A topoi characterization of Gödel's intermediate logics

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June, 2017

Abstract

A general construction for *propositional intermediate logics*, from *intuitionistic* to *classical*, is presented and it's *soundness and completeness* with respect to *heyting algebras* is proven. A review is made of basic notions of *category theory*, *topoi* and *topos semantics*. A new semantics based on *topoi* to the family of *logics* presented by *Gödel* in [4] is given and proved to be *sound* and *complete*.

1 Basic logical concepts

In this section we review the basic logical concepts needed for the rest of the work. We will define our syntax, define an algebraic semantic and show its soundness and completeness.

1.1 Syntax

Definition 1. Let $X = \{x_i\}_{i\in\mathbb{N}}$ be a set called the *set of variables*, we define the *set* $\mathcal{F}$ of all *propositional formulas inductively* as follows

- $X \subseteq \mathcal{F}$
- if $\phi \in \mathcal{F}$, then $(\neg\phi) \in \mathcal{F}$
- if $\phi, \varphi \in \mathcal{F}$, then $(\phi \wedge \varphi), (\phi \vee \varphi), (\phi \Rightarrow \varphi) \in \mathcal{F}$

Definition 2 (Propositional logic)**.** Given a *set* $\Gamma \subseteq \mathcal{F}$, we define the *propositional logic* $\mathcal{L}_\Gamma$ *axiomatized by* Γ as the *smallest subset of* $\mathcal{F}$ such that

1. $\Gamma \subseteq \mathcal{L}_\Gamma$
2. if $\varphi, (\varphi \Rightarrow \phi) \in \mathcal{L}_\Gamma$ then $\phi \in \mathcal{L}_\Gamma$
3. if $f : X \longrightarrow \mathcal{F}$ and $\varphi \in \mathcal{L}_\Gamma$, then $[[\varphi]]_f \in \mathcal{L}_\Gamma$

Where $[[\varphi]]_f$ is the *formula* obtained by substituting every instance of the *variable* x_i by $f(x_i)$. In other words, $\mathcal{L}_\Gamma$ is the *smallest subset of* $\mathcal{F}$ that is closed for *modus ponens* and *arbitrary substitutions*.

Example 1. The *propositional logics* $\mathcal{L}_{IPL}$ and $\mathcal{L}_{CPL}$ are, respectively, the *intuitionistic propositional logic* and the *classical propositional logic*, and are *axiomatized*, respectively, by the sets IPL and CPL which can be consulted in [2] (Definition 2.4 and 2.5).

Definition 3. A *propositional logic* $\mathcal{L}$ is said to be an *intermediate propositional logic* if $\mathcal{L}_{IPL} \subset \mathcal{L} \subset \mathcal{L}_{CPL}$.

1.2 Semantics

Definition 4. A *bounded lattice* (check definition 2.7 and 2.8 of [2]) $\mathcal{H}$ is said to be a *heyting algebra* if, for every $a, b \in \mathcal{H}$ there is an *element* $a \Rightarrow b \in \mathcal{H}$ such that

$$c \wedge a \leq b \qquad \text{iff} \qquad c \leq a \Rightarrow b$$

We call $a \Rightarrow b$ the *pseudo-complement of* a *relative to* b. In this context, we refer to $a \Rightarrow 0$ as the *pseudo-complement* of a and write $\neg a$ instead of $a \Rightarrow 0$.

Definition 5. A *heyting algebra* $\mathcal{B}$ is said to be a *boolean algebra* if, for every $a \in \mathcal{B}$, the *pseudo-complement* $\neg a$ is such that $a \wedge (\neg a) = 0$ and $a \vee (\neg a) = 1$. In this case, we refer to the *element* $\neg a$ as simply the *complement* of a.

Definition 6. Let $\mathcal{H}$ be a *heyting algebra*. A H-valuation is a *function* $v : X \longrightarrow \mathcal{H}$. This notion extends naturally to every *propositional formula* in the following way

- $v(\varphi \vee \phi) = v(\varphi) \vee v(\phi)$
- $v(\varphi \wedge \phi) = v(\varphi) \wedge v(\phi)$
- $v(\varphi \Rightarrow \phi) = v(\varphi) \Rightarrow v(\phi)$
- $v(\neg\varphi) = \neg v(\varphi)$

Definition 7. Consider a *heyting algebra* $\mathcal{H}$, a *formula* φ and a *propositional logic* $\mathcal{L}$. We say that $\mathcal{H}$ satisfies φ (φ is valid in $\mathcal{H}$) and write $\mathcal{H} \models \varphi$, if $v(\varphi) = 1$ for every H-valuation v. We say that $\mathcal{H}$ is a *model* for the *logic* $\mathcal{L}$ if $\mathcal{H} \models \phi$ for every $\phi \in \mathcal{L}$.

1.3 Soundness and completeness

Theorem 1. *If* $\mathcal{H}$ *is a* heyting algebra*, then* $\mathcal{H} \models \mathcal{L}_{IPL}$.

Proof. Consult [2] (theorem 2.2). □

Theorem 2. *Let* $\mathcal{H}$ *be a* heyting algebra*. Then* $\mathcal{H} \models \mathcal{L}_{CPL}$ *if and only if* $\mathcal{H}$ *is a* boolean algebra*.*

Proof. Consult [2] (theorem 2.3). □

Definition 8. Given a *propositional logic* $\mathcal{L}$, consider the following *equivalence relation* on $\mathcal{F}$

$$\varphi \equiv_{\mathcal{L}} \phi \qquad \text{iff} \qquad (\varphi \Leftrightarrow \phi) \in \mathcal{L}$$

Also, define the *partial ordering* of $\mathcal{F}/\equiv_{\mathcal{L}}$ by

$$[a] \leq_{\mathcal{L}} [b] \qquad \text{iff} \qquad (a \Rightarrow b) \in \mathcal{L}$$

The $(\mathcal{F}/\equiv_{\mathcal{L}}, \leq_{\mathcal{L}})$ is called the *Lindenbaum-Tarski algebra over* $\mathcal{L}$ and will be referred to as Lind($\mathcal{L}$).

Theorem 3. *Lind*($\mathcal{L}$) *is a* heyting algebra*.*

Proof. Consult [2] (theorems 2.4 and 2.5). □

Theorem 4. *Let* $\mathcal{L}$ *be a* propositional logic *such that* $\mathcal{L}_{IPL} \subseteq \mathcal{L}$*. Then, for every* $\varphi \in \mathcal{F}$*,*

$$\varphi \in \mathcal{L} \qquad \textit{iff} \qquad \textit{Lind}(\mathcal{L}) \models \varphi$$

Proof. Consult [2] (theorem 2.8). □

Corollary. *Let* $\mathcal{L}$ *be a* propositional logic *such that* $\mathcal{L}_{IPL} \subseteq \mathcal{L}$*. Then, for every* $\varphi \in \mathcal{F}$*,*

$$\varphi \in \mathcal{L} \qquad \textit{iff} \qquad \mathcal{H} \models \varphi$$

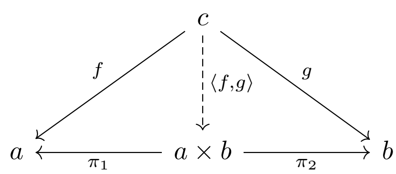
for every $\mathcal{H} \in \textit{Mod}(\mathcal{L})$*.*

2 Basic categorical concepts

This section is dedicated to the review of the most basic categorical concepts needed in the rest of this work.

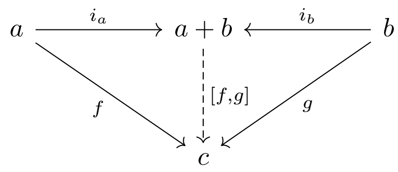
2.1 Products and co-products

Definition 9. Let $\mathcal{C}$ be a category and $a, b \in Ob(\mathcal{C})$. $a \times b \in Ob(\mathcal{C})$ together with two arrows, $\pi_1 : a \times b \longrightarrow a$ and $\pi_2 : a \times b \longrightarrow b$, is said to be *a product* of a and b if, for every $c \in Ob(\mathcal{C})$ and $(f : c \longrightarrow a), (g : c \longrightarrow b) \in Mor(\mathcal{C})$, there exists one and only one $\langle f, g \rangle \in Mor(\mathcal{C})$ such that the following diagram commutes



(1)

Definition 10. Let $\mathcal{C}$ be a category and $a, b \in Ob(\mathcal{C})$. $a + b \in Ob(\mathcal{C})$ together with two arrows, $i_a : a \longrightarrow a + b$ and $i_b : b \longrightarrow a + b$, is said to be *a co-product* of a and b if, for every $c \in Ob(\mathcal{C})$ and $(f : a \longrightarrow c), (g : b \longrightarrow c) \in Mor(\mathcal{C})$, there exists one and only one $[f, g] \in Mor(\mathcal{C})$ such that the following diagram commutes

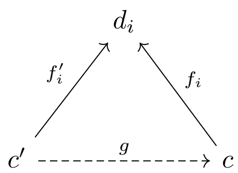


(2)

2.2 Limits and co-limits

For the definitions of *diagram* and *cone* check [2] (definitions 3.3 and 3.4).

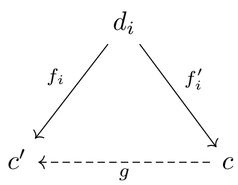
Definition 11. Let D be a diagram in the category $\mathcal{C}$. *A limit* for D is a cone $\{f_i : c \longrightarrow d_i : \}$ such that, if $\{f'_i : c' \longrightarrow d_i : \}$ is a cone for D, then there is one and only one $g \in Mor(\mathcal{C})$ such that



(3)

commutes, for every $d_i \in Ob(D)$.

Definition 12. Let D be a diagram in the category $\mathcal{C}$. *A co-limit* for D is a *co-cone* $\{f_i : d_i \longrightarrow c : \}$ such that, if $\{f_i : d_i \longrightarrow c' : \}$ is a *co-cone* for D, then there is one and only one $g \in Mor(\mathcal{C})$ such that

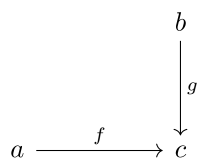


(4)

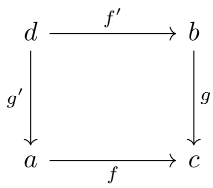
commutes, for every $d_i \in Ob(D)$.

2.3 Pullbacks

Definition 13. Given a category $\mathcal{C}$ and a pair of arrows $a \xrightarrow{f} c \xleftarrow{g} b$, a *pullback* for this diagram is a *limit* for the diagram

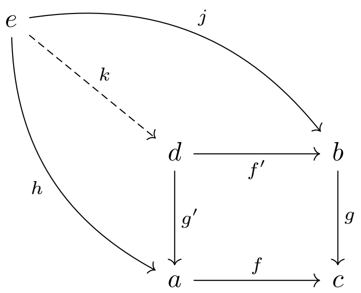


In other words, a *pullback* is a pair of arrows $a \xleftarrow{g'} d \xrightarrow{f'} b$ such that



(5)

commutes, and for every pair of arrows $a \xleftarrow{h} e \xrightarrow{j} b$ that make diagram 5 commute, there is one and only one arrow $e \dashrightarrow d$ that makes the whole diagram

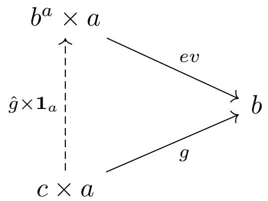


(6)

commute. Usually it is said that f' arises by *pulling back* f along g, and g' arises by *pulling back* g along f.

2.4 Exponentiation

Definition 14. A category $\mathcal{C}$ is said to have *exponentiation* if the *product* exists for any pair of $\mathcal{C} - objects$, and if given $a, b \in Ob(\mathcal{C})$ there is a $\mathcal{C} - object$ b^a and a $\mathcal{C} - arrow$ $ev : b^a \times a \longrightarrow b$ such that, for any $c \in Ob(\mathcal{C})$ and arrow $g : c \times a \longrightarrow b$, there is one and only one arrow $\hat{g} : c \longrightarrow b^a$ that makes



(7)

commute.

3 Constructing topoi

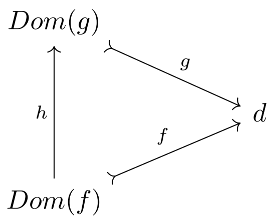
In this section we review the categorical concepts needed in the development of the topos notion.

3.1 Subobjects

Definition 15. Let $\mathcal{C}$ be a category and $d \in Ob(\mathcal{C})$. A *subobject* of d is a *monic arrow* with *co-domain* d. Furthermore, define

$$\text{Sub}(d)^* = \{f : f \text{ is a subobject of } d\}$$

Definition 16. Let $\mathcal{C}$ be a category, $d \in Ob(\mathcal{C})$ and f, g two *subobjects* of d. We say that $f \sqsubseteq g$ if there is an arrow $h : Dom(f) \longrightarrow Dom(g)$ (this arrow, when it exists will always be monic) such that the following diagram commutes



Moreover, if $f \sqsubseteq g$ and $g \sqsubseteq f$, then f and g are said to be *isomorphic* and we write $f \simeq g$. Notice that the relation $\simeq$ is a *congruence relation* in the structure $(\text{Sub}(d)^*, \sqsubseteq)$.

Definition 17. We define the *set of subobjects* of an *object* d as

$$\text{Sub}(d) = \text{Sub}(d)^* / \simeq$$

Theorem 5. *Let $\mathcal{C}$ be a category and $d \in Ob(\mathcal{C})$. Then $(Sub(d), \subseteq)$ is a* partially ordered set.

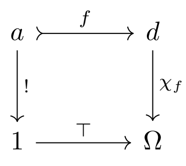
Proof. Consult [2] (theorem 4.2). □

Theorem 6. *In the category **Set**, $Sub(D) \simeq \mathcal{P}(D)$ for every object D.*

Proof. Consult [2] (theorem 4.3). □

3.2 Characteristic arrows

Definition 18. Let $\mathcal{C}$ be a category with a *terminal object* 1. A *subobject classifier* for $\mathcal{C}$ is an *object* Ω together with an *arrow* $\top : 1 \longrightarrow \Omega$ such that for every *monic arrow* $f : a \rightarrowtail d$, there is one and only one *arrow* $\chi_f : d \longrightarrow \Omega$ such that the following diagram



is a *pullback square*. The *arrow* χ_f is called the *characteristic arrow* of the *subobject* f and the elements of Ω are called the *truth values*.

Theorem 7. *Let $\mathcal{C}$ be a category with a* subobject classifier $\langle \Omega, \top \rangle$. *Also, let $d \in Ob(\mathcal{C})$ and $f, g \in Sub(d)$. Then the following equivalence is true*

$$f \simeq g \qquad \textit{iff} \qquad \chi_f = \chi_g$$

Proof. Consult [2] (theorem 4.7). □

Example 2. The existence of a *subobject classifier* $(\Omega, \top)$ and a *terminal object* 1 gives us the possibility of defining a "special" arrow, which we will call $\bot$, as the *characteristic arrow* of the *arrow* $! : 0 \longrightarrow 1$, i.e.,

$$\begin{array}{ccc} 0 & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \bot} \\ 1 & \xrightarrow{\;\top\;} & \Omega \end{array}$$

is a *pullback*. In **Set** this means that $\bot$ is the *characteristic function* of $\varnothing \subseteq 1$, i.e., it's the *function* with values $\bot(0) = 0$, as desired.
This *arrow* is often referred to as *false* or *bottom* for reasons that we will explore in the next sections.

3.3 Power objects

Definition 19. Let $\mathcal{C}$ be a category with *products*. $\mathcal{C}$ is said to have *power objects* if, for every $a \in Ob(\mathcal{C})$ there are *objects* $\mathcal{P}(a), \in_a \in Ob(\mathcal{C})$ and a *monic* $\in : \in_a \rightarrowtail \mathcal{P}(a) \times a$ such that: for every $b \in Ob(\mathcal{C})$ and *subobject* $r : Dom(r) \rightarrowtail b \times a$ of $b \times a$ there is one, and only one, *arrow* $f_r : b \longrightarrow \mathcal{P}(a)$ for which there is a *pullback* of the form

$$\begin{array}{ccc} Dom(r) & \overset{r}{\rightarrowtail} & b \times a \\ \downarrow & & \downarrow{\scriptstyle f_r \times id_a} \\ \in_a & \overset{\in}{\rightarrowtail} & \mathcal{P}(a) \times a \end{array}$$

3.4 Definition of topos

Definition 20. A category $\mathcal{C}$ is called a *Topos* if

1. $\mathcal{C}$ has a *terminal object*.
2. $\mathcal{C}$ has *pullbacks*.
3. $\mathcal{C}$ has *exponentiation*.
4. $\mathcal{C}$ has a *subobject classifier*.

As we saw already, the *category* **Set** fulfills all the requirements and so it constitutes the first example of a *topos*.

Another important property of *topoi* is that every *topos* has *power objects*.

Theorem 8. *Let $\mathcal{C}$ be a* topos. *Then $\mathcal{C}$ has* power objects. *Moreover, given $a \in Ob(\mathcal{C})$, one of its* power objects *is Ω^a.*

Proof. Consult [2] (theorem 4.12). □

4 Topoi as semantics

In this section we show how *topoi* can be used as *semantic structures* and show their results of *soundness and completeness* with respect to *intuitionistic and classical logics.*

4.1 Truth arrows

Definition 21. In **Set**, $\neg : 2 \longrightarrow 2$ is the *function* with values $\neg(0) = 1$ and $\neg(1) = 0$. In other words, $\neg$ is the *characteristic function* of $\{0 :\subseteq\}2$. On the other hand, as an *arrow*, this *subset* is the *function* $\perp$ defined in example 2. Therefore, in *categorical language*, $\neg : \Omega \longrightarrow \Omega$ is the *only arrow* such that

$$\begin{array}{ccc} 1 & \xrightarrow{\perp} & \Omega \\ \big\downarrow{!} & & \big\downarrow{\neg} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

is a *pullback.*

Definition 22. In **Set**, $\cap : \Omega \times \Omega \longrightarrow \Omega$ is the *function* that gives *output* 1 only to the *input* $\langle 1, 1\rangle$, and so it is the *characteristic function* of the *set* $\{\langle 1, 1, \subseteq\rangle 2 \times 2 : \}$, which, as an *arrow*, is the *monic* $\langle \top, \top, :\rangle 1 \times 1 \longrightarrow 2 \times 2$. Therefore, *categorically*, it is the *only arrow* such that

$$\begin{array}{ccc} 1 \times 1 & \xrightarrow{\langle \top, \top \rangle} & \Omega \times \Omega \\ \big\downarrow{!} & & \big\downarrow{\cap} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

is a *pullback.*

Definition 23. In **Set**, $\cup : \Omega \times \Omega \longrightarrow \Omega$ is the *characteristic function* of the *set* $D = \{(0, 1), (1, 0), (1, 1) :=\}\{(1, 1), (0, 1) : \cup\}\{(1, 1), (1, 0) : \}$. Now, as *arrows*, the *sets* $\{(1, 1), (0, 1) :, \}\{(1, 1), (1, 0) :\subseteq\}2 \times 2$ are, respectively, the *arrows* $\langle 1_2, \top\rangle, \langle \top, 1_2\rangle : 2 \longrightarrow 2 \times 2$, and so, D is the *characteristic arrow* of $Im(f)$, where f is the *only arrow* such that

$$\begin{array}{ccccc} \Omega & \longrightarrow & \Omega + \Omega & \longleftarrow & \Omega \\ & \searrow^{\langle 1_2, \top \rangle} & \Big\downarrow{f} & {}^{\langle \top, 1_2 \rangle}\swarrow & \\ & & \Omega \times \Omega & & \end{array}$$

commutes. In other words, $\cup = \chi_{Im(f)}$, where $Im(f)$ is obtain through the *epi-monic factorisation* of f.

Definition 24. In **Set**, $\Rightarrow : \Omega \times \Omega \longrightarrow \Omega$ is the *characteristic function* of $R = \{(0, 0), (0, 1), (1, 1) :\subseteq\}2 \times 2$. On the other hand, R is precisely the *partial order relation* in the *lattice* 2. Hence, we can rewrite it as $R = \{(x, y) \in 2 \times 2 : x \leq y : \}$ or, since 2 is a *lattice*, $R = \{(x, y) \in 2 \times 2 : x \cap y = x : \}$, and so, R is the *equalizer* of $\cap$ and π_1. With R defined this way, $\Rightarrow$ is the *only arrow* such that

$$\begin{array}{ccc} dom(R) & \xrightarrow{R} & \Omega \times \Omega \\ \downarrow ! & & \downarrow \Rightarrow \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

is a *pullback*.

Definition 25. Given any *topos* $\mathcal{C}$, a $\mathcal{C} - valuation$ is a function $V : X \longrightarrow \mathcal{C}C(1, \Omega)$ that assigns to each *logical variable* in X a *truth value*. The *truth value* of any *logical statement* can then be calculated *inductively* by the rules

- $V(\neg\varphi) = \neg \circ V(\varphi)$
- $V(\varphi \cap \phi) = \cap \circ \langle V(\varphi), V(\phi)\rangle$
- $V(\varphi \cup \phi) = \cup \circ \langle V(\varphi), V(\phi)\rangle$
- $V(\varphi \Rightarrow \phi) = \Rightarrow \circ \langle V(\varphi), V(\phi)\rangle$

As always, we say that a statement ϕ is $\mathcal{C} - valid$, and write $\mathcal{C} \models \phi$, when, for every possible $\mathcal{C} - valuation$ V, $V(\phi) = \top$.

4.2 Sub(d) as an algebraic structure

Definition 26 (Complement, intersection and union)**.** Let $f, g \in Sub(d)$ we define

Complement we define the *complement* of f as the *subobject* $-f$ whose *characteristic arrow* is $\neg \circ \chi_f$

Intersection the *intersection* of f and g is the *subobject* $f \cap g$ whose *characteristic arrow* is $\cap \circ \langle \chi_f, \chi_g\rangle$

Union the *union* of f and g is the *subobject* $f \cup g$ whose *characteristic arrow* is $\cup \circ \langle \chi_f, \chi_g\rangle$

Relative pseudo-complement the *relative pseudo-complement* of f and g is the *subobject* $f \implies g$ whose *characteristic arrow* is $\implies \circ\langle \chi_f, \chi_g\rangle$.

Theorem 9. *Let $\mathcal{C}$ be a* topos *and $d \in Ob(\mathcal{C}C)$. Then $(Sub(d), \sqsubseteq)$ is a* lattice *where $f \cap g$ and $f \cup g$ are, respectively, the* greatest lower bound *and the* least upper bound.

Proof. Consult [3]. □

Theorem 10. *Let $\mathcal{C}$ be a* topos *and $d \in Ob\mathcal{C}C$. Then $(Sub(d), \sqsubseteq)$ is a* bounded lattice.

Proof. Consult [2] (theorem 5.3). □

Theorem 11. *Let $\mathcal{C}$ be a* topos *and $d \in Ob(\mathcal{C}C)$, then $(Sub(d), \sqsubseteq)$ is a* Heyting algebra *with* top 1_d, bottom 0_d *and where $f \cap g$, $f \cup g$ and $f \Rightarrow g$ are, respectively, the* meet, join *and the* relative pseudo-complement *of f and g.*

Proof. Consult [2] (theorem 5.6). □

Theorem 12. *Let $\mathcal{C}$ be a* topos*, then $\mathcal{C}C(d, \Omega)$ is a Heyting algebra with the* truth arrows *as operations.*

Proof. Consult [2] (theorem 5.7). □

4.3 Soundness and Completeness

Theorem 13. *Let $\mathcal{C}$ be a* topos *then, for every* logical statement φ

$$\textit{if} \qquad \varphi \in \mathcal{L}_{IPL} \qquad \textit{then} \qquad \mathcal{C} \models \varphi$$

Proof. Consult [2] (theorem 5.9). □

Theorem 14. *Let $\mathcal{C}$ be a* topos *then, for every* logical statement φ

$$\textit{if} \qquad \mathcal{C} \models \varphi \qquad \textit{then} \qquad \varphi \in \mathcal{L}_{CPL}$$

Proof. Consult [2] (theorem 5.10). □

5 Gödel's family of intermediate logics

In this section we capitalise in the work of Gödel ([4]) and construct a family of *topoi* that is sound and complete with respect to *Gödel's family of intermediate logics*.

5.1 Definition and algebraic semantics

Definition 27. For every $n \in \mathbb{N}$, define $\mathcal{G}_n$ as the *derivation closure* of the set

$$G_n = IPL \cup \{((x_0 \implies x_1) \vee (x_1 \implies x_0))\} \cup \{F_{n+1}\}$$

where *IPL* is the set of *axioms* of *intuitionistic logic* (consult [2], definition 2.4) and

$$F_n = \bigvee_{0 \le i < j < n} (x_i \iff x_j)$$

Definition 28. For every $n \in \mathbb{N}$, define the *heyting algebra* S_n in the following way

$$S_n = \{k \in \mathbb{N} : k > 0 \wedge k \le n\} \qquad x \implies y = \begin{cases} 1 & \text{, if } x \le y \\ n & \text{, if } x > y \end{cases}$$

$$x \vee y = \min(x, y) \quad x \wedge y = \max(x, y) \qquad \neg x = \begin{cases} 1 & \text{, if } x = n \\ n & \text{, if } x \ne n \end{cases}$$

Theorem 15. *Let φ be any* formula*, if $S_{n+1} \models \varphi$ then $S_n \models \varphi$, for every $n \in \mathbb{N}$.*

Proof. Consult [2] (theorem 6.1). □

Theorem 16. *For every $n \in \mathbb{N}$,*

$$\mathcal{G}_{n+1} \subset \mathcal{G}_n$$

Proof. Consult [2] (theorem 6.2). □

Theorem 17.

$$\mathcal{G}_2 = \mathcal{L}_{CPL} \tag{8}$$

Proof. Consult [2] (theorem 6.3). □

Theorem 18. *Let H be a* Heyting algebra*, then $H \models \mathcal{G}_n$ if and only if $H \simeq S_k$, for some $k \leq n$.*

Proof. Consult [2] (theorem 6.4). □

Theorem 19.

$$\varphi \in \mathcal{G}_n \qquad \textit{iff} \qquad S_n \models \varphi$$

Proof. Consult [2] (theorem 6.5). □

5.2 Sheaves

Definition 29. Let $\mathcal{L}$ be a *complete lattice.* $\mathcal{L}$ is said to be a *locale* if arbitrary *joins* distribute over finite *meets*, i.e.,

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$$

holds, for every $a, \{b_i\}_{i \in I} \in \mathcal{L}$ and every *index set* I.

Theorem 20. *Let $\mathcal{L}$ be a* lattice*, then $\mathcal{L}$ is a* locale*, if and only if, it is a* complete Heyting algebra*.*

Proof. Consult [2] (theorem 6.6). □

Definition 30. Let H be a *complete Heyting algebra* and $\mathcal{C}_H$ the *category* based on the *partial ordering* of H. A *pre-sheaf* of $\mathcal{C}_H$ is a *contravariant functor* from $\mathcal{C}_H$ to **Set**. The *category* $\mathbf{PreSh}(\mathcal{C}_H)$ is the *category* with *objects pre-sheaves* in $\mathcal{C}_H$ and *morphisms* the *natural transformations* between them. When no confusion arises, we will refer to $\mathcal{C}_H$ as simply $\mathbf{H}$.

Definition 31. Let F be a *pre-sheaf* of $\mathbf{H}$ and $\{u_i\}_{i \in I}$ a family of *elements* of H. A family $\{x_i \in F(u_i)\}_{i \in I}$ is called *compatible in F* when $x_i|_{u_i \wedge u_j} = x_j|_{u_i \wedge u_j}$, for every $i, j \in I$.

Definition 32. Let F be a *pre-sheaf* in $\mathbf{H}$. F is called a *sheaf* if, given $u = \bigvee_{i \in I} u_i$ in H and $\{(x_i \in F(u_i))\}_{i \in I}$ a compatible family in F, there exists a unique $x \in F(u)$ such that $x|_{u_i} = x_i$, for every $i \in I$. The *category* $\mathbf{Sh(H)}$ is the *category* with *objects sheaves* in $\mathcal{C}_H$ and *morphisms* the *natural transformations* between them.

5.3 Topoi semantics

Theorem 21. *If* H *is a* complete Heyting algebra*, then* ***Sh(H)*** *is a* topos*.*

Proof. Consult [1] (*example 5.2.3*). ☐

Theorem 22. *If* H *is a* complete Heyting algebra*, then* $H \simeq Sub_{\boldsymbol{Sh}(H)}(1)$*.*

Proof. Consult [1] (*corollary 2.2.16*). ☐

Theorem 23. *For every* $n \in \mathbb{N}$ *and every* formula φ*,*

$$\vdash_{\mathcal{G}_n} \varphi \quad iff \quad \boldsymbol{Sh}(S_n) \models \varphi$$

Proof. Consult [2] (theorem 6.9). ☐

References

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