A topoi characterization of Gödel intermediate logics

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I dedicate this thesis to my parents, without whom nothing would be possible.

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Resumo

Apresentamos uma construção geral para lógicas proposicionais intermédias e provamos a sua correção e completude relativamente a álgebras de heyting. Revemos noções básicas de teoria de categorias, topos e semântica de topos. Finalmente apresentamos uma semântica nova, baseada em topos, para a familia de lógicas apresentadas por *Gödel* em [1] e provamos a sua correção e completude.

Palavras-chave: lógica intuicionista, lógica intermédia, topos, semântica de topos, lógicas intermédias de Gödel

Abstract

A general construction for *propositional intermediate logics*, from *intuitionistic* to *classical*, is presented and it's *soundness and completeness* with respect to *heyting algebras* is proven. A review is made of basic notions of *category theory*, *topoi* and *topoi semantics*. A new semantics based on *topoi* to the family of *logics* presented by *Gödel* in [1] is given and proved to be *sound* and *complete*.

Keywords: intuitionistic logic, intermediate logic, topoi, topos semantics, Gödel intermediate logic

Contents

Chapter 1

Introduction

This thesis is dedicated to the study of a particular kind of *logical semantics* based on *topoi*. As such, it falls in the areas of *category theory* and *logic*. In particular, as will be seen further in this text, given the *algebraic* nature of *topoi*, it is intimately related to *intuitionistic logic*.

1.1 About Intuitionism and Intuitionistic Logic

Intuitionism is a philosophy of mathematics due to L.E.J. Brouwer (1981 - 1966), which is based on the idea that mathematical constructions are a product of the human mind and possess no independent existence. In the beginning of the twentieth century the mathematical framework was philosophically dominated by Platonism and formalism under which mathematical constructions are platonic entities that are discovered and studied by mathematicians. Brouwer rejected this idea stating that mathematics was a languageless creation of the mind, based solely in intuitive ideas that were rooted in an *a priori* notion of time. He's whole philosophy was sustained upon two *acts*:

Act 1 Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics.

Act 2 Admitting two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired; secondly in the shape of mathematical species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be "equal" to it.

In a very informal way, *act 1* gives rise to the *natural numbers* as a consequence of the experience of time. The second *act*, on the other hand, gives a way of creating new mathematical entities using previously created ones. Although this paradigm shift might seem fairly innocent, it encompasses a theory of mathematics in which many basic *classical* facts aren't true.

One of the most popular interpretations of *intuitionism* is called *BHK-interpretation* and was introduced by Arend Heyting in 1932. It gives an informal way of interpreting the intuitionistic connectives:

- $\perp$ is not provable
- a proof of $\varphi \wedge \phi$ consists of a proof of φ and a proof of ϕ
- a proof of $\varphi \vee \phi$ consists of a proof of φ or a proof of ϕ
- a proof of $\varphi \Rightarrow \phi$ is a construction that transforms any proof of φ in a proof of ϕ
- a proof of $\exists_x \varphi$ is given by presenting an element d and a proof for $[\varphi]_x^d$
- a proof of $\forall_x \varphi$ is a construction that transforms every proof that d belongs to the domain into a proof of $[\varphi]_x^d$
- a proof of $\neg\varphi$ is a construction that transforms every proof of φ in a proof of $\perp$

The interpretation of the connective $\vee$ already shows a big difference between *intuitionism* and *classical mathematics*. It is very common for a mathematician, in order to prove a statement $\neg\varphi$, to check whether φ is true. If it isn't then the mathematician concludes that $\neg\varphi$ is true. In other words, $\varphi \vee \neg\varphi$ holds *classically* for every formula φ. This is called the *Law of Excluded Middle*. But in this interpretation of the *logical connective* $\vee$, $\varphi \vee \neg\varphi$ only holds if we have either a proof for φ or a proof of $\neg\varphi$, i.e., a proof that φ cannot be proven. This means that for currently open problems such as **P=NP**, the formula $\varphi \vee \neg\varphi$ doesn't hold. It is also evident from examples such as this, that the *intuitionistic* notion of truth is intimately related to a notion of the passage of time: something that isn't true in the present might be so in the future.

Although Brouwer rejected formalism and was very cautious of the use of language, he accepted mathematical language as a means of transferring information from one mathematician to another. This inevitably led to the development of a formal basis for *intuitionism* called *intuitionistic logic* that was proposed by Heyting. This formal system is widely recognised as the basis of constructivism as well. However, it's worth noting that, since Brouwer rejected language as a means of obtaining mathematical truths, a formal deduction in *intuitionistic logic* does not suffice for a statement to be *intuitionistically* true.

For more information about *intuitionism* or Brouwer, you can consult [2] or [3]. For the works of Brouwer himself check [4].

1.2 About category theory and topoi

Category theory was first introduced in a 1945 paper by Eilenberg & Maclane entitled "General theory of natural transformations". This paper was, in turn, a deepening of a previous paper where they had used some specific functors and natural transformations. As the name implies, their main objective was to study natural transformations, and in order to do so, they had to define functors and categories. At this point, category theory was seen simply as an effective language that permitted to study some constructions with much more ease than what would be permitted by it's set theoretical counterpart. Indeed, many results proved by Eilenberg in the 50's would be very intricate to prove in a set theoretical framework. Things started to change in the end of the 50's with the works of Grothendieck and Kan who, in 1958, introduced the notion of adjoint functor. The 60's would see a rapid growth in the applications of category theory. In particular, in his Ph.D thesis, Lawvere proposes category theory as a foundation for itself, but also for set theory and, therefore, for the whole of mathematics. Furthermore, he made great achievements in the field of logic and the foundation of mathematics, from axiomatizing the category of sets to showing that logical quantifiers would be captured by adjoint functors. Still during the 60's, Grothendieck introduced the notion of a topos, in order to study sheaves on a space. In the 70's Lawvere and Tierney would pick up this notion and show that topoi had a structure rich enough to develop the usual logical constructions, thus giving topoi a central foundational role. Since then, category theory has found applications in a lot of different areas, from theoretical computer science to physics.

1.3 Thesis overview

This thesis aims at giving a good introduction into *topos semantics* and to provide a new semantics (as far as we know) to *Gödel intermediate logics* based on topoi. Being an area of mathematical logic which is very multidisciplinary in itself, this thesis follows that idea and stands upon the author's deep belief that "the devil is in the details" and that in order to understand an area of knowledge, one must first understand the areas that gave rise to it. With this in mind, the first two chapters are dedicated to reviewing basic notions of *logic* and *category theory*. In chapter 2 we define essential *logical* notions of *syntax* and *semantics* and finish with the *soundness* and *completeness* results. Chapter 3 is dedicated to *category theory*, where we define basic notions like *limits* and *exponentiation*. Next, we move on to introducing the *categorical constructions* that characterize *topoi*, and we finish the chapter with their *definition*. In chapter 5 we define the notion of *topos validity*, which is the central notion of *topos semantics*. Finally, chapter 6 is dedicated to the study of a particular family of *intermediate logics* introduced by Gödel, where is presented a *topoi-based semantic characterization* for these *logics* that, as far as is known by the author, wasn't studied before.

Chapter 2

Basic Logical Concepts

This chapter is dedicated to the review of some basic concepts of mathematical logic. We will start by defining the necessary *syntactic constructions*, then we'll introduce a possible *semantics* based on *heyting algebras* and we'll finish by proving the *soundness* and *completeness* of these *semantics*. For a deeper and more complete introduction to mathematical logic consult [5] or [6].

2.1 Syntax

From here on, we fix a *set* $X = \{x_n\}_{n\in\mathbb{N}}$, which we will refer to as the *set of variables*, and four "special" symbols: $\neg, \vee, \wedge, \Rightarrow$, which we will refer to as *logical connectives*.

Definition 2.1 (Propositional formula)**.** The *set* $\mathcal{F}$ of all *propositional formulas* is *inductively defined* as follows

- $X \subseteq \mathcal{F}$
- if $\phi \in \mathcal{F}$, then $(\neg\phi) \in \mathcal{F}$
- if $\phi, \varphi \in \mathcal{F}$, then $(\phi \wedge \varphi), (\phi \vee \varphi), (\phi \Rightarrow \varphi) \in \mathcal{F}$

Definition 2.2 (Substitution)**.** Given $f : X \longrightarrow \mathcal{F}$ we define the *substitution* $[[\varphi]]_f$ by f in every *propositional formula* inductively as follows

- $[[x]]_f = f(x)$
- $[[\varphi \vee \phi]]_f = [[\varphi]]_f \vee [[\phi]]_f$
- $[[\varphi \wedge \phi]]_f = [[\varphi]]_f \wedge [[\phi]]_f$
- $[[\varphi \Rightarrow \phi]]_f = [[\varphi]]_f \Rightarrow [[\phi]]_f$
- $[[\neg\varphi]]_f = \neg\, [[\varphi]]_f$

Definition 2.3 (Propositional logic)**.** Given a *set* $\Gamma \subseteq \mathcal{F}$, we define the *propositional logic* $\mathcal{L}_\Gamma$ *axiomatized by* Γ as the *smallest subset of* $\mathcal{F}$ such that

1. $\Gamma \subseteq \mathcal{L}_\Gamma$

2. if $\varphi, (\varphi \Rightarrow \phi) \in \mathcal{L}_\Gamma$ then $\phi \in \mathcal{L}_\Gamma$

3. if $f : X \longrightarrow \mathcal{F}$ and $\varphi \in \mathcal{L}_\Gamma$, then $[[\varphi]]_f \in \mathcal{L}_\Gamma$

In other words, $\mathcal{L}_\Gamma$ is the *smallest subset of* $\mathcal{F}$ that is closed for *modus ponens* and *arbitrary substitutions*.

Although we have defined *propositional logic* in a very general way, in the rest of this work we will work almost exclusively with the following ones

Definition 2.4 (Intuitionistic propositional logic)**.** Let IPL be the *set* containing the following *propositional formulas*

1. $x_0 \Rightarrow (x_1 \Rightarrow x_0)$

2. $(x_0 \Rightarrow (x_1 \Rightarrow x_2)) \Rightarrow ((x_0 \Rightarrow x_1) \Rightarrow (x_0 \Rightarrow x_2))$

3. $(x_0 \wedge x_1) \Rightarrow x_0$

4. $(x_0 \wedge x_1) \Rightarrow x_1$

5. $(x_2 \Rightarrow x_0) \Rightarrow ((x_2 \Rightarrow x_1) \Rightarrow (x_2 \Rightarrow (x_0 \wedge x_1)))$

6. $x_0 \Rightarrow (x_0 \vee x_1)$

7. $x_1 \Rightarrow (x_0 \vee x_1)$

8. $(x_0 \Rightarrow x_2) \Rightarrow ((x_1 \Rightarrow x_2) \Rightarrow ((x_0 \vee x_1) \Rightarrow x_2)$

9. $(x_0 \Rightarrow \neg x_1) \Rightarrow (\neg x_0 \Rightarrow x_1)$

10. $(\neg(x_0 \Rightarrow x_0)) \Rightarrow x_1$

We define the *Intuitionistic propositional logic* as $\mathcal{L}_{IPL}$.

This *axiomatic* is the one used by *Wójcicki* and can be consulted in [6].

Definition 2.5 (Classical propositional logic)**.** Consider the *set* $CPL = IPL \cup \{(x_0 \vee (\neg x_0)\}$. We define the *Classical propositional logic* as the $\mathcal{L}_{CPL}$.

Definition 2.6 (Intermediate propositional logic)**.** A *propositional logic* $\mathcal{L}$ is said to be an *intermediate propositional logic* if $\mathcal{L}_{IPL} \subset \mathcal{L} \subset \mathcal{L}_{CPL}$.

If $\varphi \in \mathcal{L}$ for some *propositional logic* $\mathcal{L}$ then we write $\vdash_{\mathcal{L}} \varphi$ and say that φ is a *tautology* of $\mathcal{L}$.

2.2 Semantics

In what follows, we will define some *algebraic structures*, called *heyting algebras*, that will be the basis for our *semantics* to *propositional logics*. We end this chapter by proving the *soundness* and *completeness* of this *semantic*.

Definition 2.7 (Lattice)**.** Let $\mathcal{A} = (A, \leq)$ be a *partial ordering*. Then $\mathcal{A}$ is said to be a *lattice* if for every pair of *elements* $a, b \in A$ there are *elements* $a \vee b, a \wedge b \in A$ such that, for every $c \in A$,

- if $a, b \leq c$, then $a \vee b \leq c$
- if $c \leq a, b$, then $c \leq a \wedge b$

the *elements* $a \vee b$ and $a \wedge b$ are called, respectively, the *supremum* and the *infimum* of a and b, and the operations $\vee$ and $\wedge$ are called, respectively, the *join* and the *meet* of the *lattice* $\mathcal{A}$.

Definition 2.8 (Bounded lattice)**.** A lattice $\mathcal{A}$ is said to be *bounded* if there are two *elements* $0, 1 \in \mathcal{A}$ called, respectively, the *bottom* and the *top* of the *lattice*, such that for every $c \in \mathcal{A}$, $0 \leq c \leq 1$.

Definition 2.9 (Heyting algebra)**.** A *bounded lattice* $\mathcal{H}$ is said to be a *heyting algebra* if, for every $a, b \in \mathcal{H}$ there is an *element* $a \Rightarrow b \in \mathcal{H}$ such that

$$c \wedge a \leq b \qquad \text{iff} \qquad c \leq a \Rightarrow b$$

We call $a \Rightarrow b$ the *pseudo-complement of* a *relative to* b. In this context, we refer to $a \Rightarrow 0$ as the *pseudo-complement* of a and write $\neg a$ instead of $a \Rightarrow 0$.

Definition 2.10 (Boolean algebra)**.** A *heyting algebra* $\mathcal{B}$ is said to be a *boolean algebra* if, for every $a \in \mathcal{B}$, the *pseudo-complement* $\neg a$ is such that $a \wedge (\neg a) = 0$ and $a \vee (\neg a) = 1$. In this case, we refer to the *element* $\neg a$ as simply the *complement* of a.

The next result will be useful in short time

Theorem 2.1. *Let* $\mathcal{H}$ *be a* heyting algebra *and* $a, b \in \mathcal{H}$. *Then* $a \Rightarrow b = 1$ *if and only if* $a \leq b$.

Proof. If $a \Rightarrow b = 1$, then by the definition of the $\Rightarrow$ *operator* we know that for every $c \in \mathcal{H}$, $c \leq a \Rightarrow b = 1$ if and only if $c \wedge a \leq b$. If we choose $c = 1$ then $a = 1 \wedge a \leq b$. □

Next, we define a *semantics* based on *heyting algebras*.

Definition 2.11. Let $\mathcal{H}$ be a *heyting algebra*. A $\mathcal{H}$-valuation is a *function* $v : X \longrightarrow \mathcal{H}$. This notion extends naturally to every *propositional formula* in the following way

- $v(\varphi \vee \phi) = v(\varphi) \vee v(\phi)$
- $v(\varphi \wedge \phi) = v(\varphi) \wedge v(\phi)$
- $v(\varphi \Rightarrow \phi) = v(\varphi) \Rightarrow v(\phi)$

- $v(\neg\varphi) = \neg v(\varphi)$

Definition 2.12. Consider a *heyting algebra* $\mathcal{H}$, a *formula* φ and a *propositional logic* $\mathcal{L}$. We say that $\mathcal{H}$ satisfies φ (φ is valid in $\mathcal{H}$ or is $\mathcal{H}$-valid) and write $\mathcal{H} \models \varphi$, if $v(\varphi) = 1$ for every $\mathcal{H}$-valuation v. We say that $\mathcal{H}$ is a *model* for the *logic* $\mathcal{L}$ if $\mathcal{H} \models \phi$ for every $\phi \in \mathcal{L}$.

Given a *propositional logic* $\mathcal{L}$ and a *heyting algebra* $\mathcal{H}$ we denote by Mod$(\mathcal{L})$ the *set* of all *models* of $\mathcal{L}$ and by Th$(\mathcal{H})$ the set of all *logics* for which $\mathcal{H}$ is a *model*, i.e.,

$$\begin{aligned} \text{Mod}(\mathcal{L}) &= \{\mathcal{H} \text{ a heyting algebra } : \mathcal{H} \models \mathcal{L}\} \\ \text{Th}(\mathcal{H}) &= \{\mathcal{L} \text{ a propositional logic } : \mathcal{H} \models \mathcal{L}\} \end{aligned}$$

2.3 Soundness and completeness

We will now prove the theorems of soundness and completeness. In particular, we will prove that every *heyting algebra* is a *model* to $\mathcal{L}_{IPL}$, every *boolean algebra* is a *model* to $\mathcal{L}_{CPL}$ and that the class Mod$(\mathcal{L})$ is complete with respect to $\mathcal{L}$, whenever $\mathcal{L}_{IPL} \subset \mathcal{L}$.

Theorem 2.2. *If $\mathcal{H}$ is a* heyting algebra*, then $\mathcal{H} \models \mathcal{L}_{IPL}$.*

Proof. The proof follows by proving that every *heyting algebra* satisfies every *axiom* of $\mathcal{L}_{IPL}$ and that if $\mathcal{H} \models \varphi$ and $\mathcal{H} \models (\varphi \Rightarrow \phi)$ then $\mathcal{H} \models \phi$. Concerning the first part, we will only prove it for the first *axiom* leaving the rest to the reader. So consider a *heyting algebra* $\mathcal{H}$ and a $\mathcal{H}$-valuation v. Then

$$\begin{aligned} v(\varphi \Rightarrow (\phi \Rightarrow \varphi)) = 1 &\quad \text{iff} \\ v(\varphi) \Rightarrow (v(\phi) \Rightarrow v(\varphi)) = 1 &\quad \text{iff (by theorem 2.1)} \\ v(\varphi) \leq (v(\phi) \Rightarrow v(\varphi)) &\quad \text{iff (by definition 2.9)} \\ v(\varphi) \wedge v(\phi) \leq v(\varphi) &\quad \text{which is true in every } \textit{lattice} \end{aligned}$$

Since the choice of $\mathcal{H}$ and v was arbitrary, we conclude that for every *heyting algebra* $\mathcal{H}$, $\mathcal{H} \models (\varphi \Rightarrow (\phi \Rightarrow \varphi))$. Now suppose that $\mathcal{H}$ is such that $\mathcal{H} \models \varphi$ and $\mathcal{H} \models (\varphi \Rightarrow \phi)$. Consider also a $\mathcal{H}$-valuation v. In this case we have

$$\begin{aligned} v(\varphi) \Rightarrow v(\phi) &= v(\varphi \Rightarrow \phi) \\ &= 1 \end{aligned}$$

From theorem 2.1 we conclude that $v(\varphi) \leq v(\phi)$. Since $\mathcal{H} \models \varphi$, we know that $v(\varphi) = 1$ and so it must be the case that $v(\phi) = 1$. Since v was arbitrary we conclude that $\mathcal{H} \models \phi$. □

Theorem 2.3. *Let $\mathcal{H}$ be a* heyting algebra*. Then $\mathcal{H} \models \mathcal{L}_{CPL}$ if and only if $\mathcal{H}$ is a* boolean algebra*.*

Proof. Suppose $\mathcal{H}$ is a *boolean algebra*. Then the $\neg$ operator is actually a *complement*, i.e., $a \vee (\neg a) = 1$ for every $a \in \mathcal{H}$. So, if v is a $\mathcal{H}$-valuation,

$$\begin{aligned} v(\varphi \vee (\neg\varphi)) &= v(\varphi) \vee (\neg v(\varphi)) \\ &= 1 \end{aligned}$$

and so $\mathcal{H} \models (\varphi \vee (\neg\varphi))$ and $\mathcal{H} \models \mathcal{L}_{CPL}$. Now consider a *heyting algebra* $\mathcal{H}$, $a \in \mathcal{H}$ and suppose $\mathcal{H} \models \mathcal{L}_{CPL}$. We need to show that $a \vee (\neg a) = 1$. If $\mathcal{H} \models \mathcal{L}_{CPL}$ then, in particular, $\mathcal{H} \models \varphi \vee (\neg\varphi)$. Let v be a $\mathcal{H}$-valuation such that $v(\varphi) = a$, then

$$\begin{aligned} a \vee (\neg a) &= v(\varphi) \vee (\neg v(\varphi)) \\ &= v(\varphi \vee (\neg\varphi)) \\ &= 1 \end{aligned}$$

□

For the completeness results, we first need to construct a special *lattice* called the *Lindenbaum-Tarski algebra*.

Definition 2.13 (Lindenbaum-Tarski algebra)**.** Given a *propositional logic* $\mathcal{L}$, such that $\mathcal{L}_{IPL} \subseteq \mathcal{L}$ consider the following *equivalence relation* on $\mathcal{F}$

$$\varphi \equiv_{\mathcal{L}} \phi \qquad \text{iff} \qquad (\varphi \Leftrightarrow \phi) \in \mathcal{L}$$

Also, define the *partial ordering* of $\mathcal{F}/\equiv_{\mathcal{L}}$ by

$$[a] \leq_{\mathcal{L}} [b] \qquad \text{iff} \qquad (a \Rightarrow b) \in \mathcal{L}$$

The $(\mathcal{F}/\equiv_{\mathcal{L}}, \leq_{\mathcal{L}})$ is called the *Lindenbaum-Tarski algebra over $\mathcal{L}$* and will be referred to as Lind($\mathcal{L}$).

Theorem 2.4. *Lind($\mathcal{L}$) is a* bounded lattice *with* minimum $[\bot]$*,* maximum $[\top]$ *and with the operations of* supremum *and* infimum *defined, respectively, as follows*

$$\begin{aligned} [a] \vee [b] &= [a \vee b] \\ [a] \wedge [b] &= [a \wedge b] \end{aligned}$$

Proof. The proof that $\vee$ and $\wedge$ are well-defined operations on Lind($\mathcal{L}$) is immediate and we omit it for the sake of simplicity. Now let $[a], [b], [c], [d] \in$ Lind($\mathcal{L}$) such that $[d] \leq [a], [b] \leq [c]$. We need to show that

$[a] \vee [b] \leq [c]$ and $[d] \leq [a] \wedge [b]$. In fact:

$$\begin{aligned} [a], [b] \leq_{\mathcal{L}} [c] \quad & \text{implies (by definition)} \\ (a \Rightarrow c), (b \Rightarrow c) \in \mathcal{L} \quad & \text{implies (using axiom 8)} \\ ((a \vee b) \Rightarrow c) \in \mathcal{L} \quad & \text{implies (by definition)} \\ [a \vee b] = [a] \vee [b] \leq_{\mathcal{L}} [c] \quad & \end{aligned}$$

$$\begin{aligned} [d] \leq_{\mathcal{L}} [a], [b] \quad & \text{implies (by definition)} \\ (d \Rightarrow a), (d \Rightarrow b) \in \mathcal{L} \quad & \text{implies (using axiom 5)} \\ (d \Rightarrow (a \wedge b)) \in \mathcal{L} \quad & \text{implies (by definition)} \\ [d] \leq [a \wedge b] = [a] \wedge [b] \quad & \end{aligned}$$

The fact that $[\bot]$ and $[\top]$ are, respectively, the *minimum* and *maximum* comes from the fact that $(\bot \Rightarrow \varphi), (\varphi \Rightarrow \top) \in \mathcal{L}$ for every *formula* φ. ☐

Theorem 2.5. *Lind*$(\mathcal{L})$ *is a* heyting algebra *with* pseudo-complement *operation defined as*

$$[a] \Rightarrow [b] = [a \Rightarrow b]$$

Proof. We need to show that

$$[a] \wedge [c] \leq [b] \qquad \text{iff} \qquad [c] \leq [a] \Rightarrow [b]$$

This amounts to show that $(((a \wedge c) \Rightarrow b) \Rightarrow (c \Rightarrow (a \Rightarrow b))) \in \mathcal{L}_{IPL}$ and $((c \Rightarrow (a \Rightarrow b)) \Rightarrow ((a \wedge c) \Rightarrow b)) \in \mathcal{L}_{IPL}$. *Derivation sequences* for both results are provided in *Hilbert calculi* in appendix A. ☐

From all the possible Lind($\mathcal{L}$)-valuation, there is a particular one which will play an important role in what follows. We'll call it the *quotient projection* and it's defined by

$$\pi(x) = [x]$$

for every $x \in X$, or, in other words,

$$\pi(\varphi) = [\varphi]$$

for every $\varphi \in \mathcal{F}$. The importance of this *valuation* follows from the following fact

Theorem 2.6. *Let* $\mathcal{L}$ *be a* propositional logic*,* v *a Lind*$(\mathcal{L})$*-valuation and* π *the* quotient projection*, then for every* $\varphi \in \mathcal{F}$

$$v(\varphi) = \pi([[\varphi]]_{f_v})$$

where $f_v : X \longrightarrow$ *Lind*$(\mathcal{L})$ *is a* function *such that* $f_v(x) \in v(x)$.

Proof. The proof follows by induction on the *formula* φ. Suppose $\varphi = x$ for some $x \in X$. Then

$$\begin{aligned} \pi([[x]]_{f_v}) &= \pi(f_v(x)) \\ &= v(x) \quad \text{(since } f_v(x) \in v(x)\text{)} \end{aligned}$$

The induction step follows directly from the *inductive definition* of *valuation* and *substitution*. □

Finally

Theorem 2.7. *Let* $\mathcal{L}$ *be a* propositional logic *such that* $\mathcal{L}_{IPL} \subseteq \mathcal{L}$. *Then, for every* $\varphi \in \mathcal{F}$,

$$\varphi \in \mathcal{L} \qquad \textit{iff} \qquad \textit{Lind}(\mathcal{L}) \models \varphi$$

Proof. We start by the *only if* part, so suppose $\varphi \in \mathcal{L}$. Then $[[\varphi]]_{f_v} \in \mathcal{L}$ and $[[[\varphi]]_{f_v}] = [\top]$, since $([[\varphi]]_{f_v} \Leftrightarrow \top) \in \mathcal{L}$. Let v be a Lind$(\mathcal{L})$-valuation, by theorem 2.6 we have that

$$\begin{aligned} v(\varphi) &= \pi([[\varphi]]_{f_v}) \\ &= [[[\varphi]]_{f_v}] \\ &= [\top] \end{aligned}$$

since v was an arbitrary Lind$(\mathcal{L})$-valuation, we conclude that Lind$(\mathcal{L}) \models \varphi$.
For the *if* part suppose Lind$(\mathcal{L}) \models \varphi$. Then, for every Lind$(\mathcal{L})$-valuation v, $v(\varphi) = [\top]$. In particular

$$[\varphi] = \pi(\varphi) = [\top]$$

But this means that $(\varphi \Leftrightarrow \top) \in \mathcal{L}$, and so $\varphi \in \mathcal{L}$. □

The next result is a direct corollary of the previous result.

Theorem 2.8. *Let* $\mathcal{L}$ *be a* propositional logic *such that* $\mathcal{L}_{IPL} \subseteq \mathcal{L}$. *Then, for every* $\varphi \in \mathcal{F}$,

$$\varphi \in \mathcal{L} \qquad \textit{iff} \qquad \mathcal{H} \models \varphi$$

for every $\mathcal{H} \in$ *Mod*$(\mathcal{L})$.

Chapter 3

Basic Categorical Concepts

In this chapter we briefly introduce the concept of *category* and some of it's basic constructions. For more information on the subject the reader may consult [7] or the first chapters of [8].

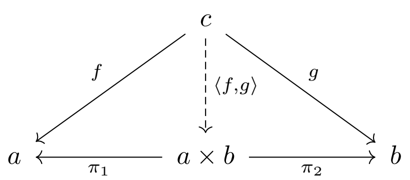
3.1 Products and Co-products

Given two sets A and B we can construct the product set $A \times B = \{\langle a, b\rangle : a \in A \text{ and } b \in B\}$. Associated with this construction there are two "special" *functions*, $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$, called *projections* that map an ordered pair $\langle a, b\rangle$ into its first and second components, respectively:

$$\pi_1(\langle a, b\rangle) = a \qquad \pi_2(\langle a, b\rangle) = b$$

These *functions* are always defined, at least as long as the product $A \times B$ exists, and so it makes sense to use them when defining the notion of *categorical product*.

Definition 3.1 (Product)**.** Let $\mathcal{C}$ be a category and $a, b \in Ob(\mathcal{C})$. $a \times b \in Ob(\mathcal{C})$ together with two arrows, $\pi_1 : a \times b \longrightarrow a$ and $\pi_2 : a \times b \longrightarrow b$, is said to be *a product* of a and b if, for every $c \in Ob(\mathcal{C})$ and $(f : c \longrightarrow a), (g : c \longrightarrow b) \in Mor(\mathcal{C})$, there exists one and only one $\langle f, g\rangle \in Mor(\mathcal{C})$ such that the following diagram commutes



(3.1)

Theorem 3.1. *Let $\mathcal{C}$ be a category and $a, b \in Ob(\mathcal{C})$. If $c, d \in Ob(\mathcal{C})$ are both products of a and b then $c \simeq d$.*

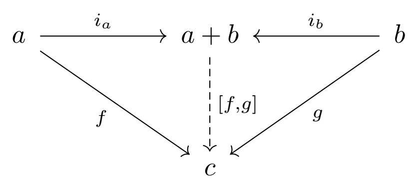
Example 3.1. In the category **Set** the product of two objects (i.e. two sets), A and B, is the *Cartesian product* $A \times B$.

Example 3.2. In the category **Top** the product of two spaces A and B is the standard product space $A \times B$.

Example 3.3. If $\mathcal{P} = (P, \sqsubseteq)$ is a *pre-order*, then it can always be viewed as a category (with $Ob(\mathcal{P}) = P$ and an arrow $f : a \longrightarrow b$ if $a \sqsubseteq b$). In this case, the product of a and b, when it exists, is the *greatest lower bound* of a and b.

The dual notion of *product* is called *co-product*. Therefore, a *co-product* in a category $\mathcal{C}$ is a *product* in the opposite category $\mathcal{C}^{op}$. For clarity sake, we'll also define *co-product* directly, without referring to the opposite category.

Definition 3.2 (Co-Product)**.** Let $\mathcal{C}$ be a category and $a, b \in Ob(\mathcal{C})$. $a + b \in Ob(\mathcal{C})$ together with two arrows, $i_a : a \longrightarrow a + b$ and $i_b : b \longrightarrow a + b$, is said to be *a co-product* of a and b if, for every $c \in Ob(\mathcal{C})$ and $(f : a \longrightarrow c), (g : b \longrightarrow c) \in Mor(\mathcal{C})$, there exists one and only one $[f, g] \in Mor(\mathcal{C})$ such that the following diagram commutes



(3.2)

As with the *product*, a dual form of theorem 3.1 is also valid. In other words, if c and d are both *co-products* of a and b, then c and d are isomorphic.

Example 3.4. In the category **Set**, the *co-product* of two sets, A and B, is the disjoint union.

Example 3.5. Let $\mathcal{P}$ be the category determined by the pre-order $(P, \sqsubseteq)$, then the *co-product* of two objects, x and y, when it exists, is the least upper bound of x and y.

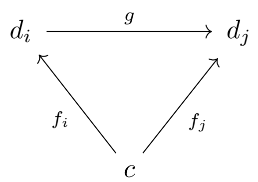
3.2 Limits and Co-limits

We start by defining some auxiliary concepts.

Definition 3.3 (Diagram)**.** Let $\mathcal{C}$ be a category. A diagram D in $\mathcal{C}$ consists of a set $Ob(D) \subseteq Ob(\mathcal{C})$ and a set $Mor(D) \subseteq Mor(\mathcal{C})$ such that, if $f \in Mor(D)$ and $f : a \longrightarrow b$, then $a, b \in Ob(D)$.

In other words, a diagram for the category $\mathcal{C}$ is just a "fragment" of $\mathcal{C}$.
Next, we define the concept of *cone* for a diagram.

Definition 3.4 (Cone)**.** Let D be a diagram in the category $\mathcal{C}$. A cone for D is a $\mathcal{C} - object$ c together with a family of $\mathcal{C} - arrows$ $\{f_i : c \longrightarrow d_i\}$, for each $d_i \in Ob(D)$, such that

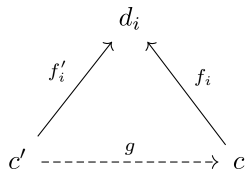


(3.3)

commutes, whenever $g \in Mor(D)$.

Now, we finally have the required tools to define the concept of limit.

Definition 3.5 (Limit)**.** Let D be a diagram in the category $\mathcal{C}$. *A limit* for D is a cone $\{f_i : c \longrightarrow d_i\}$ such that, if $\{f'_i : c' \longrightarrow d_i\}$ is a cone for D, then there is one and only one $g \in Mor(\mathcal{C})$ such that



(3.4)

commutes, for every $d_i \in Ob(D)$.

Just like with *products* and *co-products* it's trivial to show that, when they exist, the *limits* for a diagram D are uniquely determined up to isomorphism.

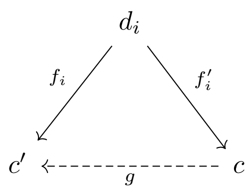
Theorem 3.2. *Let D be diagram in the category $\mathcal{C}$. If c and c' are both limits for D, then $c \simeq c'$.*

Example 3.6. Let D be a diagram in $\mathcal{C}$ with two objects, a and b, and no arrows. Then a *limit* for D, when it exists, is the *product* $a \times b$.

Example 3.7. Let D be the *empty* diagram in $\mathcal{C}$, i.e. $Ob(D) = Mor(D) = \varnothing$, then *a limit* for D is a $\mathcal{C} - object\ c$ such that, for every other $c' \in Ob(\mathcal{C})$ there is exactly one arrow $c' \dashrightarrow c$. In other words, the limits for the *empty* diagram are the *terminal objects* of $\mathcal{C}$.

Just as with *co-products*, *co-limits* are the dual notion of *limits*. This requires us to define the notion of *co-cone* as a $\mathcal{C} - object\ c$ and arrows $\{f_i : d_i \longrightarrow c\}$ for each $d_i \in Ob(D)$ with the dual property of diagram 3.4.

Definition 3.6 (Co-Limit)**.** Let D be a diagram in the category $\mathcal{C}$. *A co-limit* for D is a *co-cone* $\{f_i : d_i \longrightarrow c\}$ such that, if $\{f_i : d_i \longrightarrow c'\}$ is a *co-cone* for D, then there is one and only one $g \in Mor(\mathcal{C})$ such that



(3.5)

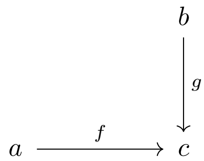
commutes, for every $d_i \in Ob(D)$.

Example 3.8. Since *co-limit* is the dual notion of *limit*, the *co-limits* for the diagrams in examples 3.6 and 3.7 are the *dual* objects of their *limits*, respectively, the *co-product* $a + b$ and the *initial objects*.

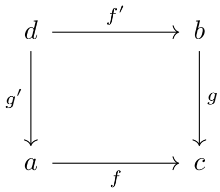
3.3 Pullbacks

We now define a particular kind of *limit* which will be used extensively through the rest of this work.

Definition 3.7. Given a category $\mathcal{C}$ and a pair of arrows $a \xrightarrow{f} c \xleftarrow{g} b$, a *pullback* for this diagram is a *limit* for the diagram

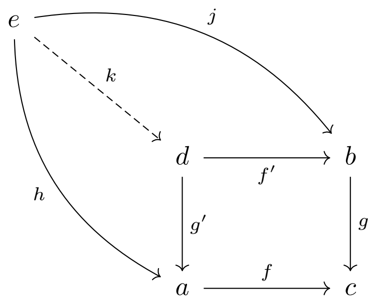


In other words, a *pullback* is a pair of arrows $a \xleftarrow{g'} d \xrightarrow{f'} b$ such that



(3.6)

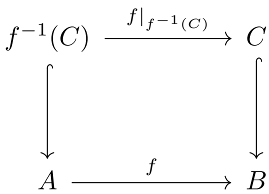
commutes, and for every pair of arrows $a \xleftarrow{h} e \xrightarrow{j} b$ that make diagram 3.6 commute, there is one and only one arrow $e \dashrightarrow d$ that makes the whole diagram



(3.7)

commute. Usually it is said that f' arises by *pulling back* f along g, and g' arises by *pulling back* g along f.

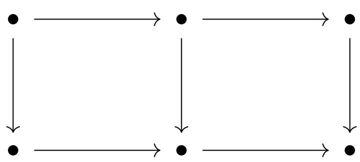
Example 3.9. Given two sets $A, B \in Ob($**Set**$)$, $C \subseteq B$ and $f : A \longrightarrow B$, then the diagram



(3.8)

is a *pullback* in **Set**, where $f|_{f^{-1}(C)}$ denotes the restriction of f to the subset $f^{-1}(C) \subseteq A$ and the tailed arrows denote the usual inclusions. In this way, we say that we get $f^{-1}(C)$ by *pulling back* C along f.

Theorem 3.3 (The Pullback Lemma)**.** *Suppose a diagram of the form*



commutes, then

- *if the two small squares are* pullbacks*, then the outer "rectangle" is a* pullback*;*

- *if the outer "rectangle" and right hand square are* pullbacks*, then so is the left hand square;*

3.4 Exponentiation

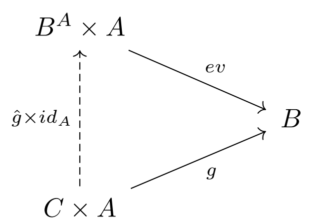
Given two sets A and B we can construct a new set B^A of all *functions* from A to B, i.e.

$$B^A = \{f|\ f \text{ is a function from } A \text{ to } B\}$$

This construction is often called *exponentiation* and can be characterized with purely categorical concepts.
First of all, observe that associated with the set B^A there is a "special" *function* $ev : B^A \times A \longrightarrow B$ called *evaluation* which is given by the rule

$$ev(\langle f, x\rangle) = f(x)$$

What makes this *function* "special" is the fact that given any set of the form $C \times A$ (where C is an arbitrary set) and a *function* $g : C \times A \longrightarrow B$ then there is one and only one *function* $\hat{g} : C \longrightarrow B^A$ such that



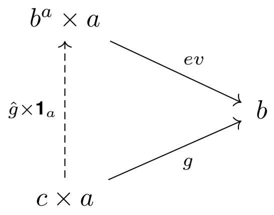
commutes. This *function* $\hat{g}$ associates to each $c \in C$ a *function* $g_c \in B^A$ given by the rule

$$g_c(a) = g(\langle c, a\rangle) \tag{3.9}$$

and so, for every $\langle c, a\rangle \in C \times A$, we get

$$(ev \circ \langle \hat{g}, id_A\rangle)(\langle c, a\rangle) = ev(\langle g_c, a\rangle) = g_c(a) = g(\langle c, a\rangle) \tag{3.10}$$

Definition 3.8 (Exponentiation)**.** A category $\mathcal{C}$ is said to have *exponentiation* if the *product* exists for any pair of $\mathcal{C} - objects$, and if given $a, b \in Ob(\mathcal{C})$ there is a $\mathcal{C} - object$ b^a and a $\mathcal{C} - arrow$ $ev : b^a \times a \longrightarrow b$ such that, for any $c \in Ob(\mathcal{C})$ and arrow $g : c \times a \longrightarrow b$, there is one and only one arrow $\hat{g} : c \longrightarrow b^a$ that makes



(3.11)

commute.

Chapter 4

Constructing Topoi

Intuitively, the notion of *topos* is that of a category which is somewhat similar to the category **Set**. This means we need to abstract to the category level, not only the set constructions like *products* and *exponentiations*, but also some fundamental notions about sets, like *subsets* and the idea of an *element of a set*. This section aims at reviewing these concepts and constructions.

4.1 Subobjects

Given *sets* A and B, if $A \subseteq B$, then there is an injective *function* $i : A \hookrightarrow B$, called the *inclusion* of A in B. On the other hand, if $f : A \longrightarrow B$ is injective, then we know A is in bijection with the subset

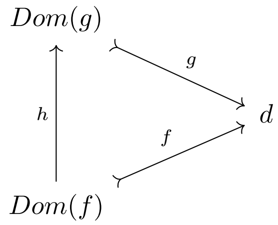
$$Im(f) = \{f(x) : x \in A\}$$

of B. These facts show us that there is a deep relation between *subsets* and *injective functions* which can be used to "lift" the concept of *subset* to the "categorical language" in the following way:

Definition 4.1 (Subobject)**.** Let $\mathcal{C}$ be a category and $d \in Ob(\mathcal{C})$. A *subobject* of d is a *monic arrow* with *co-domain* d.

Now, unlike in set theory, where a *set* and its *subsets* are objects of the same nature (they're both sets!), a *subobject* is not actually an object, but an arrow. This means that the notion of inclusion for *subobjects* cannot be generalized from the notion of *subobject*, and so as to be defined separately.

Definition 4.2 (Subobject inclusion)**.** Let $\mathcal{C}$ be a category, $d \in Ob(\mathcal{C})$ and f, g two *subobjects* of d. We say that $f \sqsubseteq g$ if there is an arrow $h : Dom(f) \longrightarrow Dom(g)$ (this arrow, when it exists will always be monic) such that the following diagram commutes



Moreover, if $f \sqsubseteq g$ and $g \sqsubseteq f$, then f and g are said to be *isomorphic* and we write $f \simeq g$.

Although the "idea" behind definition 4.1 is quite a good one, there is, however, a "technical" issue. The problem is that definition 4.1 is somewhat weaker than the notion of *subset*. Given two *sets* A and B, if $A \subseteq B$ then, not only there is an *injective function* from A to B, but one of these *functions* is actually the *inclusion function*, i.e., the restriction of id_B to the subset A. Definition 4.1 does not capture this property, which is essential for the *set inclusion* to be antisymmetric.

Example 4.1. In **Set**, the only *subsets* of a set X with exactly one element are the *empty set* and X itself. However, given any other *one element set* A, there is always a *monic arrow* $A \rightarrowtail X$.

This is a problem, because we want to be able to define the *categorical* analogous of *power set* of an *object* d - we'll call it $Sub(d)$ - in such a way that, together with the notion of *subobject inclusion*, $(Sub(d), \sqsubseteq)$ becomes a *partial ordering*, just like $(\mathcal{P}(D), \subseteq)$ in set theory.

To this end we first notice that $\simeq$ is an *equivalence relation*, and so we can *identify* all the *equivalent subobjects* and define $Sub(d)$ in the following way

$$Sub(d) = \{[f] : f \text{ is monic and } Cod(f) = d\} \tag{4.1}$$

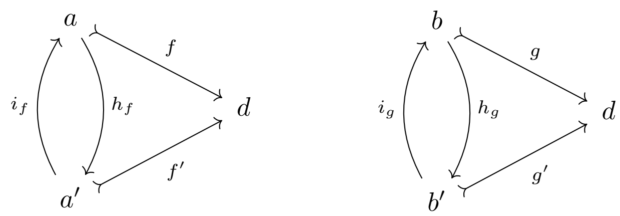
where

$$[f] = \{g \in Mor(\mathcal{C}) : g \simeq f\}$$

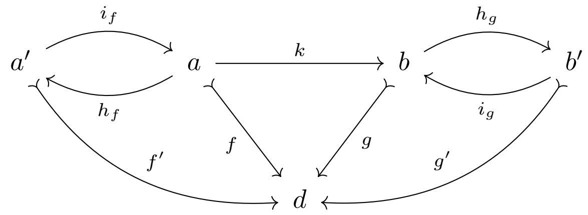
Of course this construction is only useful if the $\sqsubseteq$ relation is independent from the choice of the representant.

Theorem 4.1. *Let $\mathcal{C}$ be a category, $d \in Ob(\mathcal{C})$ and f, f', g, g' four arrows with* co-domain d. *If $[f] = [f']$ and $[g] = [g']$ then $f \sqsubseteq g$ iff $f' \sqsubseteq g'$.*

Proof. Since $[f] = [f']$ and $[g] = [g']$ the following two diagrams commute



Now, if $f \sqsubseteq g$, then there is an arrow $k : Dom(f) \longrightarrow Dom(g)$ and we can build the following diagram



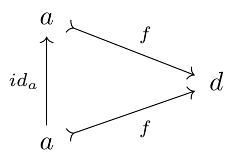
Since each of the *inner triangles* commute, we can conclude that the *outer triangle* commutes as well, and so $f' \sqsubseteq g'$. ☐

We can finally state the result we wanted in the first place.

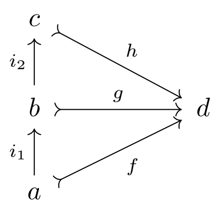
Theorem 4.2. *Let $\mathcal{C}$ be a category and $d \in Ob(\mathcal{C})$. Then $(Sub(d), \subseteq)$ is a* partially ordered set.

Proof. As we have already seen, if $(f \sqsubseteq g) \wedge (g \sqsubseteq f)$ then $[f] = [g]$. So it suffices to show that $\sqsubseteq$ is *reflexive* and *transitive* in $Sub(d)$.

Let $[f] \in Sub(d)$, then $[f] \sqsubseteq [f]$ since the following diagram always commutes



On the other hand, if $[f], [g], [h] \in Sub(d)$ with $[f] \sqsubseteq [g]$ and $[g] \sqsubseteq [h]$, then the following diagram exists



and since each of the *inner triangles* commute, the *outer triangle* commutes as well and $[f] \sqsubseteq [h]$. ☐

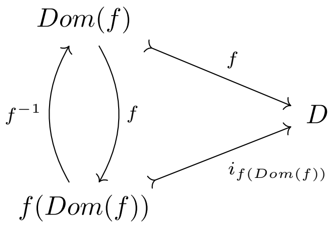
The following example testifies the "similarities" between $Sub(d)$ and the *power set*.

Theorem 4.3. *In the category **Set**, $Sub(D) \simeq \mathcal{P}(D)$ for every object D.*

Proof. We start by proving that in **Set**, if $g \sqsubseteq f$ then $g(Dom(g)) \subseteq f(Dom(f))$. For lets suppose there is $y \in g(Dom(g))$ such that $y \notin f(Dom(f))$. Then there is $x \in Dom(g)$ such that $g(x) = y$ and $f(z) \neq y$ for every $z \in Dom(f)$. In this case $g \not\sqsubseteq f$ since there is no *function* $h : Dom(g) \longrightarrow Dom(f)$ for which $f(h(x)) = g(x)$.
This means that, if $g \in [f]$ then $f(Dom(f)) = g(Dom(g))$ and so we can define a *function* $k : Sub(D) \longrightarrow \mathcal{P}(D)$ as $k([f]) = f(Dom(f))$. In the other way we define $h : \mathcal{P}(D) \longrightarrow Sub(D)$ as $h(A) = i_A$, where i_A is the *inclusion* of A in D. Let's verify that these two *function* are the inverse of each other.

$$\begin{aligned} k(h(A)) &= k(i_A) \\ &= i_A(Dom(i_A)) \\ &= i_A(A) \\ &= A \end{aligned} \qquad \begin{aligned} h(k([f])) &= h(f(Dom(f))) \\ &= [i_{f(Dom(f))}] \\ &= [f] \end{aligned}$$

to understand why $[i_{f(Dom(f))}] = [f]$, remember that, because f is injective, it is actually a bijection between $Dom(f)$ and $f(Dom(f))$ and so the following diagram exists and commutes



□

4.2 Characteristic Arrows

In set theory, given a *set* D, we can associate to every *subset* $A \subseteq D$ one, and only one, *function* $\chi : D \longrightarrow \{0,1\}$, called the *characteristic function* of A. This "relation" can be captured *categorically* in the following way

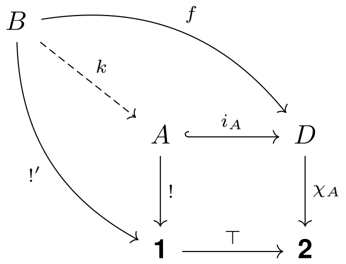
Theorem 4.4. *In* ***Set****, the following* diagram *is always a* pullback square*,*

$$\begin{array}{ccc} A & \overset{i_A}{\hookrightarrow} & D \\ \downarrow{\scriptstyle !} & & \downarrow{\scriptstyle \chi_A} \\ \mathbf{1} & \xrightarrow{\top} & \mathbf{2} \end{array} \tag{4.2}$$

*where **1** and **2** are the* sets $\{0\}$ *and* $\{0,1\}$*, respectively, and* $!$ *and* $\top$ *are defined by the rules*

$$!(x) = 0 \qquad \top(0) = 1$$

Proof. It's obvious that diagram 4.2 commutes.
Suppose that the pair of arrows $\mathbf{1} \overset{!'}{\leftarrow} B \overset{f}{\rightarrow} D$ is such that the outer square of the diagram



(4.3)

commutes. Then

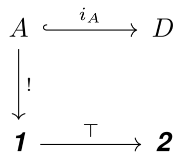
$$\chi_A(f(x)) = \top(!'(x)) = 1$$

for every $x \in B$, and so $f(x) \in A$ and we can define $k(x) = f(x)$. This makes the whole diagram commute. Moreover, suppose $h : B \longrightarrow A$ also make the whole diagram commute. Then, because the top triangle commutes, we have

$$f(x) = i_A(h(x)) = h(x)$$

and so k is the only arrow that makes the whole diagram commute. □

Theorem 4.5. *In **Set**, consider the following diagram*

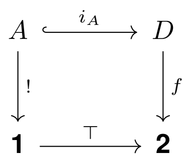


*Then $\chi_A : D \longrightarrow$ **2** is the only arrow that turn this* diagram *into a* pullback square.

Proof. Suppose $f : D \longrightarrow$ **2** makes the *diagram* a *pullback square*. Consider the set

$$A_f = \{x \in D : f(x) = 1\}$$

In other words, A_f is the set for which f is the *characteristic function*. We will show that $A_f = A$, at which point we can conclude that $f = \chi_A$.
So suppose $x \in A$. By hypothesis

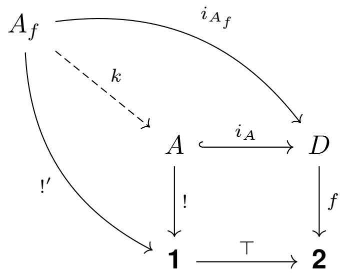


is a *pullback square*, so it commutes. Hence

$$f(x) = f(i_A(x)) = \top(!(x)) = 1$$

which means $x \in A_f$ and so $A \subseteq A_f$.

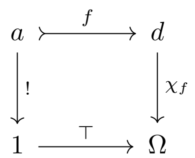
On the other hand, since it is a *pullback*, we can construct the following *diagram*



Because both i_A and i_{A_f} are inclusions, and the top triangle commutes, we conclude that k is an inclusion as well, and so $A_f \subseteq A$. □

These facts show that the two sets, **1** and **2**, together with the *function* $\top$ play a decisive roll in the characterization of *subsets*. With this in mind, we can try and "lift" this notion into categorical language

Definition 4.3. Let $\mathcal{C}$ be a category with a *terminal object* 1. A *subobject classifier* for $\mathcal{C}$ is an *object* Ω together with an *arrow* $\top : 1 \longrightarrow \Omega$ such that for every *monic arrow* $f : a \rightarrowtail d$, there is one and only one *arrow* $\chi_f : d \longrightarrow \Omega$ such that the following diagram



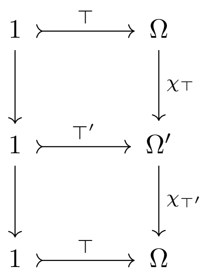
(4.4)

is a *pullback square*. The *arrow* χ_f is called the *characteristic arrow* of the *subobject* f and the elements of Ω are called the *truth values*.

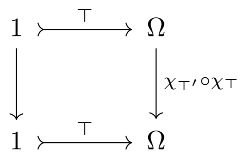
We will now study some properties of this construction.

Theorem 4.6. *Let $\mathcal{C}$ be a category with a* terminal object 1*. If $\langle \Omega, \top \rangle$ and $\langle \Omega', \top' \rangle$ are both* subobject classifiers *for $\mathcal{C}$, then $\Omega \simeq \Omega'$.*

Proof. First of all, both $\top : 1 \longrightarrow \Omega$ and $\top' : 1 \longrightarrow \Omega'$ are *monic* since their *domain* is a *terminal object*, and so we can construct the following *diagram* in $\mathcal{C}$



The top square is the *pullback* of $\chi_\top$ along $\top'$ and the bottom square is the *pullback* of $\chi_{\top'}$ along $\top$. Hence, by theorem 3.3 the *outer square*



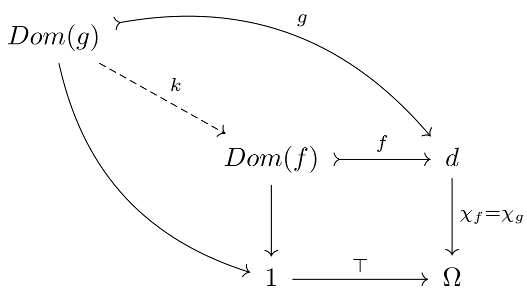
is a *pullback* as well, and so it commutes. On the other hand, due to the definition of *subobject classifier*, we know that $\chi_{\top'} \circ \chi_{\top}$ is the one and only *arrow* that makes this square a *pullback*. Since 1_Ω makes it a *pullback*, we conclude that $\chi_{\top'} \circ \chi_{\top} = 1_\Omega$.
Interchanging the roles of $\langle \Omega, \top \rangle$ and $\langle \Omega', \top' \rangle$ gives $\chi_{\top} \circ \chi_{\top'} = 1_{\Omega'}$, and so $\Omega \simeq \Omega'$. □

Now that we have categorical definitions for both *subobjects* and *characteristic functions*, the question arises: do *characteristic functions* completely characterise *subobjects*? The following theorem answers this question.

Theorem 4.7. *Let $\mathcal{C}$ be a category with a* subobject classifier $\langle \Omega, \top \rangle$*. Also, let $d \in Ob(\mathcal{C})$ and $f, g \in Sub(d)$. Then the following equivalence is true*

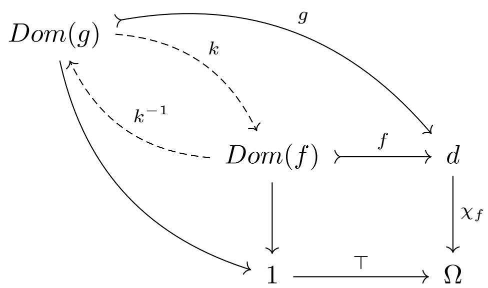
$$f \simeq g \qquad \textit{iff} \qquad \chi_f = \chi_g$$

Proof. ($\leftarrow$) Suppose $\chi_f = \chi_g$. Then in the following *diagram*



both the inner and the outer squares are *pullbacks*, and so they commute. In this case, we know there is one *arrow* $k : Dom(g) \longrightarrow Dom(f)$ that makes the whole diagram commute, and so $g \sqsubseteq f$. The symmetric argument gives $f \sqsubseteq g$ and so $f \simeq g$.

($\rightarrow$) Suppose now that $f \simeq g$. In this case we know that this k *arrow* is actually *iso*



and so, the outer square is a *pullback square*. This means that $\chi_f = \chi_g$ since χ_g is, by definition, the only *arrow* that turns the outer square into a *pullback*. □

Example 4.2. The existence of a *subobject classifier* $(\Omega, \top)$ and a *terminal object* 1 gives us the possibility of defining a "special" arrow, which we will call $\perp$, as the *characteristic arrow* of the *arrow* $! : 0 \longrightarrow 1$, i.e.,

$$\begin{array}{ccc} 0 & \xrightarrow{\;!\;} & 1 \\ {\scriptstyle !}\downarrow & & \downarrow{\scriptstyle \perp} \\ 1 & \xrightarrow{\;\top\;} & \Omega \end{array}$$

is a *pullback*. In **Set** this means that $\perp$ is the *characteristic function* of $\varnothing \subseteq 1$, i.e., it's the *function* with values $\perp(0) = 0$, as desired.
This *arrow* is often referred to as *false* or *bottom* for reasons that we will explore in the next sections.

4.3 Power Objects

We now turn to the matter of capturing the notion of *powerset*. Our objective is to show latter that, in any *category* with *exponentials*, *subobject classifiers* and *power-objects*, $\Omega^a \simeq \mathcal{P}(a)$. The next results aim at "capturing" the "*categorical* properties" of the *powerset* construction in **Set**.

Theorem 4.8. *Every* relation $R \subseteq B \times A$ *determines one and only one* function $f_R : B \longrightarrow \mathcal{P}(A)$ *defined by the rule*

$$f_R(x) = \{a \in A : \langle x, a\rangle \in R\}$$

On the other hand, every injection $r : R \rightarrowtail B \times A$ *determines a* subset *of* $B \times A$*, and therefore a* function $f_r : B \longrightarrow \mathcal{P}(A)$ *defined by*

$$f_r(x) = \{a \in A : \text{ there is } y \in R \text{ such that } r(y) = (x, a)\}$$

Theorem 4.9. *Consider the following diagram in **Set***

$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & B \times A \\ & & \downarrow{\scriptstyle f_r \times id_A} \\ \in_A & \overset{i}{\hookrightarrow} & \mathcal{P}(A) \times A \end{array}$$

where $\in_A = \{\langle U, a\rangle : a \in U\}$ *and* i *is the usual* inclusion. *Then there exists an* arrow $g : R \longrightarrow \in_A$ *such that the diagram*

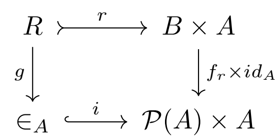
$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & B \times A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_r \times id_A} \\ \in_A & \overset{i}{\hookrightarrow} & \mathcal{P}(A) \times A \end{array}$$

is a pullback.

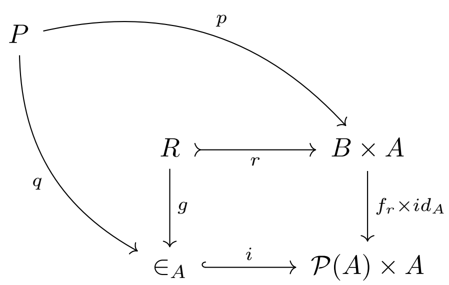
Proof. Let $x \in R$ and $r(x) = (b, a)$. By the definition of f_r we know that $a \in f_r(b)$ and so $(f_r(b), a) \in \in_A$. We can, therefore, define a *function* $g : R \longrightarrow \in_A$ as

$$g(x) = (f_r \times id_A)(r(x))$$

We will now show that



is a *pullback*. The fact that the diagram commutes is trivial, so suppose there is $\in_A \leftarrow P \rightarrow B \times A$ such that the *outer square*



commutes. Then for every $x \in P$, let $p(x) = (b, a)$, we have

$$\begin{aligned}(f_r(b), a) &= (f_r \times id_A)(b, a)\\ &= (f_r \times id_A)(p(x))\\ &= i(q(x))\\ &= q(x) \in \in_A\end{aligned}$$

and so, by the definition of f_r, there is $y \in R$ such that $r(y) = (b, a) = p(x)$. More over, because r is *injective*, we know that this y is unique, and so we can define $\alpha : P \longrightarrow R$ as $\alpha(x) = r^{-1}(p(x))$, making the whole diagram commute.

The uniqueness of this function comes from the fact that r is *injective*, and so, *left-cancellable*. In fact, suppose that β also makes the diagram commute. Then, because the top triangle commutes,

$$p = r \circ \alpha = r \circ \beta \Rightarrow \alpha = \beta$$

□

Theorem 4.10. *Consider the following diagram in **Set**.*

$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & B \times A \\ & & \big\downarrow{\scriptstyle f \times id_A} \\ \in_A & \overset{i}{\hookrightarrow} & \mathcal{P}(A) \times A \end{array}$$

If $f \neq f_r$ then there is no arrow $g : R \longrightarrow \in_A$ *such that*

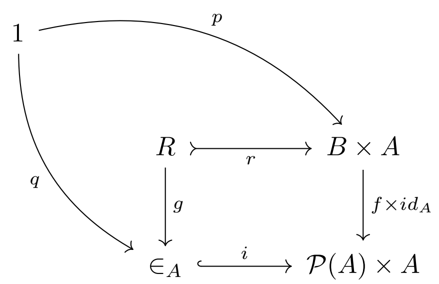
$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & B \times A \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f \times id_A} \\ \in_A & \overset{i}{\hookrightarrow} & \mathcal{P}(A) \times A \end{array}$$

is a pullback.

Proof. If $f \neq f_r$ then there is $b \in B$ such that $f(b) \neq f_r(b)$. Then there are two possibilities

1. either there is $a \in f(b)$ such that $a \notin f_r(b)$
2. or there is $a \in f_r(b)$ such that $a \notin f(b)$

In the first case, since $a \notin f_r(b)$, we know that there is no $x \in R$ such that $r(x) = (b, a)$. Then consider the diagram



where

$$p(0) = (b, a)$$
$$q(0) = (f(b), a)$$

note that q is well defined since, by hypothesis, $a \in f(b)$ and so $(f(b), a) \in \in_A$. Then the *outer square* commutes trivially but, because $(b, a) \notin r(R)$, there is no *function* $\alpha : 1 \longrightarrow R$ making the *top triangle* commute. Hence, the *inner square* cannot be a *pullback*.

Consider now the second possibility. Since $a \in f_r(b)$, there is $x \in R$ such that $r(x) = (b, a)$, and so $f \times id_A(r(x)) = (f(b), a)$. On the other hand, because $a \notin f(b)$, we know that $(f(b), a) \notin \in_A$ and so there is no $g : R \longrightarrow \in_a$ such that

$$i(g(x)) = g(x) = (f(b), a) = f \times id_A(r(x))$$

Hence

$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & B \times A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f \times id_A} \\ \in_A & \overset{i}{\hookrightarrow} & \mathcal{P}(A) \times A \end{array}$$

Does not commute and cannot be a *pullback*. □

These two results show us that there is an *universal property* associated with $\mathcal{P}(A)$: given a diagram (in **Set**) of the form

$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & B \times A \\ & & \downarrow{\scriptstyle \_\_ \times id_A} \\ \in_A & \overset{i}{\hookrightarrow} & \mathcal{P}(A) \times A \end{array}$$

there is one, and only one, *arrow* $f_r : B \longrightarrow \mathcal{P}(A)$ for which there is an *arrow* $g : R \longrightarrow \in_A$ such that

$$\begin{array}{ccc} R & \overset{r}{\rightarrowtail} & B \times A \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f_r \times id_A} \\ \in_A & \overset{i}{\hookrightarrow} & \mathcal{P}(A) \times A \end{array}$$

is a *pullback*.
This leads us to the following definition of $powerobject$.

Definition 4.4. Let $\mathcal{C}$ be a category with *products*. $\mathcal{C}$ is said to have *power objects* if, for every $a \in Ob(\mathcal{C})$ there are *objects* $\mathcal{P}(a), \in_a \in Ob(\mathcal{C})$ and a *monic* $\in : \in_a \rightarrowtail \mathcal{P}(a) \times a$ such that: for every $b \in Ob(\mathcal{C})$ and *subobject* $r : Dom(r) \rightarrowtail b \times a$ of $b \times a$ there is one, and only one, *arrow* $f_r : b \longrightarrow \mathcal{P}(a)$ for which there is a *pullback* of the form

$$\begin{array}{ccc} Dom(r) & \overset{r}{\rightarrowtail} & b \times a \\ \downarrow & & \downarrow{\scriptstyle f_r \times id_a} \\ \in_a & \overset{\in}{\rightarrowtail} & \mathcal{P}(a) \times a \end{array}$$

As usual, given an *object* a, its *power object* $\mathcal{P}(a)$ is unique up to *isomorphism*.

Theorem 4.11. *Let $\mathcal{C}$ be a* category *with* power objects*. If $\mathcal{P}(a)$ and $\mathcal{P}'(a)$ are two* power objects *of the same $a \in Ob(\mathcal{C})$ then $\mathcal{P}(a) \simeq \mathcal{P}'(a)$.*

Proof. Since both $\mathcal{P}(a)$ and $\mathcal{P}'(a)$ are *power objects* of a, there is only one $\alpha : \mathcal{P}'(a) \longrightarrow \mathcal{P}(a)$ and one $\beta : \mathcal{P}(a) \longrightarrow \mathcal{P}'(a)$ such that

$$
\begin{array}{ccc}
\in'_a & \overset{\in'}{\rightarrowtail} & \mathcal{P}'(a) \times a \\
\downarrow & & \downarrow{\scriptstyle \alpha \times id_a} \\
\in_a & \overset{\in}{\rightarrowtail} & \mathcal{P}(a) \times a \\
\downarrow & & \downarrow{\scriptstyle \beta \times id_a} \\
\in'_a & \overset{\in'}{\rightarrowtail} & \mathcal{P}'(a) \times a
\end{array}
$$

both *inner squares* are *pullbacks*, and hence the *outer square* as well (by theorem 3.3). On the other hand, by the definition of *power object* there is only one *arrow* $f : \mathcal{P}'(a) \longrightarrow \mathcal{P}'(a)$ for which there is a *pullback* of the form of the *outer square*. But $id_{\mathcal{P}'(a)}$ does the trick, and so we conclude that $\beta \circ \alpha = id_{\mathcal{P}'(a)}$.
Interchanging the roles of $\mathcal{P}(a)$ and $\mathcal{P}'(a)$ wields $\alpha \circ \beta = id_{\mathcal{P}(a)}$, and so $\mathcal{P}(a) \simeq \mathcal{P}'(a)$. □

4.4 Definition of Topos

We finally have the required tools for defining a *Topos*.

Definition 4.5. A category $\mathcal{C}$ is called a *Topos* if

1. $\mathcal{C}$ has a *terminal object* 1.

2. $\mathcal{C}$ has *pullbacks*.

3. $\mathcal{C}$ has *exponentiation*.

4. $\mathcal{C}$ has a *subobject classifier* $(\Omega, \top)$.

As we saw already, the *category* **Set** fulfills all the requirements and so it constitutes the first example of a *topos*.

Another important property of *topoi* is that every *topos* has *power objects*.

Theorem 4.12. *Let* $\mathcal{C}$ *be a* topos*. Then* $\mathcal{C}$ *has* power objects*. Moreover, given* $a \in Ob(\mathcal{C})$*, one of its* power objects *is* Ω^a*.*

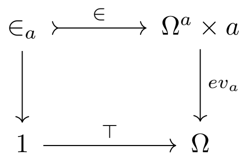
Proof. Given $a \in Ob(\mathcal{C})$, consider as a possible *power object*

- $\mathcal{P}(a) = \Omega^a$

- $\in$ is the *subobject* of $\Omega^a \times a$ whose *character* is ev_a

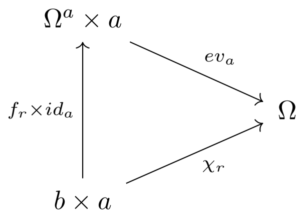
- $\in_a = Dom(\in)$

then the following diagram is a *pullback*.



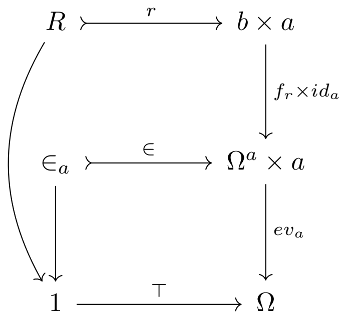
(4.5)

We will show that this construction is a *power object* of a. To this end, consider a *subobject* $r : R \rightarrowtail b \times a$ and let $\chi_r : b \times a \longrightarrow \Omega$ be its *character*. Moreover, let f_r be the unique *arrow* making the *exponentiation diagram*



(4.6)

commute. Now consider the diagram



(4.7)

by diagram 4.6 we know that $ev_a \circ (f_r \times id_a) = \chi_r$ and so the *outer square* commutes (in fact, it is a *pullback*). Hence, since the *bottom square* is a *pullback*, we conclude that there is one, and only one, *arrow* $R \rightarrow \in_a$ that makes the whole diagram commute. Moreover, since the *outer square* is also a *pullback*, we conclude by theorem 3.3 that the *top square* is also a *pullback*.

Now suppose $\alpha : b \longrightarrow \Omega^a$ is such that, substituting f_r by α in diagram 4.7, wields the *top square* as a *pullback*. Then again, by theorem 3.3, we have that the *outer square* is a *pullback*, and so, by the definition of *subobject classifier*, $ev_a \circ (\alpha \times id_a) = \chi_r$. Hence $\alpha = f_r$, since f_r is, by definition, the only *arrow* such that $ev_a \circ (f_r \times id_a) = \chi_r$. □

4.5 Bundles

We will now take some time to define a construction that will constitute another example of a *topos*.

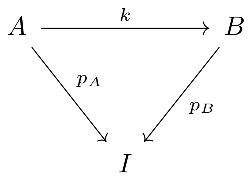
Definition 4.6 (Bundle over a set)**.** Let I be a set. A *bundle over* I is a pair $\langle A, p \rangle$, where A is a *set* and $p : A \longrightarrow I$. The *set* I is called the *base space* and A is called the *stalk space*. We also define the *sets*

$$A_i = p^{-1}(i), \forall_{i \in I}$$

which will be referred to as the *stalks over i*, respectively. Each element of the set A_i is called a *germ at i*.

Now, given a *set* I, taking the *bundles over* I as *objects*, and choosing the *arrows* carefully, we can construct a *category*.

Definition 4.7 (Category of bundles over I)**.** Given a *set* I, we define the category **Bn(**I**)**, where the *objects* are the *bundles* over I, and an *arrow* $k : (A, p_A) \longrightarrow (B, p_B)$ is a *function* $k : A \longrightarrow B$ such that the following diagram commutes



Notice that, if $x \in A_i$, then $p_A(x) = p_B(k(x)) = i$, and so $k(x) \in B_i$. This show us that the *arrows* of **Bn(**I**)** are essentially the *functions* that map the *germ* of A_i into *germs* of B_i.

We will now show that **Bn(**I**)** is a *topos*.

Theorem 4.13. *Let I be a* set*. A* terminal object *of* ***Bn(**I**)** *is* (I, id_I)*.*

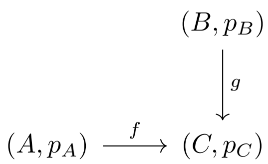
Theorem 4.14. *Let I be a* Set*.* ***Bn(**I**)** *has* products *for every pair of* objects *and one of them can be constructed as follows: let* $(A, p_A), (B, p_B) \in Ob($***Bn(**$I$**)**$)$ *we define the* object (D, p_D) *as*

$$D = \bigcup_{i \in I} A_i \times B_i$$

$$p_D = p_A \circ \pi_1 = p_B \circ \pi_2$$

Then the bundle (D, p_D) *together with the usual* projections*,* π_1 *and* π_2*, constitutes a* product *of* (A, p_A) *and* (B, p_B)*.*

Theorem 4.15. *Let I be a* set*. A* pullback *for*



in ***Bn(**I**)**, *is*

$$\begin{array}{ccc} (P, p_p) & \xrightarrow{q} & (B, p_B) \\ \Big\downarrow p & & \Big\downarrow g \\ (A, p_A) & \xrightarrow{f} & (C, p_C) \end{array}$$

where

$$\begin{array}{ccc} P & \xrightarrow{q} & B \\ \Big\downarrow p & & \Big\downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

is a pullback square *in **Set** and*

$$p_P = p_A \circ p$$

Theorem 4.16. *Let I be a* set*. A* subobject classifier *in **Bn(I)** is the* bundle $(2 \times I, \pi_2)$ *together with the* arrow $\top_{Bn(I)} : I \longrightarrow 2 \times I$ *defined as*

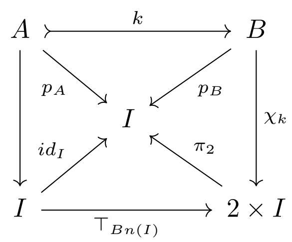
$$\top_{Bn(I)}(x) = \langle 1, x \rangle$$

where π_2 is the usual projection *onto the second component.*

Proof. Let $k : (A, p_A) \longrightarrow (B, p_B)$ be a *monic arrow*, we will prove the theorem in two steps. First, we will show how to construct an *arrow* $\chi_k : (B, p_B) \longrightarrow (2 \times I, \pi_2)$ such that the diagram

$$\begin{array}{ccc} (A, p_A) & \overset{k}{\rightarrowtail} & (B, p_B) \\ \Big\downarrow & & \Big\downarrow \chi_k \\ (I, id_I) & \xrightarrow{\top_{Bn(I)}} & (2 \times I, \pi_2) \end{array} \tag{4.8}$$

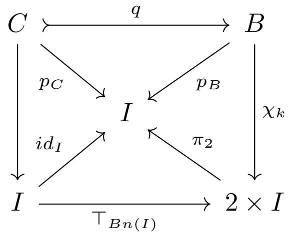
is a *pullback square*. We will then show that this is the only arrow that that can do this.
So, let's start by pointing out that the diagram 4.8 in **Bn(I)** can be "translated" into the following *whole commuting diagram* in **Set**



(4.9)

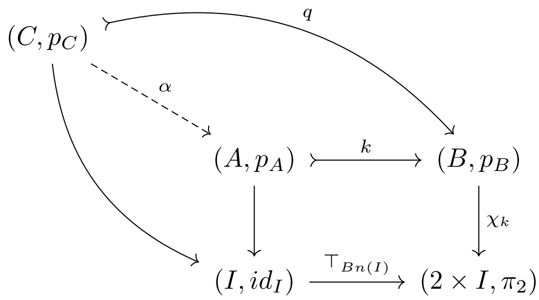
We will define χ_k as $\chi_k = \langle \chi_k^{\textbf{Set}}, p_B \rangle$, where $\chi_k^{\textbf{Set}}$ is the usual *characteristic function*. Note that this is an *arrow* in **Bn(I)** since $\pi_2 \circ \langle \chi_k^{\textbf{Set}}, p_B \rangle = p_B$.

Now suppose there is a *bundle* (C, p_C) and a *monic* $q : (C, p_C) \longrightarrow (B, p_B)$ such that



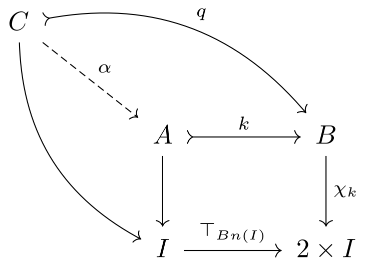
(4.10)

commutes (in **Set**). We need to show that there is one, and only one, arrow $\alpha : (C, p_c) \longrightarrow (A, p_A)$ that makes the whole diagram



(4.11)

commute in **Bn(**I**)**. Recall that, because the *outer square* of 4.9 is a *pullback* in **Set**, there is one, and only one, *arrow* $\alpha : C \longrightarrow A$ in **Set** that makes the whole

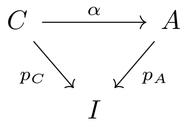


(4.12)

commute. Since both 4.10 and 4.12 commute, we have that

$$
\begin{aligned}
p_C &= p_B \circ q && \text{by (4.10)}\\
&= p_B \circ k \circ \alpha && \text{by (4.12)}\\
&= p_A \circ \alpha && \text{by (4.9)}
\end{aligned}
$$

which mean that



(4.13)

and so $\alpha : (C, p_C) \longrightarrow (A, p_A)$ is an arrow in **Bn(**I**)**. To see that this is the only one, recall that, by definition, every arrow in **Bn(**I**)** is an arrow in **Set** and so, if there was another arrow, $\alpha : C \longrightarrow A$ would not be the only arrow in **Set** making 4.8 commute. We conclude 4.8 is a *pullback*.

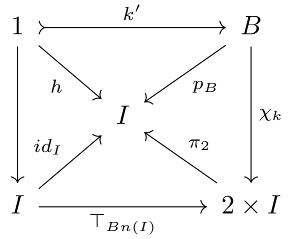
We know turn to proving that χ_k is the only one making 4.8 a *pullback*. Suppose $\beta : B \longrightarrow 2 \times I$ is such that the whole 4.9 commute. β will be of the form $\beta = \langle \beta_1, \beta_2 \rangle$ and, because the right triangle commutes, we have

$$\pi_2 \circ \beta = p_B \tag{4.14}$$

and so

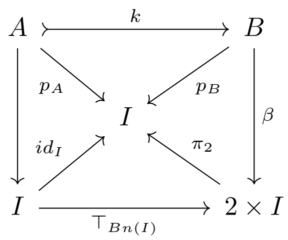
$$\beta_2 = p_B \tag{4.15}$$

The only thing left to show at this point is that $\beta_1 = \chi_k^{\textbf{Set}}$ Because the outer square commutes, we know that, for every $a \in A$, $\beta_1 \circ k = 1$, and so, for every $b \in k(A)$, $\beta_1(b) = 1$. Now suppose there is $b \notin k(A)$ such that $\beta_1(b) = 1$, and consider $k' : 1 \longrightarrow B$ defined by the rule $\beta_1(x) = b$. Then the diagram



(4.16)

commutes. But since $b \notin k(A)$ there cannot be any *function* $\alpha : 1 \longrightarrow A$ such that $k \circ \alpha = k'$, which means

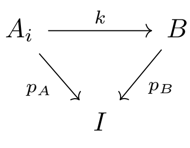


(4.17)

is not a *pullback*.

We conclude that, in order for 4.17 to be a *pullback*, β must be χ_k. □

Theorem 4.17. *Let I be a set. Then **Bn(**I**)** has* exponentials *for every pair of objects $(A, p_A), (B, p_B) \in Ob($**Bn(**$I$**)**$)$, and one can be constructed as follows. Let D_i be the* set *of* functions *from A_i to B such that*



commutes, i.e., $k(a) \in B_i$. Note that these sets *are disjoint since every* arrow *in D_i has a different* domain *from any arrow in D_j (if $i \neq j$ of course!). Then we can define the* bundle

$$D = \bigcup_{i \in I} D_i$$

$$p_D(d) = i, \text{ when } d \in D_i$$

This bundle*, together with the empharrow* $ev : D \longrightarrow B$ *defined by the rule*

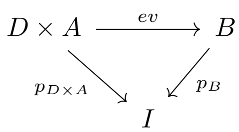
$$ev(f, a) = f(a)$$

is an exponential $(B, p_B)^{(A, p_A)}$*.*

Proof. First, let us show that ev is in fact an *arrow* in **Bn(I).** So supose $(f, a) \in D \times A$. Then $f \in D_i$ and $a \in A_i$ for some $i \in I$. And since f preserves indexes,

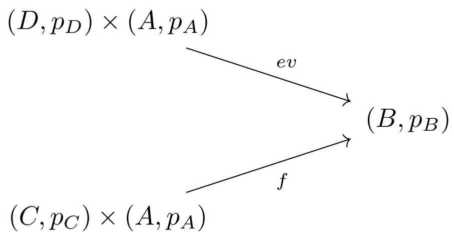
$$p_B(ev(f, a)) = p_B(f(a)) = I = p_A(\pi_2(f, a)) = p_{D \times A}$$

and so the diagram

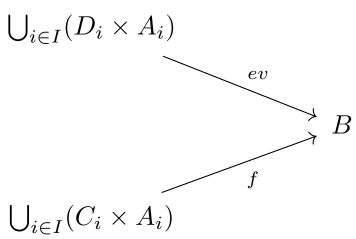


commutes and ev is an *arrow* in **Bn(I).**

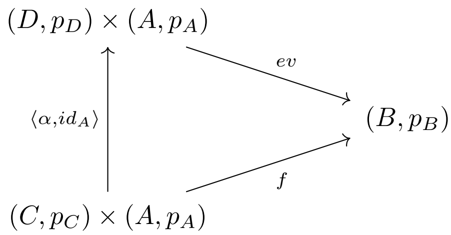
Now consider the following *diagram*



this means we can construct the following diagram in **Set**



Now, if $c \in C$ then $c \in C_i$ for some $i \in I$. Then the *function* $f_c(x) = f(c, x)$ is a *function* from A_i to B. Moreover, because f is an *arrow* in **Set**, if $c \in C_i$ then $f(c, a) \in B_i$, and so $f_c \in D_i$. We then define $\alpha : C \longrightarrow D$ as $\alpha(c) = f_c$ and, since α preserves *fibers*, it is an *arrow* in **Bn(I).** This means that the following *diagram* is a *commutative diagram* in **Bn(I)**



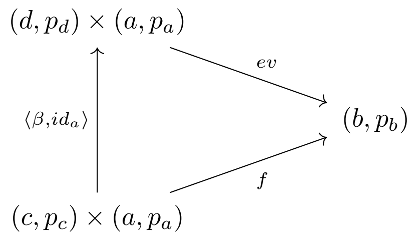
To see that α is the only such arrow, suppose $\beta : (C, p_C) \longrightarrow (D, p_D)$ such that $\beta \neq \alpha$. Then there is $c \in C_i$ such that

$$f'_c = \beta(c) \neq \alpha(c) = f_c$$

in other words, there is $a \in a_i$ such that

$$(ev \circ \langle \beta, id_A \rangle)(c, a) = ev(f'_c, a) = f'_c(a) \neq f_c(a) = ev(f_c, a) = f(c, a)$$

and so the diagram



does not commute. $\square$

Chapter 5

Topos as Semantics

In this chapter we define a way of using *topoi* as semantic structures, so that we can use them to calculate the truth value of *logic statements*.

5.1 Truth Arrows

The idea behind using *topoi* as *semantic structures* is to attribute to each *variable* a *truth value*, i.e., an *element* of Ω, and to each *logical connective* an *arrow*. We will refer to these last entities as *truth arrows*. As always, we will start by observing the "arrows only" constructions in **Set** and then generalise them to any *topos*.

Definition 5.1 (Negation)**.** In **Set**, $\neg : 2 \longrightarrow 2$ is the *function* with values $\neg(0) = 1$ and $\neg(1) = 0$. In other words, $\neg$ is the *characteristic function* of $\{0\} \subseteq 2$. On the other hand, as an *arrow*, this *subset* is the *function* $\perp$ defined in 4.2. Therefore, in *categorical language*, $\neg : \Omega \longrightarrow \Omega$ is the *only arrow* such that

$$\begin{array}{ccc} 1 & \xrightarrow{\perp} & \Omega \\ \big\downarrow{\scriptstyle !} & & \big\downarrow{\scriptstyle \neg} \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

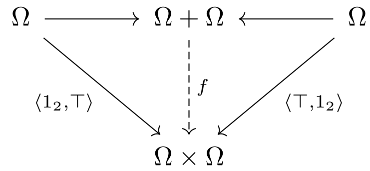
is a *pullback*.

Definition 5.2 (Conjunction)**.** In **Set**, $\wedge : \Omega \times \Omega \longrightarrow \Omega$ is the *function* that gives *output* 1 only to the *input* $\langle 1, 1\rangle$, and so it is the *characteristic function* of the *set* $\{\langle 1, 1\rangle \subseteq 2 \times 2\}$, which, as an *arrow*, is the *monic* $\langle \top, \top\rangle : 1 \times 1 \longrightarrow 2 \times 2$. Therefore, *categorically*, it is the *only arrow* such that

$$\begin{array}{ccc} 1 \times 1 & \xrightarrow{\langle \top, \top\rangle} & \Omega \times \Omega \\ \big\downarrow{\scriptstyle !} & & \big\downarrow{\scriptstyle \wedge} \\ 1 & \xrightarrow{\quad\top\quad} & \Omega \end{array}$$

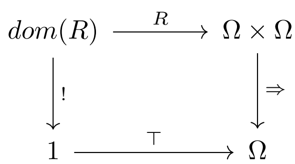
is a *pullback*.

Definition 5.3 (Disjunction)**.** In **Set**, $\vee : \Omega \times \Omega \longrightarrow \Omega$ is the *characteristic function* of the *set* $D = \{(0,1),(1,0),(1,1)\} = \{(1,1),(0,1)\} \cup \{(1,1),(1,0)\}$. Now, as *arrows*, the *sets* $\{(1,1),(0,1)\}, \{(1,1),(1,0)\} \subseteq 2 \times 2$ are, respectively, the *arrows* $\langle 1_2, \top \rangle, \langle \top, 1_2 \rangle : 2 \longrightarrow 2 \times 2$, and so, D is the *characteristic arrow* of $Im(f)$, where f is the *only arrow* such that



commutes. In other words, $\vee = \chi_{Im(f)}$, where $Im(f)$ is obtain through the *epi-monic factorisation* of f.

Definition 5.4 (Implication)**.** In **Set**, $\Rightarrow : \Omega \times \Omega \longrightarrow \Omega$ is the *characteristic function* of $R = \{(0,0),(0,1),(1,1)\} \subseteq 2 \times 2$. On the other hand, R is precisely the *partial order relation* in the *lattice* 2. Hence, we can rewrite it as $R = \{(x,y) \in 2 \times 2 : x \leq y\}$ or, since 2 is a *lattice*, $R = \{(x,y) \in 2 \times 2 : x \wedge y = x\}$, and so, R is the *equalizer* of $\wedge$ and π_1. With R defined this way, $\Rightarrow$ is the *only arrow* such that



is a *pullback*.

Now that we have defined the *truth arrows* we can define a way to calculate the *logical value* of any *logical statement*

Definition 5.5 (Topos validity)**.** Given any *topos* $\mathcal{C}$, a $\mathcal{C} - valuation$ is a function $V : X \longrightarrow \mathbf{C}(1, \Omega)$ that assigns to each *logical variable* in X a *truth value*. The *truth value* of any *logical statement* can then be calculated *inductively* by the rules

- $V(\neg\varphi) = \neg \circ V(\varphi)$
- $V(\varphi \wedge \phi) = \wedge \circ \langle V(\varphi), V(\phi) \rangle$
- $V(\varphi \vee \phi) = \vee \circ \langle V(\varphi), V(\phi) \rangle$
- $V(\varphi \Rightarrow \phi) = \Rightarrow \circ \langle V(\varphi), V(\phi) \rangle$

As always, we say that a statement ϕ is $\mathcal{C} - valid$, and write $\mathcal{C} \models \phi$, when, for every possible $\mathcal{C} - valuation$ V, $V(\phi) = \top$.

5.2 $Sub(d)$ as an algebraic structure

The *arrows* defined in the last section can be used to define operations in $Sub(d)$. In this section we will define these operations and study the algebraic properties of $Sub(d)$. Our goal will be to show that in every *topos* and for every *object* d, $Sub(d)$ will always be a Heyting algebra (although it might not be a Boolean algebra). Consider first the following result in **Set**

Theorem 5.1. *Let D be a set and $A, B \subseteq D$ then*

1. $\chi_{-A} = \neg \circ \chi_A$
2. $\chi_{A \cap B} = \wedge \circ \langle \chi_A, \chi_B \rangle$
3. $\chi_{A \cup B} = \vee \circ \langle \chi_A, \chi_B \rangle$

We will define the operations in $Sub(d)$ according with this result

Definition 5.6 (Complement, intersection and union)**.** Let $f, g \in Sub(d)$ we define

Complement we define the *complement* of f as the *subobject* $-f$ whose *characteristic arrow* is $\neg \circ \chi_f$

Intersection the *intersection* of f and g is the *subobject* $f \cap g$ whose *characteristic arrow* is $\wedge \circ \langle \chi_f, \chi_g \rangle$

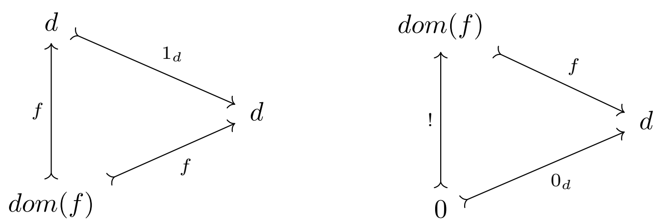
Union the *union* of f and g is the *subobject* $f \cup g$ whose *characteristic arrow* is $\vee \circ \langle \chi_f, \chi_g \rangle$

Our first main result of this section has a somewhat lengthy proof which we will skip for simplicity sake. A sketch of the proof can be checked in [8].

Theorem 5.2. *Let **C** be a* topos *and $d \in Ob($**C**$)$. Then $(Sub(d), \sqsubseteq)$ is a* lattice *where $f \cap g$ and $f \cup g$ are, respectively, the* greatest lower bound *and the* least upper bound*.*

Theorem 5.3. *Let **C** be a* topos *and $d \in Ob$**C***. Then $(Sub(d), \sqsubseteq)$ is a* bounded lattice*.*

Proof. The *unit* and the *bottom* will be, respectively, the *identity morphism* of d and the *only arrow* from the *initial object* to d. From now on they will be called, respectively, 1_d and 0_d. We have to show that, for every *subobject* f, $0_d \sqsubseteq f \sqsubseteq 1_d$. The commutativity of the following diagrams accounts for each of these *inclusions*.



□

In the same way we defined *complements*, *intersections* and *unions*, we can define a "new" operation

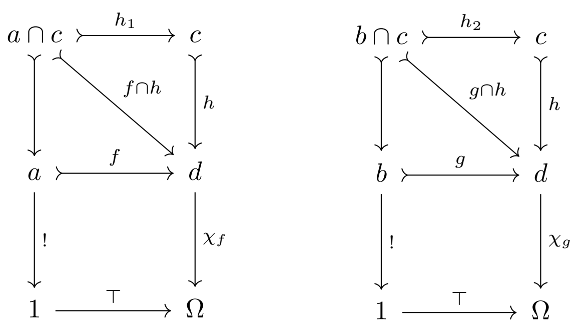
Definition 5.7. Let $f, g \in Sub(d)$, define $f \Rightarrow g$ as the *subobject* whose *characteristic arrow* is $\Rightarrow \circ \langle \chi_f, \chi_g \rangle$.

Theorem 5.4. *Let **C** be a* topos*, $d \in Ob($**C**$)$ and $f, g, h \in Sub(d)$, then*

1. $f \cap h \simeq g \cap h$ *iff* $\chi_f \circ h = \chi_g \circ h$

2. $f \cap h \sqsubseteq g$ *iff* $\chi_{f \cap g} \circ h = \chi_f \circ h$

Proof. 1. Consider the following diagrams



The bottom squares are *pullbacks* by definition, and the top ones are *pullbacks* since $\cap$ is the *lattice meet*. Therefor, by the *pullback lemma*, the *outer square* is a *pullback* and so

$$\chi_f \circ h = \chi_{h_1} \qquad \chi_g \circ h = \chi_{h_2}$$

Therefor we have

$$\begin{aligned}
\chi_{f \circ h} = \chi_{g \circ h} \quad & iff \quad \chi_{h_1} = \chi_{h_2} \\
& iff \quad h_1 \simeq h_2 \\
& iff \quad h_1 \circ k = h_2 && \text{(for some iso } k\text{)} \\
& iff \quad h \circ h_1 \circ k = h \circ h_2 && (h \text{ is monic}) \\
& iff \quad (f \cap h) \circ k = g \cap h && \text{(commutativity of the diagrams)} \\
& iff \quad f \cap h \simeq g \cap h
\end{aligned}$$

2.

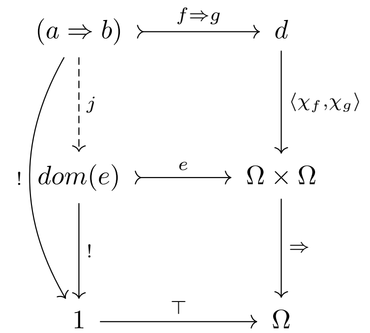
$$\begin{aligned}
f \cap h \sqsubseteq g \quad & iff \quad (f \cap h) \cap g \simeq f \cap h && \text{(inclusion property)} \\
& iff \quad (f \cap g) \cap h \simeq f \cap h && \text{(lattice properties)} \\
& iff \quad \chi_{f \cap g} \circ h = \chi_f \circ h && \text{(by (1))}
\end{aligned}$$

□

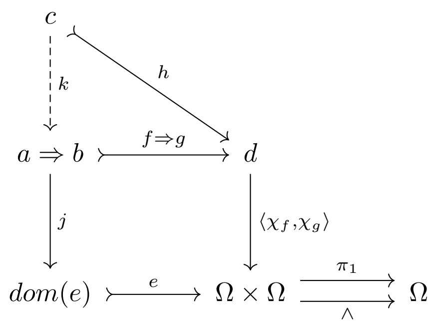
Theorem 5.5. *Let **C** be a* topos*,* $d \in Ob(\boldsymbol{C})$ *and* $f, g, h \in Sub(d)$*, then*

$$h \sqsubseteq (f \Rightarrow g) \quad iff \quad f \cap h \sqsubseteq g$$

Proof. Consider the following diagram



where e is the *equaliser* of π_1 and $\wedge$. The *bottom square* is a *pullback* by the definition of $\Rightarrow$ and the *outer square* commutes by the definition of $f \Rightarrow g$. Therefore, there is j such that the whole diagram commutes. Furthermore, since both the *outer* and *bottom squares* are *pullbacks* , the *pullback lemma* gives the *top square* as a *pullback* as well. Now consider the next diagram



Now, $h \sqsubseteq f \Rightarrow g$ if and only if there is an *arrow* k that makes the top triangle commute. Since the square is a *pullback*, this *arrow* exists if and only if $\langle \chi_f, \chi_g \rangle \circ h$ *factors* through e. On the other hand, since e is an *equaliser*, this happens if and only if $\pi_1 \circ \langle \chi_f, \chi_g \rangle \circ h = \wedge \circ \langle \chi_f, \chi_g \rangle \circ h$, i.e., $\chi_f \circ h = \chi_{f \cap g} \circ h$. But by the last theorem, this happens if and only if $f \cap h \sqsubseteq g$. □

This last result is important because it shows us that, in $Sub(d)$, $\Rightarrow$ is actually a *relative pseudo-complement operation*, and since the *subobject* $f \Rightarrow g$ exists for every $f, g \in Sub(d)$, we get the main result of this section

Theorem 5.6. *Let **C** be a* topos *and* $d \in Ob(\mathbf{C})$*, then* $(Sub(d), \sqsubseteq)$ *is a* Heyting algebra *with* top 1_d*,* bottom 0_d *and where* $f \cap g$*,* $f \cup g$ *and* $f \Rightarrow g$ *are, respectively, the* meet*,* join *and the* relative pseudo-complement *of* f *and* g*.*

The next results are a consequence of the results of this section, and will be useful later on

Theorem 5.7. *Let **C** be a* topos*, then* $\mathbf{C}(d, \Omega)$ *is a Heyting algebra with the* truth arrows *as operations.*

Proof. Let $\Delta : Sub(d) \longrightarrow \mathbf{C}(d, \Omega)$ be the function that assigns to each $f \in Sub(d)$ it's *characteristic arrow* $\chi_f \in \mathbf{C}(d, \Omega)$. Then it's easy to see that, not only is Δ a *bijection*, but it also preserves the algebraic structure, i.e.

$$\Delta(\neg f) = \neg \circ \Delta(f)$$
$$\Delta(f \cap g) = \wedge \circ \langle \Delta(f), \Delta(g) \rangle$$
$$\Delta(f \cup g) = \vee \circ \langle \Delta(f), \Delta(g) \rangle$$
$$\Delta(f \Rightarrow g) = \Rightarrow \circ \langle \Delta(f), \Delta(g) \rangle$$

This is a direct consequence of the fact that we have defined our operations in $Sub(d)$ "on top" of the *truth arrows*, i.e., the operations in **C**(d, Ω). In other words, $Sub(d)$ and **C**(d, Ω) are *isomorphic* as *algebras*, and since $Sub(d)$ is *Heyting algebra*, **C**(d, Ω) must be as well. □

Theorem 5.8. *In any* topos ***C****,* $\top$ *and* $\bot$ *exhibit the following behavior*

x	$\neg x$
$\top$	$\bot$
$\bot$	$\top$

$\wedge$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

$\vee$	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$

$\Rightarrow$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\top$	$\top$

Proof. We'll construct only the table for $\wedge$ since the others are analogous. Remember that

$$\begin{array}{ccc} 1 & \xrightarrow{1_1} & 1 \\ \downarrow & & \downarrow \top \\ 1 & \xrightarrow{\top} & \Omega \end{array} \qquad \begin{array}{ccc} 0 & \xrightarrow{0_1} & 1 \\ \downarrow & & \downarrow \bot \\ 1 & \xrightarrow{\top} & \Omega \end{array}$$

are *pullbacks*, therefore $\top = \chi_{1_1}$ and $\bot = \chi_{0_1}$. On the other hand, since $Sub(1)$ is a *lattice* we have $1_1 \wedge 1_1 = 1_1$ and $0_1 \wedge 1_1 = 1_1 \wedge 0_1 = 0_1 \wedge 0_1 = 0_1$. At this point we can conclude that

$$\top \wedge \top = \chi_{1_1} \wedge \chi_{1_1} = \chi_{1_1 \cap 1_1} = \chi_{1_1} = \top$$
$$\top \wedge \bot = \chi_{1_1} \wedge \chi_{0_1} = \chi_{1_1 \cap 0_1} = \chi_{0_1} = \bot$$
$$\bot \wedge \top = \chi_{0_1} \wedge \chi_{1_1} = \chi_{0_1 \cap 1_1} = \chi_{0_1} = \bot$$
$$\bot \wedge \bot = \chi_{0_1} \wedge \chi_{0_1} = \chi_{0_1 \cap 0_1} = \chi_{0_1} = \bot$$

□

5.3 Soundness and completeness

We finally arrive at the main results about *topos semantics*. The first thing we'll see is that *topos semantics* are sound with respect to *intuitionistic logic*.

Theorem 5.9. *Let* ***C*** *be a* topos *then, for every* logical statement φ

$$\textit{if} \qquad \varphi \in \mathcal{L}_{IPL} \qquad \textit{then} \qquad \textbf{\textit{C}} \models \varphi \tag{5.1}$$

Proof. Notice first the following equivalences

$$\mathbf{C} \models \varphi \qquad \text{iff} \qquad \mathbf{C}(1,\Omega) \models \varphi \qquad \text{iff} \qquad Sub(1) \models \varphi \tag{5.2}$$

The first equivalence comes from the very definition of $\mathbf{C} - validity$. The second one comes from the fact that $\mathbf{C}(1,\Omega)$ and $Sub(1)$ are *isomorphic* as we have shown in 5.7.

Now, if $\varphi \in \mathcal{L}_{IPL}$, then $H \models \varphi$ for every *Heyting algebra* H. In particular $Sub(1) \models \varphi$ and so $\mathbf{C} \models \varphi$. □

The converse is obviously not true since, for example, **Set** $\models (\varphi \wedge \neg\varphi)$. However, *topos semantics* are actually *complete* for *classical propositional logic*.

Theorem 5.10. *Let **C** be a* topos *then, for every* logical statement φ

$$\textit{if} \qquad \textbf{\textit{C}} \models \varphi \qquad \textit{then} \qquad \varphi \in \mathcal{L}_{CPL} \tag{5.3}$$

Proof. Let $V : X \longrightarrow 2$ be a *boolean valuation*, define the following ***C**-valuation*

$$V'(x_i) = \begin{cases} \top & \text{, if } V(x_i) = 1 \\ \bot & \text{, if } V(x_i) = 0 \end{cases}$$

Notice that, because of it's definition and because of the tables derived in 5.8, $V'(\varphi)$ takes only one of two possible values: $\top$ or $\bot$, for every *logical statement* φ. Furthermore, since the *truth tables* in 5.8 behave in the exact same way as the *truth tables* in any *boolean algebra*, we have that

$$V'(\varphi) = \top \quad iff \quad V(\varphi) = 1$$

Now, if **C** $\models \varphi$, then $V'(\varphi) = \top$ and, by the last result, $V(\varphi) = 1$. Since this happens for every possible *valuation* V, we conclude that $\vdash_{\mathcal{L}_{IPL}} \varphi$. □

Chapter 6

Gödel's family of intermediate logics

In the last chapter we have shown that the *semantics of topoi* lie between *intuitionistic logic* and *propositional logic*, being stronger than the first and weaker than the second. Logics with this property are called *intermediate logics* and we dedicate this chapter to their study. We'll focus ourselves in a particular family of these *logics* who were first studied by Gödel in 1932. More about this work can be consulted in [9].

6.1 Algebraic semantics

In 1932, Gödel showed that there are at least *countably* many intermediate *logics*. To this end, he defined a particular family of them which were latter *axiomatized* by *I. Thomas* in the following way

Definition 6.1. For every $n \in \mathbb{N}$, define $\mathcal{G}_n$ as the *derivation closure* of the set

$$G_n = IPL \cup \{((x_0 \Rightarrow x_1) \vee (x_1 \Rightarrow x_0))\} \cup \{F_{n+1}\}$$

where *IPL* is the set of *axioms* of *intuitionistic logic* and

$$F_n = \bigvee_{0 \leq i < j < n} (x_i \Leftrightarrow x_j)$$

It is evident that $\mathcal{L}_{IPL} \subseteq \mathcal{G}_n$ for every $n \in \mathbb{N}$ since every axiom of $\mathcal{L}_{IPL}$ is, by definition, an axiom of $\mathcal{G}_n$. On the other hand, the axiom $((x_0 \Rightarrow x_1) \vee (x_1 \Rightarrow x_0))$ helps us by restricting the space of possible *models* to only those *heyting algebras* which are build on top of a *total order*. Next, we will see that the logics $\mathcal{G}_n$ constitute a *descending chain*, but in order to do that, let us first define the following family of *models*

Definition 6.2. For every $n \in \mathbb{N}$, define the *heyting algebra* S_n in the following way

$$S_n = \{k \in \mathbb{N} : k > 0 \wedge k \leq n\} \qquad x \Rightarrow y = \begin{cases} 1 & \text{, if } x \leq y \\ n & \text{, if } x > y \end{cases}$$

$$x \vee y = \min(x, y) \quad x \wedge y = \max(x, y) \qquad \neg x = \begin{cases} 1 & \text{, if } x = n \\ n & \text{, if } x \neq n \end{cases}$$

Theorem 6.1. *Let φ be any* formula*, if $S_{n+1} \models \varphi$ then $S_n \models \varphi$, for every $n \in \mathbb{N}$.*

Proof. First, consider $h : S_n \longrightarrow S_{n+1}$ defined by

$$h(x) = \begin{cases} x & \text{, if } x \neq n \\ n+1 & \text{, if } x = n \end{cases} \tag{6.1}$$

it is easy to show that this *function* is actually a *homomorphism* (a proof can be consulted in appendix B). So suppose that φ is a *formula* such that $S_{n+1} \models \varphi$. Then, for every S_{n+1}-valuation v', $v'(\varphi) = 1$. Now suppose v is any S_n-valuation. Because h is a *homomorphism*, $h \circ v$ is a S_{n+1}-valuation and so $h(v(\varphi)) = 1$, but since h is *injective*, it must be the case that $v(\varphi) = 1$. We conclude that $S_n \models \varphi$. □

Theorem 6.2. *For every $n \in \mathbb{N}$,*

$$\mathcal{G}_{n+1} \subset \mathcal{G}_n$$

Proof. The fact that $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$ is pretty evident, since $(F_n \Rightarrow F_{n+1}) \in \mathcal{L}_{IPL}$ (F_{n+1} is a *weakening of* F_n).

To see why this *inclusion* is *strict*, first notice that if $k \leq n$ then $S_k \models \mathcal{G}_n$ since there is no way of choosing $n+1$ *elements* of S_k, without repeating some of them. On the other hand, if $k > n$, then it is always possible to choose values for the *variables* $x_1, \cdots, x_n$ without repeating any of them, and so, any *valuation* of this kind does not fulfill F_{n+1}, and so $S_k \not\models \mathcal{G}_n$. Thus we have, for every $n \in \mathbb{N}$,

$$S_n \in Mod(\mathcal{G}_n) \quad \text{and} \quad S_n \notin Mod(\mathcal{G}_{n+1})$$

and so

$$\mathcal{G}_n \not\subseteq \mathcal{G}_{n+1}$$

□

Theorem 6.3.

$$\mathcal{G}_2 = \mathcal{L}_{CPL} \tag{6.2}$$

Proof. We will show that $(\varphi \vee \neg\varphi) \in \mathcal{G}_2$, for every formula φ. In fact, we will show that $(\varphi \vee \neg\varphi)$ is valid in every *model* of $\mathcal{G}_2$, at which point the *syntactic entailment* will follow by the *soundness* of *heyting algebras* in relation to *propositional logic*.

Consider a *heyting algebra* H such that $H \models \mathcal{G}_2$. Then $H \models F_3$ which means that it isn't possible to choose three *distinct elements* of H. This means that H has exactly two *elements*, since we are not considering the possibility of a *degenerate heyting algebra*. Now, let v be a *H-valuation* and φ a formula, then

$$v(\varphi \vee \neg\varphi) = v(\varphi) \cup \neg v(\varphi) = \begin{cases} \top & \text{if } v(\varphi) = \top \\ \top & \text{if } v(\varphi) = \bot \end{cases}$$

since the choice of v was arbitrary, we conclude that for every *H-valuation* v, $v(\varphi) = \top$, and so $H \models (\varphi \vee \neg\varphi)$. □

These results show that the *family* $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is in fact a *countable non-degenerated descending chain* of *intermediate logics*.

$$\mathcal{L}_{CPL} = \mathcal{G}_2 \supset \mathcal{G}_3 \supset \cdots \supset \mathcal{L}_{IPL}$$

We can now completely characterize the *logics* $\mathcal{G}_n$

Theorem 6.4. *Let H be a* Heyting algebra*, then $H \models \mathcal{G}_n$ if and only if $H \simeq S_k$, for some $k \leq n$.*

Proof. As we have already seen, if $H \models \mathcal{G}_n$, then H must have, at most, n *elements*. Moreover, the $((x_0 \Rightarrow x_1) \vee (x_1 \Rightarrow x_0))$ *axiom* guarantees us that H is built on top of a *total order*. Now, if H is a *total order* of k elements, then $H \simeq S_k$. The proof is straightforward (just notice that in any *total order* $a \wedge b$ will always be the "smallest" *element* of the set $\{a, b\}$ and $a \vee b$ the "greatest"). □

Theorem 6.5.

$$\varphi \in \mathcal{G}_n \qquad \textit{iff} \qquad S_n \models \varphi$$

Proof. As we have already seen in theorem 6.2, $S_n \models \mathcal{G}_n$, so the "only if" implication is trivial. For the "if" part, notice that *theorem* 6.4 completely characterizes the *algebraic models* of $\mathcal{G}_n$. In fact

$$Mod(\mathcal{G}_n) = \{S_k : k \leq n\}$$

On the other hand, let H be a *heyting algebra*, define $H^{\models} = \{\varphi : H \models \varphi\}$. Then, by theorem 6.1, $S_{n+1}^{\models} \subset S_n^{\models}$, i.e., if some *formula* φ is *true* in S_n then it is also *true* in every S_k with smaller index. This means that φ is *true* in every model of $\mathcal{G}_n$ if and only if it is true in S_n. This, coupled with *theorem* 2.8, proves the desired result. □

6.2 Sheaves

The use of *sheaves* started in *topology*, as a way of formalising the process of restricting certain kinds of *functions*, defined in an *open set*, to its *open subsets*, and of "glueing" the restrictions in an *open cover* in order to obtain the *restricted function* again. In categorical terms, a *sheaf* is a special case of a *pre-sheaf*, and its reaches go way beyond what will be presented here. For more information on the subject, the interested reader may consult [10] and [11]. Traditionally, given a *topological space* X, a *pre-sheaf* is a *functor* from $\mathcal{O}(X)$ (the *algebra of open subsets* of X) to **Sets**. A more "general" theory of *sheaves* can be obtained if, instead of starting with a *topological space*, we start with a suitable *lattice* that "mimics" the properties of an *algebra of open subsets*.

Definition 6.3 (Locale)**.** Let $\mathcal{L}$ be a *complete lattice*. $\mathcal{L}$ is said to be a *locale* if arbitrary *joins* distribute over finite *meets*, i.e.,

$$a \wedge (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \wedge b_i)$$

holds, for every $a, \{b_i\}_{i \in I} \in \mathcal{L}$ and every *index set* I.

To see why this definition makes sense, remember that the *lattice of open subsets* is a *sublattice* of the *lattice of subsets* that is closed to arbitrary *joins* and finite *meets*. The reason why *locales* are important in this context follows from the next result.

Theorem 6.6. *Let $\mathcal{L}$ be a* lattice*, then $\mathcal{L}$ is a* locale*, if and only if, it is a* complete Heyting algebra*.*

Proof. A *proof* of this result can be consulted in [10] (*proposition 1.3.2*). □

This shows us that *locales* are essentially *complete Heyting algebras* and so, the whole theory of *sheaves* in a *locale* can be "applied" to *sheaves* in *complete Heyting algebras*.

Definition 6.4 (Pre-sheaf)**.** Let H be a *complete Heyting algebra* and $\mathbf{C}_H$ the *category* based on the *partial ordering* of H. A *pre-sheaf* of $\mathbf{C}_H$ is a *contravariant functor* from $\mathbf{C}_H$ to **Set**. The *category* **PreSh(**$\mathbf{C}_H$**)** is the *category* with *objects pre-sheaves* in $\mathbf{C}_H$ and *morphisms* the *natural transformations* between them. When no confusion arises, we will refer to $\mathbf{C}_H$ as simply **H**.

Definition 6.5. Let F be a *pre-sheaf* of **H** and $\{u_i\}_{i \in I}$ a family of *elements* of H. A family $\{x_i \in F(u_i)\}_{i \in I}$ is called *compatible in F* when $x_i|_{u_i \wedge u_j} = x_j|_{u_i \wedge u_j}$, for every $i, j \in I$.

Definition 6.6 (Sheaf)**.** Let F be a *pre-sheaf* in **H**. F is called a *sheaf* if, given $u = \bigvee_{i \in I} u_i$ in H and $\{(x_i \in F(u_i))\}_{i \in I}$ a compatible family in F, there exists a unique $x \in F(u)$ such that $x|_{u_i} = x_i$, for every $i \in I$. The *category* **Sh(H)** is the *category* with *objects sheaves* in $\mathbf{C}_H$ and *morphisms* the *natural transformations* between them.

6.3 Topos semantics

We will now *characterize semantically* the family $\{\mathcal{G}_n\}_{n \in \mathbb{N}}$ of *logics*, using *topoi*.

Theorem 6.7. *If H is a* complete Heyting algebra*, then **Sh(H)** is a* topos*.*

Proof. A *proof* of this result can be consulted in [10] (*example 5.2.3*). □

Theorem 6.8. *If H is a* complete Heyting algebra*, then $H \simeq Sub_{\mathbf{Sh(H)}}(1)$.*

Proof. A *proof* of this result can be consulted in [10] (*corollary 2.2.16*). □

Now consider the family $\{S_k\}_{k\in\mathbb{N}}$ of *Heyting algebras* defined in the previous sections. Notice that, for every $k \in \mathbb{N}$, S_k is a *complete Heyting algebra* since it is finite.

Theorem 6.9. *For every $n \in \mathbb{N}$ and every* formula φ*,*

$$\vdash_{\mathcal{G}_n} \varphi \quad iff \quad \textbf{\textit{Sh(}}\mathbf{S_n}\textbf{\textit{)}} \models \varphi$$

Proof.

$$\begin{aligned} \vdash_{\mathcal{G}_n} \varphi \qquad & iff \qquad S_n \models \varphi && \text{(theorem 6.5)} \\ & iff \qquad Sub_{\mathbf{Sh(S_n)}}(1) \models \varphi && \text{(theorem 6.8)} \\ & iff \qquad \mathbf{Sh(S_n)} \models \varphi && \text{(equivalence 5.2)} \end{aligned}$$

□

Chapter 7

Conclusions

Given the broad nature of topoi, I could not have imagined, when I first opened the book *Topoi* from Goldblatt, the multiplicity of areas that I would touch in the making of this thesis. Category theory was assured, which was a good thing (I had fallen in love with the subject since the moment it was introduced to me by professor Cristina Sernadas), and logic as well, at least in some general form, but from the meeting of these two amazing subjects I had the privilege to study a vastness of subjects that I had never imagined would be connected. In *category theory* I was able to see how concepts that seem intrinsic to *set theory*, like the notions of *subset* or *characteristic function*, could be abstracted into the categorical level bringing with them all the richness of *set theory*. *Topoi* are indeed one of the most amazing structures I had the opportunity to study, and their relation to the *category of sheaves* (and therefore to *topology*) just evidence the richness of their structure. The *lift* of the *logical constructions* is another evidence of this richness, and after the results of *completeness* relative to *intuitionistic logic* and *soundness* relative to *classical logic*, their intimate relation to *intermediate logics* could not be overlooked.
From a *logics* point of view, I had the opportunity to study new *formal systems* and their relation to *algebraic structures* unknown to me at the time. I had also the opportunity of redefining some of the usual *logical structures* which inevitably led me to a deeper understanding of the their underlying mechanisms. *Intermediate logics* proved to be a very interesting area of research from a practical standpoint, like their applications to automated proof systems, but also from a theoretical one, like the amount of logics that lay between *intuitionistic* and *classical logics*.

Topoi are very conceptually rich. In one hand, they are *categories*, so their study may very well lay primarily in *category theory*. On the other hand, having their inspiration in the field of *algebraic topology* and being a generalization of the *category of sets*, they are deeply linked with *algebra*, *topology* and the *foundations of mathematics*. This already suffices to link it to *mathematical logic* but when it becomes possible to define *semantics* based on these structures in such a way that every *topos* has its own *logic*, their connection to *mathematical logic* becomes impossible to ignore. From a *categorical* point of view, *topoi* show us how to lift concepts that seem intrinsic to *set theory*, like the notions of *subset* or *characteristic function*, into the *categorical* realm. With this lift comes a deeper understanding of the notions themselves. From a *logical* perspective it's amazing how almost every concept of *mathematical logic* can be coded intro the structure of a *topos* , and after the results of *completeness* relative to *intuitionistic logic* and *soundness* relative to *classical logic*, their intimate relation to *intermediate logics* could not be overlooked.

Intermediate logics also proved to be a very interesting area of research from a practical standpoint, like their applications to automated proof systems, but also from a theoretical one, like the amount of logics that lay between *intuitionistic* and *classical logics*.

7.1 Future Work

A lot of things come to mind when thinking about possible extensions of this work. The most obvious one would be to extend the *categorical semantics* to *first-order* or even *higher-order logics*. There is already a considerable amount of work in this area, check for example [12]. In particular, I'm very curious to see how *models* of *sublogics* relate to each other categorically. Next, there is a result from *Umezawa* ([13]) that shows the existence of a *non-countable chain* of *intermediate logics*. It would be interesting to study their *categorical models*. Finally, it would be very interesting to study how some usual *categorical* constructions that preserve *topoi* could be "translated" into *logical language*. For example, the *bundle* over a *topos* is still a *topos*, but it's *subobject classifier* has much more *truth values* - how does the *logic* of the first, relate to the logic of the second?

Appendix A

Here we present two *derivation sequences* in a *Hilbert-style calculi*. The first one testifies that if $((x0 \wedge x1) \Rightarrow x2) \in \mathcal{L}_{IPL}$, then $(x0 \Rightarrow (x1 \Rightarrow x2)) \in \mathcal{L}_{IPL}$, and the second one testifies the converse.

Remember the following *axioms* of $\mathcal{L}_{IPL}$

Ax$_1$ $(x_0 \Rightarrow (x_1 \Rightarrow x_0))$

Ax$_2$ $((x_0 \Rightarrow (x_1 \Rightarrow x_2)) \Rightarrow ((x_0 \Rightarrow x_1) \Rightarrow (x_0 \Rightarrow x_2)))$

Ax$_3$ $((x_0 \wedge x_1) \Rightarrow x_0)$

Ax$_4$ $((x_0 \wedge x_1) \Rightarrow x_1)$

We will also assume that the following formulas belong to $\mathcal{L}_{IPL}$. This can easily be proven by the interested reader or consulted in the references.

Ins$_\wedge$ $(x_0 \Rightarrow (x_1 \Rightarrow (x_0 \wedge x_1)))$

Trans$_\Rightarrow$ $((x_0 \Rightarrow x_1) \Rightarrow ((x_1 \Rightarrow x_2) \Rightarrow (x_0 \Rightarrow x_2)))$

Theorem A.1. *If* $((x_0 \wedge x_1) \Rightarrow x_2) \in \mathcal{L}_{IPL}$ *then* $(x_0 \Rightarrow (x_1 \Rightarrow x_2)) \in \mathcal{L}_{IPL}$.

Proof. Consider the following *derivation sequence*:

1.	$(x_0 \wedge x_1) \Rightarrow x_2$	Hip
2.	$x_0 \Rightarrow (x_1 \Rightarrow (x_0 \wedge x_1))$	Ins$_\wedge$
3.	$(x_1 \Rightarrow (x_0 \wedge x_1)) \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2))$	Trans$_\Rightarrow$
4.	$(x_0 \Rightarrow ((x_1 \Rightarrow (x_0 \wedge x_1)) \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2))))$ $\Rightarrow ((x_0 \Rightarrow (x_1 \Rightarrow (x_0 \wedge x_1))) \Rightarrow (x_0 \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2))))$	Ax$_2$
5.	$((x_1 \Rightarrow (x_0 \wedge x_1)) \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2)))$ $\Rightarrow (x_0 \Rightarrow ((x_1 \Rightarrow (x_0 \wedge x_1)) \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2))))$	Ax$_1$
6.	$(x_0 \Rightarrow ((x_1 \Rightarrow (x_0 \wedge x_1)) \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2))))$	MP$_{3,5}$
7.	$(x_0 \Rightarrow (x_1 \Rightarrow (x_0 \wedge x_1))) \Rightarrow (x_0 \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2)))$	MP$_{4,6}$
8.	$(x_0 \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2)))$	MP$_{2,7}$
9.	$(x_0 \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_1 \Rightarrow x_2))) \Rightarrow ((x_0 \Rightarrow ((x_0 \wedge x_1) \Rightarrow x_2)) \Rightarrow (x_0 \Rightarrow (x_1 \Rightarrow x_2)))$	Ax$_2$
10.	$(x_0 \Rightarrow ((x_0 \wedge x_1) \Rightarrow x_2)) \Rightarrow (x_0 \Rightarrow (x_1 \Rightarrow x_2))$	MP$_{8,9}$
11.	$((x_0 \wedge x_1) \Rightarrow x_2) \Rightarrow (x_0 \Rightarrow ((x_0 \wedge x_1) \Rightarrow x_2))$	Ax$_1$
12.	$x_0 \Rightarrow ((x_0 \wedge x_1) \Rightarrow x_2)$	MP$_{1,11}$
13.	$x_0 \Rightarrow (x_1 \Rightarrow x_2)$	MP$_{12,10}$

□

Theorem A.2. *If* $(x_0 \Rightarrow (x_1 \Rightarrow x_2)) \in \mathcal{L}_{IPL}$ *then* $((x_0 \wedge x_1) \Rightarrow x_2) \in \mathcal{L}_{IPL}$.

Proof. Consider the following *derivation sequence*:

1.	$x_0 \Rightarrow (x_1 \Rightarrow x_2)$	Hip
2.	$(x_0 \Rightarrow (x_1 \Rightarrow x_2)) \Rightarrow ((x_0 \wedge x_1) \Rightarrow (x_0 \Rightarrow (x_1 \Rightarrow x_2)))$	Ax$_1$
3.	$(x_0 \wedge x_1) \Rightarrow (x_0 \Rightarrow (x_1 \Rightarrow x_2))$	MP$_{1,2}$
4.	$((x_0 \wedge x_1) \Rightarrow (x_0 \Rightarrow (x_1 \Rightarrow x_2))) \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_0) \Rightarrow ((x_0 \wedge x_1) \Rightarrow (x_1 \Rightarrow x_2)))$	Ax$_2$
5.	$(((x_0 \wedge x_1) \Rightarrow x_0) \Rightarrow ((x_0 \wedge x_1) \Rightarrow (x_1 \Rightarrow x_2)))$	MP$_{3,4}$
6.	$(x_0 \wedge x_1) \Rightarrow x_0$	Ax$_3$
7.	$((x_0 \wedge x_1) \Rightarrow (x_1 \Rightarrow x_2))$	MP$_{6,5}$
8.	$((x_0 \wedge x_1) \Rightarrow (x_1 \Rightarrow x_2)) \Rightarrow (((x_0 \wedge x_1) \Rightarrow x_1) \Rightarrow ((x_0 \wedge x_1) \Rightarrow x_2))$	Ax$_2$
9.	$((x_0 \wedge x_1) \Rightarrow x_1) \Rightarrow ((x_0 \wedge x_1) \Rightarrow x_2)$	MP$_{7,8}$
10.	$(x_0 \wedge x_1) \Rightarrow x_1$	Ax$_4$
11.	$11.(x_0 \wedge x_1) \Rightarrow x_2$	MP$_{10,9}$

□

Appendix B

Consider the *function* $h : S_n \longrightarrow S_{n+1}$ defined by

$$h(x) = \begin{cases} x & \text{, if } x \neq n \\ n+1 & \text{, if } x = n \end{cases} \tag{B.1}$$

We will show that h is an *homomorphism*, i.e., that it preserves the *heyting algebra* structure. Notice that $h(1) = 1$ and $h(n) = n + 1$ by definition.

Join Consider $x, y \in S_n$. If $x \neq n$ or $y \neq n$ then $x \vee_{S_n} y = \min(x, y) \neq n$, and so

$$h(x \vee_{S_n} y) = h(\min(x, y)) = \min(x, y) = \min(h(x), h(y)) = h(x) \vee_{S_{n+1}} h(y)$$

On the other hand, if $x = y = n$ then

$$h(x \vee_{S_n} y) = h(n) = n + 1 = \min(n + 1, n + 1) = \min(h(x), h(y)) = h(x) \vee_{S_{n+1}} h(y)$$

Meet Consider $x, y \in S_n$. If $x \neq n$ and $y \neq n$ then $x \wedge_{S_n} y = \max(x, y) \neq n$, and so

$$h(x \wedge_{S_n} y) = h(\max(x, y)) = \max(x, y) = \max(h(x), h(y)) = h(x) \wedge_{S_{n+1}} h(y)$$

On the other hand, if $x = n$ or $y = n$ then

$$h(x \wedge_{S_n} y) = h(\max(x, y)) = h(n) = n + 1 = \max(h(x), h(y)) = h(x) \wedge_{S_{n+1}} h(y)$$

Pseudo-complement Notice that h preserves order, i.e., if $x \leq_{S_n} y$ then $h(x) \leq_{S_{n+1}} h(y)$. Now consider $x, y \in S_n$. If $x \leq y$ then

$$h(x \Rightarrow_{S_n} y) = h(1) = 1 = h(x) \Rightarrow_{S_{n+1}} h(y)$$

Now suppose $x > y$, then

$$h(x \Rightarrow_{S_n} y) = h(n) = n + 1 = h(x) \Rightarrow_{S_{n+1}} h(y)$$

Negation Let $x \in S_n$. If $x \neq n$ then

$$h(\neg_{S_n} x) = h(n) = n + 1 = \neg_{S_{n+1}} h(x)$$

On the other hand, if $x = n$, then

$$h(\neg_{S_n} x) = h(1) = 1 = \neg_{S_{n+1}} (n + 1) = \neg_{S_{n+1}} h(x)$$

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