mathbf{S}\left(-\mathbf{d}_{1/2}\right)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^2\mathbf{d}_2^* - \sin(\theta_2)\mathbf{d}_1^T \ \mathbf{S}\left(-\mathbf{d}_{1/2}\right)\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\mathbf{d}_2^* = 0 \ ,$$

or equivalently

$$\cos(\theta_2)\mathbf{d}_1^T\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^3\mathbf{d}_2^* = -\sin(\theta_2)\mathbf{d}_1^T\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^2\mathbf{d}_2^* \ . \tag{3.48}$$

Next, from Lemma 25, shown in Appendix B.2, *i.e.*,

$$\mathbf{S}\left(\mathbf{x}\right)^3 = -\mathbf{S}\left(\mathbf{x}\right) \ ,$$

(3.48) can be rewritten as

$$\mathbf{d}_1^T\mathbf{S}\left(-\mathbf{d}_{1/2}\right)\mathbf{d}_2^*\cos(\theta_2) = \mathbf{d}_1^T\mathbf{S}\left(-\mathbf{d}_{1/2}\right)^2\mathbf{d}_2^*\sin(\theta_2) \ . \tag{3.49}$$

Making use of the coefficients (3.29) in (3.49) gives

$$a_{s_{12}}\cos(\theta_2) = a_{c_{12}}\sin(\theta_2) \ ,$$

which is rewritten as

$$\theta_2 = \operatorname{atan2}\left(a_{s_{12}}, a_{c_{12}}\right) + \pi\mathrm{k} \ , \ \mathrm{k} \in \mathbb{Z} \ . \tag{3.50}$$

Comparing (3.50) with (3.27), it is concluded that

$$\arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) = \pi k \quad , \quad k \in \mathbb{Z}$$

which means

$$\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}} = \pm 1 \ .$$

□

Remark. Note that the case with a coplanar vector configuration is a particular case of the general problem that is addressed here. Moreover, in the problem shown in [16] and [17], which have only 1 solution, the vectors not only are coplanar, but also have a specific direction, in order to form a triangle. Therefore, an immediate unique solution can be found for a wider range of scenarios, just using the partial solution, than in these particular cases, hence adding to the contributions of these works.

3.2 Solution and characterization: inertial disambiguation

As it was shown, for the general case of the partial problem, there are two solutions. So, $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$, are not determined uniquely. However, only one of the candidate matrices for each branch is correct by construction, considering non degenerate cases.

A method to disambiguate the problem was deduced and it consists in, firstly, using the partial solution candidates, for $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$, to find candidates for the inertial rotation matrix, $\mathbf{R}_1^I$.

After having found the different $\mathbf{R}_1^I$ candidates, the idea is to compare the results from one branch with the results from the other, since the correct results from the partial problems will yield the same $\mathbf{R}_1^I$, hence the final solution will be reached. This process can be visualized in Fig. 3.3. Some special cases may arise, which will be considered in the characterization of the solution.

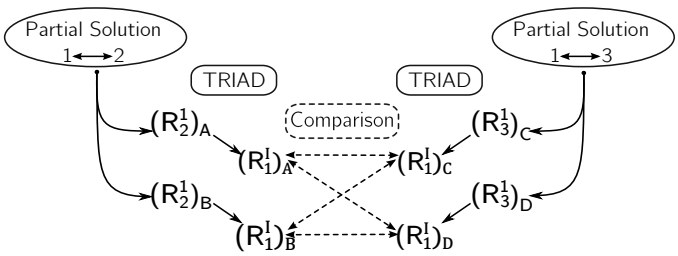


Figure 3.3: Disambiguation procedure

A new notation is now introduced to deal with the origin of the different candidates of the problem: the two candidates for the $\mathbf{R}_2^1$ are denoted by $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$ and the two candidates for $\mathbf{R}_3^1$ are expressed as $\left(\mathbf{R}_3^1\right)_C$ and $\left(\mathbf{R}_3^1\right)_D$. Furthermore, each candidate for $\mathbf{R}_1^I$ has the same subscript as the relative rotation matrix candidate that was used in its determination, that is, $\left(\mathbf{R}_1^I\right)_A$ is determined using $\left(\mathbf{R}_2^1\right)_A$, $\left(\mathbf{R}_1^I\right)_B$ is determined using $\left(\mathbf{R}_2^1\right)_B$, $\left(\mathbf{R}_1^I\right)_C$ is determined using $\left(\mathbf{R}_3^1\right)_C$ and $\left(\mathbf{R}_1^I\right)_D$ is determined using $\left(\mathbf{R}_3^1\right)_D$. This notation is used whenever the candidates need to be distinguished.
The problem of finding $\mathbf{R}_1^I$ is addressed using convenient relations of the problem, that are given by

$$^I\mathbf{d}_1 = \mathbf{R}_1^I \ \mathbf{d}_1 \ , \tag{3.51}$$

$$^I\mathbf{d}_2 = \mathbf{R}_1^I \ \mathbf{R}_2^1 \ \mathbf{d}_2 \tag{3.52}$$

and

$$ {}^{I}\mathbf{d}_3 = \mathbf{R}_1^I \, \mathbf{R}_3^1 \, \mathbf{d}_3 \ . \tag{3.53} $$

Firstly, the uniqueness of $\mathbf{R}_1^I$ is addressed in the next lemmas. As for the partial problem, this gives some hints about the solution of the problem.

Lemma 5. *Let* $\mathbf{R}_1^I$ *be the rotation matrix that satisfies both* (3.51) *and* (3.52)*. If*

$$ {}^{I}\mathbf{d}_1 = \pm {}^{I}\mathbf{d}_2 \tag{3.54} $$

then $\mathbf{R}_1^I$ *has an infinite number of solutions.*

Proof. Fix $\mathbf{R}_1^I$ such that both (3.51) and (3.52) are verified and consider a new candidate defined by

$$ \mathbf{A}(\theta) := \mathbf{R}\left(\theta, {}^{I}\mathbf{d}_1\right) \mathbf{R}_1^I \ , \tag{3.55} $$

where $\theta \in \mathbb{R}$ is an arbitrary angle. It will be shown by direct computation that this arbitrary rotation satisfies both (3.51) and (3.52). To that end, compute

$$ \mathbf{A}(\theta)\, \mathbf{d}_1 = \mathbf{R}\left(\theta, {}^{I}\mathbf{d}_1\right) \mathbf{R}_1^I \mathbf{d}_1 \ , $$

which, using (3.51) becomes

$$ \mathbf{A}(\theta)\, \mathbf{d}_1 = \mathbf{R}\left(\theta, {}^{I}\mathbf{d}_1\right) {}^{I}\mathbf{d}_1 \ . \tag{3.56} $$

Next, Lemma 20, in Appendix A.2, is applied to (3.56), which gives

$$ \mathbf{A}(\theta)\, \mathbf{d}_1 = {}^{I}\mathbf{d}_1 \ . $$

Therefore, any θ satisfies (3.51). Next, take the right side of (3.52) and substitute $\mathbf{R}_1^I$ by $\mathbf{A}(\theta)$ which substituting (3.55), yields

$$ \mathbf{A}(\theta)\, \mathbf{R}_2^1 \mathbf{d}_2 = \mathbf{R}\left(\theta, {}^{I}\mathbf{d}_1\right) \mathbf{R}_1^I \mathbf{R}_2^1 \mathbf{d}_2 \ . \tag{3.57} $$

Applying (3.52) in (3.57) results in

$$ \mathbf{A}(\theta)\, \mathbf{R}_2^1 \mathbf{d}_2 = \mathbf{R}\left(\theta, {}^{I}\mathbf{d}_1\right) {}^{I}\mathbf{d}_2 \ . $$

Without loss of generality the assumption with positive sign, *i.e.*, ${}^{I}\mathbf{d}_1 = {}^{I}\mathbf{d}_2$, is now used, which gives

$$ \mathbf{A}(\theta)\, \mathbf{R}_2^1 \mathbf{d}_2 = \mathbf{R}\left(\theta, {}^{I}\mathbf{d}_1\right) {}^{I}\mathbf{d}_1 \ . \tag{3.58} $$

From Lemma 20, in Appendix A.2, (3.58) is given as

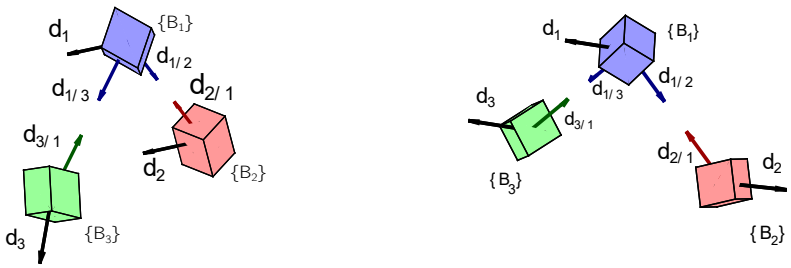
$$ \mathbf{A}(\theta)\, \mathbf{R}_2^1 \mathbf{d}_2 = {}^{I}\mathbf{d}_1 \ , $$

which using ${}^{I}\mathbf{d}_1 = {}^{I}\mathbf{d}_2$ again, gives

$$ \mathbf{A}(\theta)\, \mathbf{R}_2^1 \mathbf{d}_2 = {}^{I}\mathbf{d}_2 \ . $$

This concludes the proof, since there are infinite possibilities for $\mathbf{R}_1^I$ that satisfy both (3.51) and (3.52), when ${}^{I}\mathbf{d}_1 = \pm {}^{I}\mathbf{d}_2$. □

Remark. Note that (3.54) implies that partial problem's vectors, of the the branch with vehicles 1 and



(a) ${}^I\mathbf{d}_1 = {}^I\mathbf{d}_2$ (b) ${}^I\mathbf{d}_1 = {}^I\mathbf{d}_3$

Figure 3.4: Degenerate cases in the determination of the inertial attitude

2, are coplanar. Which means that (3.54) is equivalent to $\mathbf{R}_2^1\mathbf{d}_2 = \mathbf{d}_1$, since both candidates for $\mathbf{R}_2^1$ are equal.

Lemma 6. *Let* $\mathbf{R}_1^I$ *be the rotation matrix that satisfies both* (3.51) *and* (3.53)*. If*

$$ {}^I\mathbf{d}_1 = \pm {}^I\mathbf{d}_3 \tag{3.59} $$

then there are infinite solutions for $\mathbf{R}_1^I$*.*

Proof. This proof is similar to the one of Lemma 5 and thus it is omitted.

Remark. Lemmas 5 and 6 are cases where there is an undetermined degree of freedom. Fig. 3.4 shows the situations considered in Lemmas 5 and 6. Furthermore, analogously to Lemma 5, (3.59) implies that the partial problem's vectors, of the the branch with vehicles 1 and 3, are coplanar. Which means that (3.59) is equivalent to $\mathbf{R}_3^1\mathbf{d}_3 = \mathbf{d}_1$, since both candidates for $\mathbf{R}_3^1$ are equal.

3.2.1 Determination of chief's inertial attitude

In the general case, the method described can disambiguate which candidates are correct or not. For that matter, notice that the relations to determine $\mathbf{R}_1^I$ only consider two vectors expressed in two different frames for each branch, consequently, these two vectors are coplanar. Thus, each candidate for $\mathbf{R}_2^1$ and for $\mathbf{R}_3^1$ gives a single candidate for $\mathbf{R}_1^I$, considering that the problem is not degenerate.

Moreover, it corresponds to the TRIAD framework and hence, a widely known solution can be applied to find $\mathbf{R}_1^I$. Therefore, $\left(\mathbf{R}_1^I\right)_A$, $\left(\mathbf{R}_1^I\right)_B$, $\left(\mathbf{R}_1^I\right)_C$ and $\left(\mathbf{R}_1^I\right)_D$ are obtained using the TRIAD algorithm [8]. A brief description of this algorithm is now given to better illustrate how it works.

Consider the rotation matrix $\mathbf{R}_R^B$, that transforms a vector in a reference frame, $\{\mathcal{R}\}$, to a body-fixed frame, $\{\mathcal{B}\}$. Consider as well, two non-parallel unit vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, in the reference frame, and their corresponding vectors in the body-fixed frame, $\mathbf{b}_1$ and $\mathbf{b}_2$. The rotation matrix is such that

$$ \mathbf{R}_R^B\mathbf{r}_i = \mathbf{b}_i \ , \ \text{for } i = 1, 2 \ . \tag{3.60} $$

Notice that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are corresponding orthonormal right-handed triads of unit vectors, the first in the reference frame and the second in the body-fixed frame, then the rotation matrix

that verifies $\mathbf{R}_r^b \mathbf{v}_i = \mathbf{w}_i$ for $i = 1, 2, 3$, is given directly by [7]

$$\mathbf{R}_R^B \equiv \sum_{i=1}^{3} \mathbf{w}_i \mathbf{v}_i^T \ . \tag{3.61}$$

The algorithm reconstructs an orthonormal right-handed triad of vectors in the reference frame and another in the body-fixed frame from two non-parallel vectors in each frame, to meet the conditions to apply (3.61). The triad for the vectors of (3.60) is built using the cross product characteristics. For the body-fixed frame it is given by

$$\left\{ \mathbf{b}_1, \ \frac{(\mathbf{b}_1 \times \mathbf{b}_2)}{\| (\mathbf{b}_1 \times \mathbf{b}_2) \|}, \ \mathbf{b}_1 \times \left(\frac{(\mathbf{b}_1 \times \mathbf{b}_2)}{\| (\mathbf{b}_1 \times \mathbf{b}_2) \|} \right) \right\} \ ,$$

whereas for the reference frame it is given by

$$\left\{ \mathbf{r}_1, \ \frac{(\mathbf{r}_1 \times \mathbf{r}_2)}{\| (\mathbf{r}_1 \times \mathbf{r}_2) \|}, \ \mathbf{r}_1 \times \left(\frac{(\mathbf{r}_1 \times \mathbf{r}_2)}{\| (\mathbf{r}_1 \times \mathbf{r}_2) \|} \right) \right\} \ .$$

Thus, the TRIAD algorithm is given by

$$\mathbf{R}_R^B = \mathbf{b}_1 \mathbf{r}_1^T + (\mathbf{b}_1 \times \mathbf{b}_x) (\mathbf{r}_1 \times \mathbf{r}_x)^T + \mathbf{b}_x \mathbf{r}_x^T \ , \tag{3.62}$$

with

$$\mathbf{b}_x := \frac{\mathbf{b}_1 \times \mathbf{b}_2}{\|\mathbf{b}_1 \times \mathbf{b}_2\|} \quad \text{and} \quad \mathbf{r}_x := \frac{\mathbf{r}_1 \times \mathbf{r}_2}{\|\mathbf{r}_1 \times \mathbf{r}_2\|} \ .$$

Remark. Note that the cases where there are infinite solutions from Lemmas 5 and 6 are not applicable to the TRIAD, because it would mean that either $\mathbf{b}_1$ and $\mathbf{b}_2$ or $\mathbf{r}_1$ and $\mathbf{r}_2$ are parallel.

The direct application of (3.62) to the corresponding inertial vectors of the chief and each of the deputies, in the inertial frame and in the chief body-fixed frame, allows to find $\mathbf{R}_1^I$ candidates using either $\mathbf{R}_2^1$ or $\mathbf{R}_3^1$. Using $\mathbf{R}_2^1$ the corresponding vectors are $\mathbf{d}_1$ and ${}^I\mathbf{d}_1$, and also $\mathbf{R}_2^1 \mathbf{d}_2$ and ${}^I\mathbf{d}_2$. Therefore, the application of (3.62) gives

$$\mathbf{R}_1^I = {}^I\mathbf{d}_1 \mathbf{d}_1^T + \left({}^I\mathbf{d}_1 \times \frac{{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_2}{\|{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_2\|} \right) \left(\mathbf{d}_1 \times \frac{\mathbf{d}_1 \times \mathbf{R}_2^1 \mathbf{d}_2}{\|\mathbf{d}_1 \times \mathbf{R}_2^1 \mathbf{d}_2\|} \right)^T + \frac{{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_2}{\|{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_2\|} \left(\frac{\mathbf{d}_1 \times \mathbf{R}_2^1 \mathbf{d}_2}{\|\mathbf{d}_1 \times \mathbf{R}_2^1 \mathbf{d}_2\|} \right)^T \ ,$$

where $\left(\mathbf{R}_1^I\right)_A$ and $\left(\mathbf{R}_1^I\right)_B$ are, respectively, obtained by replacing $\left(\mathbf{R}_2^1\right)$ with $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$. On the other hand, using $\mathbf{R}_3^1$ the corresponding vectors are $\mathbf{d}_1$ and ${}^I\mathbf{d}_1$, and also $\mathbf{R}_3^1 \mathbf{d}_3$ and ${}^I\mathbf{d}_3$. So, using (3.62) gives

$$\mathbf{R}_1^I = {}^I\mathbf{d}_1 \mathbf{d}_1^T + \left({}^I\mathbf{d}_1 \times \frac{{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_3}{\|{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_3\|} \right) \left(\mathbf{d}_1 \times \frac{\mathbf{d}_1 \times \mathbf{R}_3^1 \mathbf{d}_3}{\|\mathbf{d}_1 \times \mathbf{R}_3^1 \mathbf{d}_3\|} \right)^T + \frac{{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_3}{\|{}^I\mathbf{d}_1 \times {}^I\mathbf{d}_3\|} \left(\frac{\mathbf{d}_1 \times \mathbf{R}_3^1 \mathbf{d}_3}{\|\mathbf{d}_1 \times \mathbf{R}_3^1 \mathbf{d}_3\|} \right)^T \ ,$$

where $\left(\mathbf{R}_1^I\right)_C$ and $\left(\mathbf{R}_1^I\right)_D$ are, respectively, obtained by replacing $\left(\mathbf{R}_3^1\right)$ with $\left(\mathbf{R}_3^1\right)_C$ and $\left(\mathbf{R}_3^1\right)_D$.

An alternative method based on the one used for the partial problem is described in Appendix D. Contrary to what happens in the determination of $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$, there is only one solution for each $\mathbf{R}_1^I$ candidate, because the measurements used are coplanar, when expressed in the same frame. Therefore, the alternative method is in accordance with the TRIAD method results.

All four candidates for $\mathbf{R}_1^I$ are related to each other and that is evident in the construction of $\mathbf{R}_1^I$ with the alternative method in Appendix D, because these can be decomposed into a product of two matrices, where one of them is common to all four decompositions. For the sake of representation simplicity,

consider the notation in (D.1) and the angles defined by

$$\theta_A := \operatorname{atan2}\left({}^I\mathbf{d}_2^T \mathbf{S}\left({}^I\mathbf{d}_1\right) \mathbf{R}^* \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2, {}^I\mathbf{d}_2^T \mathbf{S}\left({}^I\mathbf{d}_1\right)^2 \mathbf{R}^* \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right) \ ; \tag{3.63a}$$

$$\theta_B := \operatorname{atan2}\left({}^I\mathbf{d}_2^T \mathbf{S}\left({}^I\mathbf{d}_1\right) \mathbf{R}^* \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2, {}^I\mathbf{d}_2^T \mathbf{S}\left({}^I\mathbf{d}_1\right)^2 \mathbf{R}^* \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right) \ ; \tag{3.63b}$$

$$\theta_C := \operatorname{atan2}\left({}^I\mathbf{d}_3^T \mathbf{S}\left({}^I\mathbf{d}_1\right) \mathbf{R}^* \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3, {}^I\mathbf{d}_3^T \mathbf{S}\left({}^I\mathbf{d}_1\right)^2 \mathbf{R}^* \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right) \ ; \tag{3.63c}$$

$$\theta_D := \operatorname{atan2}\left({}^I\mathbf{d}_3^T \mathbf{S}\left({}^I\mathbf{d}_1\right) \mathbf{R}^* \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3, {}^I\mathbf{d}_3^T \mathbf{S}\left({}^I\mathbf{d}_1\right)^2 \mathbf{R}^* \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\right) \ . \tag{3.63d}$$

Next, the relation between some useful candidates for $\mathbf{R}_1^I$ is described in the following lemmas.

Lemma 7. *The relation between the inertial rotation matrices* $\left(\mathbf{R}_1^I\right)_A$ *and* $\left(\mathbf{R}_1^I\right)_C$ *is given by*

$$\left(\mathbf{R}_1^I\right)_C = \mathbf{R}\left(\theta_C - \theta_A, {}^I\mathbf{d}_1\right) \left(\mathbf{R}_1^I\right)_A \ . \tag{3.64}$$

Proof. Consider the alternative determination of $\mathbf{R}_1^I$ described in Appendix D and take (D.2) and (D.3), which can both be, respectively, rearranged as is given by

$$\mathbf{R}^* = \mathbf{R}\left(\theta_A, {}^I\mathbf{d}_1\right)^T \left(\mathbf{R}_1^I\right)_A \tag{3.65a}$$

$$\mathbf{R}^* = \mathbf{R}\left(\theta_C, {}^I\mathbf{d}_1\right)^T \left(\mathbf{R}_1^I\right)_C \ . \tag{3.65b}$$

Comparing (3.65a) with (3.65b) gives

$$\mathbf{R}\left(\theta_C, {}^I\mathbf{d}_1\right)^T \left(\mathbf{R}_1^I\right)_C = \mathbf{R}\left(\theta_A, {}^I\mathbf{d}_1\right)^T \left(\mathbf{R}_1^I\right)_A \ ,$$

which is rewritten as

$$\left(\mathbf{R}_1^I\right)_C = \mathbf{R}\left(\theta_C, {}^I\mathbf{d}_1\right) \mathbf{R}\left(\theta_A, {}^I\mathbf{d}_1\right)^T \left(\mathbf{R}_1^I\right)_A \ . \tag{3.66}$$

From Lemma 21 in Appendix A.3 and having in mind that the transpose of a rotation is equal to that rotation with a symmetric angle instead, it is concluded that (3.66) is equivalent to (3.64). □

Lemma 8. *The relation between the inertial rotation matrices* $\left(\mathbf{R}_1^I\right)_B$ *and* $\left(\mathbf{R}_1^I\right)_D$ *is given by*

$$\left(\mathbf{R}_1^I\right)_D = \mathbf{R}\left(\theta_D - \theta_B, {}^I\mathbf{d}_1\right) \left(\mathbf{R}_1^I\right)_B \ .$$

Proof. This proof is similar to the one from Lemma 7 and hence, it is omitted.

3.2.2 Comparison

Consider that the problem presented is not degenerate, then from the four $\mathbf{R}_1^I$ candidates there is at least a pair of equal matrices, that emerge from the two different branches. This situation is inferred by construction of the solution, since there is a matrix coming from each branch that corresponds to the real rotation between those two reference frames, in a non degenerate situation. Therefore, the solution for the overall problem is reached by comparing the $\mathbf{R}_1^I$ candidates. This comparison is made resorting to the rotation defined by

$$\tilde{\mathbf{R}_1^I}\left(\theta, \mathbf{x}\right) := \left(\mathbf{R}_1^I\right)_X \left(\mathbf{R}_1^I\right)_Y^T \tag{3.67}$$

This rotation is equal to the identity matrix when both $\left(\mathbf{R}_1^I\right)_X$ and $\left(\mathbf{R}_1^I\right)_Y$ are equal. This means that

$$\theta = 0$$

is equivalent to

$$\tilde{\mathbf{R}}_1^I(\theta, \mathbf{x}) = \mathbf{I} \ .$$

Therefore, the absolute value of θ is used as the comparison parameter that gives the proximity between each pair of $\mathbf{R}_1^I$ candidates. Using the trace of (3.67) to find θ, since the trace of a square matrix is the sum of its eigenvalues [21], which in the case of this rotation matrix is given as

$$\begin{aligned} \operatorname{trace}\left(\tilde{\mathbf{R}}_1^I(\theta, \mathbf{x})\right) &= 1 + \mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta} \\ &= 1 + 2\cos(\theta) \ , \end{aligned}$$

which, defining $\phi = |\theta|$ and rearranging, is expressed as

$$\phi = \left| \arccos\left(\frac{\operatorname{trace}\left(\tilde{\mathbf{R}}_1^I(\theta, \mathbf{x})\right) - 1}{2} \right) \right| \ . \tag{3.68}$$

The information of the axis is not used, because $\tilde{\mathbf{R}}_1^I(\theta, \mathbf{x})$ is expected to be approximately equal to the identity matrix which is fully defined by $\phi = 0$. Moreover, by construction, at least one of the four comparisons has $\phi = 0$, when the problem is not degenerate.

In general, there are only two matrices that match. One is determined by one of the $\mathbf{R}_2^1$ candidates and the other from one of the $\mathbf{R}_3^1$ candidates. Thus, it is concluded that those two identical $\mathbf{R}_1^I$ are the correct inertial rotation matrix, furthermore, the candidates for $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$, that were involved in the construction of the correct $\mathbf{R}_1^I$, are also the correct ones. Nonetheless, there are certain situations where the outcome is different, as it will be discussed in the next section.

3.2.3 Characterization of the Disambiguation

There are some formation configurations that do not give an unique solution. These either have infinite solutions or make the disambiguation impossible. The different situations are described afterwards to give an understanding about the different possibilities of the solution of the problem.

In the general case, there is only one pair of identical candidates for $\mathbf{R}_1^I$, which occurs by construction. This pair can be given by $\left(\mathbf{R}_1^I\right)_A = \left(\mathbf{R}_1^I\right)_C$, $\left(\mathbf{R}_1^I\right)_A = \left(\mathbf{R}_1^I\right)_D$, $\left(\mathbf{R}_1^I\right)_B = \left(\mathbf{R}_1^I\right)_C$ or $\left(\mathbf{R}_1^I\right)_B = \left(\mathbf{R}_1^I\right)_D$, because each pair has a candidate that was found using $\mathbf{R}_2^1$ and the other that was found using $\mathbf{R}_3^1$. This general case also includes the situations where one or both branches' vectors are coplanar. As Lemma 4 shows, the coplanar configuration case has a unique solution to the partial problem, thus the one candidate for $\mathbf{R}_1^I$ originated in that branch corresponds, by construction, to the real value of $\mathbf{R}_1^I$.[2]

In the case of degenerate formations, in the determination of $\mathbf{R}_2^1$, $\mathbf{R}_3^1$ or $\mathbf{R}_1^I$, generally a unique solution cannot be found, because there are, respectively, infinite solutions for $\mathbf{R}_2^1$, $\mathbf{R}_3^1$ or $\mathbf{R}_1^I$, as shown in Lemmas 1, 5 and 6 and Corollary 3. However, in some specific cases it is still possible to find some or even all of the rotation matrices of the problem uniquely. For example, when both branches' vectors are coplanar, which means both have an unique solution, and only one of the branches is degenerate, when determining $\mathbf{R}_1^I$, which allows to use the other branch to find the correct $\mathbf{R}_1^I$. The measurements of this case follow, as an example, the relations given by

$$\begin{cases} {}^I\mathbf{d}_2 = \pm {}^I\mathbf{d}_1 \\ {}^I\mathbf{d}_3 \neq \pm {}^I\mathbf{d}_1 \\ \mathbf{d}_1^T \ \mathbf{S}\left(-\mathbf{d}_{1/3}\right) \mathbf{R}_3^1 \mathbf{d}_3 = 0 \end{cases} \ .$$

[2] By abuse of language the coplanar configuration case of the partial solution shown in Lemma 4 is referred both as having only one solution or as having two identical solutions.

Finally, there is a case where there are two pairs of identical candidates for $\mathbf{R}_1^I$, moreover, each pair is different from the other, otherwise the partial problems' vectors would be coplanar, which has already been shown to have an unique solution. In these cases, however, the disambiguation of the solution is not possible, so a unique solution cannot be found. The only two possibilities to have an ambiguous double solution are given as

$$\begin{cases} \left(\mathbf{R}_1^I\right)_A = \left(\mathbf{R}_1^I\right)_D \\ \left(\mathbf{R}_1^I\right)_B = \left(\mathbf{R}_1^I\right)_C \\ \left(\mathbf{R}_1^I\right)_A \neq \left(\mathbf{R}_1^I\right)_B \end{cases} \quad \text{or} \quad \begin{cases} \left(\mathbf{R}_1^I\right)_A = \left(\mathbf{R}_1^I\right)_C \\ \left(\mathbf{R}_1^I\right)_B = \left(\mathbf{R}_1^I\right)_D \\ \left(\mathbf{R}_1^I\right)_A \neq \left(\mathbf{R}_1^I\right)_B \end{cases} .$$

This only happens in specific cases, which are determined afterwards. Before addressing this situation, for the sake of simplicity of representation, the rotation angles from (3.37) and (3.44) are defined as

$$\theta_{12} := 2 \arccos\left(\frac{a_{p_{12}}}{\sqrt{a_{s_{12}}^2 + a_{c_{12}}^2}}\right) \tag{3.69a}$$

and

$$\theta_{13} := 2 \arccos\left(\frac{a_{p_{13}}}{\sqrt{a_{s_{13}}^2 + a_{c_{13}}^2}}\right) . \tag{3.69b}$$

Ambiguous double solution: necessary and sufficient condition

First, the rotation that aligns the vector $\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2$ with $\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$ is described by a rotation about $\mathbf{d}_1$. This rotation is useful because it gives one more constraint to these vectors and is related to the conditions associated with the ambiguous double solution. Analogously, the rotation of $\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3$ to $\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3$ is also described by a rotation about $\mathbf{d}_1$. The definition and existence of such rotation is then addressed by the following lemma.

Lemma 9. *Let $\lambda_2 \in \mathbb{R}$ represent an angle of a rotation, recall the relation between $\left(\mathbf{R}_2^1\right)_A$ and $\left(\mathbf{R}_2^1\right)_B$, and the respective conditions described in Corollary 2. Recall as well the angle θ_{12} defined in* (3.69a). *Then*

$$\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\theta_{12}, -\mathbf{d}_{1/2}\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 . \tag{3.70}$$

Proof. Recall that both candidates for $\mathbf{R}_2^1$ are obtained applying (3.2). Hence,

$$\mathbf{d}_1^T \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{d}_1^T \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = {}^I\mathbf{d}_1^T \, {}^I\mathbf{d}_2 ,$$

which, from (3.37), is expressed as

$$\mathbf{d}_1^T \mathbf{R}\left(\theta_{12}, -\mathbf{d}_{1/2}\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{d}_1^T \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = {}^I\mathbf{d}_1^T \, {}^I\mathbf{d}_2 .$$

The inner product is represented as $\mathbf{a}^T\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\left(\hat{\mathbf{ab}}\right)$, because the inner product space of the measurements is real [22]. Therefore, introducing α and $\beta \in \mathbb{R}$, where α is the angle between $\mathbf{d}_1$ and $\mathbf{R}\left(\theta_{12}, -\mathbf{d}_{1/2}\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$, and β is the angle between $\mathbf{d}_1$ and $\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$, gives

$$\|\mathbf{d}_1\|\|\mathbf{R}\left(\theta_{12}, -\mathbf{d}_{1/2}\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\| \cos(\alpha) = \|\mathbf{d}_1\|\| \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\| \cos(\beta) ,$$

which, because the vectors involved are unit vectors, is given as

$$\cos(\alpha) = \cos(\beta) .$$

It is concluded that there is a rotation about $\mathbf{d}_1$ that takes $\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$ to $\mathbf{R}\left(\theta_{12}, \mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$, because the absolute value of the angle between each of them and $\mathbf{d}_1$ is the same, *i.e.*, $|\alpha| = |\beta|$. □

Remark. Geometrically this lemma can be seen as both $\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2$ and $\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$ being in the same circumference normal to $\mathbf{d}_1$. Notice that Fig. 3.5 shows α, β and λ_2, evidencing that $|\alpha| = |\beta|$.

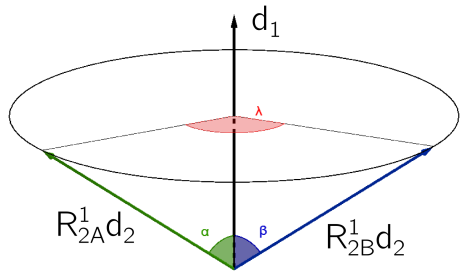


Figure 3.5: Representation of $|\alpha|$ and $|\beta|$ along with λ_2.

Lemma 10. *Let $\lambda_3 \in \mathbb{R}$ represent an angle of a rotation, recall the relation between $\left(\mathbf{R}_3^1\right)_C$ and $\left(\mathbf{R}_3^1\right)_D$, and the respective conditions described in Corollary 5. Recall as well the angle θ_{13} defined in* (3.69b)*. Then*

$$\mathbf{R}\left(\lambda_3, \mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 = \mathbf{R}\left(\theta_{13}, -\mathbf{d}_{1/3}\right)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ . \tag{3.71}$$

Proof. This proof is similar to the one of Lemma 9, thus it is omitted.

It is not enough to ensure that λ_2 and λ_3 exist. It is also convenient to have some insight about the parameters that influence these angles. Hence, the values of λ_2 and λ_3 are determined afterwards.

Lemma 11. *Consider λ_2 and the conditions defined in Lemma 9. Furthermore, consider that the determination of $\mathbf{R}_1^I$ is not degenerate, which means*

$$\mathbf{R}_2^1 \mathbf{d}_2 \neq \pm \mathbf{d}_1 \ . \tag{3.72}$$

Then

$$\lambda_2 = \operatorname{atan2}\left(a_{s_{\lambda_2}}, a_{c_{\lambda_2}}\right) + \pi \ , \tag{3.73}$$

where

$$\begin{cases} a_{s_{\lambda_2}} := \left[\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \\ a_{c_{\lambda_2}} := \left[\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \end{cases} . \tag{3.74}$$

Proof. This proof is constructed rewriting (3.70) as an inner product, which yields

$$\left(\mathbf{R}\left(\theta_{12}, -\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = 1 \ .$$

Then, $\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)$ is expanded using (3.31), which gives

$$\left(\mathbf{R}\left(\theta_{12}, -\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \left[\mathbf{d_1}\mathbf{d_1}^T - \cos(\lambda_2)\mathbf{S}\left(\mathbf{d}_1\right)^2 - \sin(\lambda_2)\mathbf{S}\left(\mathbf{d}_1\right)\right]\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = 1 \ ,$$

or equivalently, from (3.37),

$$\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \left[\mathbf{d_1}\mathbf{d_1}^T - \cos(\lambda_2)\mathbf{S}\left(\mathbf{d}_1\right)^2 - \sin(\lambda_2)\mathbf{S}\left(\mathbf{d}_1\right)\right]\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = 1 \ . \tag{3.75}$$

Using the distributive property, (3.75) is given as

$$\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{d_1}\mathbf{d_1}^T \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 - \cos(\lambda_2)\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2 \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 - \sin(\lambda_2)\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = 1 \ ,$$

or equivalently,

$$\cos(\lambda_2)\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2 \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 + \sin(\lambda_2)\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{d_1}\mathbf{d_1}^T \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 - 1 \ . \quad (3.76)$$

From (3.2) and recalling the commutative property of the real inner product space [22], it follows that by construction of the problem $\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{d}_1 = \left(\mathbf{d}_1\right)^T \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \left(\mathbf{d}_1\right)^T \mathbf{R}_2^1\mathbf{d}_2$, hence (3.76) is expressed as

$$\cos(\lambda_2)\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2 \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 + \sin(\lambda_2)\left(\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \left(\mathbf{d_1}^T\mathbf{R}_2^1\mathbf{d}_2\right)^2 - 1 \ . \quad (3.77)$$

Considering the coefficients in (3.74) and also the coeffiecient given as

$$a_{p_{\lambda_2}} = \left(\mathbf{d_1}^T\mathbf{R}_2^1\mathbf{d}_2\right)^2 - 1 \ , \quad (3.78)$$

then (3.77) is rewritten as

$$\cos(\lambda_2)a_{c_{\lambda_2}} + \sin(\lambda_2)a_{s_{\lambda_2}} = a_{p_{\lambda_2}} \ . \quad (3.79)$$

Recall that the solution for this trigonometric equation is addressed in appendix C, by Lemma 29, which applied to (3.79) gives

$$\lambda_2 = \operatorname{atan2}(a_{s_{\lambda_2}}, a_{c_{\lambda_2}}) \pm \arccos\left(\frac{a_{p_{\lambda_2}}}{\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2}}\right) \ .$$

Next, it is shown that $\arccos\left(\frac{a_{p_{\lambda_2}}}{\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2}}\right) = \pi$, by taking $\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2}$ and expanding the coefficients using (3.74), which gives

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \sqrt{\left(\left[(\mathbf{R}_2^1)_A \mathbf{d}_2\right]^T \mathbf{S}\left(\mathbf{d}_1\right)(\mathbf{R}_2^1)_B \mathbf{d}_2\right)^2 + \left(\left[(\mathbf{R}_2^1)_A \mathbf{d}_2\right]^T \mathbf{S}\left(\mathbf{d}_1\right)^2(\mathbf{R}_2^1)_B \mathbf{d}_2\right)^2} \ . \quad (3.80)$$

Substituting (3.70) in (3.80) gives

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \left[\left(\left(\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^2 + \left(\left(\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^2\right]^{\frac{1}{2}} \ . \quad (3.81)$$

Expanding both squared terms of (3.81) gives

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \Big[\left(\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \left(\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$$
$$+ \left(\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \left(\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\Big]^{\frac{1}{2}} .$$

Next, because the inner product in a real vector space is commutative [22], it follows that

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \Big[\left(\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$$
$$+ \left(\mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \left(\mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\Big]^{\frac{1}{2}} ,$$

or, equivalently,

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \Big[\mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)^T \mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$$
$$+ \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \left(\mathbf{S}\left(\mathbf{d}_1\right)^2\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\Big]^{\frac{1}{2}} . \quad (3.82)$$

By definition $\mathbf{S}\left(\mathbf{d}_1\right)^T = -\mathbf{S}\left(\mathbf{d}_1\right)$, hence (3.82) is given as

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \Big[-\mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)^T \mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$$
$$+ \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2 \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\Big]^{\frac{1}{2}} ,$$

which, rearranging, is rewritten as

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \Big[\mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)^T \Big(-\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{S}\left(\mathbf{d}_1\right)$$
$$+ \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\Big) \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\Big]^{\frac{1}{2}} . \quad (3.83)$$

From Lemma 35, in Appendix E.2, (3.83) is equivalent to

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \sqrt{\mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2} , \quad (3.84)$$

Then, from Lemma 24, where it is shown that $\mathbf{S}\left(\mathbf{d}_1\right)^2 = \mathbf{d}_1\mathbf{d}_1^T - \mathbf{I}$, which substituting in (3.84) gives

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \sqrt{\mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \left(\mathbf{d}_1\mathbf{d}_1^T - \mathbf{I}\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_B{}^T \left(\mathbf{d}_1\mathbf{d}_1^T - \mathbf{I}\right)\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2} . \quad (3.85)$$

Expanding (3.85) and considering that $\mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_B^T \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = 1$, yields

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \sqrt{\left(\mathbf{d}_2^T \left(\left(\mathbf{R}_2^1\right)_B\right)^T \mathbf{d}_1\mathbf{d}_1^T \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right)^2 - 2\left(\mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_B^T \mathbf{d}_1\mathbf{d}_1^T \mathbf{R}_2^1 \mathbf{d}_2\right) + 1} ,$$

which, since all the terms are scalar, is the square of the difference. Therefore, it is given as

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \sqrt{\left(\mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_B^T \mathbf{d}_1\mathbf{d}_1^T \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 - 1\right)^2} ,$$

which, simplifying, is expressed as

$$\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2} = \left| \left(\mathbf{d}_1^T \left(\mathbf{R}_2^1 \right)_B \mathbf{d}_2 \right)^2 - 1 \right| \ . \tag{3.86}$$

Notice that (3.86) is equal to $\left| a_{p_{\lambda_2}} \right|$, because $\mathbf{d}_1^T \left(\mathbf{R}_2^1 \right)_B \mathbf{d}_2 = \mathbf{d}_1^T \mathbf{R}_2^1 \mathbf{d}_2$ by construction. Furthermore, $a_{p_{\lambda_2}} \leq 0$, since it is defined as the difference between the square of the inner product of two unit vectors and one, as it is seen from (3.78). Also, because the determination of $\mathbf{R}_1^I$ is not degenerate, as given by (3.72), $a_{p_{\lambda_2}}$ is in the interval $[-1, 0[$. Hence, it follows that

$$\frac{a_{p_{\lambda_2}}}{\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2}} = \frac{a_{p_{\lambda_2}}}{\left| a_{p_{\lambda_2}} \right|} = -1 \ ,$$

therefore,

$$\arccos \left(\frac{a_{p_{\lambda_2}}}{\sqrt{a_{s_{\lambda_2}}^2 + a_{c_{\lambda_2}}^2}} \right) = \pi \ . \tag{3.87}$$

This completes the proof, because (3.87) shows that the second term of (3.73) is equal to π and if π is summed or subtracted to an angle the result differs in 2π, which is the period of the trigonometric functions where λ_2 is applied. Hence the result of the inverse cosine can be considered positive without loss of generality, because it will not influence the value of λ_2. □

Lemma 12. *Consider λ_3 and the conditions defined in Lemma 10. Furthermore, consider that the determination of $\mathbf{R}_1^I$ is not degenerate, which means*

$$\mathbf{R}_3^1 \mathbf{d}_3 \neq \pm \mathbf{d}_1 \ ,$$

then

$$\lambda_3 = \operatorname{atan2} \left(a_{s_{\lambda_3}}, a_{c_{\lambda_3}} \right) + \pi \ , \tag{3.88}$$

where

$$\begin{cases} a_{s_{\lambda_3}} := \left[\left(\mathbf{R}_3^1 \right)_C \mathbf{d}_3 \right]^T \mathbf{S} \left(\mathbf{d}_1 \right) \ \left(\mathbf{R}_3^1 \right)_D \mathbf{d}_3 \\ a_{c_{\lambda_3}} := \left[\left(\mathbf{R}_3^1 \right)_C \mathbf{d}_3 \right]^T \mathbf{S} \left(\mathbf{d}_1 \right)^2 \left(\mathbf{R}_3^1 \right)_D \mathbf{d}_3 \end{cases} \ . \tag{3.89}$$

Proof. This proof is similar to the one of Lemma 11 and thus, it is omitted.

The definitions of λ_2 and λ_3 are used to find a necessary and sufficient conditions for the situations where there are two pairs of identical candidates for $\mathbf{R}_1^I$. These conditions depend on which candidates are paired and are considered in the next lemmas.

Lemma 13. *Suppose that the conditions of Lemmas 11 and 12 hold. Further suppose that*

$$\left(\mathbf{R}_1^I \right)_A = \left(\mathbf{R}_1^I \right)_C \ . \tag{3.90}$$

Then

$$\left(\mathbf{R}_1^I \right)_B = \left(\mathbf{R}_1^I \right)_D \ , \tag{3.91}$$

if and only if

$$\lambda_2 = \lambda_3 \ .$$

Proof. Firstly, notice that all candidates for $\mathbf{R}_1^I$ verify, by construction, (3.52) and (3.53), *i.e.*,

$$^I\mathbf{d}_2 = \left(\mathbf{R}_1^I\right)_A \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 \ ,$$

$$^I\mathbf{d}_2 = \left(\mathbf{R}_1^I\right)_B \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \ ,$$

$$^I\mathbf{d}_3 = \left(\mathbf{R}_1^I\right)_C \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ ,$$

and

$$^I\mathbf{d}_3 = \left(\mathbf{R}_1^I\right)_D \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ .$$

Consequently, the inner products between $^I\mathbf{d}_2$ and $^I\mathbf{d}_3$ can be given by

$$^I\mathbf{d}_2^T \, ^I\mathbf{d}_3 = \left[\left(\mathbf{R}_1^I\right)_A \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \left(\mathbf{R}_1^I\right)_C \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ ,$$

and

$$^I\mathbf{d}_2^T \, ^I\mathbf{d}_3 = \left[\left(\mathbf{R}_1^I\right)_B \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right]^T \left(\mathbf{R}_1^I\right)_D \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ ,$$

or equivalently,

$$^I\mathbf{d}_2^T \, ^I\mathbf{d}_3 = \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_A^T \left(\mathbf{R}_1^I\right)_A^T \left(\mathbf{R}_1^I\right)_C \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ , \tag{3.92a}$$

and

$$^I\mathbf{d}_2^T \, ^I\mathbf{d}_3 = \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_B^T \left(\mathbf{R}_1^I\right)_B^T \left(\mathbf{R}_1^I\right)_D \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ . \tag{3.92b}$$

Next, substituting (D.2) and (D.3) from Appendix D in (3.92), recalling the angles defined in (3.63), yields

$$^I\mathbf{d}_2^T \, ^I\mathbf{d}_3 = \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_A^T \left[\mathbf{R}\left(\theta_A, ^I\mathbf{d}_1\right) \mathbf{R}^*\right]^T \mathbf{R}\left(\theta_C, ^I\mathbf{d}_1\right) \mathbf{R}^* \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ ,$$

and

$$^I\mathbf{d}_2^T \, ^I\mathbf{d}_3 = \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_B^T \left[\mathbf{R}\left(\theta_B, ^I\mathbf{d}_1\right) \mathbf{R}^*\right]^T \mathbf{R}\left(\theta_D, ^I\mathbf{d}_1\right) \mathbf{R}^* \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ .$$

or, equivalently,

$$^I\mathbf{d}_2^T \, ^I\mathbf{d}_3 = \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_A^T \left(\mathbf{R}^*\right)^T \mathbf{R}\left(\theta_A, ^I\mathbf{d}_1\right)^T \mathbf{R}\left(\theta_C, ^I\mathbf{d}_1\right) \mathbf{R}^* \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ , \tag{3.93a}$$

and, noticing Corollaries 2 and 5,

$$^I\mathbf{d}_2^T \, ^I\mathbf{d}_3 = \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_B^T \left(\mathbf{R}^*\right)^T \mathbf{R}\left(\theta_B, ^I\mathbf{d}_1\right)^T \mathbf{R}\left(\theta_D, ^I\mathbf{d}_1\right) \mathbf{R}^* \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ . \tag{3.93b}$$

As $\mathbf{R}^*$ is a rotation of an angle equal to π, as defined in (D.1), then $\mathbf{R}^* = \left(\mathbf{R}^*\right)^T$ as concluded in Appendix A.5. Recall as well, from Appendix D, that $\mathbf{R}^* \, ^I\mathbf{d}_1 = \mathbf{d}_1$, hence

$$\mathbf{R}^* \, ^I\mathbf{d}_1 = \left(\mathbf{R}^*\right)^T \, ^I\mathbf{d}_1 = \mathbf{d}_1 \ . \tag{3.94}$$

Consequently, from Lemma 22, described in Appendix A.4 and using (3.94), it is concluded that

$$\mathbf{R}\left(\theta_C, ^I\mathbf{d}_1\right) \mathbf{R}^* = \mathbf{R}^* \mathbf{R}\left(\theta_C, \mathbf{R}^* \, ^I\mathbf{d}_1\right) = \mathbf{R}^* \mathbf{R}\left(\theta_C, \mathbf{d}_1\right) \tag{3.95}$$

and that

$$\mathbf{R}\left(\theta_D, ^I\mathbf{d}_1\right) \mathbf{R}^* = \mathbf{R}^* \mathbf{R}\left(\theta_D, \mathbf{R}^* \, ^I\mathbf{d}_1\right) = \mathbf{R}^* \mathbf{R}\left(\theta_D, \mathbf{d}_1\right) \ . \tag{3.96}$$

Also notice that $\mathbf{R}\left(\theta_A,{}^I\mathbf{d}_1\right)^T = \mathbf{R}\left(-\theta_A,{}^I\mathbf{d}_1\right)$ and that $\mathbf{R}\left(\theta_B,{}^I\mathbf{d}_1\right)^T = \mathbf{R}\left(-\theta_B,{}^I\mathbf{d}_1\right)$, thus it follows that

$$(\mathbf{R}^*)^T\mathbf{R}\left(\theta_A,{}^I\mathbf{d}_1\right)^T = (\mathbf{R}^*)^T\mathbf{R}\left(-\theta_A,{}^I\mathbf{d}_1\right) \ ,$$

and that

$$(\mathbf{R}^*)^T\mathbf{R}\left(\theta_B,{}^I\mathbf{d}_1\right)^T = (\mathbf{R}^*)^T\mathbf{R}\left(-\theta_B,{}^I\mathbf{d}_1\right) \ ,$$

hence applying Lemma 22 and recalling (3.94) gives

$$(\mathbf{R}^*)^T\mathbf{R}\left(-\theta_A,{}^I\mathbf{d}_1\right) = \mathbf{R}\left(-\theta_A,(\mathbf{R}^*)^T\,{}^I\mathbf{d}_1\right)(\mathbf{R}^*)^T = \mathbf{R}\left(-\theta_A,\mathbf{d}_1\right)(\mathbf{R}^*)^T \tag{3.97}$$

and also

$$(\mathbf{R}^*)^T\mathbf{R}\left(-\theta_B,{}^I\mathbf{d}_1\right) = \mathbf{R}\left(-\theta_B,(\mathbf{R}^*)^T\,{}^I\mathbf{d}_1\right)(\mathbf{R}^*)^T = \mathbf{R}\left(-\theta_B,\mathbf{d}_1\right)(\mathbf{R}^*)^T \ . \tag{3.98}$$

Therefore, substituting (3.95) and (3.97) in (3.93a), gives

$${}^I\mathbf{d}_2^T\,{}^I\mathbf{d}_3 = \mathbf{d}_2^T\left(\mathbf{R}_2^1\right)_A^T\mathbf{R}\left(-\theta_A,\mathbf{d}_1\right)(\mathbf{R}^*)^T\mathbf{R}^*\mathbf{R}\left(\theta_C,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3 \ ,$$

and also substituting (3.96) and (3.98) in (3.93b) gives

$${}^I\mathbf{d}_2^T\,{}^I\mathbf{d}_3 = \mathbf{d}_2^T\left(\mathbf{R}_2^1\right)_B^T\mathbf{R}\left(-\theta_B,\mathbf{d}_1\right)(\mathbf{R}^*)^T\mathbf{R}^*\mathbf{R}\left(\theta_D,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3 \ .$$

which is simplified to

$${}^I\mathbf{d}_2^T\,{}^I\mathbf{d}_3 = \mathbf{d}_2^T\left(\mathbf{R}_2^1\right)_A^T\mathbf{R}\left(-\theta_A,\mathbf{d}_1\right)\mathbf{R}\left(\theta_C,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3 \ , \tag{3.99a}$$

and

$${}^I\mathbf{d}_2^T\,{}^I\mathbf{d}_3 = \mathbf{d}_2^T\left(\mathbf{R}_2^1\right)_B^T\mathbf{R}\left(-\theta_B,\mathbf{d}_1\right)\mathbf{R}\left(\theta_D,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3 \ . \tag{3.99b}$$

Observe that from Corollaries 2 and 5, the right hand side of (3.70) and (3.71) are equivalent to $\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2$ and $\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3$, respectively. This means that (3.70) and (3.71) can be written, respectively, as

$$\mathbf{R}\left(\lambda_2,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2 = \left(\mathbf{R}_2^1\right)_A\mathbf{d}_2 \ . \tag{3.100}$$

and

$$\mathbf{R}\left(\lambda_3,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3 = \left(\mathbf{R}_3^1\right)_C\mathbf{d}_3 \tag{3.101}$$

Recalling Lemma 23, then (3.100) and (3.101) are applied to (3.99b), which gives

$${}^I\mathbf{d}_2^T\,{}^I\mathbf{d}_3 = \mathbf{d}_2^T\left(\mathbf{R}_2^1\right)_A^T\mathbf{R}\left(-\lambda_2,\mathbf{d}_1\right)^T\mathbf{R}\left(-\theta_B,\mathbf{d}_1\right)\mathbf{R}\left(\theta_D,\mathbf{d}_1\right)\mathbf{R}\left(-\lambda_3,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3 \ ,$$

which, from Lemma 23, gives

$${}^I\mathbf{d}_2^T\,{}^I\mathbf{d}_3 = \mathbf{d}_2^T\left(\mathbf{R}_2^1\right)_A^T\mathbf{R}\left(\lambda_2,\mathbf{d}_1\right)\mathbf{R}\left(-\theta_B,\mathbf{d}_1\right)\mathbf{R}\left(\theta_D,\mathbf{d}_1\right)\mathbf{R}\left(-\lambda_3,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3 \ . \tag{3.102}$$

Thus, it follows, from Lemma 21, described in Appendix A.3, that (3.99a) and (3.102) are, respectively, given as

$${}^I\mathbf{d}_2^T\,{}^I\mathbf{d}_3 = \mathbf{d}_2^T\left(\mathbf{R}_2^1\right)_A^T\mathbf{R}\left(-\theta_A+\theta_C,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3 \ ,$$

and

$${}^I\mathbf{d}_2^T\,{}^I\mathbf{d}_3 = \mathbf{d}_2^T\left(\mathbf{R}_2^1\right)_A^T\mathbf{R}\left(\lambda_2-\theta_B+\theta_D-\lambda_3,\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3 \ .$$

Next, from (3.90) it follows

$$ {}^{I}\mathbf{d}_2^T\, {}^{I}\mathbf{d}_3 = \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_A^T \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ , \tag{3.103a} $$

and

$$ {}^{I}\mathbf{d}_2^T\, {}^{I}\mathbf{d}_3 = \mathbf{d}_2^T \left(\mathbf{R}_2^1\right)_A^T \mathbf{R}\left(\lambda_2 - \theta_B + \theta_D - \lambda_3, \mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ . \tag{3.103b} $$

Since the problem is assumed to be non degenerate, it is possible to write the conditions from Lemmas 5 and 6 in $\{\mathcal{B}_1\}$, respectively, as $\mathbf{d}_1 \neq \mathbf{R}_2^1\mathbf{d}_2$ and $\mathbf{d}_1 \neq \mathbf{R}_3^1\mathbf{d}_3$. Therefore, (3.103a) implies that $\mathbf{R}\left(\lambda_2 - \theta_B + \theta_D - \lambda_3, \mathbf{d}_1\right) = \mathbf{I}$, which means that the parameters of the rotation about $\mathbf{d}_1$ in (3.103b) are given as

$$ \lambda_2 - \theta_B + \theta_D - \lambda_3 = 0 \ . $$

Therefore, if $\theta_B = \theta_D$, then $\lambda_2 = \lambda_3$ and if $\lambda_2 = \lambda_3$, then $\theta_B = \theta_D$. □

Remark. Note that the proof of Lemma 13 would have followed similarly, if (3.91) had been supposed instead of (3.90).

Lemma 14. *Suppose that the conditions Lemmas 11 and 12 hold. Further suppose that*

$$ \left(\mathbf{R}_1^I\right)_A = \left(\mathbf{R}_1^I\right)_D \ . \tag{3.104} $$

Then,

$$ \left(\mathbf{R}_1^I\right)_B = \left(\mathbf{R}_1^I\right)_C \ , \tag{3.105} $$

if and only if

$$ \lambda_2 = -\lambda_3 \ . $$

Proof. The proof follows analogously to the one of Lemma 13, thus it is omitted. □

Remark. Analogously to Lemma 13 (3.105) can be supposed instead of (3.104). Also, notice that when the partial problems' vectors are not coplanar $\left(\mathbf{R}_1^I\right)_A \neq \left(\mathbf{R}_1^I\right)_B$ and $\left(\mathbf{R}_1^I\right)_C \neq \left(\mathbf{R}_1^I\right)_D$, consequently the conditions from Lemmas 13 and 14 cannot happen simultaneously, that is, if $\left(\mathbf{R}_1^I\right)_A = \left(\mathbf{R}_1^I\right)_C$, then $\left(\mathbf{R}_1^I\right)_A \neq \left(\mathbf{R}_1^I\right)_D$ and so on. Furthermore, Lemmas 13 and 14 do not distinguish between the cases where there are two different $\mathbf{R}_1^I$ pairs and the situation where all $\mathbf{R}_1^I$ candidates are identical, because the latter is given when $\lambda_2 = \lambda_3 = 0$.

Ambiguous double solution: formation geometry

The configurations where the ambiguous double solution happens are found from recalling the conditions obtained for λ_2 and λ_3. These cases are addressed in the next lemmas.

Lemma 15. *In the conditions of Lemma 13, consider that the partial problem vector configuration is not coplanar. If*

$$ \lambda_2 = \lambda_3 \ , \tag{3.106} $$

then

$$ \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ , \tag{3.107} $$

$$ \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\right\|}\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ , \tag{3.108} $$

and

$$\mathbf{d}_{1/2} = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\beta, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}\right\|}\right) \mathbf{d}_{1/3} \ , \tag{3.109}$$

for some α, β, $\gamma \in \mathbb{R}$.

Proof. Consider the rotation $\mathbf{R}\left(\epsilon, \mathbf{d}_1\right)$, $\epsilon \in \mathbb{R}$. Now, notice that for some $\epsilon \in \mathbb{R}$

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right\|} = \pm \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|} \ , \tag{3.110}$$

because $\frac{\mathbf{S}(\mathbf{d}_1)\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2}{\left\|\mathbf{S}(\mathbf{d}_1)\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right\|}$ and $\frac{\mathbf{S}(\mathbf{d}_1)\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}(\mathbf{d}_1)\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}$ are coplanar unit vectors and $\mathbf{d}_1$ is orthogonal to that plane. Recalling that the norm of a vector is not changed by a rotation about a unitary axis, then (3.110) is expressed as

$$\frac{\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2}{\left\|\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right\|} = \pm \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|} \ ,$$

which, recalling that $\mathbf{d}_1 = \mathbf{R}\left(\epsilon, \mathbf{d}_1\right)^T \mathbf{d}_1$ and applying Lemma 26, is rewritten as

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right\|} = \pm \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|} \ . \tag{3.111}$$

Next, applying Lemma 33 from Appendix E.1 to (3.111), with $\mathbf{a} = \mathbf{d}_1$, $\mathbf{b} = \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2$, and $\mathbf{c} = \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3$, and introducing $\gamma \in \mathbb{R}$ gives

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ . \tag{3.112}$$

Now, substituting (3.37) and (3.44) in (3.112), recalling θ_{12} and θ_{13}, gives

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{R}\left(\theta_{12}, -\mathbf{d}_{1/2}\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\theta_{13}, -\mathbf{d}_{1/3}\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right) \mathbf{R}\left(\theta_{13}, -\mathbf{d}_{1/3}\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ ,$$

which, from Lemmas 9 and 10, is rewritten as

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\lambda_3, \mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right) \mathbf{R}\left(\lambda_3, \mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ . \tag{3.113}$$

Applying Lemma 26, recalling that $\mathbf{R}\left(\lambda_3, \mathbf{d}_1\right) \mathbf{d}_1 = \mathbf{d}_1$, to (3.113) yields

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\gamma, \mathbf{R}\left(\lambda_3, \mathbf{d}_1\right) \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right) \mathbf{R}\left(\lambda_3, \mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ ,$$

which, making use Lemma 41, from Appendix F.1, is expressed as

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\gamma, \mathbf{R}\left(\lambda_3, \mathbf{d}_1\right) \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\right\|}\right) \mathbf{R}\left(\lambda_3, \mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ .$$

Next, applying (A.9a) from Lemma 22 in Appendix A.4, with $\mathbf{R}\left(\theta, \mathbf{x}\right) = \mathbf{R}\left(\lambda_3, \mathbf{d}_1\right)$ and $\mathbf{R}\left(\lambda, \mathbf{y}\right) =$

$\mathbf{R}\left(\gamma, \frac{\mathbf{S}(\mathbf{d}_1)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\|\mathbf{S}(\mathbf{d}_1)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\|}\right)$, gives

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\lambda_3, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\|}\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ ,$$

which, recalling (3.106) and noticing that $\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right) = \mathbf{R}\left(\lambda_2, \mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right)$, is rewritten as

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\|}\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ . \tag{3.114}$$

Applying Lemma 34 from Appendix E.1, with $\mathbf{a} = \mathbf{d}_1$, $\mathbf{b} = \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2$, and $\mathbf{c} = \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3$, to (3.114) gives

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2}{\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\|} = \pm \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\|} \ . \tag{3.115}$$

Notice that the remark about the sign of Lemma 34 allows to conclude that the sign of (3.111) and (3.115) is the same, because (3.112) and (3.114) have the same rotation angle and also because $\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3$ and $\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3$ are in the same cone around the axis $\mathbf{d}_1$, as noticed in Lemma 10. Therefore, (3.111) and (3.115) are applied to Lemma 42 from Appendix F.2, yielding

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{d}_{1/2}}{\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{d}_{1/2}\|} = \pm \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}\|} \ ,$$

which, applying Lemma 33 from Appendix E.1, with $\mathbf{a} = \mathbf{d}_1$, $\mathbf{b} = \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{d}_{1/2}$, and $\mathbf{c} = \mathbf{d}_{1/3}$, and introducing $\beta \in \mathbb{R}$ gives

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{d}_{1/2} = \mathbf{R}\left(\beta, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}\right) \mathbf{d}_{1/3} \ . \tag{3.116}$$

Introducing $\alpha \in \mathbb{R}$, defined as $\alpha = -\epsilon$, and multiplying both sides of (3.112), (3.114) and (3.116) by $\mathbf{R}\left(\alpha, \mathbf{d}_1\right)$, gives (3.107), (3.108), and (3.109), thus concluding the proof. □

Lemma 16. *In the conditions of Lemma 14, consider that the partial problem vector configuration is not coplanar. If*

$$\lambda_2 = -\lambda_3 \ ,$$

then

$$\mathbf{d}_{1/2} = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\beta, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}\|}\right) \mathbf{d}_{1/3} \ ,$$

$$\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\|}\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ ,$$

and

$$\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\|}\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ ,$$

for some α, β and $\gamma \in \mathbb{R}$.

Proof. The proof follows analogously to the one of Lemma 15 and thus it is omitted.

Finally, the following lemmas show that the configurations in the last lemmas not only result from, but also result in $\lambda_2 = \lambda_3$ or $\lambda_2 = -\lambda_3$, accordingly.

Lemma 17. *In the conditions of Lemma 13, consider that the partial problem vector configuration is not coplanar. If*

$$\mathbf{d}_{1/2} = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\beta, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}\right\|}\right) \mathbf{d}_{1/3} \tag{3.117}$$

and

$$\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ , \tag{3.118}$$

for some α, β and $\gamma \in \mathbb{R}$, then

$$\lambda_2 = \lambda_3 \ .$$

Proof. First, note that (3.117) and (3.118) are equivalent to

$$\left(\mathbf{R}\left(\alpha, \mathbf{d}_1\right)\right)^T \mathbf{d}_{1/2} = \mathbf{R}\left(\beta, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}\right\|}\right) \mathbf{d}_{1/3} \ ,$$

and

$$\left(\mathbf{R}\left(\alpha, \mathbf{d}_1\right)\right)^T \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ .$$

Then introduce $\epsilon \in \mathbb{R}$ defined as $\epsilon = -\alpha$ and rearrange the conditions, which gives

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{d}_{1/2} = \mathbf{R}\left(\beta, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}\right\|}\right) \mathbf{d}_{1/3} \tag{3.119}$$

and

$$\mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ . \tag{3.120}$$

Next, take (3.119) and (3.120) and apply Lemma 34 from Appendix E.1, which gives, respectively,

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}\left(\epsilon, \mathbf{d}_1\right)\right) \mathbf{d}_{1/2}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}\left(\epsilon, \mathbf{d}_1\right)\right) \mathbf{d}_{1/2}\right\|} = \pm \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}\right\|} \tag{3.121}$$

and

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right\|} = \pm \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|} \ . \tag{3.122}$$

Then, (3.121) and (3.122) are applied to Lemma 43 from Appendix F.2, which gives

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2\right\|} = \pm \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\right\|} \ , \tag{3.123}$$

with the same sign as (3.122). Next, rewriting (3.122) and (3.123), respectively, as

$$\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 = \mu_1 \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 \tag{3.124}$$

and

$$\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 = \mu_2 \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \ , \tag{3.125}$$

with

$$\mu_1 = \pm \frac{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right\|}$$

and

$$\mu_2 = \pm \frac{\left\| \mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \right\|}{\left\| \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \right\|} \ .$$

Next, recall the definition of λ_3 and the respective coefficients from (3.88) and (3.89), which is expressed as

$$\lambda_3 = \operatorname{atan2}\left(\left[\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right]^T \mathbf{S}\left(\mathbf{d}_1\right) \ \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3, \left[\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right]^T \mathbf{S}\left(\mathbf{d}_1\right)^2 \ \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \right) \ ,$$

which, substituting (3.125), is given as

$$\lambda_3 = \operatorname{atan2}\Big(\mu_2 \left[\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right]^T \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2, \\ \mu_2 \left[\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right]^T \mathbf{S}\left(\mathbf{d}_1\right)^2 \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \Big) \ ,$$

or, equivalently, recalling that by definition $\mathbf{S}\left(\mathbf{d}_1\right)^T = -\mathbf{S}\left(\mathbf{d}_1\right)$,

$$\lambda_3 = \operatorname{atan2}\Big(\mu_2 \left[-\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right]^T \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2, \\ \mu_2 \left[-\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right]^T \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \Big) \ .$$

Then substituting (3.124) gives

$$\lambda_3 = \operatorname{atan2}\Big(\mu_1\mu_2 \left[-\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2, \\ \mu_1\mu_2 \left[-\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \Big) \ ,$$

which, again from $\mathbf{S}\left(\mathbf{d}_1\right)^T = -\mathbf{S}\left(\mathbf{d}_1\right)$, is expressed as

$$\lambda_3 = \operatorname{atan2}\Bigg(\mu_1\mu_2 \left[\mathbf{S}\left(\mathbf{d}_1\right)^T \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2, \\ \mu_1\mu_2 \left[\mathbf{S}\left(\mathbf{d}_1\right)^T \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \Bigg) \ ,$$

or, equivalently,

$$\lambda_3 = \operatorname{atan2}\Big(\mu_1\mu_2 \left[\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{R}\left(\epsilon, \mathbf{d}_1\right)^T \mathbf{S}\left(\mathbf{d}_1\right) \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2, \\ \mu_1\mu_2 \left[\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{R}\left(\epsilon, \mathbf{d}_1\right)^T \mathbf{S}\left(\mathbf{d}_1\right)^2 \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \Big) \ . \quad (3.126)$$

Applying Lemma 26 to (3.126), recalling that $\mathbf{d}_1 = \mathbf{R}\left(\epsilon, \mathbf{d}_1\right) \mathbf{d}_1$, gives

$$\lambda_3 = \operatorname{atan2}\left(\mu_1\mu_2 \left[\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2, \mu_1\mu_2 \left[\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2\right]^T \mathbf{S}\left(\mathbf{d}_1\right)^2 \left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 \right) \ . \quad (3.127)$$

Notice that the sign of μ_1 and μ_2 is the same, because the sign of (3.122) and (3.123) is the same. Finally, it is concluded that (3.127) is equivalent to (3.73), recalling (3.74), firstly, because $\mu_1\mu_2 \geq 0$ and also because $\mu_1\mu_2$ is in both arguments of the function. Hence the value of atan2, recalling its properties, is the same regardless of the $\mu_1\mu_2$. Therefore the proof is complete. □

Remark. Lemma 43 implies that, it is equivalent to consider the condition given as

$$\left(\mathbf{R}_2^1\right)_B \mathbf{d}_2 = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\left\| \mathbf{S}\left(\mathbf{d}_1\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \right\|}\right) \left(\mathbf{R}_3^1\right)_D \mathbf{d}_3$$

instead of

$$\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3\right\|}\right)\left(\mathbf{R}_3^1\right)_C \mathbf{d}_3 \ .$$

Lemma 18. *In the conditions of Lemma 14, consider that the partial problem vector configuration is not coplanar. If*

$$\mathbf{d}_{1/2} = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\beta, \frac{\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right) \mathbf{d}_{1/3}\right\|}\right) \mathbf{d}_{1/3}$$

and

$$\left(\mathbf{R}_2^1\right)_A \mathbf{d}_2 = \mathbf{R}\left(\alpha, \mathbf{d}_1\right) \mathbf{R}\left(\gamma, \frac{\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3\right\|}\right)\left(\mathbf{R}_3^1\right)_D \mathbf{d}_3 \ ,$$

for some α, β and $\gamma \in \mathbb{R}$, then

$$\lambda_2 = -\lambda_3 \ .$$

Proof. The proof follows analogously to the one of Lemma 17 and thus, it is omitted.

To gain insight to the formation geometries that lead to an ambiguous solution, start by noticing that in the collinear formation with parallel inertial vectors of the deputies, represented in Fig. 3.6, both branches are equivalent in the inertial frame, because the analogous vectors are parallel. Thus, the two candidates for $\mathbf{R}_1^I$ from both branches coincide. That is even more evident when $\mathbf{R}_2^1 = \mathbf{R}_3^1$, because the two candidates for $\mathbf{R}_2^1$ are identical to both candidates for $\mathbf{R}_3^1$, which means that the candidates for $\mathbf{R}_1^I$ are ambiguous.

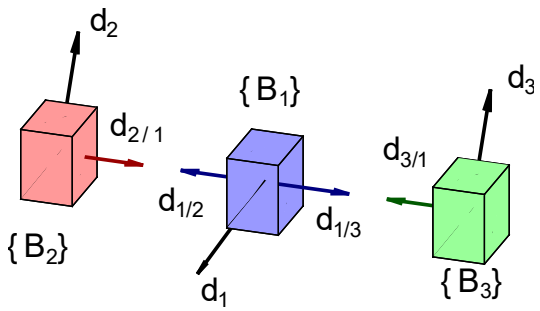


Figure 3.6: Ambiguous double solution: collinear formation

Then, the measurements of the collinear formation with parallel inertial vector measurements of the deputies can be transformed according to three degrees of freedom given by α, β, and γ as defined in Lemmas 15, 16, 17, and 18, which give the other possible formations for the ambiguous solution. These degrees of freedom are seen for the collinear configuration in Fig. 3.7, where the red and violet dotted lines represent possible positions of the pink vehicle and the black continuous line represents the possible directions of $\mathbf{d}_2$. Also, in the next figures the LOS are represented in the same color of the body where they are originated and the inertial measurement vectors are represented in black.

In fact, the red vehicle can be in any position of the formation as the rotations governed by α or γ have their axis orthogonal to each other meaning that any given formation shape can originate an ambiguous double solution. Hence, the choice of inertial references in the formation becomes important to avoid this ambiguity. Four different configurations, for four different values of α are visible in Fig.3.8. The possible values of γ are depicted for each configuration, depending on α. Also, four different configurations, for four different values of γ are visible in Fig.3.9. The possible values of α are depicted for each configuration, depending on γ.

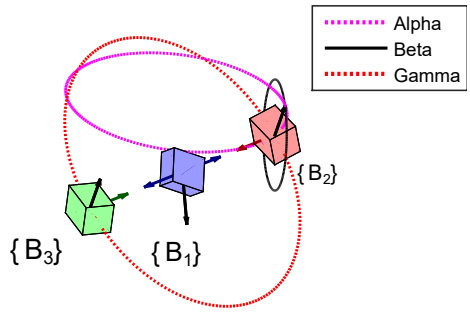
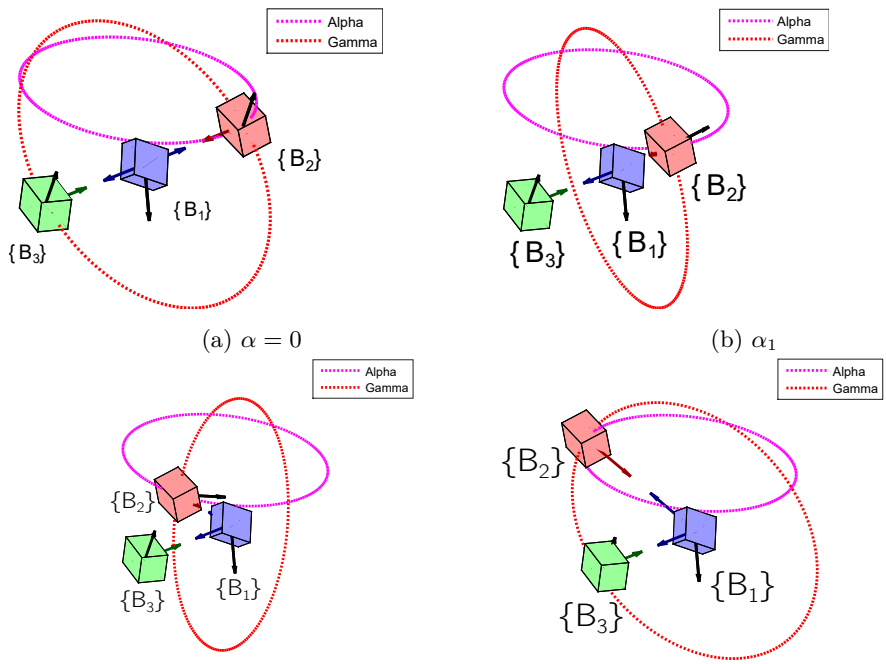


Figure 3.7: Ambiguous double solution: degrees of freedom



(a) $\alpha = 0$ (b) α_1

(c) α_2 (d) α_3

Figure 3.8: Ambiguous double solution: different α examples

The rotation governed by β does not influence the position of the vehicles, because it refers only to the inertial vector, which can be transformed to any direction within that plane. Its variation can be seen in Fig. 3.10.

Note that the ambiguous double solution is influenced not by the LOS or inertial vectors alone, but by the combination of both.

This concludes the characterization of the solution, which includes the general unique solution, the degenerate cases with infinite solutions, and the ambiguous case with two ambiguous candidates for $\mathbf{R}_1^I$.

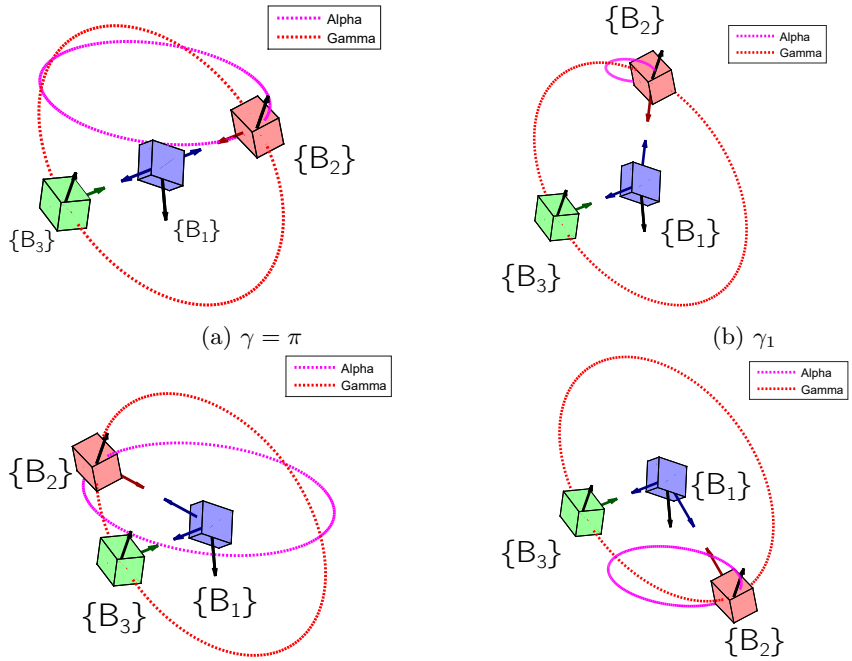
3.3 Solution completion

After having determined and disambiguated $\mathbf{R}_2^1$, $\mathbf{R}_3^1$ and $\mathbf{R}_1^I$, the remaining attitude matrices are found from the relations expressed as

$$\mathbf{R}_3^2 = \mathbf{R}_1^2 \mathbf{R}_3^1 \ ;$$

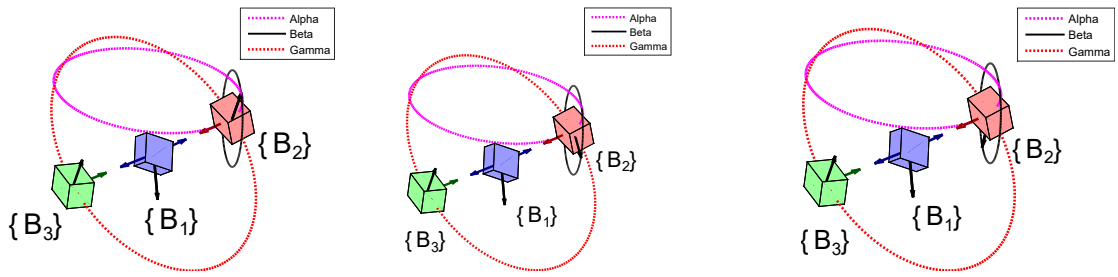
$$\mathbf{R}_2^I = \mathbf{R}_1^I \mathbf{R}_2^1 \ ;$$

$$\mathbf{R}_3^I = \mathbf{R}_1^I \mathbf{R}_3^1 \ .$$



(a) $\gamma = \pi$ (b) γ_1 (c) γ_2 (d) γ_3

Figure 3.9: Ambiguous double solution: different γ examples



(a) β_1 (b) β_2 (c) β_3

Figure 3.10: Ambiguous double solution: different β examples

Recall that the rotations that represent the transformations in the opposite sense are obtained from the respective transpose. Hence, the solution has been fully characterized and the deterministic solution of the problem has been concluded.

3.4 Solution in the presence of noise

Real sensors have a noise component in the measurements that they make. As a consequence, the rotation matrices determined in the problem will not be exactly equal to the real solutions. This deviation creates a new problem for the solution described before.

The criteria for the comparison of the $\mathbf{R}_1^I$ candidates assumed that when both $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$ candidates are the real solution they generate the same $\mathbf{R}_1^I$. However, the presence of noise makes both $\mathbf{R}_1^I$ candidates only approximately equal.

Therefore, the criteria must be changed in order to work in the presence of noise. As described before, the comparison parameter, ϕ, determined in (3.68), is closer to zero, since the two rotations compared get nearer. So, instead of looking for $\phi = 0$, the correct solution is defined by the smallest ϕ, noticing that $\phi \geq 0$.

Furthermore, to improve the accuracy of $\mathbf{R}_1^I$ the average of the two candidates with the smallest ϕ is used as the solution, whereas the correspondent candidates for $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$ are used as firstly calculated.

The average used is matrix-based and is based on the SVD of the sum of both candidates [23]. Hence, considering the SVD given by

$$\frac{\left(\mathbf{R}_1^I\right)_A + \left(\mathbf{R}_1^I\right)_C}{2} = \mathbf{XDY} \ ,$$

the average $\mathbf{R}_1^I$ is given by

$$\left(\mathbf{R}_1^I\right)_{avg} = \mathbf{XY} \ .$$

Chapter 4

Simulations

The solution for the problem proposed is now tested in a simulated environment, where noise was added to reproduce an approximation to the real application. This process allows the evaluation and analysis of the solution.

4.1 Sensor model

The sensor measurement model used in the simulations is based on the FPD, thus it is assumed that the sensors used are vision based, even for the inertial measurements which can be, for example, from a star tracker. These kind of devices measure the elevation of a light source projected into mutually perpendicular planes that contain the sensor boresight axis. Denoting the vector $\mathbf{m} \equiv [\chi \ \psi]^T$, as the image-space observation[1], then the measurement model is given as

$$\tilde{\mathbf{m}} = \mathbf{m} + \mathbf{n} \ , \tag{4.1}$$

with $\tilde{\mathbf{m}}$ denoting a measurement and $\mathbf{n}$ denoting the random noise. The noise model that describes the uncertainty of the image-space observations assumed to follow a zero mean Gaussian distribution,

$$\mathbf{n} \sim \mathcal{N}\left(\mathbf{0}, R_{\text{FOCAL}}\right) \ , \tag{4.2}$$

with the covariance of the focal plane given as [20]

$$R_{\text{FOCAL}} = \frac{\sigma^2}{1 + d\left(\chi^2 + \psi^2\right)} \begin{bmatrix} \left(1 + d\chi^2\right)^2 & \left(d\chi\psi\right)^2 \\ \left(d\chi\psi\right)^2 & \left(1 + d\psi^2\right)^2 \end{bmatrix} \ , \tag{4.3}$$

where σ^2 is the variance of the measurement errors associated to χ and ψ, and d as a parameter on the order of 1. It is noticed that this model predicts that the measurements are more accurate as the distance to the boresight decreases.

The focal length is assumed to be unitary and the sensor boresight is assumed to be in the z-axis which means that the measurement vector in the object space and in the sensor frame is given as

$${}^{s}\mathbf{d} = \frac{1}{\sqrt{1 + \chi^2 + \psi^2}} \begin{bmatrix} \chi \\ \psi \\ 1 \end{bmatrix} \ . \tag{4.4}$$

[1] The image-space frame is the 2D coordinate system of the FPD, whereas the object-space frame is the vehicle body coordinate system.

4.1.1 Sensor boresight

In practice (4.4) allows to obtain the image space vector $\mathbf{m}$, after having transformed the measurement vector to the respective boresight frame. Therefore, the simulations require that the boresight frames are defined, in order to apply the sensor measurement model.

The boresight axis were chosen differently for the chief and for the deputies, since the first makes more measurements than the former. For convenience, the chief has three sensors, one with the boresight axis aligned with the body frame x-axis, another aligned with the x-axis but in the opposite direction and the third aligned with the opposite direction of the y-axis. As for the deputies both have two sensors, one with the boresight axis aligned with y-axis and the other aligned with the z-axis.

The transformations from the boresight frame to the respective body frame depend on the attitude of the vehicle, hence these are presented in the formation configuration.

4.1.2 Sampling

The noise of the measurement model follows a Gaussian distribution whose mean and covariance have already been defined according to (4.2). The random sampling from this distribution is made by, firstly, sampling a standard normal distribution, since there are several widely known sampling algorithms for this distribution. Then the values sampled are transformed with a matrix $\mathbf{A}$, as given by [24]

$$X = \mathbf{A}Y \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_{\text{FOCAL}}\right) \ ,$$

where $Y \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right)$ and $\mathbf{R}_{\text{FOCAL}} = \mathbf{A}\mathbf{A}^T$. The matrix $\mathbf{A}$ can be found with a Cholesky factorization [25] of $\mathbf{R}_{\text{FOCAL}}$, since this variable is known.

4.2 Static simulation setup

This simulation uses a fixed formation chosen a priori for 1000 Monte Carlo trials, which means that at each iteration random noise is sampled from the sensor model and added to the measurement vectors, which are then used to find the estimates for the rotation matrices of the problem.

4.2.1 Formation configuration

The configuration chosen in this simulation uses the true LOS vectors given by

$$^I\mathbf{d}_{1/2} = \begin{bmatrix} 0 \\ -\sin\left(30^\circ\right) \\ -\cos\left(30^\circ\right) \end{bmatrix} \ , \ ^I\mathbf{d}_{1/3} = \begin{bmatrix} 0 \\ \sin\left(30^\circ\right) \\ -\cos\left(30^\circ\right) \end{bmatrix} \ , \tag{4.5a}$$

the inertial vectors given by

$$^I\mathbf{d}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ , \ ^I\mathbf{d}_2 = \begin{bmatrix} 0 \\ \cos\left(60^\circ\right) \\ -\sin\left(60^\circ\right) \end{bmatrix} \ , \ ^I\mathbf{d}_3 = \begin{bmatrix} \cos\left(60^\circ\right)\sin\left(30^\circ\right) \\ \cos\left(60^\circ\right)\cos\left(30^\circ\right) \\ -\sin\left(60^\circ\right) \end{bmatrix} \tag{4.5b}$$

and also, the true rotation matrices defined by

$$\mathbf{R}_I^1 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ , \ \mathbf{R}_I^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \ , \ \mathbf{R}_I^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \ . \tag{4.5c}$$

The remaining matrices are a combination of those shown in (4.5c), which are expressed as

$$\begin{matrix} \mathbf{R}_I^1 = \mathbf{R}_I^{1T} \\ \mathbf{R}_I^2 = \mathbf{R}_I^{2T} \\ \mathbf{R}_I^3 = \mathbf{R}_I^{3T} \end{matrix} \quad , \quad \begin{matrix} \mathbf{R}_2^1 = \mathbf{R}_I^1 \mathbf{R}_I^{2T} \\ \mathbf{R}_3^1 = \mathbf{R}_I^1 \mathbf{R}_I^{3T} \\ \mathbf{R}_3^2 = \mathbf{R}_I^2 \mathbf{R}_I^{3T} \end{matrix} \quad , \quad \begin{matrix} \mathbf{R}_1^2 = \mathbf{R}_I^2 \mathbf{R}_I^{1T} \\ \mathbf{R}_1^3 = \mathbf{R}_I^3 \mathbf{R}_I^{1T} \\ \mathbf{R}_1^3 = \mathbf{R}_I^3 \mathbf{R}_I^{2T} \end{matrix} \quad .$$

The remaining vectors of the problems are generated from (4.5b), (4.5a) and (4.5c) with the relations given as

$$\begin{matrix} \mathbf{d}_1 = \mathbf{R}_I^1 \, {}^I\mathbf{d}_1 \\ \mathbf{d}_2 = \mathbf{R}_I^2 \, {}^I\mathbf{d}_2 \\ \mathbf{d}_3 = \mathbf{R}_I^3 \, {}^I\mathbf{d}_3 \end{matrix} \quad , \quad \begin{matrix} \mathbf{d}_{1/2} = \mathbf{R}_I^1 \, {}^I\mathbf{d}_{1/2} \\ \mathbf{d}_{1/3} = \mathbf{R}_I^1 \, {}^I\mathbf{d}_{1/3} \\ \mathbf{d}_{2/1} = -\mathbf{R}_1^2 \, {}^I\mathbf{d}_{1/2} \\ \mathbf{d}_{3/1} = -\mathbf{R}_1^3 \, {}^I\mathbf{d}_{1/3} \end{matrix} \quad .$$

The measurement vectors are computed in the focal plane which contains noise has described by the sensor model. Therefore, the true vectors have to be transformed into the focal plane frame and then noise is added to each of them sampling using the covariance given in (4.3) with $\sigma = 17 \times 10^{-6}$ rad. The transformation from the body frame to the sensor frame makes use of a rotation matrix. The orthogonal transformations, from the respective body frame to the sensor frame, used in this simulation are shown afterwards

$$\mathbf{R}_1^{s_{\mathbf{d}_{1/2}}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} , \mathbf{R}_1^{s_{\mathbf{d}_{1/3}}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} , \mathbf{R}_1^{s_{\mathbf{d}_1}} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \tag{4.6a}$$

$$\mathbf{R}_2^{s_{\mathbf{d}_{2/1}}} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \mathbf{R}_2^{s_{\mathbf{d}_2}} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} , \tag{4.6b}$$

$$\mathbf{R}_3^{s_{\mathbf{d}_{3/1}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \mathbf{R}_3^{s_{\mathbf{d}_3}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} , \tag{4.6c}$$

with the superscript representing each sensor, since there is a sensor for each vector measurement, and the subscript representing the body frame of the vehicle. Note that the inertial vectors in the inertial frame are assumed to be known exactly as these are the reference for the inertial sensor, hence no noise is added to them.

The transformations shown in (4.6a), (4.6b) and (4.6c) allow to obtain each of the measurement vectors in the form given by (4.4). Then, it is possible to get the respective image-space variables χ and ψ and consequently the respective vector $\mathbf{m}$. The image-space variables are then used in (4.3) to find each individual R_{FOCAL}, which allows the sampling of the noise, $\mathbf{n}$, for each measurement. Finally, the measurements in the object-space are obtained after applying (4.1) and reusing (4.4) with the components of $\tilde{\mathbf{m}}$ instead of χ and ψ.

4.3 Dynamic simulation setup

A variable formation was also simulated two observe how the rotation estimates errors varied, namely near a degenerate geometry.

4.3.1 Formation configuration

The motion of the formation starts from the geometry described for the static simulation and rotates vehicle 2 around vehicle 1 about the axis given by $\mathbf{d}_1$, which in this case, is perpendicular to the plane formed by the three vehicles. Therefore, introducing an angle given by $\omega \in \mathbb{R}$, the LOS between vehicle 1 and vehicle 2 are given by

$$\mathbf{d}_{1/2}(t) = \mathbf{R}\left(\omega(t), \mathbf{d}_1\right) \mathbf{d}_{1/2}(t=0) \ ,$$

$$\mathbf{d}_{2/1}(t) = -\mathbf{R}_1^2(t)\mathbf{d}_{1/2}(t) \ .$$

Note that t represents the time. The total simulation time was 100 seconds with a sampling frequency of 10 Hz and the ω varied exactly 360 degrees during that period. All the other vectors remain constant throughout this simulation. This motion creates two degenerate cases, one when $\omega = 120°$ and another when $\omega = 300°$. This case happens because $\mathbf{d}_3$ is constant and is in the plane perpendicular to $\mathbf{d}_1$. Hence for those angles $\mathbf{d}_3 = \pm\mathbf{d}_{3/1}$, which from Corollary 4 is a degenerate case. For simplicity all boresight axis of all the sensors are kept as given by (4.6a), (4.6b) and (4.6c). Moreover, all the attitude matrices are kept as constant and equal to the static simulation. The variance of the measurement errors is also $\sigma = 17 \times 10^{-6}$ rad and the same operations as in the static simulation are made in order to add the noise in each iteration.

4.4 Results

The results for the simulation with the static and dynamic formation configurations are shown in this section along with the statistical values that characterize them. For the static simulation, the results are shown only for half the rotation matrices, since the other rotations correspond to the transposes of this half, which have the same accuracy and precision outcome. For the dynamic simulation, a selection of interesting rotation matrices are shown.

Euler angle errors

It is convenient to show as a result the error between the rotation matrices estimates and the true rotation matrices. Furthermore, relating the error with a physical value like an angle, also gives some insight about the situation, hence the errors in terms of Euler angles have been obtained from the estimated error matrices.

The estimated error matrices are given as

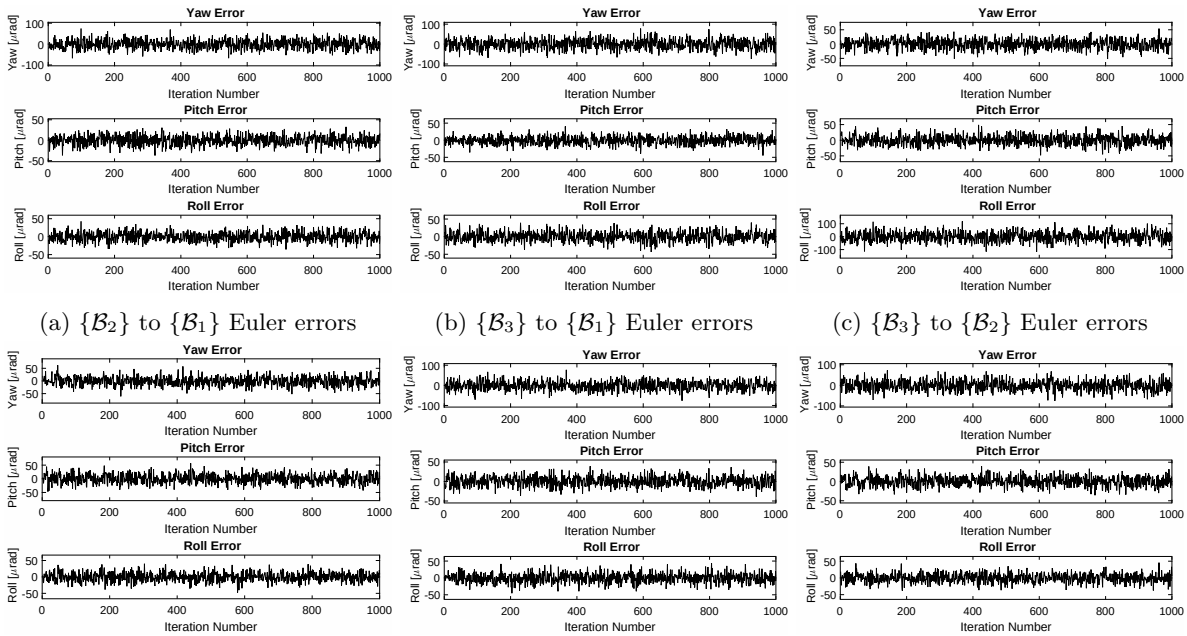
$$\mathbf{E} = \tilde{\mathbf{R}}\mathbf{R}^T \ , \tag{4.7}$$

where $\mathbf{E}$ is the rotation matrix representing the errors, $\tilde{\mathbf{R}}$ is the estimated rotation matrix and $\mathbf{R}$ is the true rotation matrix. Then, the Euler angles are obtained by decomposing (4.7) into three rotations as given by

$$\mathbf{E} = \mathbf{R}\left(\mathbf{n}_3, \psi_E\right) \mathbf{R}\left(\mathbf{n}_2, \theta_E\right) \mathbf{R}\left(\mathbf{n}_1, \phi_E\right) \ ,$$

where ψ_E is the yaw, θ_E is the pitch and ϕ_E is the roll of the vehicle, which means that the axis are given as

$$\mathbf{n}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{n}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{n}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$



(a) $\{\mathcal{B}_2\}$ to $\{\mathcal{B}_1\}$ Euler errors (b) $\{\mathcal{B}_3\}$ to $\{\mathcal{B}_1\}$ Euler errors (c) $\{\mathcal{B}_3\}$ to $\{\mathcal{B}_2\}$ Euler errors

(d) $\{\mathcal{B}_1\}$ to $\{\mathcal{I}\}$ Euler errors (e) $\{\mathcal{B}_2\}$ to $\{\mathcal{I}\}$ Euler errors (f) $\{\mathcal{B}_3\}$ to $\{\mathcal{I}\}$ Euler errors

Figure 4.1: Relative and inertial attitude estimation errors

The angles are extracted from the rotation matrix with the relations given by

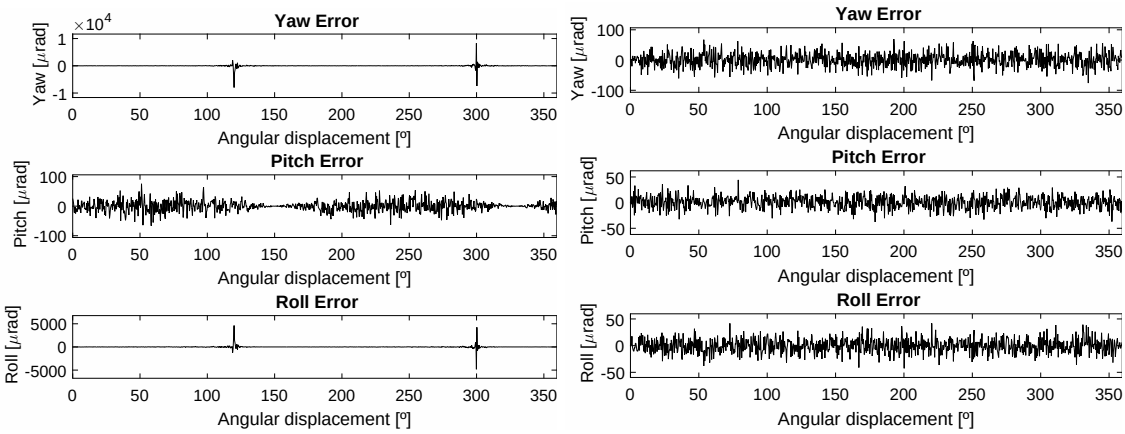
$$\begin{aligned}\theta_E &= \sin^{-1}\left(-\mathbf{E}_{13}\right)\\ \psi_E &= \operatorname{atan2}\left(\lambda_E\mathbf{E}_{12}, \lambda_E\mathbf{E}_{11}\right)\\ \phi_E &= \operatorname{atan2}\left(\lambda_E\mathbf{E}_{23}, \lambda_E\mathbf{E}_{33}\right)\ ,\end{aligned}$$

where λ_E is the sign of $\cos(\theta_E)$, $\mathbf{E}_{13}$ is the element of the rotation $\mathbf{E}$ in the first line and third column, and so on.

The Euler angle errors for each matrix of the static simulation are shown in Fig. 4.1. The dynamic simulation Euler angle errors are shown only for $\mathbf{R}_2^1$ and $\mathbf{R}_3^1$, respectively in Fig. 4.2a and Fig. 4.2b. These show how the error increases greatly near the the degenerate cases in $\mathbf{R}_2^1$, whereas $\mathbf{R}_3^1$ is not affected by this situation. Note that, in Fig. 4.2, the x-axis represents ω in degrees.

Mean and standard deviation

The mean and standard deviation values are given in Tables 4.1 and 4.2 for the static simulation and in Tables 4.3 and 4.4 for the dynamic simulation. It is expected that the mean value of the error is very close to zero. As for the standard deviation the smaller it is, the better the estimate quality is. Comparing the standard deviation of the static simulation with the results from [16] and [17], it is seen that it is within the same order of magnitude. As for the standard deviation of the dynamic simulation, the values of the rotations related with vehicle 2 have greater values due to the situation near the degenerate case.



(a) $\{\mathcal{B}_2\}$ to $\{\mathcal{B}_1\}$ Euler errors

(b) $\{\mathcal{B}_3\}$ to $\{\mathcal{B}_1\}$ Euler errors

Figure 4.2: Dynamic simulation: estimation errors

	Yaw [μrad]	Pitch [μrad]	Roll [μrad]
$\mathbf{R}_2^1$	-0.7708	-0.2257	-0.0654
$\mathbf{R}_3^1$	0.0544	-0.2678	-0.1805
$\mathbf{R}_3^2$	0.0423	-0.1151	-0.8252
$\mathbf{R}_1^I$	-0.4391	0.6010	-0.2656
$\mathbf{R}_2^I$	-1.2099	0.5356	-0.0400
$\mathbf{R}_3^I$	-0.3847	0.4205	0.0023

Table 4.1: Static simulation: error mean value

	Yaw [μrad]	Pitch [μrad]	Roll [μrad]
$\mathbf{R}_2^1$	22.1135	11.9154	12.7887
$\mathbf{R}_3^1$	24.6756	11.8781	13.7495
$\mathbf{R}_3^2$	16.9766	15.2472	37.7859
$\mathbf{R}_1^I$	17.3074	17.0761	13.7146
$\mathbf{R}_2^I$	22.6272	12.4893	14.0033
$\mathbf{R}_3^I$	24.5916	12.5199	13.2469

Table 4.2: Static simulation: error standard deviation

	Yaw [μrad]	Pitch [μrad]	Roll [μrad]
$\mathbf{R}_2^1$	-12.7148	0.0641	5.7952
$\mathbf{R}_3^1$	0.5687	0.7233	-0.2791
$\mathbf{R}_3^2$	-0.6587	-6.0743	-13.2835
$\mathbf{R}_1^I$	0.0865	0.9937	-0.1946
$\mathbf{R}_2^I$	-12.6283	6.7889	-0.2588
$\mathbf{R}_3^I$	0.6552	0.7146	-0.9178

Table 4.3: Dynamic simulation: error mean value

	Yaw [μrad]	Pitch [μrad]	Roll [μrad]
$\mathbf{R}_2^1$	483.5578	17.2559	279.8784
$\mathbf{R}_3^1$	23.0402	12.0644	13.2217
$\mathbf{R}_3^2$	20.4484	280.5301	485.4916
$\mathbf{R}_1^I$	17.3089	16.2846	14.6804
$\mathbf{R}_2^I$	483.8340	280.1460	14.7386
$\mathbf{R}_3^I$	22.6416	12.6046	14.6273

Table 4.4: Dynamic simulation: error standard deviation

Chapter 5

Conclusion

In this work, the topic of deterministic attitude determination is addressed, with a main focus on constrained and distributed problems. As a starting point, the static attitude determination methods available, the constrained problems considered and the noise models for the sensors were concisely reviewed.

The goal of solving the attitude problem in a distributed manner, such that each vehicle can relate its attitude to another without ever sensing it, motivated the formulation of a new problem that represents a heterogeneous distributed simple formation. It consists of a three-vehicle formation, composed by one chief and two deputies, where the LOS vectors are only measured between each deputy and the chief vehicle. Moreover, each vehicle can measure an inertial reference independently.

The rotation matrices between the different frames of the problem were found deterministically and this solution was characterized as having generally a unique solution. Furthermore, in the degenerate cases there are infinite solutions and in some specific cases there are two solutions because of an ambiguity. The solution consists in, firstly, finding the relation between each of the deputies and the chief vehicle, which is called a branch. In general, each branch has two solutions. Then, this ambiguity is resolved by determining the inertial attitude of the chief vehicle using each of the candidates, which will generally result in an unique rotation that satisfies both branches. Furthermore, it was shown that the relation between the chief and one of the deputies could be solved unambiguously if the vector measurements of that branch are in the same plane.

The cases where there are infinite solutions were shown to be the situations where the LOS and inertial measurements of some vehicle were parallel or when the inertial measurements of a deputy was parallel to the inertial measurement of the chief. It was also shown that the situations where there is an ambiguity are situations where there is a rotation of the LOS and inertial measurements of one of the deputies about the inertial measurement vector of the chief that makes all LOS measurements and the inertial measurement of the chief vehicle coplanar and simultaneously all the inertial measurements coplanar. These specific cases of the characterization were defined mathematically.

Simulations of the solution developed were carried out, including noise in the measurements, according, to a model based on the FPD sensors. The simulations resulted in mean error near zero and standard deviation was coherent with the estimation error of other state-of-the-art solutions to constrained formation problems.

5.1 Future work

The work done so far opens up several new problems and challenges. Firstly, there are problems directly related with the formulated problem such as the computation of the theoretical covariance, for

example using perturbation theory, and the determination of the optimal solution, which is interesting for obtaining the best results out of the problem. Naturally, it is also of interest to experimentally evaluate the proposed solution.

Other problems can be inspired in the one that is defined here, such as the generalization to a bigger formation and the addition of weights to the measurements according to their importance.

Bibliography

[1] Y. Cheng, J. L. Crassidis, and F. L. Markley, "Attitude determination for large field-of-view sensors," *The Journal of the Astronautical Sciences*, vol. 54, no. 3-4, pp. 433–448, 2006.

[2] J. M. Marr, R. L. Snell, and S. E. Kurtz, *Fundamentals of Radio Astronomy: Observational Methods*. CRC Press, 1 ed., 2015.

[3] R. Science and T. O. Group., "Radioastron user handbook." RadioAstron Science and Technical Operations Group., 2016. Online; accessed 1-August-2017; `http://www.asc.rssi.ru/radioastron/documents/rauh/en/rauh.pdf`.

[4] N. R. A. Observatory., "Very large array." National Radio Astronomy Observatory., 2017. Online; accessed 1-August-2017; `http://www.vla.nrao.edu/`.

[5] A. S. C. of Lebedev Physical Institute., "Spektr-r radio telescope." Astro Space Center of Lebedev Physical Institute., 2017. Online; accessed 1-Spetember-2017; `http://www.asc.rssi.ru`.

[6] N. R. A. Observatory., "Vla south." National Radio Astronomy Observatory., 2017. Online; accessed 1-Spetember-2017; `http://images.nrao.edu`.

[7] F. L. Markley and J. L. Crassidis, *Fundamentals of Spacecraft Attitude Determination and Control*. New York: Microcosm Press and Springer, 1 ed., 2014.

[8] M. D. Shuster and S. D. Oh, "Three-axis attitude determination from vector observations," *Journal of Guidance, Control and Dynamics*, vol. 4, pp. 70–77, Jan. 1981.

[9] G. Wahba, "A least-squares estimate of satellite attitude," *SIAM Review*, vol. 7, p. 409, July 1965.

[10] J. L. Farrell and J. C. Stuelpnagel, "A least-squares estimate of satellite attitude," *SIAM Review*, vol. 7, no. 3, pp. 384–386, 1966.

[11] F. L. Markley, "Attitude determination using vector observations: A fast optimal matrix," *Journal of the Astronautical Sciences*, vol. 41, no. 2, pp. 261–280, 1993.

[12] G. M. Lerner, "Three-axis attitude determination," in *Spacecraft Attitude Determination and Control* (J. Wertz, ed.), ch. 12, Dordrecht, Netherlands: Kluwer Academic, 4 ed., 1978.

[13] M. D. Shuster, "The quest for better attitudes," *Journal of the Astronautical Sciences*, vol. 54, no. 3/4, pp. 657–683, 2006.

[14] D. Mortari, "Esoq: A closed-form solution of the wahba problem," *Journal of the Astronautical Sciences*, vol. 45, no. 2, pp. 195–204, 1997.

[15] D. Mortari, "Second estimator of the optimal quaternion," *Journal of Guidance, Control and Dynamics*, vol. 23, no. 5, pp. 885–888, 2000.

[16] R. Linares, J. L. Crassidis, and Y. Cheng, "Constrained relative attitude determination for two-vehicle formations," *Journal of Guidance, Control and Dynamics*, vol. 34, pp. 543–553, Mar. 2011.

[17] M. S. Andrle, B. Hyun, J. L. Crassidis, and R. Linares, "Deterministic relative attitude determination of formation flying spacecraft," *IAA/AAS Astrodynamics Specialist Conference and Exhibit Honolulu, Hawaii*, pp. 18–21, Aug. 2008.

[18] V. L. Pisacane, *Fundamentals of Space Systems.* New York: Oxford University Press, 1 ed., 2005.

[19] D. L. Light, *"Satellite Photogrammetry," Manual of Photogrammetry.* Falls Church, VA: edited by C. C. Slama, chap. 17, American Society of Photogrammetry, 4 ed., 1980.

[20] M. D. Shuster, "Kalman filtering of spacecraft attitude and the quest model," *The Journal of the Astronautical Sciences*, vol. 38, pp. 377–393, Sept. 1990.

[21] G. Arfken, *Mathematical Methods for Physicists.* Academic Press, 3 ed., 1985.

[22] L. Han and M. Neumann, "Basic linear algebra: Inner product spaces," in *Handbook of Linear Algebra* (L. Hogben, ed.), ch. 5, Chapman and Hall/CRC, 1 ed., 2006.

[23] W. D. Curtis, A. Janin, and K. Zikan, "A note on averaging rotations," *Virtual Reality Annual International Symposium, 1993 IEEE, Seattle, WA, USA*, pp. 377–385, Sept. 1993.

[24] J. E. Gentle, *Computational Statistics.* Satistics and Computing, New York: Springer-Verlag, 1 ed., 2009.

[25] J. E. Gentle, *Numerical Linear Algebra for Applications in Statistics.* New York: Springer-Verlag, 1 ed., 1998.

Appendix A

Rotation Matrices

A.1 Properties of the Product Between Proper Rotations

The product between two proper rotation matrices keeps the properties that define a proper rotation matrix. These properties are orthogonality and the determinant equal to 1. The determinant is constant because

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

and the orthogonality is kept as addressed in the next lemma.

Lemma 19. *If* $\mathbf{AA}^T = \mathbf{I}$ *and* $\mathbf{BB}^T = \mathbf{I}$ *then*

$$\mathbf{AB}(\mathbf{AB})^T = \mathbf{I} \tag{A.1}$$

and

$$(\mathbf{AB})^T\mathbf{AB} = \mathbf{I}\ . \tag{A.2}$$

Proof. By direct computation, taking the left expression of (A.1) and applying the transpose properties of the product between two matrices gives

$$\mathbf{AB}(\mathbf{AB})^T = \mathbf{ABB}^T\mathbf{A}^T\ .$$

As $\mathbf{BB}^T = \mathbf{I}$, it follows that

$$\mathbf{AB}(\mathbf{AB})^T = \mathbf{AA}^T\ .$$

As $\mathbf{AA}^T = \mathbf{I}$ it follows that

$$\mathbf{AB}(\mathbf{AB})^T = \mathbf{I}\ .$$

Next, taking the left expression of (A.2) and applying the transpose properties of the product between two matrices gives

$$(\mathbf{AB})^T\mathbf{AB} = \mathbf{B}^T\mathbf{A}^T\mathbf{AB}\ .$$

As $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, it follows that

$$(\mathbf{AB})^T\mathbf{AB} = \mathbf{BB}^T\ .$$

As $\mathbf{B}^T\mathbf{B} = \mathbf{I}$ it follows that

$$(\mathbf{AB})^T\mathbf{AB} = \mathbf{I}\ .$$

□

A.2 Rotation of a Vector about a Parallel Axis

The next lemma addresses the rotation of a vector about an axis parallel to itself.

Lemma 20. *Let* $\mathbf{n} \in \mathbb{R}^3$ *and theta* $\in \mathbb{R}$. *Then the rotation of a vector about an axis parallel to the vector itself results in the same vector, i.e.,*

$$\mathbf{R}\left(\theta, \frac{\mathbf{n}}{\|\mathbf{n}\|}\right)\mathbf{n} = \mathbf{n} \ . \tag{A.3}$$

Proof. By direct computation, using (2.1) the left expression of (A.3) is rewritten as

$$\mathbf{R}\left(\theta, \frac{\mathbf{n}}{\|\mathbf{n}\|}\right)\mathbf{n} = \cos(\theta)\mathbf{I}\ \mathbf{n} + (1-\cos(\theta))\frac{\mathbf{n}}{\|\mathbf{n}\|}\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right)^T\mathbf{n} - \sin(\theta)\mathbf{S}\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right)\mathbf{n} \ ,$$

or equivalently,

$$\mathbf{R}\left(\theta, \frac{\mathbf{n}}{\|\mathbf{n}\|}\right)\mathbf{n} = \cos(\theta)\mathbf{n} + (1-\cos(\theta))\mathbf{n}\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right)^T\frac{\mathbf{n}}{\|\mathbf{n}\|} - \sin(\theta)\mathbf{S}\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right)\mathbf{n} \ . \tag{A.4}$$

Since $\frac{\mathbf{n}}{\|\mathbf{n}\|}$ has unit length, then $\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right)^T\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right) = 1$. Moreover, the cross product between two parallel vectors results in a null vector, *i.e.*, $\mathbf{S}\left(\frac{\mathbf{n}}{\|\mathbf{n}\|}\right)\mathbf{n} = \mathbf{0}$. Consequently, (A.4) yields

$$\begin{aligned}\mathbf{R}\left(\theta, \tfrac{\mathbf{n}}{\|\mathbf{n}\|}\right)\mathbf{n} &= \cos(\theta)\mathbf{n} + (1-\cos(\theta))\mathbf{n} \\ &= \cos(\theta)\mathbf{n} - \cos(\theta)\mathbf{n} + \mathbf{n} \\ &= \mathbf{n} \ .\end{aligned}$$

□

A.3 Product of Rotation Matrices about the Same Axis

Lemma 21. *Let* ϕ *and* $\lambda \in \mathbb{R}$ *represent any two angles , and* $\mathbf{n} \in \mathbb{R}^3$ *represent any unit length axis. Then,*

$$\mathbf{R}(\phi+\lambda, \mathbf{n}) = \mathbf{R}(\phi, \mathbf{n})\,\mathbf{R}(\lambda, \mathbf{n}) \tag{A.5a}$$

$$\mathbf{R}(\phi+\lambda, \mathbf{n}) = \mathbf{R}(\lambda, \mathbf{n})\,\mathbf{R}(\phi, \mathbf{n}) \tag{A.5b}$$

Proof. This proof is given by direct computation. First, take any of the right expressions of (A.5) and expand using (2.1). Without loss of generality the right expression of(A.5a) is used, which gives

$$\begin{aligned}\mathbf{R}(\phi, \mathbf{n})\,\mathbf{R}(\lambda, \mathbf{n}) = \big(\cos(\phi)\,\mathbf{I} + (1-\cos(\phi))\,\mathbf{n}\mathbf{n}^T - \sin(\phi)\,\mathbf{S}(\mathbf{n})\big) \\ \big(\cos(\lambda)\,\mathbf{I} + (1-\cos(\lambda))\,\mathbf{n}\mathbf{n}^T - \sin(\lambda)\,\mathbf{S}(\mathbf{n})\big) \ ,\end{aligned}$$

or equivalently, computing the product,

$$\begin{aligned}\mathbf{R}(\phi, \mathbf{n})\,\mathbf{R}(\lambda, \mathbf{n}) = \ & \cos(\phi)\cos(\lambda)\,\mathbf{I} + \cos(\phi)(1-\cos(\lambda))\,\mathbf{n}\mathbf{n}^T - \cos(\phi)\sin(\lambda)\,\mathbf{S}(\mathbf{n}) \\ & + (1-\cos(\phi))\cos(\lambda)\,\mathbf{n}\mathbf{n}^T + (1-\cos(\phi))(1-\cos(\lambda))\,\mathbf{n}\mathbf{n}^T\mathbf{n}\mathbf{n}^T \\ & - (1-\cos(\phi))\sin(\lambda)\,\mathbf{n}\mathbf{n}^T\mathbf{S}(\mathbf{n}) - \sin(\phi)\cos(\lambda)\,\mathbf{S}(\mathbf{n}) \\ & - \sin(\phi)(1-\cos(\lambda))\,\mathbf{S}(\mathbf{n})\,\mathbf{n}\mathbf{n}^T + \sin(\phi)\sin(\lambda)\,\mathbf{S}(\mathbf{n})\,\mathbf{S}(\mathbf{n}) \ .\end{aligned} \tag{A.6}$$

Then, noticing that $\mathbf{S}(\mathbf{n})\,\mathbf{n}\mathbf{n}^T = \mathbf{0}$ and, because $\mathbf{n}$ is a unit vector, that $\mathbf{n}\mathbf{n}^T\mathbf{n}\mathbf{n}^T = \mathbf{n}\mathbf{n}^T$, (A.6) is

rewritten as

$$\begin{aligned}\mathbf{R}(\phi,\mathbf{n})\,\mathbf{R}(\lambda,\mathbf{n}) = \quad & \cos(\phi)\cos(\lambda)\,\mathbf{I} + \cos(\phi)\,(1-\cos(\lambda))\,\mathbf{n}\mathbf{n}^T - \cos(\phi)\sin(\lambda)\,\mathbf{S}(\mathbf{n}) \\ & + (1-\cos(\phi))\cos(\lambda)\,\mathbf{n}\mathbf{n}^T + (1-\cos(\phi))\,(1-\cos(\lambda))\,\mathbf{n}\mathbf{n}^T \\ & - \sin(\phi)\cos(\lambda)\,\mathbf{S}(\mathbf{n}) + \sin(\phi)\sin(\lambda)\,\mathbf{S}(\mathbf{n})\,\mathbf{S}(\mathbf{n}) \ .\end{aligned} \tag{A.7}$$

Next, from Lemma 24, in Appendix B, it follows that (A.7) is given as

$$\begin{aligned}\mathbf{R}(\phi,\mathbf{n})\,\mathbf{R}(\lambda,\mathbf{n}) = \quad & \cos(\phi)\cos(\lambda)\,\mathbf{I} + \cos(\phi)\,(1-\cos(\lambda))\,\mathbf{n}\mathbf{n}^T - \cos(\phi)\sin(\lambda)\,\mathbf{S}(\mathbf{n}) \\ & + (1-\cos(\phi))\cos(\lambda)\,\mathbf{n}\mathbf{n}^T + (1-\cos(\phi))\,(1-\cos(\lambda))\,\mathbf{n}\mathbf{n}^T \\ & - \sin(\phi)\cos(\lambda)\,\mathbf{S}(\mathbf{n}) + \sin(\phi)\sin(\lambda)\,\mathbf{n}\mathbf{n}^T - \sin(\phi)\sin(\lambda)\,\mathbf{I}\ ,\end{aligned}$$

or equivalently,

$$\begin{aligned}\mathbf{R}(\phi,\mathbf{n})\,\mathbf{R}(\lambda,\mathbf{n}) = & \left[\cos(\phi)\cos(\lambda) - \sin(\phi)\sin(\lambda)\right]\ \mathbf{I} \\ & + \left[1 + \sin(\phi)\sin(\lambda) - \cos(\phi)\cos(\lambda)\right]\ \mathbf{n}\mathbf{n}^T \\ & - \left[\sin(\phi)\cos(\lambda) + \cos(\phi)\sin(\lambda)\right]\ \mathbf{S}(\mathbf{n})\ .\end{aligned} \tag{A.8}$$

Making use of the trigonometric identities of the sum

$$\cos(\phi+\lambda) = \cos(\phi)\cos(\lambda) - \sin(\phi)\sin(\lambda)$$

and

$$\sin(\phi+\lambda) = \sin(\phi)\cos(\lambda) + \cos(\phi)\sin(\lambda)\ ,$$

(A.8) is expressed as

$$\mathbf{R}(\phi,\mathbf{n})\,\mathbf{R}(\lambda,\mathbf{n}) = \mathbf{I}\cos(\phi+\lambda) + \mathbf{n}\mathbf{n}^T\left[1-\cos(\phi+\lambda)\right] - \mathbf{S}(\mathbf{n})\sin(\phi+\lambda)\ ,$$

which, using (2.1), gives (A.5a). □

A.4 Product between Rotation Matrices

In some situations it is useful to write the product between two rotations in a different order. The next lemma addresses that same problem.

Lemma 22. *The following are equivalent*

$$\mathbf{R}(\theta,\mathbf{x})\,\mathbf{R}(\lambda,\mathbf{y}) = \mathbf{R}(\lambda,\mathbf{R}(\theta,\mathbf{x})\,\mathbf{y})\,\mathbf{R}(\theta,\mathbf{x})\ ; \tag{A.9a}$$

$$\mathbf{R}(\theta,\mathbf{x})\,\mathbf{R}(\lambda,\mathbf{y}) = \mathbf{R}(\lambda,\mathbf{y})\,\mathbf{R}(\theta,\mathbf{R}(\lambda,\mathbf{y})\,\mathbf{x})\ ; \tag{A.9b}$$

where θ, $\lambda \in \mathbb{R}$ represent rotation angles, and $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$ are unit vectors.

Proof. This proof is given by direct computation. Without loss of generality the proof is given for (A.9a), since the proof for (A.9b) is analogous. Thus, taking the left expression of (A.9a) and expanding $\mathbf{R}(\lambda,\mathbf{y})$ using (2.1) gives

$$\mathbf{R}(\theta,\mathbf{x})\,\mathbf{R}(\lambda,\mathbf{y}) = \mathbf{R}(\theta,\mathbf{x})\left[\cos(\lambda)\ \mathbf{I} + (1-\cos(\lambda))\,\mathbf{y}\mathbf{y}^{\mathbf{T}} - \sin(\lambda)\mathbf{S}(\mathbf{y})\right]\ . \tag{A.10}$$

Distributing $\mathbf{R}(\theta, \mathbf{x})$, it follows that (A.10) is given as

$$\mathbf{R}(\theta, \mathbf{x})\,\mathbf{R}(\lambda, \mathbf{y}) = \cos(\lambda)\mathbf{R}(\theta, \mathbf{x}) + (1 - \cos(\lambda))\,\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y}\mathbf{y}^{\mathbf{T}} - \sin(\lambda)\mathbf{R}(\theta, \mathbf{x})\,\mathbf{S}(\mathbf{y}) \ ,$$

or equivalently, multiplying by $\mathbf{R}(\theta, \mathbf{x})^T\,\mathbf{R}(\theta, \mathbf{x}) = \mathbf{I}$ on the right,

$$\begin{aligned}\mathbf{R}(\theta, \mathbf{x})\,\mathbf{R}(\lambda, \mathbf{y}) = \cos(\lambda)\mathbf{R}(\theta, \mathbf{x}) + (1 - \cos(\lambda))\,\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y}\mathbf{y}^{\mathbf{T}}\mathbf{R}(\theta, \mathbf{x})^T\,\mathbf{R}(\theta, \mathbf{x}) \\ - \sin(\lambda)\mathbf{R}(\theta, \mathbf{x})\,\mathbf{S}(\mathbf{y})\,\mathbf{R}(\theta, \mathbf{x})^T\,\mathbf{R}(\theta, \mathbf{x}) \ .\end{aligned} \tag{A.11}$$

It follows that $\mathbf{R}(\theta, \mathbf{x})$ is on the right of every term, hence (A.11) is expressed as

$$\begin{aligned}\mathbf{R}(\theta, \mathbf{x})\,\mathbf{R}(\lambda, \mathbf{y}) = \Big[\cos(\lambda)\mathbf{I} + (1 - \cos(\lambda))\,\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y}\mathbf{y}^{\mathbf{T}}\mathbf{R}(\theta, \mathbf{x})^T \\ - \sin(\lambda)\mathbf{R}(\theta, \mathbf{x})\,\mathbf{S}(\mathbf{y})\,\mathbf{R}(\theta, \mathbf{x})^T\Big]\,\mathbf{R}(\theta, \mathbf{x}) \ .\end{aligned} \tag{A.12}$$

From Lemma 26 in Appendix B, it follows that (A.12) is given by

$$\mathbf{R}(\theta, \mathbf{x})\,\mathbf{R}(\lambda, \mathbf{y}) = \Big[\cos(\lambda)\mathbf{I} + (1 - \cos(\lambda))\,\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y}\mathbf{y}^{\mathbf{T}}\mathbf{R}(\theta, \mathbf{x})^T - \sin(\lambda)\mathbf{S}(\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y})\Big]\,\mathbf{R}(\theta, \mathbf{x}) \ ,$$

or equivalently,

$$\mathbf{R}(\theta, \mathbf{x})\,\mathbf{R}(\lambda, \mathbf{y}) = \Big[\cos(\lambda)\mathbf{I} + (1 - \cos(\lambda))\,\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y}\,(\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y})^T - \sin(\lambda)\mathbf{S}(\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y})\Big]\,\mathbf{R}(\theta, \mathbf{x}) \ . \tag{A.13}$$

Finally, it is noticed that the left term of of the right expression of (A.13) has the same structure as (2.1) with $\mathbf{R}(\theta, \mathbf{x})\,\mathbf{y}$ as the axis of rotation and λ as the angle of rotation, thus yielding (A.9a).

□

A.5 Transpose of a Rotation Matrix

Lemma 23. *Consider the rotation in S(3) parametrized by as* $\mathbf{R}(\theta, \mathbf{n})$*, with* $\theta \in \mathbb{R}$ *and* $\mathbf{n} \in \mathbb{R}^3$*. Then,*

$$\mathbf{R}(\theta, \mathbf{n})^T = \mathbf{R}(-\theta, \mathbf{n}) \ . \tag{A.14}$$

Proof. This proof is given by direct computation. Take the left hand side of (A.14) and expand it using (2.1), as given by

$$\mathbf{R}(\theta, \mathbf{n})^T = \big[\cos(\theta)\,\mathbf{I} + (1 - \cos(\theta))\mathbf{n}\,\mathbf{n}^T - \sin(\theta)\,\mathbf{S}(n)\big]^T \ ,$$

or equivalently,

$$\mathbf{R}(\theta, \mathbf{n})^T = [\cos(\theta)\,\mathbf{I}]^T + \big[(1 - \cos(\theta))\,\mathbf{n}\,\mathbf{n}^T\big]^T - [\sin(\theta)\,\mathbf{S}(n)]^T \ .$$

Then, applying the transpose properties gives

$$\mathbf{R}(\theta, \mathbf{n})^T = \cos(\theta)\,\mathbf{I}^T + (1 - \cos(\theta))\,\mathbf{n}\,\mathbf{n}^T - \sin(\theta)\,\mathbf{S}(n)^T \ ,$$

which, recalling that by definition $\mathbf{S}(n)^T = -\mathbf{S}(n)$ and the transpose of an identity matrix, is expressed as

$$\mathbf{R}(\theta, \mathbf{n})^T = \cos(\theta)\,\mathbf{I} + (1 - \cos(\theta))\,\mathbf{n}\,\mathbf{n}^T + \sin(\theta)\,\mathbf{S}(n) \ . \tag{A.15}$$

Notice that $\cos(\theta) = \cos(-\theta)$ and that $\sin(\theta) = -\sin(-\theta)$, which means that (A.15) is equivalent to

$$\mathbf{R}\left(\theta, \mathbf{n}\right)^T = \cos\left(-\theta\right)\mathbf{I} + \left(1 - \cos(-\theta)\right)\mathbf{n}\ \mathbf{n}^T - \sin\left(-\theta\right)\mathbf{S}\left(n\right)\ . \tag{A.16}$$

Since (A.16) has the same structure of (2.1) but with the angle $-\theta$, the proof is complete. □

Corollary 6. *Consider the rotation in S(3) parametrized by* $\mathbf{R}\left(\theta, \mathbf{n}\right)$*, with* $\theta \in \mathbb{R}$ *and* $\mathbf{n} \in \mathbb{R}^3$*. If* $\theta = \pi$ *then*

$$\mathbf{R}\left(\pi, \mathbf{n}\right) = \mathbf{R}\left(\pi, \mathbf{n}\right)^T\ . \tag{A.17}$$

Proof. This proof is given by direct computation taking (A.17) and applying Lemma 23 to the left hand side of (A.17), which gives

$$\mathbf{R}\left(\pi, \mathbf{n}\right) = \mathbf{R}\left(-\pi, \mathbf{n}\right)\ .$$

The proof is complete because the angles are applied to trigonometric functions with a period of 2π, as seen in (2.1), and $\pi - (-\pi) = 2\pi$. □

Appendix B

Skew-symmetric matrix

B.1 Square skew-symmetric matrix

In this appendix, some properties of the skew symmetric matrix are shown considering unit vectors in $\mathbb{R}^3$, which is the case of the measurements of the scenario studied in this work.

Lemma 24. *Let* $\mathbf{x} = [x_1, x_2, x_3]^T$ *be a unit vector in* $\mathbb{R}^3$*. Then*

$$\mathbf{S}(\mathbf{x})^2 = -\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + \mathbf{x}\mathbf{x}^T \ . \tag{B.1}$$

Proof. Given by direct computation. Firstly, observe that $\mathbf{S}(\mathbf{x})^2 = \mathbf{S}(\mathbf{x})\,\mathbf{S}(\mathbf{x})$, which representing the matrix elements considering Definition (2.1), is rewritten as

$$\mathbf{S}(\mathbf{x})^2 = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} ,$$

or equivalently, by computing the matrix product,

$$\mathbf{S}(\mathbf{x})^2 = \begin{bmatrix} -x_2^2 - x_3^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & -x_1^2 - x_3^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & -x_1^2 - x_2^2 \end{bmatrix} . \tag{B.2}$$

Next, expand the matrix elements of the right expression of (B.1), which gives

$$-\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + \mathbf{x}\mathbf{x}^T = \begin{bmatrix} \text{-1} & 0 & 0 \\ 0 & \text{-1} & 0 \\ 0 & 0 & \text{-1} \end{bmatrix} + \begin{bmatrix} x_1x_1 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2x_2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3x_3 \end{bmatrix} . \tag{B.3}$$

Notice that for unit vectors in $\mathbb{R}^3$

$$x_1^2 + x_2^2 + x_3^2 = 1 \ ,$$

which means that

$$\begin{cases} x_1^2 - 1 = -x_2^2 - x_3^2 \\ x_2^2 - 1 = -x_1^2 - x_3^2 \\ x_3^2 - 1 = -x_2^2 - x_1^2 \end{cases} . \tag{B.4}$$

Hence, (B.3) becomes

$$-\mathbf{I}_{\mathbf{3}\times\mathbf{3}} + \mathbf{x}\mathbf{x}^T = \begin{bmatrix} -x_2^2 - x_3^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & -x_1^2 - x_3^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & -x_1^2 - x_2^2 \end{bmatrix} ,$$

which is equal to (B.2). □

B.2 Third power of a skew-symmetric matrix

Lemma 25. *Let* $\mathbf{x} = [x_1, x_2, x_3]^T$ *be a unit vector in* $\mathbb{R}^3$*. Then*

$$\mathbf{S}(\mathbf{x})^3 = -\mathbf{S}(\mathbf{x}) \ , \tag{B.5}$$

Proof. Given by direct computation taking the left expression of (B.5) and making use of (B.2), which gives

$$\mathbf{S}(\mathbf{x})^3 = \mathbf{S}(\mathbf{x})^2 \mathbf{S}(\mathbf{x}) = \begin{bmatrix} -x_2^2 - x_3^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & -x_1^2 - x_3^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & -x_2^2 - x_1^2 \end{bmatrix} \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} . \tag{B.6}$$

Using (B.4), (B.6) becomes

$$\mathbf{S}(\mathbf{x})^3 = \begin{bmatrix} x_1^2 - 1 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 - 1 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 - 1 \end{bmatrix} \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} ,$$

which, by computing the matrix product, yields

$$\mathbf{S}(\mathbf{x})^3 = \begin{bmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{bmatrix} .$$

This equation is equivalent to $-\mathbf{S}(\mathbf{x})$ as seen from (2.1). □

B.3 Rotation of skew-symmetric matrix

Lemma 26. *Let* $\mathbf{x} \in \mathbb{R}^3$ *and* $\tilde{\mathbf{R}}$ *be any rotation matrix in* SO(3)*. Then,*

$$\mathbf{S}\left(\tilde{\mathbf{R}}\mathbf{x}\right) = \tilde{\mathbf{R}}\mathbf{S}(\mathbf{x})\tilde{\mathbf{R}}^T \ . \tag{B.7}$$

Proof. Define the components of $\mathbf{x}$ and $\tilde{\mathbf{R}}$ as

$$\mathbf{x} = [x_1 \ x_2 \ x_3]^T \tag{B.8}$$

and

$$\tilde{\mathbf{R}} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} , \tag{B.9}$$

where all components are in $\mathbb{R}$. It follows that the elements of $\tilde{\mathbf{R}}\mathbf{x}$ are given by

$$\tilde{\mathbf{R}}\mathbf{x} = \begin{bmatrix} x_1 r_1 + x_2 r_2 + x_3 r_3 \\ x_1 r_4 + x_2 r_5 + x_3 r_6 \\ x_1 r_7 + x_2 r_8 + x_3 r_9 \end{bmatrix} .$$

Then, using Definition 2.1 one has that

$$\mathbf{S}\left(\tilde{\mathbf{R}}\mathbf{x}\right) = \begin{bmatrix} 0 & -x_1 r_7 - x_2 r_8 - x_3 r_9 & x_1 r_4 + x_2 r_5 + x_3 r_6 \\ x_1 r_7 + x_2 r_8 + x_3 r_9 & 0 & -x_1 r_1 - x_2 r_2 - x_3 r_3 \\ -x_1 r_4 - x_2 r_5 - x_3 r_6 & x_1 r_1 + x_2 r_2 + x_3 r_3 & 0 \end{bmatrix} . \tag{B.10}$$

Now, taking the right expression of (B.7) and substituing (B.9) and using Definition 2.1 with (B.8) gives

$$\tilde{\mathbf{R}}\mathbf{S}\left(\mathbf{x}\right)\tilde{\mathbf{R}}^T = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix}^T .$$

Computing the first product and applying the transpose yields

$$\tilde{\mathbf{R}}\mathbf{S}\left(\mathbf{x}\right)\tilde{\mathbf{R}}^T = \begin{bmatrix} r_2 x_3 - r_3 x_2 & r_3 x_1 - r_1 x_3 & r_1 x_2 - r_2 x_1 \\ r_5 x_3 - r_6 x_2 & r_6 x_1 - r_4 x_3 & r_4 x_2 - r_5 x_1 \\ r_8 x_3 - r_9 x_2 & r_9 x_1 - r_7 x_3 & r_7 x_2 - r_8 x_1 \end{bmatrix} \begin{bmatrix} r_1 & r_4 & r_7 \\ r_2 & r_5 & r_8 \\ r_3 & r_6 & r_9 \end{bmatrix} . \tag{B.11}$$

Applying the remaining matrix product, the first, second and third columns of (B.11) are, respectively, given as

$$\begin{bmatrix} r_1(r_2 x_3 - r_3 x_2) + r_2(r_3 x_1 - r_1 x_3) + r_3(r_1 x_2 - r_2 x_1) \\ r_1(r_5 x_3 - r_6 x_2) + r_2(r_6 x_1 - r_4 x_3) + r_3(r_4 x_2 - r_5 x_1) \\ r_1(r_8 x_3 - r_9 x_2) + r_2(r_9 x_1 - r_7 x_3) + r_3(r_7 x_2 - r_8 x_1) \end{bmatrix} ,$$

$$\begin{bmatrix} r_4(r_2 x_3 - r_3 x_2) + r_5(r_3 x_1 - r_1 x_3) + r_6(r_1 x_2 - r_2 x_1) \\ r_4(r_5 x_3 - r_6 x_2) + r_5(r_6 x_1 - r_4 x_3) + r_6(r_4 x_2 - r_5 x_1) \\ r_4(r_8 x_3 - r_9 x_2) + r_5(r_9 x_1 - r_7 x_3) + r_6(r_7 x_2 - r_8 x_1) \end{bmatrix} ,$$

$$\begin{bmatrix} r_7(r_2 x_3 - r_3 x_2) + r_8(r_3 x_1 - r_1 x_3) + r_9(r_1 x_2 - r_2 x_1) \\ r_7(r_5 x_3 - r_6 x_2) + r_8(r_6 x_1 - r_4 x_3) + r_9(r_4 x_2 - r_5 x_1) \\ r_7(r_8 x_3 - r_9 x_2) + r_8(r_9 x_1 - r_7 x_3) + r_9(r_7 x_2 - r_8 x_1) \end{bmatrix} .$$

Then, it follows that the diagonals are zero, as is seen by, respectively, computing the first, the second and the third elements of the first, second and third columns, as given by

$$r_1 r_2 x_3 - r_1 r_3 x_2 + r_2 r_3 x_1 - r_2 r_1 x_3 + r_3 r_1 x_2 - r_3 r_2 x_1 = 0 ,$$

$$r_4 r_5 x_3 - r_4 r_6 x_2 + r_5 r_6 x_1 - r_5 r_4 x_3 + r_6 r_4 x_2 - r_6 r_5 x_1 = 0$$

and

$$r_7 r_8 x_3 - r_7 r_9 x_2 + r_8 r_9 x_1 - r_8 r_7 x_3 + r_9 r_7 x_2 - r_9 r_8 x_1 = 0 .$$

Substituting the previous values and rearranging, the first, second and third columns are, respectively,

rewritten as

$$\begin{bmatrix} 0 \\ x_1(r_2r_6 - r_3r_5) + x_2(r_3r_4 - r_1r_6) + x_3(r_1r_5 - r_2r_4) \\ x_1(r_2r_9 - r_3r_8) + x_2(r_3r_7 - r_1r_9) + x_3(r_1r_8 - r_2r_7) \end{bmatrix} , \tag{B.12a}$$

$$\begin{bmatrix} x_1(r_3r_5 - r_2r_6) + x_2(r_1r_6 - r_3r_4) + x_3(r_2r_4 - r_1r_5) \\ 0 \\ x_1(r_5r_9 - r_6r_8) + x_2(r_4r_9 - r_6r_7) + x_3(r_4r_8 - r_5r_7) \end{bmatrix} , \tag{B.12b}$$

$$\begin{bmatrix} x_1(r_3r_8 - r_2r_9) + x_2(r_1r_9 - r_3r_7) + x_3(r_2r_7 - r_1r_8) \\ x_1(r_6r_8 - r_5r_9) + x_2(r_4r_9 - r_6r_7) + x_3(r_5r_7 - r_4r_8) \\ 0 \end{bmatrix} . \tag{B.12c}$$

Furthermore, as the rotation matrix is in $SO(3)$, then its elements can be written in terms of three angles, α, β and $\gamma \in \mathbb{R}$, that represent the angles of the decomposition of the rotation in three rotations with orthogonal axis as given by [7]

$$\tilde{\mathbf{R}} = \mathbf{R}\left(\alpha, \mathbf{z}\right) \mathbf{R}\left(\beta, \mathbf{y}\right) \mathbf{R}\left(\gamma, \mathbf{z}\right) \ ,$$

where $\mathbf{z}$ and $\mathbf{y} \in \mathbb{R}^3$ are orthogonal unit vectors aligned with two reference frame axes. α, β and γ represent a symmetric Euler angle set. Then, the elements of $\tilde{\mathbf{R}}$ are also written as follows

$$\tilde{\mathbf{R}} = \begin{bmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & -\cos\alpha\cos\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\gamma & -\sin\alpha\cos\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta \\ -\sin\beta\cos\gamma & \sin\beta\sin\gamma & \cos\beta \end{bmatrix} .$$

Hence, each element can be written as function of other elements of the matrix, as expressed by

$$\begin{aligned} r_1 &= r_5r_9 - r_6r_8 \ , \\ r_2 &= r_6r_7 - r_4r_9 \ , \\ r_3 &= r_4r_8 - r_5r_7 \ , \\ r_4 &= r_3r_8 - r_2r_9 \ , \\ r_5 &= r_1r_9 - r_3r_7 \ , \\ r_6 &= r_2r_7 - r_1r_8 \ , \\ r_7 &= r_2r_6 - r_3r_5 \ , \\ r_8 &= r_3r_4 - r_1r_6 \ , \\ r_9 &= r_1r_5 - r_2r_4 \ . \end{aligned}$$

So, (B.12a), (B.12b) and (B.12c), again in matrix form, are expressed as

$$\tilde{\mathbf{R}}\mathbf{S}\left(\mathbf{x}\right)\tilde{\mathbf{R}}^T = \begin{bmatrix} 0 & -x_1r_7 - x_2r_8 - x_3r_9 & x_1r_4 + x_2r_5 + x_3r_6 \\ x_1r_7 + x_2r_8 + x_3r_9 & 0 & -x_1r_1 - x_2r_2 - x_3r_3 \\ -x_1r_4 - x_2r_5 - x_3r_6 & x_1r_1 + x_2r_2 + x_3r_3 & 0 \end{bmatrix} . \tag{B.13}$$

Finally, it is noticed that (B.13) and (B.10) are equivalent. □

B.4 Skew Symmetric of a Cross Product

Lemma 27. *Consider* $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^3$. *Then,*

$$\mathbf{S}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right) = \mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right) \ . \tag{B.14}$$

Proof. Given by substitution. Firstly, consider the element representation of $\mathbf{a}$ and $\mathbf{b}$ given as

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad , \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \ . \tag{B.15}$$

Then, compute $\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}$ applying (2.1), which gives

$$\mathbf{S}\left(\mathbf{a}\right)\mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} ,$$

which applying Definition 2.1, yields the left expression of (B.14), given as

$$\mathbf{S}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right) = \begin{bmatrix} 0 & a_2b_1 - a_1b_2 & a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 & 0 & a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 & 0 \end{bmatrix} \ . \tag{B.16}$$

Next, representing the right expression of using (B.14) Definition 2.1 and (B.15) gives

$$\begin{aligned} \mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right) = & \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \\ & - \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} , \end{aligned}$$

which after computing the matrix products is expressed as

$$\begin{aligned} \mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right) = & \begin{bmatrix} -a_3b_3 - a2b_2 & a_2b_1 & a_3b_1 \\ a_1b_2 & -a_1b_1 - a_3b_3 & a_3b_2 \\ a_1b_3 & a_2b_3 & -a_1b_1 - a_2b_2 \end{bmatrix} \\ & - \begin{bmatrix} -a_3b_3 - a2b_2 & a_1b_2 & a_1b_3 \\ a_2b_1 & -a_1b_1 - a_3b_3 & a_2b_3 \\ a_3b_1 & a_3b_2 & -a_1b_1 - a_2b_2 \end{bmatrix} , \end{aligned}$$

or equivalently,

$$\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right) = \begin{bmatrix} 0 & a_2b_1 - a_1b_2 & a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 & 0 & a_3b_2 - a_2b_3 \\ a_1b_3 - a_3b_1 & a_2b_3 - a_3b_2 & 0 \end{bmatrix} \ . \tag{B.17}$$

The proof is concluded because (B.16) and (B.17) are equal. □

Appendix C

Trigonometric Function Solution

In this appendix, two different approaches are provided to solve the trigonometric equation given by

$$a_p = a_c \ \cos(\theta) + a_s \ \sin(\theta) \ , \tag{C.1}$$

where a_p, a_s and a_c are known scalars.

C.1 Square Method

A first method to solve (C.1) is by squaring both sides of the equation, after a convenient rearrangement. This operation will cause the number of solutions to double from two to four, however, two of them can be ruled out by substituting the results in the initial equation and verifying which are true.

Lemma 28. *Consider the trigonometric equation* (C.1) *and assume that*

$$a_s^2 + a_c^2 \neq 0$$

and

$$\left| \frac{a_p}{\sqrt{a_s^2 + a_c^2}} \right| \leq 1 \ .$$

Then, the solutions for θ that satisfy (C.1) *are a subset of*

$$\begin{cases} \arcsin\left(\frac{a_s a_p \pm a_c \sqrt{a_s^2 + a_c^2 - a_p^2}}{a_s^2 + a_c^2}\right) + 2\pi k \\ \pi - \arcsin\left(\frac{a_s a_p \pm a_c \sqrt{a_s^2 + a_c^2 - a_p^2}}{a_s^2 + a_c^2}\right) + 2\pi k \end{cases} , k \in \mathbb{Z} \ .$$

Proof. Rearranging (C.1) gives

$$a_p - a_s \ \sin(\theta) = a_c \ \cos(\theta) \ . \tag{C.2}$$

Squaring both sides of (C.2) yields

$$[a_p - a_s \ \sin(\theta)]^2 = [a_c \ \cos(\theta)]^2 \ ,$$

or equivalently,

$$a_p^2 - a_p a_s \ \sin(\theta) + a_s^2 \ \sin^2(\theta) = a_c^2 \ \cos^2(\theta) \ . \tag{C.3}$$

Next, using the trigonometric identity

$$\sin^2(\theta) + \cos^2(\theta) = 1 \ ,$$

it follows that (C.3) is given as

$$a_p^2 - 2a_p a_s \ \sin(\theta) + a_s^2 \ \sin^2(\theta) = a_c^2 \ \left(1 - \sin^2(\theta)\right) \ ,$$

which can be rearranged into

$$\left(a_s^2 + a_c^2\right) \ \sin^2(\theta) - 2a_p a_s \ \sin(\theta) + a_p^2 - a_c^2 = 0 \ .$$

This rearrangement emphasizes the quadratic function form, where $\sin(\theta)$ is the variable. Consequently, the quadratic formula is applied, so that a solution can be derived, yielding

$$\sin(\theta) = \frac{2a_s a_p \pm \sqrt{\left(-2a_s a_p\right)^2 - 4\left(a_s^2 + a_c^2\right)\left(a_p^2 - a_c^2\right)}}{2\left(a_s^2 + a_c^2\right)} \ ,$$

which is simplified to

$$\sin(\theta) = \frac{a_s a_p \pm |a_c|\sqrt{a_s^2 + a_c^2 - a_p^2}}{a_s^2 + a_c^2} \ .$$

□

Applying the inverse sine function results, in general, in four angles, but only two of them verify (C.1), considering $\theta \in [0, 2\pi[$. So, after substituting the angles in (C.1) and checking if it is verified, only two solutions remain, generally.

The verification step to find the correct solutions of the equation is more complex to analyse, hence, the second method, where both solutions are found directly from the expression, was derived to simplify future analysis.

C.2 Wave Equation Method

The second method to solve (C.1) is found by simplifying the right side of the trigonometric equation (C.1) into a wave equation.

Lemma 29. *Consider the trigonometric equation* (C.1) *and assume that*

$$a_s^2 + a_c^2 \neq 0$$

and that

$$\left|\frac{a_p}{\sqrt{a_s^2 + a_c^2}}\right| \leq 1 \ .$$

Then

$$\theta = \operatorname{atan2}(a_s, a_c) \pm \arccos\left(\frac{a_p}{\sqrt{a_s^2 + a_c^2}}\right) .$$

Proof. Let

$$\lambda := \operatorname{atan2}\left(a_s, \ a_c\right) \ ,$$

which means that

$$\sin(\lambda) = \frac{a_s}{\sqrt{a_s^2 + a_c^2}} \ , \tag{C.4a}$$

$$\cos(\lambda) = \frac{a_c}{\sqrt{a_s^2 + a_c^2}} \ . \tag{C.4b}$$

Next, rewrite (C.1) as given by [1]

$$a_p = \sqrt{a_s^2 + a_c^2} \left(\frac{a_c}{\sqrt{a_s^2 + a_c^2}} \cos(\theta) + \frac{a_s}{\sqrt{a_s^2 + a_c^2}} \sin(\theta) \right) \ . \tag{C.5}$$

Rearranging (C.5) gives

$$\frac{a_p}{\sqrt{a_s^2 + a_c^2}} = \frac{a_c}{\sqrt{a_s^2 + a_c^2}} \cos(\theta) + \frac{a_s}{\sqrt{a_s^2 + a_c^2}} \sin(\theta) \ . \tag{C.6}$$

It becomes clear from (C.4) that (C.6) can be expressed as

$$\frac{a_p}{\sqrt{a_s^2 + a_c^2}} = \cos(\lambda)\cos(\theta) + \sin(\lambda)\sin(\theta) \ . \tag{C.7}$$

Then, from the trigonometric identity of the cosine of the difference given as

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \ ,$$

(C.7) yields

$$\cos(\theta - \lambda) = \frac{a_p}{\sqrt{a_s^2 + a_c^2}} \ . \tag{C.8}$$

Finally, applying the inverse cosine to (C.8) leads to the solution for the angle, which given as

$$\theta = \lambda \pm \arccos\left(\frac{a_p}{\sqrt{a_s^2 + a_c^2}} \right) = \operatorname{atan2}(a_s, a_c) \pm \arccos\left(\frac{a_p}{\sqrt{a_s^2 + a_c^2}} \right) \ .$$

□

Remark. As can be seen in (C.5), the construction of this solution requires that

$$\left(a_s^2 + a_c^2\right) \neq 0 \ ,$$

because the denominator $\sqrt{a_s^2 + a_c^2}$ is introduced. This condition is satisfied whenever both $a_s \neq 0$ and $a_c \neq 0$. Moreover, from the analysis of the coefficients definition in (3.29), the only cases when $a_s = 0$ and $a_c = 0$ are either when $\mathbf{d}_1 = \pm\mathbf{d}_{1/2}$ or when $\mathbf{d}_2 = \pm\mathbf{d}_{2/1}$. These two situations correspond to the degenerate cases of the problem, as seen in Lemma 1, which have infinite solutions. Hence, this method can be applied as long as the problem has a finite number of solutions.

[1] The right side of (C.5) corresponds to a wave equation with phase λ and amplitude $\left(\sqrt{a_s^2 + a_c^2}\right)$.

Appendix D

Alternative Inertial Attitude Determination

Another option to obtain $\mathbf{R}_1^I$ is using the same method employed for the partial solution. So, the rotation matrix is decomposed into two separate rotations analogously to (3.3). In order to adapt the problem to this solution, the inner products are used instead of (3.52) and (3.53), which become analogous to (3.2). So, (3.52) and (3.53), respectively, become

$$^I\mathbf{d}_2 \mathbf{R}_1^I \ \mathbf{R}_2^1 \ \mathbf{d}_2 = 1 \ ,$$

$$^I\mathbf{d}_3 \mathbf{R}_1^I \ \mathbf{R}_3^1 \ \mathbf{d}_3 = 1 \ .$$

In this case, there is the particularity that the vectors are always coplanar when represented in the same frame, because there are only two vectors that are known in two different frames. That means that, by construction, both

$$^I\mathbf{d}_2^T \mathbf{S}\left(^I\mathbf{d}_1\right) \mathbf{R}_1^I \ \mathbf{R}_2^1 \ \mathbf{d}_2 = 0$$

and

$$^I\mathbf{d}_3^T \mathbf{S}\left(^I\mathbf{d}_1\right) \mathbf{R}_1^I \ \mathbf{R}_3^1 \ \mathbf{d}_3 = 0$$

are verified. This method is derived from the partial problem solution and therefore, an analogous result to Corollary 1 is obtained. Firstly, to simplify the notation define

$$\mathbf{R}^* := \begin{cases} \mathbf{R}\left(\pi, \frac{^I\mathbf{d}_1 + \mathbf{d}_1}{\|^I\mathbf{d}_1 + \mathbf{d}_1\|}\right) & , ^I\mathbf{d}_1 \neq -\mathbf{d}_1 \\ \mathbf{R}\left(\pi, \frac{^I\mathbf{d}_1 \times ^I\mathbf{d}_2}{\|^I\mathbf{d}_1 \times ^I\mathbf{d}_2\|}\right) & , ^I\mathbf{d}_1 = -\mathbf{d}_1 \end{cases} . \tag{D.1}$$

Note that the use of $^I\mathbf{d}_2$ in the second branch of (D.1) is a vector choice, which the only requirement is to be different from $^I\mathbf{d}_1$. Furthermore, note that $\mathbf{R}^*$ verifies (3.51) when it substitutes $\mathbf{R}_1^I$, analogously to Corollary 1.

The next lemmas give an alternative expression for the candidates of $\mathbf{R}_1^I$.

Lemma 30. *Let $\mathbf{R}_2^1$ be one of the candidates obtained from Corollary 1. Then, the inertial rotation matrix candidate for $\mathbf{R}_1^I$ is given by*

$$\mathbf{R}_1^I = \mathbf{R}\left(\operatorname{atan2}\left(^I\mathbf{d}_2^T \mathbf{S}\left(^I\mathbf{d}_1\right) \mathbf{R}^* \mathbf{R}_2^1 \mathbf{d}_2, ^I\mathbf{d}_2^T \mathbf{S}\left(^I\mathbf{d}_1\right)^2 \mathbf{R}^* \mathbf{R}_2^1 \mathbf{d}_2\right), ^I\mathbf{d}_1\right) \mathbf{R}^* \ . \tag{D.2}$$

Proof. An analogous construction to Corollary 1 can be made, but the result is slightly simpler because

of the coplanarity of the measurements. Hence, the proof is omitted.

Lemma 31. *Let* $\mathbf{R}_3^1$ *be one of the candidates obtained from Corollary 4. Then, the inertial rotation matrix candidate for* $\mathbf{R}_1^I$ *is given by*

$$\mathbf{R}_1^I = \mathbf{R}\left(\operatorname{atan2}\left({}^I\mathbf{d}_3^T\mathbf{S}\left({}^I\mathbf{d}_1\right)\mathbf{R}^*\mathbf{R}_3^1\mathbf{d}_3, {}^I\mathbf{d}_3^T\mathbf{S}\left({}^I\mathbf{d}_1\right)^2\mathbf{R}^*\mathbf{R}_3^1\mathbf{d}_3\right), {}^I\mathbf{d}_1\right)\mathbf{R}^* \ . \tag{D.3}$$

Proof. Analogous to Lemma 30 and thus it is omitted.

Appendix E

Vector Calculus

E.1 Normal Vectors Generation

This appendix addresses the relation between the vectors that are orthogonal to the same normal. Firstly, the different possible vectors that generate the same normal vector are considered in the next lemma.

Lemma 32. *Let* $\mathbf{a},\ \mathbf{b} \in \mathbb{R}^3$ *and* $\theta \in \mathbb{R}$*. Then,*

$$\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|} = \pm\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{R}\left(\theta, \frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{R}\left(\theta, \frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b}\right\|} \ . \tag{E.1}$$

Proof. This proof is given by direct computation. Taking $\mathbf{S}\left(a\right)\mathbf{R}\left(\theta, \frac{\mathbf{S}(a)b}{\mathbf{S}(a)b}\right)b$ from (E.1) and expanding $\mathbf{R}\left(\theta, \frac{\mathbf{S}(a)b}{\mathbf{S}(a)b}\right)$ according to (3.31) gives

$$\mathbf{S}\left(\mathbf{a}\right)\mathbf{R}\left(\theta, \frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}\right)\mathbf{b} = \mathbf{S}\left(\mathbf{a}\right)\left[\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right)^T}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|^2} - \frac{\cos\theta}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|^2}\mathbf{S}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right)^2 - \frac{\sin\theta}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|}\mathbf{S}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right)\right]\mathbf{b} \ . \tag{E.2}$$

From Lemma 27, $\mathbf{S}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right) = \mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right)$. Hence, (E.2) is rewritten as

$$\begin{aligned}\mathbf{S}\left(\mathbf{a}\right)\mathbf{R}\left(\theta, \frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}\right)\mathbf{b} = \mathbf{S}\left(\mathbf{a}\right)\Bigg[&\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right)^T}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|^2} - \frac{\cos\theta}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|^2}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right)\right)^2 \\ &- \frac{\sin\theta}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right)\right)\Bigg]\mathbf{b}\end{aligned}$$

or, equivalently,

$$\begin{aligned}\mathbf{S}\left(\mathbf{a}\right)\mathbf{R}\left(\theta, \frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}\right)\mathbf{b} = \mathbf{S}\left(\mathbf{a}\right)\Bigg[&\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right)^T}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|^2} - \frac{\cos\theta}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|^2}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right)\right. \\ &\left.-\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) + \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right)\right) \\ &- \frac{\sin\theta}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|}\left(\mathbf{S}\left(\mathbf{a}\right)\mathbf{S}\left(\mathbf{b}\right) - \mathbf{S}\left(\mathbf{b}\right)\mathbf{S}\left(\mathbf{a}\right)\right)\Bigg]\mathbf{b} \ .\end{aligned}$$

Distributing $\mathbf{S}(\mathbf{a})$ and $\mathbf{b}$ gives

$$\mathbf{S}(\mathbf{a})\mathbf{R}\left(\theta, \frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b} = \frac{\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{a})\mathbf{b}(\mathbf{S}(\mathbf{a})\mathbf{b})^T\mathbf{b}}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2} - \frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}[\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{b}$$
$$-\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b} - \mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{b} + \mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}]$$
$$-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}(\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{b} - \mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}) \quad . \tag{E.3}$$

Next, notice that $\mathbf{S}(\mathbf{b})\mathbf{b} = \mathbf{0}$, since the cross product between two collinear vectors is zero, and notice also that $\mathbf{b}^T(\mathbf{S}(\mathbf{a})\mathbf{b}) = 0$, since $\mathbf{b}$ is orthogonal to $(\mathbf{S}(\mathbf{a})\mathbf{b})$. Thus, (E.3) is given by

$$\mathbf{S}(\mathbf{a})\mathbf{R}\left(\theta, \frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b} = -\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}(-\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}$$
$$+\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}) - \frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}(-\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}) \quad ,$$

or equivalently

$$\mathbf{S}(\mathbf{a})\mathbf{R}\left(\theta, \frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b} = \frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}\left(\mathbf{S}(\mathbf{a})^2\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b} - \mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}\right) +$$
$$\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}(\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}) \quad . \tag{E.4}$$

Now, each of the cross product terms will be simplified using the vector triple product, which is given, for three random vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbb{R}^3$, as [21]

$$\mathbf{S}(\mathbf{x})\mathbf{S}(\mathbf{y})\mathbf{z} = \mathbf{y}\left(\mathbf{x}^T\mathbf{z}\right) - \mathbf{z}\left(\mathbf{x}^T\mathbf{y}\right) \quad . \tag{E.5}$$

Firstly, taking $\mathbf{S}(\mathbf{a})^2\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b}$ and applying the triple product, with $\mathbf{x} = \mathbf{a}$, $\mathbf{y} = \mathbf{a}$, and $\mathbf{z} = \mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b}$, gives

$$\mathbf{S}(\mathbf{a})^2\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b} = \mathbf{a}\left(\mathbf{a}^T\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b}\right) - \mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b}\left(\mathbf{a}^T\mathbf{a}\right) \quad ,$$

which, because $\mathbf{a}$ and $\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b}$ are orthogonal, and also $\mathbf{a}^T\mathbf{a} = 1$, is given as

$$\mathbf{S}(\mathbf{a})^2\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b} = -\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b} \ . \tag{E.6}$$

Applying again (E.5), this time to (E.6), gives

$$\mathbf{S}(\mathbf{a})^2\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b} = -\left[\mathbf{b}\left(\mathbf{b}^T\mathbf{S}(\mathbf{a})\mathbf{b}\right) - \mathbf{S}(\mathbf{a})\mathbf{b}\left(\mathbf{b}^T\mathbf{b}\right)\right] \quad ,$$

which, because $\mathbf{b}$ and $\mathbf{S}(\mathbf{a})\mathbf{b}$ are orthogonal, and also $\mathbf{b}^T\mathbf{b} = 1$, is expressed as

$$\mathbf{S}(\mathbf{a})^2\mathbf{S}(\mathbf{b})^2\mathbf{S}(\mathbf{a})\mathbf{b} = \mathbf{S}(\mathbf{a})\mathbf{b} \ . \tag{E.7}$$

Now, taking $\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}$ and applying the vector triple product as shown in (E.5), with $\mathbf{x} = \mathbf{a}$, $\mathbf{y} = \mathbf{b}$, and $\mathbf{z} = \mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}$, gives

$$\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b} = \mathbf{b}\left(\mathbf{a}^T\mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}\right) - \mathbf{S}(\mathbf{a})\mathbf{S}(\mathbf{b})\mathbf{S}(\mathbf{a})\mathbf{b}\left(\mathbf{a}^T\mathbf{b}\right) \quad ,$$

which, because $\mathbf{a}$ and $\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}$ are orthogonal, *i.e.*, $\mathbf{a}^T\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}=0$, is expressed as

$$\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}=-\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}\left(\mathbf{a}^T\mathbf{b}\right)\ . \tag{E.8}$$

Applying again (E.5) to (E.8), with $\mathbf{x}=\mathbf{a}$, $\mathbf{y}=\mathbf{b}$, and $\mathbf{z}=\mathbf{S}(\mathbf{a})\,\mathbf{b}$, gives

$$\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}=-\left[\mathbf{b}\left(\mathbf{a}^T\mathbf{S}(\mathbf{a})\,\mathbf{b}\right)-\mathbf{S}(\mathbf{a})\,\mathbf{b}\left(\mathbf{a}^T\mathbf{b}\right)\right]\left(\mathbf{a}^T\mathbf{b}\right)\ ,$$

which, because $\mathbf{a}^T\mathbf{S}(\mathbf{a})\,\mathbf{b}=0$, since both arguments of the inner product are orthogonal, is simplified to

$$\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}=\mathbf{S}(\mathbf{a})\,\mathbf{b}\left(\mathbf{a}^T\mathbf{b}\right)^2\ . \tag{E.9}$$

Finally taking $\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}$ and applying (E.5), with $\mathbf{x}=\mathbf{a}$, $\mathbf{y}=\mathbf{b}$, and $\mathbf{z}=\mathbf{S}(\mathbf{a})\,\mathbf{b}$, gives

$$\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}=\mathbf{b}\left(\mathbf{a}^T\mathbf{S}(\mathbf{a})\,\mathbf{b}\right)-\mathbf{S}(\mathbf{a})\,\mathbf{b}\left(\mathbf{a}^T\mathbf{b}\right)\ ,$$

which, because $\mathbf{a}^T\mathbf{S}(\mathbf{a})\,\mathbf{b}=0$ from the orthogonal relation between $\mathbf{a}$ and $\mathbf{S}(\mathbf{a})\,\mathbf{b}$, is expressed as

$$\mathbf{S}(\mathbf{a})\,\mathbf{S}(\mathbf{b})\,\mathbf{S}(\mathbf{a})\,\mathbf{b}=-\mathbf{S}(\mathbf{a})\,\mathbf{b}\left(\mathbf{a}^T\mathbf{b}\right)\ . \tag{E.10}$$

Substituting (E.7), (E.9) and (E.10) in (E.4) gives

$$\mathbf{S}(\mathbf{a})\,\mathbf{R}\left(\theta,\frac{\mathbf{S}(\mathbf{a})\,\mathbf{b}}{\mathbf{S}(\mathbf{a})\,\mathbf{b}}\right)\mathbf{b}=\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\,\mathbf{b}\|^2}\left(\mathbf{S}(\mathbf{a})\,\mathbf{b}-\mathbf{S}(\mathbf{a})\,\mathbf{b}\left(\mathbf{a}^T\mathbf{b}\right)^2\right)+\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\,\mathbf{b}\|}\left(-\mathbf{S}(\mathbf{a})\,\mathbf{b}\left(\mathbf{a}^T\mathbf{b}\right)\right)\ ,$$

or equivalently

$$\mathbf{S}(\mathbf{a})\,\mathbf{R}\left(\theta,\frac{\mathbf{S}(\mathbf{a})\,\mathbf{b}}{\mathbf{S}(\mathbf{a})\,\mathbf{b}}\right)\mathbf{b}=\left[\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\,\mathbf{b}\|^2}\left(1-\left(\mathbf{a}^T\mathbf{b}\right)^2\right)-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\,\mathbf{b}\|}\left(\left(\mathbf{a}^T\mathbf{b}\right)\right)\right]\mathbf{S}(\mathbf{a})\,\mathbf{b}\ . \tag{E.11}$$

Now, taking the right expression of (E.1) and substituting (E.11) gives

$$\frac{\mathbf{S}(\mathbf{a})\,\mathbf{R}\left(\theta,\frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b}}{\left\|\mathbf{S}(\mathbf{a})\,\mathbf{R}\left(\theta,\frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b}\right\|}=\frac{\left[\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}\left(1-\left(\mathbf{a}^T\mathbf{b}\right)^2\right)-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}\left(\left(\mathbf{a}^T\mathbf{b}\right)\right)\right]\mathbf{S}(\mathbf{a})\,\mathbf{b}}{\left\|\left[\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}\left(1-\left(\mathbf{a}^T\mathbf{b}\right)^2\right)-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}\left(\left(\mathbf{a}^T\mathbf{b}\right)\right)\right]\mathbf{S}(\mathbf{a})\,\mathbf{b}\right\|}\ ,$$

which, because of the term, given by $\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}\left(1-\left(\mathbf{a}^T\mathbf{b}\right)^2\right)-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}\left(\left(\mathbf{a}^T\mathbf{b}\right)\right)$, being a scalar, is rewritten as

$$\frac{\mathbf{S}(\mathbf{a})\,\mathbf{R}\left(\theta,\frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b}}{\left\|\mathbf{S}(\mathbf{a})\,\mathbf{R}\left(\theta,\frac{\mathbf{S}(\mathbf{a})\mathbf{b}}{\mathbf{S}(\mathbf{a})\mathbf{b}}\right)\mathbf{b}\right\|}=\frac{\left[\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}\left(1-\left(\mathbf{a}^T\mathbf{b}\right)^2\right)-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}\left(\left(\mathbf{a}^T\mathbf{b}\right)\right)\right]}{\left|\left[\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}\left(1-\left(\mathbf{a}^T\mathbf{b}\right)^2\right)-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}\left(\left(\mathbf{a}^T\mathbf{b}\right)\right)\right]\right|}\frac{\mathbf{S}(\mathbf{a})\,\mathbf{b}}{\|\mathbf{S}(\mathbf{a})\,\mathbf{b}\|}\ . \tag{E.12}$$

Furthermore, the quotient between the scalar components of (E.12) is given as

$$\frac{\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}\left(1-\left(\mathbf{a}^T\mathbf{b}\right)^2\right)-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}\left(\left(\mathbf{a}^T\mathbf{b}\right)\right)}{\left|\frac{\cos\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|^2}\left(1-\left(\mathbf{a}^T\mathbf{b}\right)^2\right)-\frac{\sin\theta}{\|\mathbf{S}(\mathbf{a})\mathbf{b}\|}\left(\left(\mathbf{a}^T\mathbf{b}\right)\right)\right|}=\pm1\ ,$$

therefore, (E.12) is equivalent to the left expression of (E.1). □

A second lemma, showing the relation between vectors that originate the same normal, is considered.

Lemma 33. *Let* $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c} \in \mathbb{R}^3$ *and* $\alpha \in \mathbb{R}$. *If*

$$\frac{\mathbf{S}(\mathbf{a})\,\mathbf{b}}{\|\mathbf{S}(\mathbf{a})\,\mathbf{b}\|} = \pm \frac{\mathbf{S}(\mathbf{a})\,\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\,\mathbf{c}\|} \ ,$$

then

$$\mathbf{b} = \mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\,\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\,\mathbf{c}\|}\right)\mathbf{c} \ . \tag{E.13}$$

Proof. This proof is given by direct computation, noticing that having two equal normals to the planes respectively generated by $\mathbf{a}$ and $\mathbf{b}$ and by $\mathbf{a}$ and $\mathbf{c}$, means that the three vectors are coplanar, *i.e.*,

$$\mathbf{b}^T\mathbf{S}(\mathbf{c})\,\mathbf{a} = 0 \ . \tag{E.14}$$

Then, writing $\mathbf{b}$ as rotation of $\mathbf{c}$, *i.e.*,

$$\mathbf{b} = \tilde{\mathbf{R}}\mathbf{c} \ , \tag{E.15}$$

where $\tilde{\mathbf{R}} \in SO(3)$ represents any rotation, and substituting (E.15) in (E.14) gives

$$\left(\tilde{\mathbf{R}}\mathbf{c}\right)^T\mathbf{S}(\mathbf{c})\,\mathbf{a} = 0 \ ,$$

or equivalently,

$$\mathbf{c}^T\tilde{\mathbf{R}}^T\mathbf{S}(\mathbf{c})\,\mathbf{a} = 0 \ . \tag{E.16}$$

As $\mathbf{c}^T\mathbf{S}(\mathbf{c})\,\mathbf{a} = 0$, then (E.16) is only verified when, introducing γ and $\beta \in \mathbb{R}$, $\tilde{\mathbf{R}}$ is given as

$$\tilde{\mathbf{R}} = \begin{cases} \mathbf{R}\left(\gamma, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\mathbf{S}(\mathbf{a})\mathbf{c}}\right) \\ \mathbf{R}(\beta, \mathbf{c}) \end{cases} \ , \tag{E.17}$$

which, noticing that $\mathbf{R}(\beta, \mathbf{c})$ implies that $\mathbf{b} = \mathbf{c}$ and, since this situation is included in the other case of $\tilde{\mathbf{R}}$, then (E.17) is reduced to

$$\tilde{\mathbf{R}} = \mathbf{R}\left(\gamma, \frac{\mathbf{S}(\mathbf{a})\,\mathbf{c}}{\mathbf{S}(\mathbf{a})\,\mathbf{c}}\right) \ . \tag{E.18}$$

This concludes the proof, since substituting (E.18) in (E.15) gives (E.13). □

In Fig. E.1 the different possible locations for $\mathbf{b}$ are represented by the red circle.

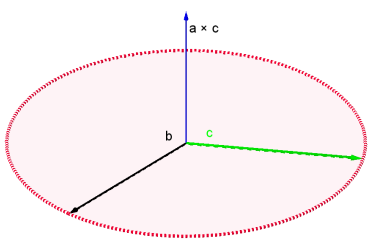


Figure E.1: Possibilities for $\mathbf{b}$

Lemma 34. *Consider* $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c} \in \mathbb{R}^3$ *and* $\alpha \in \mathbb{R}$. *If*

$$\mathbf{b} = \mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\,\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\,\mathbf{c}\|}\right)\mathbf{c} \ , \tag{E.19}$$

then

$$\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|} = \pm\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}\right\|} \ . \tag{E.20}$$

Proof. This proof is given by direct computation. Take the left hand side of (E.20) and substitute (E.19), which gives

$$\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|} = \frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\mathbf{c}\|}\right)\mathbf{c}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\mathbf{c}\|}\right)\mathbf{c}\right\|} \ ,$$

which, applying Lemma 26, yields

$$\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|} = \frac{\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\mathbf{c}\|}\right)\mathbf{S}\left(\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\mathbf{c}\|}\right)^T\mathbf{a}\right)\mathbf{c}}{\left\|\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\mathbf{c}\|}\right)\mathbf{S}\left(\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\mathbf{c}\|}\right)^T\mathbf{a}\right)\mathbf{c}\right\|} \ . \tag{E.21}$$

Notice that there exists $\mu \in \mathbb{R}$ such that

$$\mathbf{S}\left(\mathbf{R}\left(\alpha, \frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}\right\|}\right)^T\mathbf{a}\right)\mathbf{c} = \mu\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}$$

because $\mathbf{a}$ and $\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\mathbf{S}(\mathbf{a})\mathbf{c}}\right)^T\mathbf{a}$ are coplanar. Hence, (E.21) can be rewritten as

$$\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|} = \mu\frac{\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\mathbf{c}\|}\right)\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}}{\left\|\mu\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\|\mathbf{S}(\mathbf{a})\mathbf{c}\|}\right)\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}\right\|} \ ,$$

which, noticing that $\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\mathbf{S}(\mathbf{a})\mathbf{c}}\right)\mathbf{S}\left(\mathbf{a}\right)\mathbf{c} = \mathbf{S}\left(\mathbf{a}\right)\mathbf{c}$, because the rotation axis is parallel to the vector, gives

$$\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|} = \mu\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}}{\left\|\mu\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}\right\|} \ .$$

Since μ is a scalar, then the norm of the expression can be decomposed into a product of a norms and the absolute value of the scalar, yielding

$$\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{b}\right\|} = \frac{\mu}{|\mu|}\frac{\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}}{\left\|\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}\right\|} \ . \tag{E.22}$$

Finally, (E.22) gives (E.20), because $\frac{\mu}{|\mu|} = \pm 1$ according to the sign of μ.

□

Remark. The sign μ is influenced by two parameters. One is the angle between $\mathbf{a}$ and $\mathbf{c}$ and the second is the angle of the rotation, that is, α. These will determine the magnitude of $\mathbf{R}\left(\alpha, \frac{\mathbf{S}(\mathbf{a})\mathbf{c}}{\mathbf{S}(\mathbf{a})\mathbf{c}}\right)\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}$ relative to $\mathbf{S}\left(\mathbf{a}\right)\mathbf{c}$.

E.2 Technical Lemma

This appendix addresses a technical lemma that is needed to determine the value of λ_2 and λ_3 in Section 3.2.3.

Lemma 35. *Consider any* $\mathbf{a}$ *and* $\mathbf{c} \in \mathbb{R}^3$ *and any* $\theta \in \mathbb{R}$*. Then,*

$$\mathbf{a}^T\mathbf{R}\left(\theta,\mathbf{c}\right)^T\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)\mathbf{R}\left(\theta,\mathbf{c}\right)\mathbf{a}=\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\ . \tag{E.23}$$

Proof. This proof is given by direct computation, taking the left expression of (E.23) and expanding the rotations according to (2.1), which gives

$$\begin{aligned}\mathbf{a}^T\mathbf{R}\left(\theta,\mathbf{c}\right)^T\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)\mathbf{R}\left(\theta,\mathbf{c}\right)\mathbf{a}=\mathbf{a}^T&\left[\cos(\theta)\mathbf{I}+(1-\cos(\theta))\mathbf{c}\mathbf{c}^T\right.\\ \left.-\sin(\theta)\mathbf{S}\left(\mathbf{c}\right)\right]^T\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)&\left[\cos(\theta)\mathbf{I}+(1-\cos(\theta))\mathbf{c}\mathbf{c}^T-\sin(\theta)\mathbf{S}\left(\mathbf{c}\right)\right]\mathbf{a}\ ,\end{aligned}$$

or equivalently,

$$\begin{aligned}\mathbf{a}^T\mathbf{R}\left(\theta,\mathbf{c}\right)^T\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)\mathbf{R}\left(\theta,\mathbf{c}\right)\mathbf{a}=\mathbf{a}^T&\left[\cos(\theta)\mathbf{I}+(1-\cos(\theta))\left(\mathbf{c}\mathbf{c}^T\right)^T\right.\\ \left.-\sin(\theta)\mathbf{S}\left(\mathbf{c}\right)^T\right]\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)&\left[\cos(\theta)\mathbf{I}+(1-\cos(\theta))\mathbf{c}\mathbf{c}^T-\sin(\theta)\mathbf{S}\left(\mathbf{c}\right)\right]\mathbf{a}\ .\end{aligned} \tag{E.24}$$

Using the transpose properties, (E.24) can be written as

$$\begin{aligned}\mathbf{a}^T\mathbf{R}\left(\theta,\mathbf{c}\right)^T\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)\mathbf{R}\left(\theta,\mathbf{c}\right)\mathbf{a}=\mathbf{a}^T&\left[\cos(\theta)\mathbf{I}+(1-\cos(\theta))\mathbf{c}\mathbf{c}^T\right.\\ \left.+\sin(\theta)\mathbf{S}\left(\mathbf{c}\right)\right]\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)&\left[\cos(\theta)\mathbf{I}+(1-\cos(\theta))\mathbf{c}\mathbf{c}^T-\sin(\theta)\mathbf{S}\left(\mathbf{c}\right)\right]\mathbf{a}\ .\end{aligned} \tag{E.25}$$

Then, noticing that the term $(1-\cos(\theta))\mathbf{c}\mathbf{c}^T$ is multiplied by $\mathbf{S}\left(\mathbf{c}\right)$ and, because $\mathbf{S}\left(\mathbf{c}\right)\mathbf{c}=\mathbf{c}^T\mathbf{S}\left(\mathbf{c}\right)=0$, (E.25) is rewritten as

$$\begin{aligned}\mathbf{a}^T\mathbf{R}\left(\theta,\mathbf{c}\right)^T\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)\mathbf{R}\left(\theta,\mathbf{c}\right)\mathbf{a}=\mathbf{a}^T&\left[\cos(\theta)\mathbf{I}+\sin(\theta)\mathbf{S}\left(\mathbf{c}\right)\right]\\ &\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)\left[\cos(\theta)\mathbf{I}-\sin(\theta)\mathbf{S}\left(\mathbf{c}\right)\right]\mathbf{a}\ ,\end{aligned}$$

which distributing all the terms is expressed as

$$\begin{aligned}\mathbf{a}^T\mathbf{R}\left(\theta,\mathbf{c}\right)^T\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)\mathbf{R}\left(\theta,\mathbf{c}\right)\mathbf{a}=&\\ \cos(\theta)^2&\left(-\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}+\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\right)\\ -\sin(\theta)\cos(\theta)&\left(-\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}+\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\right)\\ +\sin(\theta)\cos(\theta)&\left(-\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}+\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\right)\\ -\sin(\theta)^2&\left(-\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}+\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\right)\ ,\end{aligned}$$

or equivalently,

$$\begin{aligned}\mathbf{a}^T\mathbf{R}\left(\theta,\mathbf{c}\right)^T\left(-\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)+\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\right)\mathbf{R}\left(\theta,\mathbf{c}\right)\mathbf{a}=&\\ \cos(\theta)^2&\left(-\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}+\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\right)\\ -\sin(\theta)\cos(\theta)&\left(-\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}+\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^3\mathbf{a}\right)\\ +\sin(\theta)\cos(\theta)&\left(-\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)\mathbf{a}+\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^3\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\right)\\ -\sin(\theta)^2&\left(-\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^2\mathbf{a}+\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^3\mathbf{a}\mathbf{a}^T\mathbf{S}\left(\mathbf{c}\right)^3\mathbf{a}\right)\ .\end{aligned} \tag{E.26}$$

From Lemma 25, $\mathbf{S}(\mathbf{c})^3 = -\mathbf{S}(\mathbf{c})$, so (E.26) is given as

$$\begin{aligned}\mathbf{a}^T\mathbf{R}(\theta,\mathbf{c})^T\left(-\mathbf{S}(\mathbf{c})\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})+\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\right)\mathbf{R}(\theta,\mathbf{c})\mathbf{a} = \\ \cos(\theta)^2\left(-\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a}+\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\right) \\ -\sin(\theta)\cos(\theta)\left(-\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}-\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a}\right) \\ +\sin(\theta)\cos(\theta)\left(-\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a}-\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\right) \\ -\sin(\theta)^2\left(-\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}+\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a}\right)\end{aligned} ,$$

which, because $\mathbf{a}^T\mathbf{S}(\mathbf{c})\mathbf{a} = 0$, is rewritten as

$$\begin{aligned}\mathbf{a}^T\mathbf{R}(\theta,\mathbf{c})^T\left(-\mathbf{S}(\mathbf{c})\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})+\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\right)\mathbf{R}(\theta,\mathbf{c})\mathbf{a} = \\ \cos(\theta)^2\left(\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\right)-\sin(\theta)^2\left(-\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\right) .\end{aligned} \tag{E.27}$$

Rearranging (E.27) yields

$$\begin{aligned}\mathbf{a}^T\mathbf{R}(\theta,\mathbf{c})^T\left(-\mathbf{S}(\mathbf{c})\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})+\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\right)\mathbf{R}(\theta,\mathbf{c})\mathbf{a} = \\ \left(\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\mathbf{a}^T\mathbf{S}(\mathbf{c})^2\mathbf{a}\right)\left[\cos(\theta)^2+\sin(\theta)^2\right] ,\end{aligned}$$

which, substituting the trigonometric identity $\sin(\theta)^2 + \cos(\theta)^2 = 1$, gives (E.23). □

E.3 Inner Product of Coplanar Vectors

Lemma 36. *Consider the coplanar unit vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{R}^3$ *and assume that*

$$\mathbf{b} \neq \mathbf{c} . \tag{E.28}$$

If

$$\begin{cases}\mathbf{a}^T\mathbf{b} = \mathbf{a}^T\mathbf{c} \\ \mathbf{d}^T\mathbf{b} = \mathbf{d}^T\mathbf{c}\end{cases} \tag{E.29}$$

then

$$\mathbf{a} = \pm\mathbf{d} .$$

Proof. This proof is given by direct computation, rewriting (E.29) as a relation between cosines, because the vectors are unit vectors, which gives

$$\begin{cases}\cos\left(\widehat{\mathbf{ab}}\right) = \cos\left(\widehat{\mathbf{ac}}\right) \\ \cos\left(\widehat{\mathbf{db}}\right) = \cos\left(\widehat{\mathbf{dc}}\right)\end{cases} , \tag{E.30}$$

with $\left(\widehat{\mathbf{ab}}\right)$, $\left(\widehat{\mathbf{ac}}\right)$, $\left(\widehat{\mathbf{db}}\right)$ and $\left(\widehat{\mathbf{dc}}\right)$ representing the angles between the respective vectors. Then, from the characteristics of the cosine function, (E.30) is given as

$$\begin{cases}\left(\widehat{\mathbf{ab}}\right) = \pm\left(\widehat{\mathbf{ac}}\right) \\ \left(\widehat{\mathbf{db}}\right) = \pm\left(\widehat{\mathbf{dc}}\right)\end{cases} .$$

From (E.28) and because the vectors are coplanar, then it can be concluded that both $\mathbf{a}$ and $\mathbf{d}$ are

bisectors of $\pm\left(\hat{\mathbf{bc}}\right)$, which means that $\mathbf{a} = \pm\mathbf{d}$. $\square$

Lemma 37. *Consider the coplanar unit vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d} \in \mathbb{R}^3$. *If*

$$\begin{cases} \mathbf{a}^T\mathbf{b} = \mathbf{a}^T\mathbf{d} \\ \mathbf{a}^T\mathbf{c} = \mathbf{a}^T\mathbf{e} \end{cases} , \tag{E.31}$$

with

$$\mathbf{d} = \pm\mathbf{e} \ . \tag{E.32}$$

Consider also that

$$\mathbf{b} \neq \mathbf{d} \tag{E.33}$$

and

$$\mathbf{c} \neq \mathbf{e} \ . \tag{E.34}$$

Then,

$$\mathbf{b} = \pm\mathbf{c} \ ,$$

with the same sign as (E.32).

Proof. This proof is given by rewriting (E.31) as a relation between cosines, because the vectors are unit vectors, which gives

$$\begin{cases} \cos\left(\hat{\mathbf{ab}}\right) = \cos\left(\hat{\mathbf{ad}}\right) \\ \cos\left(\hat{\mathbf{ac}}\right) = \cos\left(\hat{\mathbf{ae}}\right) \end{cases} , \tag{E.35}$$

with $\left(\hat{\mathbf{ab}}\right)$, $\left(\hat{\mathbf{ac}}\right)$ $\left(\hat{\mathbf{ad}}\right)$, and $\left(\hat{\mathbf{ae}}\right)$ representing the angles between the respective vectors. Then, from the characteristics of the cosine function, (E.35) is given as

$$\begin{cases} \left(\hat{\mathbf{ab}}\right) = \pm\left(\hat{\mathbf{ad}}\right) \\ \left(\hat{\mathbf{ac}}\right) = \pm\left(\hat{\mathbf{ae}}\right) \end{cases} .$$

Choosing an angle direction without loss of generality, from (E.33) and (E.34) and as the vectors are coplanar, then

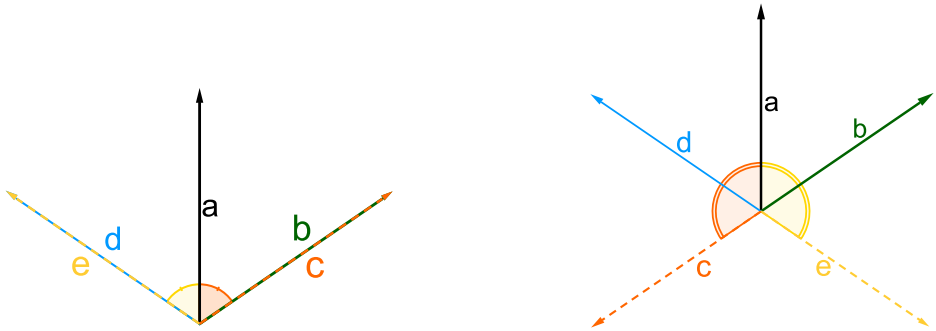
$$\begin{cases} \left(\hat{\mathbf{ab}}\right) = -\left(\hat{\mathbf{ad}}\right) \\ \left(\hat{\mathbf{ac}}\right) = -\left(\hat{\mathbf{ae}}\right) \end{cases} . \tag{E.36}$$

There are two possible outcomes, from (E.36), that depend on the sign of (E.32). These are given as

$$\begin{cases} \mathbf{b} = \mathbf{c} & \text{, if } \mathbf{d} = \mathbf{e} \\ \mathbf{b} = -\mathbf{c} & \text{, if } \mathbf{d} = -\mathbf{e} \end{cases}$$

which is the conclusion taken in this Lemma, thus the proof is concluded. $\square$

The possible outcomes of lemma 37 are shown geometrically in Fig. E.2.



(a) $\mathbf{d} = \mathbf{e}$ (b) $\mathbf{d} = -\mathbf{e}$

Figure E.2: Geometric visualization of Lemma 37

Appendix F

Specific Relations of the Problem

F.1 Inner Product between Normals

Lemma 38. *Consider* $\mathbf{d}_1$, $\mathbf{d}_{1/2}$, *as described in Section 2.2 and* $\left(\mathbf{R}_2^1\right)_A$ *and* $\left(\mathbf{R}_2^1\right)_B$ *as in Section 3.2.3. Then,*

$$\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)=\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\ . \tag{F.1}$$

Proof. This proof is given by direct computation. First, apply Corollary 2 to the left side of (F.1), which gives

$$\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)=\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\theta_{12},-\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)$$

Then, notice that applying the scalar triple product property gives [21]

$$\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)=\mathbf{d}_1^T\mathbf{S}\left(\mathbf{R}\left(\theta_{12},-\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\ \mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\ . \tag{F.2}$$

Using the vector triple product with $\mathbf{R}\left(\theta_{12},-\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2=\mathbf{x}$, $\mathbf{d}_1=\mathbf{y}$ and $\mathbf{d}_{1/2}=\mathbf{z}$, shown in (E.5), it follows that (F.2) is rewritten as

$$\begin{aligned}&\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)=\\&\qquad\mathbf{d}_1^T\left[\mathbf{d}_1\left(\mathbf{d}_{1/2}^T\mathbf{R}\left(\theta_{12},-\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)-\mathbf{d}_{1/2}\left(\mathbf{d}_1^T\mathbf{R}\left(\theta_{12},-\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\right]\ .\end{aligned} \tag{F.3}$$

As

$$\mathbf{d}_{1/2}^T\mathbf{R}\left(\theta_{12},-\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2=\mathbf{d}_{1/2}^T\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2$$

and also, from Lemma 9,

$$\mathbf{d}_1^T\mathbf{R}\left(\theta_{12},-\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2=\mathbf{d}_1^T\mathbf{R}\left(\lambda_2,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2=\mathbf{d}_1^T\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\ ,$$

then (F.3) is expressed as

$$\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)=\mathbf{d}_1^T\left[\mathbf{d}_1\left(\mathbf{d}_{1/2}^T\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)-\mathbf{d}_{1/2}\left(\mathbf{d}_1^T\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\right]\ . \tag{F.4}$$

Since (F.4) maintains the form of the triple product from (E.5), it is given as

$$\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)=\mathbf{d}_1^T\left[\mathbf{S}\left(\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right]\ ,$$

which is equivalent to the right member of (F.1), from the properties of the scalar triple product. □

Lemma 39. *Consider* $\mathbf{d}_1,\ \mathbf{d}_{1/3}$, *as described in Section 2.2 and* $\left(\mathbf{R}_3^1\right)_C$ *and* $\left(\mathbf{R}_3^1\right)_D$ *as in Section 3.2.3. Then,*

$$\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\right)=\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right)^T\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right)\ .$$

Proof. This proof follows analogously to the one of Lemma 38 and thus, it is omitted.

Lemma 40. *Consider* $\mathbf{d}_1,\ \mathbf{d}_{1/2}$, *as described in Section 2.2 and* $\left(\mathbf{R}_2^1\right)_A$ *and* $\left(\mathbf{R}_2^1\right)_B$ *as in Section 3.2.3. Then,*

$$\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|=\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right\|\ . \tag{F.5}$$

Proof. This proof is given by direct computation. First take the left hand side of (F.5) and apply Corollary 2 which gives

$$\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|=\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\theta_{12},-\mathbf{d}_{1/2}\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right\|\ .$$

Then, from Lemma 9 it follows that

$$\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|=\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\lambda_2,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right\|\ ,$$

which, using Lemma 26 and recalling that $\mathbf{R}\left(\lambda_2,\mathbf{d}_1\right)\mathbf{d}_1=\mathbf{d}_1$, is given as

$$\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|=\left\|\mathbf{R}\left(\lambda_2,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right\|\ . \tag{F.6}$$

It is concluded that (F.6) is equivalent to (F.5), because the norm of a vector is kept when it is rotated about an unitary axis. □

Lemma 41. *Consider* $\mathbf{d}_1,\ \mathbf{d}_{1/3}$, *as described in Section 2.2 and* $\left(\mathbf{R}_3^1\right)_C$ *and* $\left(\mathbf{R}_3^1\right)_D$ *as in Section 3.2.3. Then,*

$$\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\right\|=\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right\|\ .$$

Proof. This proof follows analogously to the one of Lemma 40 and thus, it is omitted.

F.2 Dependencies between Planes

Lemma 42. *Consider that*

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|}=\pm\frac{\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\right\|}\ , \tag{F.7}$$

and

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right\|}=\pm\frac{\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right\|}\ , \tag{F.8}$$

with some $\epsilon\in\mathbb{R}$, *and both have the same sign. Furthermore, consider that the partial problems' vectors are not coplanar, which, in* (F.8), *is given as,*

$$\theta_{12}\neq 0\ \wedge\ \theta_{13}\neq 0\ , \tag{F.9}$$

where θ_{12} *and* θ_{13} *are respectively defined in* (3.69a) *and* (3.69b). *Then,*

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}=\pm\frac{\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right\|}\ . \tag{F.10}$$

Proof. First take Lemma 38 and write it, recalling Lemma 40, as

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)}{\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\|}=\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)}{\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\|}\ , \tag{F.11a}$$

and take also Lemma 39 and rewrite it, recalling Lemma 41, as

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\right)}{\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\|}=\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right)}{\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right)\|}\ . \tag{F.11b}$$

Substitute (F.7) and (F.8) in (F.11b), which gives

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\|}=\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)}{\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\|}\ , \tag{F.12}$$

and next, introduce $\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)^T\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)$ in (F.11a) as given by

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\|}\frac{\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)^T\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)}{\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)^T\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)}{\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\|}=$$
$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\|}\frac{\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)^T\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)}{\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)^T\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)}{\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\|}\ ,$$

which, rearranging, using Lemma 26 and noticing that if both vector norms are rotated equally, then the value of their product is kept, gives

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\|}=$$
$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)}{\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\|}\ , \tag{F.13}$$

Then, recalling (F.9), which means that $\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\neq\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2$ and $\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\neq\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3$, and applying (F.12) and (F.13) in Lemma 36, shown in Appendix E.3, it is concluded that (F.10) is verified. □

Lemma 43. *Consider that*

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\|}=\pm\frac{\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3}{\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\|}\ , \tag{F.14}$$

and that

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\|}=\pm\frac{\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\|} \tag{F.15}$$

with some $\epsilon\in\mathbb{R}$. Furthermore, consider that the partial problems' vectors are not coplanar, which, in (F.8)*, is given as,*

$$\theta_{12}\neq 0\ \wedge\ \theta_{13}\neq 0\ , \tag{F.16}$$

where θ_{12} and θ_{13} are respectively defined in (3.69a) *and* (3.69b)*. Then,*

$$\frac{\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2}{\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\|}=\pm\frac{\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3}{\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\|}\ , \tag{F.17}$$

with the same sign as (F.14)*.*

Proof. First take Lemma 38 and rewrite it, recalling Lemma 40, as

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|}=\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)}{\left\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\right\|}\ , \tag{F.18a}$$

and take also Lemma 39 and rewrite it, recalling Lemma 41, as

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right)^T}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\right)}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\right\|}=\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right)^T}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/3}\right\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right)}{\left\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right)\right\|}\ . \tag{F.18b}$$

Then, noticing that $\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)^T\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)=\mathbf{I}$ and that the norm of a vector is not changed by a rotation about a unitary vector, (F.18a) is rewritten as

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)^T\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)}{\left\|\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|}=$$
$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)^T\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)}{\left\|\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\right\|}\ ,$$

or, equivalently,

$$\frac{\left(\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\frac{\left(\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)}{\left\|\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|}=$$
$$\frac{\left(\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\frac{\left(\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)}{\left\|\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\right\|}\ . \tag{F.19}$$

Then, from Lemma 26 and recalling that $\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_1=\mathbf{d}_1$, (F.19) is given as

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right)}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_A\mathbf{d}_2\right\|}=$$
$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)}{\left\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\left(\mathbf{R}_2^1\right)_B\mathbf{d}_2\right)\right\|}\ . \tag{F.20}$$

Taking (F.18b) and substituting (F.15) gives

$$\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\right)}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_C\mathbf{d}_3\right\|}=\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right)^T}{\left\|\mathbf{S}\left(\mathbf{d}_1\right)\mathbf{R}\left(\epsilon,\mathbf{d}_1\right)\mathbf{d}_{1/2}\right\|}\frac{\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right)}{\left\|\left(\mathbf{S}\left(\mathbf{d}_1\right)\left(\mathbf{R}_3^1\right)_D\mathbf{d}_3\right)\right\|}\ . \tag{F.21}$$

Then, applying (F.20) and (F.21) in Lemma 37, shown in Appendix E.3, recalling (F.14) and (F.16), it is concluded that (F.17) is verified. Furthermore, (F.17) has the same sign as (F.14). □