Optimal Stopping Time Involving Polynomial Profit Functions

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Abstract

In this paper we study three different problems that are common in real options: the "exit problem", the "investment problem" and the "the changing market problem". We assume that the market demand is modelled by a geometric Brownian motion. We consider the profit functions polynomials. Using the Hamilton-Jacobi-Bellman equations we solve the exit problem. One can easily study the investment problem and changing market problem, since they are related with the exit problem. We end up analysing the influence of several parameters on our solutions to these problems.

Keywords: Real options, Hamilton-Jacobi-Bellman equations, Geometric Brownian motion, Exit problem, Investment problem, Changing market problem.

1 Introduction

In this work we address three profit maximization problems that are often studied in relation to real options. First, we determine optimal conditions under which a firm should permanently exit a market. We then look at optimal conditions again, but for irreversible investment decisions. Finally, we study the problem of establishing the optimal conditions for a firm to switch irreversibly from one market to another.

We begin using a linear profit function and find an optimal stopping strategy to the exit problem and investment problem. Later we find an optimal stopping strategy for a firm that has the option to switch from one market characterized by a linear profit function to another market characterized by a different linear profit function. To conclude this section, we discuss the influence of several parameters in the value function. Then we find an optimal stopping strategy for a firm that has the option to switch from one market characterized by a monomial profit function to a market characterized by a linear profit function and, conversely, from linear to monomial. Again we study the influence of several parameters on the value function.

We consider a firm which produces an established product in a stochastic environment, which is characterized by the demand process $X = \{X_t : t \geq 0\}$, defined on a complete filtered space $(\Omega, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$. Moreover, we assume that τ is a $\mathcal{F}_t$-stopping time if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all t ≥ 0. We denote by S the set of all $\mathcal{F}_t$-stopping times. On this work X is a geometric Brownian motion, solution of the stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

with $X_0 = x$, drift $\mu \in \mathbb{R}$, volatility $\sigma \geq 0$, and $\{W_t : t \geq 0\}$ being a Winner process.

In this paper the firm's profit function is given by $\Pi :]0, \infty[\to \mathbb{R}$, which depends only on the demand level of the process X.

In this paper we study three different problems that are common in the real options:

- *Exit problem:*

  The firm has the possibility to completely abandon the production. When the firm decides to exit the market at time τ and the current demand is x, its value is given by

$$^1J(x,\tau) = E_x\left[\int_0^\tau e^{-\gamma s}\Pi(X_s)ds + e^{-\gamma\tau}C\right]. \quad (1)$$

[1]From now on we use the short notation: $E_x[\star] = E[\star | X_0 = x]$.

where γ is a positive interest rate and $C \in \mathbb{R}$ is the terminal cost or profit. The function

$$\mathcal{V}(x) = \sup_{\tau \in S} J(x, \tau)$$

is called value function. Thus we have an optimal stopping problem, where the main goal is to maximize the expected total pay-off.

- *Investment problem:*

  The firm has the possibility of choosing the moment of entry in the market. In the literature the problem is often called the "entry problem" [1]. If the firm decides to enter in the market at time τ, and the current demand is x, its value is given by

$$\tilde{J}(x, \tau) = E_x \left[\int_{\tau}^{\infty} e^{-\gamma s} \Pi(X_s) ds + e^{-\gamma \tau} K \right] \tag{2}$$

  where $K \leq 0$ represents, in this case, an entry or profit cost. Therefore we want to find an optimal strategy for a firm that decides to enter in the market. The firm's value is given by the function

$$\tilde{\mathcal{V}}(x) = \sup_{\tau \in S} \tilde{J}(x, \tau)$$

  therefore we have an optimal stopping problem, where the main goal is to maximize the expected total pay-off.

- *Changing market problem:*

  The firm's is currently producing the product 1 in the market 1 with the profit function Π_1. Moreover, at any moment τ the firm may decide to switch from market 1 to market 2, in order to produce a different product that we call product 2. Associated with this second product, the firm's profit function is Π_2. We assume that both markets are modelled by a geometric Brownian motion with the same parameters. In this case, the firm's value, when the current level of demand is x, is given by

$$I(x, \tau) = E_x \left[\int_{0}^{\tau} e^{-\gamma s} \Pi_1(X_s) ds + \int_{\tau}^{\infty} e^{-\gamma s} \Pi_2(X_s) ds + e^{-\gamma \tau} Q \right] \tag{3}$$

  where $Q \in \mathbb{R}$ is the switching cost or profit from one market to the other. Therefore, we want to know when is the right moment τ to change from the market 1 to market 2. The value function in this case is given by

$$\mathcal{G}(x) = \sup_{\tau \in S} I(x, \tau)$$

  again we have an optimal stopping problem, where the main goal is to maximize the expected total pay-off.

It can be proved that the investment problem and changing market problem are related with the exit problem, therefore for the moment we solve the exit problem. The solution to the exit problem defined in (1) can be written as follows

$$J(x, \tau^\star) = \mathcal{V}(x)$$

where $\tau^\star \in S$. In an optimal stopping problem we want to find an optimal action, continue or stop, for each state. Consequently, our state space is split into two regions: a continuation region, which we denote by $\mathcal{C}$, and a stopping region, denoted by $\mathcal{D}$. As expected, in the continuation region the optimal action is to continue. Hence, for our case, an optimal stopping time should be

$$\tau^\star = inf\{t \geq 0 : X_t \in \mathcal{D}\}.$$

So, it should be intuitive that

$$\mathcal{C} = \{x \in \mathbb{R}^+ : \mathcal{V}(x) \geq C\}.$$

One way to address the optimal stopping problem is to solve the following variational inequalities:

$$\max\{-\gamma\mathcal{V}(x) + \mu x f'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) + \Pi(x), C - \mathcal{V}(x)\} = 0 \tag{4}$$

This is the Hamilton-Jacobi-Bellman (HJB) equation that we need to solve in order to get a solution to our stopping problem. To solve the HJB equation, first we use some intuition from the problem and guess the form of the continuation region. Our continuation region will have a threshold denoted by $x^\star$. Then we solve the differential equation $\Pi(x) + \mu x\mathcal{V}'(x) + \frac{1}{2}\sigma^2 x^2 \mathcal{V}''(x) - \gamma\mathcal{V}(x) = 0$.
According to the literature [2], [3], the value function $\mathcal{V}(x)$ should be $C_1(]0, \infty[)$. To ensure this we use:

- The fit condition $\lim_{x \to x^\star} \mathcal{V}(x) = \mathcal{V}(x^\star)$.
- The smooth condition $\lim_{x \to x^\star} \mathcal{V}'(x) = \mathcal{V}'(x^\star)$.

To solve the HJB equation (4) we start by solving the homogeneous differential equation

$$-\gamma V(x) + \mu x V'(x) + \frac{1}{2}\sigma^2 x^2 V''(x) = 0 \tag{5}$$

The solution of (5) is $V(x) = A_1 x^{\beta_1} + A_2 x^{\beta_2}$ where, β_1 and β_2 are given by

$$\beta_1 = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}}{\sigma^2} < 0$$

$$\beta_2 = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}}{\sigma^2} > 0$$

2 Affine profit function

In this section we study the problems above described when the profit function is linear, that is, our market is characterized by the profit function $\Pi(x) = ax - b$. We assume that $C = K = Q = 0$.

- Exit problem

  To avoid trivial cases, that is $\tau = 0$ and $\tau = \infty$, we choose $a \geq 0$ and $b \geq 0$ such that the function $ax - b$ takes positive or negative values.

  The condition $E_x\left[\int_0^\infty e^{-\gamma s}\Pi(X_s)ds\right] < \infty$ holds true if and only if $\gamma - \mu \geq 0$. This condition guarantees that $J(x, \tau)$ is well defined, and finite for all x and τ.

  The continuation region for the exit problem is $\mathcal{C} = \{x : x \geq x^\star\}$.

  The solution for our problem is: $\mathcal{V}(x) = \begin{cases} 0 & : x < x^\star \\ \frac{ax}{\gamma - \mu} - \frac{b}{\gamma} + A_1 x^{\beta_1} & : x \geq x^\star \end{cases}$
  where $A_1 = \frac{\frac{b}{\gamma} - \frac{a}{\gamma-\mu}x^\star}{x^{\star\beta_1}} \geq 0$, and $x^\star = \frac{b}{a}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1 - 1}\right) x^\star \leq \frac{b}{a}$.

  With this analysis we are also studying the situation of a monomial case. As we can see in Guerra, Nunes and Oliveira [4] that the exit problem with $\Pi(X_s) = aX_s^\theta - b$ can be reduced to this problem. To see this we use Ito's Lemma and conclude that X_s^θ is a GBM with volatility $\sigma_\theta = \sigma\theta$ and drift $\mu_\theta = \mu\theta + \frac{1}{2}\sigma^2\theta(\theta - 1)$.

- Investment problem

  Similarly to the exit problem, we need to assume $a \geq 0$ and $b \geq 0$ in order to avoid trivial problems.

The continuation region for the investment problem is $\mathcal{C} = \{x : x \le x^\star\}$.

The solution for our problem is $\tilde{\mathcal{V}}(x) = \begin{cases} A_2 x^{\beta_2} & : x < x^\star \\ \frac{ax}{\gamma-\mu} - \frac{b}{\gamma} & : x \ge x^\star \end{cases}$
where $A_2 = \frac{-\frac{b}{\gamma} + \frac{a}{\gamma-\mu}x^\star}{x^{\star\beta_2}} \ge 0$ and $x^\star = \frac{b}{a}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_2}{\beta_2-1}\right) \ge \frac{b}{a}$.

- Changing market problem

  We now consider that the firm has the option to switch from one market characterized by the profit function $\Pi_1(x) = a_1 x - b_1$, to another one characterized by the profit function $\Pi_2(x) = a_2 x - b_2$ at time τ. We are assuming that $a_1, a_2, b_1, b_2 \ge 0$, for the same reasons as before.

  We will say that Π_1 is more risky than Π_2, if $a_1 \ge a_2$. To avoid trivial cases we choose $b_1 \ge b_2$.

  This problem may be solved using the results derived in the exit problem. Thus the value function for such case is given by:

$$\mathcal{G}(x) = \begin{cases} \frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma} & : x < x^\star \\ \frac{a_1}{\gamma-\mu}x - \frac{b_1}{\gamma} + A_1 x^{\beta_1} & : x \ge x^\star \end{cases}$$

  where $A_1 = \frac{\frac{\tilde{b}}{\gamma} - \frac{\tilde{a}}{\gamma-\mu}x^\star}{x^{\star\beta_1}} \ge 0$ and $x^\star = \frac{\tilde{b}}{\tilde{a}}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right) \le \frac{\tilde{b}}{\tilde{a}} = c$ where $\tilde{a} := a_1 - a_2 \ge 0$, $\tilde{b} := b_1 - b_2 \ge 0$, and $c = \frac{\tilde{b}}{\tilde{a}}$. Note that c is precisely the point where Π_1 and Π_2 intersect, therefore the value x where the firm should optimally change from the first market to the second one is smaller or equal to the point where both markets are equally profitable.

  The other relevant case occurs when $a_1 \le a_2$, in which case we say that Π_2 is more risky than Π_1. Again to avoid trivial cases, we choose $b_2 \ge b_1$.

  This problem can be solved using the results derived in the investment problem, the value function for such case is given by:

$$\mathcal{G}(x) = \begin{cases} A_2 x^{\beta_2} + \frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma} & : x < x^\star \\ \frac{a_1}{\gamma-\mu}x - \frac{b_1}{\gamma} & : x \ge x^\star \end{cases}$$

  where $x^\star = \frac{\tilde{b}}{\tilde{a}}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_2}{\beta_2-1}\right) \ge \frac{\tilde{b}}{\tilde{a}} = c$ and $A_2 = \frac{\frac{\tilde{b}}{\gamma} - \frac{\tilde{a}}{\gamma-\mu}x^\star}{x^{\star\beta_2}} \ge 0$, with $\tilde{a} := a_1 - a_2 \le 0$ and $\tilde{b} := b_1 - b_2 \le 0$. Therefore the value x where the firm should optimally change from the first market to the second one is bigger or equal to the point where both markets are equally profitable.

We now study the influence of the market expectations, μ and σ, and the parameters a and b on the decision to exit. Simple calculations show that:

$$\frac{\partial x^\star}{\partial \sigma} = \frac{b}{a}\frac{\gamma-\mu}{\gamma}\frac{-\frac{\partial \beta_1}{\partial \sigma}}{(\beta_1-1)^2} \le 0.$$
$$\frac{\partial x^\star}{\partial \mu} = \frac{b}{a}\frac{1}{\gamma}\frac{\frac{\partial \beta_1}{\partial \mu}}{(\beta_1-1)^2} \le 0.$$

Therefore if the drift or the volatility of the uncertainty process X increases, the decision to exit the market is postponed.
Regarding the behaviour of $x^\star$ with the parameters of the gain function, we have

$$\frac{\partial x^\star}{\partial b} = \frac{1}{a}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right) \ge 0.$$
$$\frac{\partial x^\star}{\partial a} = -\frac{b}{a^2}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right) \le 0.$$

it follows that $x^\star$ increases with b (i.e. the decision to exit the market is postponed if the profit flow shifts upward) and decreases with the slope.

Again as for the exit problem for the investment problem we study the influence of the market expectations, μ and σ, and the parameters a and b on the decision to invest. Simple calculations show that:

$$\frac{\partial x^\star}{\partial \sigma} = \frac{b}{a}\frac{\gamma-\mu}{\gamma}\frac{-\frac{\partial \beta_2}{\partial \sigma}}{(\beta_2-1)^2} \geq 0.$$
$$\frac{\partial x^\star}{\partial \mu} = \frac{b}{a}\frac{1}{\gamma}\frac{\frac{\partial \beta_2}{\partial \mu}}{(\beta_2-1)^2} \leq 0.$$

Therefore if σ increases, the decision to invest is anticipated whereas if μ increases, it is postponed. Regarding the behaviour of $x^\star$ with the parameters of the profit function, we have

$$\frac{\partial x^\star}{\partial b} = \frac{1}{a}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_2}{\beta_2-1}\right) \geq 0.$$
$$\frac{\partial x^\star}{\partial a} = -\frac{b}{a^2}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_2}{\beta_2-1}\right) \leq 0.$$

It follows that $x^\star$ increases with b (i.e. the decision to invest the market is postponed if the profit flow shifts upward) and decreases with the slope.

When one faces the decision to change from a first market to a second market, besides the influence of $\tilde{a} = a_1 - a_2$ and $\tilde{b} = b_1 - b_2$ (which follows the same pattern as the dependency of $x^\star$ with respect to a and b in the exit and investment problems), it is also interesting to study how $x^\star$ varies when we rotate the function Π_1 around Π_2, keeping the intersection point fixed. The point where Π_1 and Π_2 intersects is $c = \frac{b_1-b_2}{a_1-a_2} = \frac{\tilde{b}}{\tilde{a}}$. Then since $x^\star = \frac{\tilde{b}}{\tilde{a}}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right) = Kc$, where $K = \left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right)$ we conclude that $x^\star$ remains the same when we rotate the function Π_1 around Π_2, keeping the intersection point fixed.

3 Polynomial profit functions

In this section we consider a polynomial profit function , that is, our market is characterized by the profit function $\Pi(x) = a_1 x^\theta - a_2 x - b$. We assume that $C = K = Q = 0$. Mathematically, this kind of functions increases the difficulty of the problem because sometimes we cannot obtain explicitly the threshold $x^\star$.
Financially, this kind of function allow us to consider more realistic scenarios as well as changes of the market with different types of profit functions. We consider $a_1, a_2 \geq 0$, and to avoid trivial solutions to our problem, we consider $b \geq 0$ and $\theta \geq 1$.

- Exit problem
  We assume $\theta \in]1, \beta_2[$ to guarantee that $E_x\left[\int_0^\infty e^{-\gamma s}\Pi(X_s)ds\right]$ is finite. This condition guarantees that $J(x,\tau)$ is well defined, and finite for all x and τ. The solution for this problem is:

$$\mathcal{V}(x) = \begin{cases} 0 & : x < x^\star \\ \frac{a_1 x^\theta}{\gamma-\mu_\theta} - \frac{a_2 x}{\gamma-\mu} - \frac{b}{\gamma} + A_1 x^{\beta_1} & : x \geq x^\star \end{cases}$$

  with $\mu_\theta = \mu\theta + \frac{1}{2}\sigma^2\theta(\theta-1)$, $A_1 = \frac{\frac{a_2 x^\star}{\gamma-\mu} - \frac{a_1 (x^\star)^\theta}{\gamma-\mu_\theta} + \frac{b}{\gamma}}{x^{\star\beta_1}} \geq 0$ and $x^\star$ is a zero of the polynomial

$$f(x) := \frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}(x)^\theta - \frac{a_2(1-\beta_1)}{\gamma-\mu}x + \frac{\beta_1 b}{\gamma}. \quad (6)$$

Unfortunately, by Abel Ruffini theorem, f does not have an algebraic solution for all $\theta \in (1, \beta_2)$. Even though we cannot find explicitly the exit threshold $x^\star$ we can prove that $x^\star \in [y^\star, c]$ where $y^\star = \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}}$ and c such that $\Pi(c) = 0$.

- Investment problem

  The solution for this problem is:

$$\tilde{\mathcal{V}}(x) = \begin{cases} A_2 x^{\beta_2} & : x < x^\star \\ \frac{a_1 x^\theta}{\gamma-\mu_\theta} - \frac{a_2 x}{\gamma-\mu} - \frac{b}{\gamma} & : x \geq x^\star \end{cases}$$

  where $A_2 = \frac{\frac{a_1 (x^\star)^\theta}{\gamma-\mu_\theta} - \frac{a_2 x^\star}{\gamma-\mu} - \frac{b}{\gamma}}{x^{\star\beta_2}} \geq 0$ and the value of $x^\star$ is a zero of the polynomial

$$f(x) := \frac{a_1(\beta_2-\theta)}{\gamma-\mu_\theta}(x)^\theta - \frac{a_2(\beta_2-1)}{\gamma-\mu}x - \frac{\beta_2 b}{\gamma}. \tag{7}$$

  Again, the investment threshold $x^\star$ is the zero of $f(x)$ which cannot be found analytically. We can prove that $x^\star \geq y^\star$ where $y^\star = \sqrt[\theta-1]{\frac{a_2}{a_1}\frac{\theta-\beta_1}{1-\beta_1}}$.

- Changing market problem

  We now consider that the firm has the option to switch from one market characterized by the profit function $\Pi_1(x) = a_1 x^\theta - b_1$, to another one characterized by the profit function $\Pi_2(x) = a_2 x - b_2$ at time τ. We are assuming that $a_1, a_2 \geq 0$, for the same reasons before. To avoid trivial cases we assume $b_1 \geq b_2$.

  This problem can be solved using the results derived in the exit problem with polynomial profit functions, with $b = b_1 - b_2$. Thus the value function for such case is given by:

$$\mathcal{G}(x) = \begin{cases} \frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma} & : x < x^\star \\ \frac{a_1}{\gamma-\mu_\theta}x^\theta - \frac{b_1}{\gamma} + A_1 x^{\beta_1} & : x \geq x^\star \end{cases}$$

  where $A_1 = \frac{\frac{a_2 x^\star}{\gamma-\mu} - \frac{a_1 (x^\star)^\theta}{\gamma-\mu_\theta} + \frac{b_1-b_2}{\gamma}}{x^{\star\beta_1}} \geq 0$ and $x^\star \leq c$ is such that $f(x^\star) = 0$, with $f(x)$ given by (6), and c such that $\Pi_1(c) = \Pi_2(c)$. Note that c is precisely the point where Π_1 and Π_2 intersects, therefore the value x where the firm should optimally change from the first market to the second one is smaller or equal to the point where both markets are equally profitable.

  Now we consider that the firm has the option to switch from one market characterized by the profit function $\Pi_1(x) = a_1 x - b_1$, to another one characterized by the profit function $\Pi_2(x) = a_2 x^\theta - b_2$ at time τ. We are assuming that $a_1, a_2 \geq 0$, for the same reasons as before. To avoid trivial cases we assume $b_1 \leq b_2$.

  Therefore this problem can be solved using the results derived in the investment problem with polynomial profit functions, with $b = b_2 - b_1$. Thus the value function for such case is given by:

$$\mathcal{G}(x) = \begin{cases} A_2 x^{\beta_2} + \frac{a_1}{\gamma-\mu}x - \frac{b_1}{\gamma} & : x < x^\star \\ \frac{a_2}{\gamma-\mu_\theta}x^\theta - \frac{b_2}{\gamma} & : x \geq x^\star \end{cases}$$

  where $A_2 = \frac{\frac{a_2 (x^\star)^\theta}{\gamma-\mu_\theta} - \frac{a_1 x^\star}{\gamma-\mu} - \frac{b_2-b_1}{\gamma}}{x^{\star\beta_2}} \geq 0$ and $x^\star \geq c$ is such that $f(x^\star) = 0$, with $f(x)$ given by (7), and c such that $\Pi_1(c) = \Pi_2(c)$. Therefore the value x where the firm should optimally change from the first market to the second one is larger or equal to the point where both markets are equally profitable.

We now study the influence of the market expectation of the parameters μ, σ and b on the decision to exit.

To show that $x^\star$ decreases if μ increases we use the implicit derivative theorem

$$\frac{\partial x^\star}{\partial \mu} = -\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial x^\star}}.$$

Since $\frac{\partial f}{\partial x^\star} \geq -\frac{\beta_1 b}{\gamma x^\star} \geq 0$, then we only have to prove that $\frac{\partial f}{\partial \mu} \geq 0$.
Simple calculations prove that $\frac{\partial f}{\partial \mu} = p_\theta(x^\star) := \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{\theta-\beta_1}{\gamma-\mu_\theta}\right)(x^\star)^\theta + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} - \frac{1-\beta_1}{\gamma-\mu}\right)x^\star$.
Let $a^\star$ be such that $p_\theta(a^\star) = 0$. After proving that $a^\star \leq y^\star$ and since $y^\star$ is a lower bound of $x^\star$, then by transitivity $a^\star \leq x^\star$. Studying the function $p_\theta(x)$ we conclude that $\frac{\partial f}{\partial \mu} = p_\theta(x^\star) \geq p_\theta(a^\star) = 0$.

Again simple calculations prove that $\frac{\partial f}{\partial \sigma} = q_\theta(x^\star) := \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma} + \frac{(\theta-\beta_1)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)x^{\star\theta} + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma}\right)x^\star$.
Let $b^\star$ such that $q_\theta(b^\star) = 0$. Since $b^\star \leq y^\star$, then studying the function $q_\theta(x)$ we conclude that $\frac{\partial f}{\partial \sigma} = q_\theta(x^\star) \geq q_\theta(b^\star) = 0$. Note that, since $\frac{\partial f}{\partial b} = \frac{\beta_1}{\gamma} \leq 0$ then $\frac{\partial x^\star}{\partial b} \geq 0$.

As for the exit problem we study the influence of the market expectations, μ and σ, and the parameters b on the decision to invest.
If $b = 0$ then $x^\star = c\sqrt[\theta-1]{\frac{\theta-\beta_1}{1-\beta_1}} = y^\star$ where $c = \sqrt[\theta-1]{\frac{a_2}{a_1}}$. In this case we can solve $x^\star$ analytically, and therefore it follows easily that $\frac{\partial x^\star}{\partial \mu} \leq 0$ and $\frac{\partial x^\star}{\partial \sigma} \geq 0$. By the implicit derivative theorem, we have

$$\frac{\partial x^\star}{\partial \mu} = -\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial x^\star}}.$$

with $\frac{\partial f}{\partial x^\star} \geq \frac{\beta_2 b}{\gamma x^\star} \geq 0$. Since for $b = 0$, $\frac{\partial x^\star}{\partial \mu} \leq 0$ then $0 \leq \left.\frac{\partial f}{\partial \mu}\right|_{b=0}$. If $b \geq 0$, then

$$\frac{\partial f}{\partial \mu} = p_\theta(x^\star) := \frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{(\beta_2-\theta)}{\gamma-\mu_\theta}\right)x^{\star\theta} - \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{(\beta_2-1)}{\gamma-\mu}\right)x^\star.$$

Studying the function $p_\theta(x)$ we have

$$0 \leq \left.\frac{\partial f}{\partial \mu}\right|_{b=0} = p_\theta(y^\star) \leq p_\theta(x^\star) = \left.\frac{\partial f}{\partial \mu}\right|_{b>0}.$$

We conclude that $\frac{\partial x^\star}{\partial \mu} \leq 0$.

Again by the implicit derivative theorem

$$\frac{\partial x^\star}{\partial \sigma} = -\frac{\frac{\partial f}{\partial \sigma}}{\frac{\partial f}{\partial x^\star}}.$$

If $b = 0$, then $\frac{\partial x^\star}{\partial \sigma} \geq 0$ therefore $\left.\frac{\partial f}{\partial \sigma}\right|_{b=0} \leq 0$. If $b \geq 0$, then

$$\frac{\partial f}{\partial \sigma} = q_\theta(x^\star) := \frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} + \frac{(\beta_2-\theta)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)x^{\star\theta} - \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma}\right)x^\star.$$

Studying the function $q_\theta(x)$ we have

$$\left.\frac{\partial f}{\partial \sigma}\right|_{b>0} = q_\theta(x^\star) \leq q_\theta(y^\star) = \left.\frac{\partial f}{\partial \sigma}\right|_{b=0} \leq 0.$$

We conclude that $\frac{\partial x^\star}{\partial \sigma} \geq 0$.

Note that, since $\frac{\partial f}{\partial b} = -\frac{\beta_2}{\gamma} \leq 0$, then $\frac{\partial x^\star}{\partial b} \geq 0$.

4 Conclusion

In this work we analysed three profit maximization problems: the "exit problem", the "investment problem" and the "changing market problem". We assumed that the market demand followed a Geometric Brownian Motion. We solved the exit and investment problems for a class of profit functions of the form $\Pi(x) = a_1 x^\theta - a_2 x - b$. Using the Hamilton-Jacobi-Bellman equation, first we found the value function for the exit problem. Even though the exit threshold could not be calculated analytically, we determined an upper and lower bound. We then presented comparative statistics with respect to the drift and the volatility and concluded that if the drift or the volatility of the uncertainty process X increases, then the decision to exit the market is postponed. We then determined the value function for the investment problem. In this case we could only calculate a lower bound for the investment threshold. Again we presented comparative statistics with respect to the drift and the volatility and concluded that if the volatility increases, the decision to invest is anticipated whereas if the drift increases, it is postponed. Using the results derived in the exit problem we calculated the value function for the changing market problem of a firm that has the option to switch from one market characterised by a monomial profit function to a market characterised by a linear profit function. We then presented comparative statistics with respect to the drift (denoted by μ) and the volatility (denoted by σ) and concluded that the influence of μ and σ of the uncertainty process X, follows the same pattern as the dependency of $x^\star$ with respect to μ and σ in the exit problem. Conversely, we calculated the value function for the changing market problem of a firm that has the option to switch from one market characterised by a linear profit function to a market characterised by a monomial profit function. Again we presented comparative statistics with respect to μ and σ and concluded that the influence of μ and σ of the uncertainty process X, follows the same pattern as the dependency of $x^\star$ with respect to μ and σ in the investment problem.

References

[1] Avinash K Dixit and Robert S Pindyck. *Investment under uncertainty.* Princeton university press, 1994.

[2] Tomas Björk. *Arbitrage theory in continuous time.* Oxford university press, 2009.

[3] Kevin Ross. Stochastic control in continuous time. *Lecture Notes on Continuous Time Stochastic Control, Spring*, 2008.

[4] Manuel Guerra, Claudia Nunes, and Carlos Oliveira. Exit option for a class of profit functions. *International Journal of Computer Mathematics*, pages 1–16, 2016.