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Optimal Stopping time involving polynomial profit functions

Francisco Stefano Sobrito de Almeida

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Supervisor: Prof. Claudia Rita Ribeiro Coelho Nunes Philippart

Examination Committee

Chairperson: Prof. António Manuel Pacheco Pires
Supervisor: Prof. Cláudia Rita Ribeiro Coelho Nunes Philippart
Members of the Committee: Prof. António Manuel Pacheco Pires,
Prof. Cláudia Rita Ribeiro Coelho Nunes Philippart,
Dr. José Carlos Dias

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Abstract

In this thesis we study three different problems that are common in real options: the exit problem, the investment problem and the the changing market problem. We assume that the market demand is modelled by a geometric Brownian motion. We consider the profit functions polynomials. Using the Hamilton-Jacobi-Bellman equations we solve the exit problem. One can then easily study the investment problem and the changing market problem, since they are related with the exit problem. We end up analysing the influence of several parameters on our solutions to these problems.

Keywords: Real options, Hamilton-Jacobi-Bellman equations, Geometric Brownian motion, Exit problem, Investment problem, Changing market problem.

Resumo

Nesta tese estudamos 3 problemas diferentes que são comuns nas opções reais: O problema de saída, o problema de investimento e o problema de mudança de mercados. Assumimos que a procura do mercado é modelada por um Movimento Geométrico Browniano, e que a função lucro é polinomial. Usando a equação de Hamilton-Jacobi-Bellman resolvemos o problema de saída. Facilmente se estuda o problema de investimento e o problema de mudança de mercados uma vez que estão relacionados com o problema de saída. Por fim analisamos a influência de vários parâmetros nas soluções dos nosso problemas.

Palavras chave: Opções reais, Equação de Hamilton-Jacobi-Bellman, Movimento Geométrico Browniano, Problema de saída, Problema de investimento, Problema de mudança de mercados .

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Chapter 1

Introduction

1.1 Introduction

A common problem in mathematics is to determine the optimal conditions for undertaking a particular action in order to maximize or minimize a certain function [1]. This type of problem arises naturally in many situations in which a timing decision needs to be made, as illustrated by the popular secretary problem [2], or for example, in the domain of real options, for deciding when to buy or sell stock [3].
In this work we address three profit maximising problems that are often studied in relation to real options. First, we determine optimal conditions under which a firm should permanently exit a market. We then look at optimal conditions again, but for irreversible investment decisions. Finally, our main goal is to establish the optimal conditions for a firm to switch irreversibly from one market characterized by the profit function Π_1 to another one characterized by the profit function Π_2.
These problems generally pose difficulties due to the randomness of markets. Such difficulties can nevertheless be readily overcome by using the mathematical tools of stochastic control and optimal stopping. Solutions arising from using these tools in turn enable us to carry out comparative statics, thus framing our problems in a more financial perspective.

The work is organised as follows: We start defining what an exit problem, an investment problem and a changing market problem are, and present some general assumptions. In Chapter 2 we use a linear profit function and find an optimal stopping strategy to the exit problem and investment problem. Later we find an optimal stopping strategy for a firm that has the option to switch from one market characterized by a linear profit function to another market characterized by a different linear profit function. To conclude the chapter we discuss the influence of several parameters in the value function. In Chapter 3, since in many financial problems the profit functions are monomial, we find an optimal stopping strategy for a firm that has the option to switch from one market characterized by a monomial profit function to a market characterized by a linear profit function and, conversely, from linear to monomial. Finally, as in Chapter 2, we study the influence of the drift and volatility on the value function.

1.2 Model setup

In this section we present the model that we consider in this work. We consider a firm which produces an established product in a stochastic environment, which is characterized by the demand process $X = \{X_t : t \geq 0\}$, defined on a complete filtered space $(\Omega, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$. Moreover, we assume that τ is a $\mathcal{F}_t$-stopping time, i.e. $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. We denote by S the set of all $\mathcal{F}_t$-stopping times. On this work X is a geometric Brownian motion, solution of the stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

with $X_0 = x$, drift $\mu \in \mathbb{R}$, volatility $\sigma \geq 0$, and $\{W_t : t \geq 0\}$, being a Winner process.

On this thesis the firm's profit function is given by $\Pi :]0, \infty[\to \mathbb{R}$, which depends only on the demand level of the process X.
We study three different problems that are common in the real options: the "exit problem", the "investment problem" and the "the changing market problem".

- *Exit problem:*

  The firm has the possibility to completely abandon the production. When the firm decides to exit the market at time τ and the current demand is x, its value is given by

  $$^{1}J(x,\tau) = E_x\left[\int_0^{\tau} e^{-\gamma s}\Pi(X_s)ds + e^{-\gamma\tau}C\right]. \tag{1.1}$$

  where γ is a positive interest rate and $C \in \mathbb{R}$ is the terminal cost or profit. The function $\mathcal{V} :]0, \infty[\to [0, \infty[$, henceforward called value function, is defined as

  $$\mathcal{V}(x) = \sup_{\tau \in S} J(x, \tau) \tag{1.2}$$

  Thus we have an optimal stopping problem, where the main goal is to maximize the expected total pay-off.

- *Investment problem:*

  The firm has the possibility of choosing the moment of entry in the market. In the literature the problem is often called the "entry problem" [4]. If the firm decides to enter in the market at time τ, and the current demand is x, its value is given by

  $$\tilde{J}(x,\tau) = E_x\left[\int_{\tau}^{\infty} e^{-\gamma s}\Pi(X_s)ds + e^{-\gamma\tau}K\right] \tag{1.3}$$

[1] From now on we use the short notation: $E_x[\star] = E[\star | X_0 = x]$.

where $K \leq 0$ represents, in this case, an entry cost. Therefore we want to find an optimal strategy for a firm that decides to enter in the market. The firm's value is given by the function $\tilde{\mathcal{V}} :]0, \infty[\to [0, \infty[$, which is defined by

$$\tilde{\mathcal{V}}(x) = \sup_{\tau \in S} \tilde{J}(x, \tau) \tag{1.4}$$

Therefore we have an optimal stopping problem, where the main goal is to maximize the expected total pay-off.

- *Changing market problem:*

  The firm has the option to switch from a market characterized by the profit function Π_1 to another market characterized by the profit function Π_2. We assume that both markets are modelled by a geometric Brownian motion with the same parameters. In this case, the firm's value, when the current level of demand is x, is given by

$$I(x, \tau) = E_x \left[\int_0^\tau e^{-\gamma s} \Pi_1(X_s) ds + \int_\tau^\infty e^{-\gamma s} \Pi_2(X_s) ds + e^{-\gamma \tau} Q \right] \tag{1.5}$$

  where $Q \in \mathbb{R}$ is the switching cost or profit from one market to the other. Therefore, we want to know when it is the right moment τ to change from market 1 to market 2. The value function in this case is given by

$$\mathcal{G}(x) = \sup_{\tau \in S} I(x, \tau)$$

  Again we have an optimal stopping problem, where the main goal is to maximize the expected total pay-off.

  Without loss of generality we will assume $Q = 0$. This is because

$$I(x, \tau) = E_x \left[\int_0^\tau e^{-\gamma s} \Pi_1(X_s) ds + \int_\tau^\infty e^{-\gamma s} (\Pi_2(X_s) + \gamma Q) ds \right].$$

1.3 Hamilton-Jacobi-Bellman equations

In this section, we introduce the main mathematical tools that are used along the thesis. We do not intend to do an exhaustive study, as it is not the propose of this section. For a detailed and comprehensive presentation of the dynamics programming principle and Hamilton-Jacobi-Bellman equations, we refer, for instance [5, 6, 7].
Some notions of stochastic optimization and diffusion processes as well knowledge of Ito's Lemma and options analysis, are needed for what follows. In the upcoming chapters we will see that the investment problem and changing market problem are related with the exit problem. Therefore for the moment we solve the exit problem. The exit problem defined in (1.1) can be written as follows

$$J(x, \tau^\star) = \mathcal{V}(x)$$

where $\tau^\star \in S$. In an optimal stopping problem we want to find an optimal action, continue or stop, for each state. Consequently, our state space is split into two regions: a continuation region, which we denote by $\mathcal{C}$, and a stopping region, denoted by $\mathcal{D}$. As expected, in the continuation region the optimal action is to continue. Hence, for our case, an optimal stopping time should be $\tau^\star = inf\{t \geq 0 : X_t \notin \mathcal{C}\}$. So, it should be intuitive that
$\mathcal{C} = \{x \in \mathbb{R}^+ : \mathcal{V}(x) \geq C\}$ and $\mathcal{D} = \{x \in \mathbb{R}^+ : \mathcal{V}(x) \leq C\}$.
One way to address the optimal stopping problem is to solve the following variational inequalities:

$$\max\{-\gamma\mathcal{V}(x) + \mathcal{L}(\mathcal{V}(x)) + \Pi(x), C - \mathcal{V}(x)\} = 0 \tag{1.6}$$

where $\mathcal{L}$ is the infinitesimal generator defined as follows:

$$\mathcal{L}(f(x)) = \mu x f'(x) + \frac{1}{2}\sigma^2 x^2 f''(x) \tag{1.7}$$

for $f \in C^2(\mathbb{R}^+)$, assuming that the stochastic process X is a geometric Brownian motion (GBM) with drift μ and volatility σ.
This is the Hamilton-Jacobi-Bellman (HJB) equation that we need to solve in order to get a solution to our stopping problem. To solve the HJB equation, first we use some intuition from the problem and guess the form of the continuation region. Our continuation region will have a threshold denoted by $x^\star$. Then we solve the ordinary differential equation (ODE)

$$\Pi(x) + \mu x\mathcal{V}'(x) + \frac{1}{2}\sigma^2 x^2 \mathcal{V}''(x) - \gamma\mathcal{V}(x) = 0 \tag{1.8}$$

According to the literature [5, 6], the value function $\mathcal{V}(x)$ should be $C_1(]0, \infty[)$. To ensure this we use:

- The fit condition $\lim_{x \to x^\star} \mathcal{V}(x) = \mathcal{V}(x^\star)$.
- The smooth condition $\lim_{x \to x^\star} \mathcal{V}'(x) = \mathcal{V}'(x^\star)$.

We assume that

$$E_x\left[\int_0^{\infty} e^{-\gamma s}\Pi(X_s)ds\right] < \infty. \tag{1.9}$$

This assumption will be used during this thesis and guarantees that $J(x,\tau)$ is well defined, and finite for all x and τ.

To solve the ODE (1.8) we start by solving the homogeneous differential equation

$$-\gamma V(x) + \mu x V'(x) + \frac{1}{2}\sigma^2 x^2 V''(x) = 0 \tag{1.10}$$

This is a Cauchy-Euler equation of order 2. Making $x = e^u$ this ODE becomes

$$\begin{aligned}
&\mu e^u V'(e^u) + \frac{1}{2}\sigma^2 e^{2u} V''(e^u) - \gamma V(e^u) = 0 \Rightarrow \\
&\mu(V(e^u))' + \frac{1}{2}\sigma^2[(V(e^u))'' - (V(e^u))'] - \gamma V(e^u) = 0 \Rightarrow \\
&-\gamma V(e^u) + (\mu - \frac{1}{2}\sigma^2)(V(e^u))' + \frac{1}{2}\sigma^2(V(e^u))'' = 0
\end{aligned} \tag{1.11}$$

We have, therefore, reduced (1.10) to a linear second order ODE with constant coefficients. The characteristic polynomial of (1.11) is

$$\frac{1}{2}\sigma^2 t^2 + (\mu - \frac{1}{2}\sigma^2)t - \gamma = 0$$

The solutions of the characteristic polynomial are

$$t = \frac{-(\mu - \frac{1}{2}\sigma^2) \pm \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}}{\sigma^2}. \tag{1.12}$$

Let β_1 and β_2 be the roots of the characteristic polynomial (1.12)

$$\beta_1 = \frac{-(\mu - \frac{1}{2}\sigma^2) - \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}}{\sigma^2} < 0 \tag{1.13}$$

$$\beta_2 = \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}}{\sigma^2} > 1 \tag{1.14}$$

We conclude that $V(e^u) = A_1 e^{\beta_1 u} + A_2 e^{\beta_2 u}$. Reversing the change of variable, the solution of the homogeneous differential equation (1.10) is

$$V(x) = A_1 x^{\beta_1} + A_2 x^{\beta_2}. \tag{1.15}$$

Therefore the solution for the ODE (1.10) is $V_0(x) + A_1 x^{\beta_1} + A_2 x^{\beta_2}$ with $V_0(x)$ a particular solution of (1.10).

Chapter 2

Affine profit functions

In this section we study the problems previously described when the profit function is linear, that is, the profit function is of the form $ax - b$.

2.1 Exit problem

Here we solve the exit problem as described in (1.1) considering an affine profit function. Since the firm wants to maximize it is profit, then if currently the demand is x and if $ax - b \geq 0$, for all $x > 0$, then the firm should never exit the market and therefore the optimal strategy is to choose $\tau = \infty$. On the other hand, if $ax - b \leq 0$ for all x, then the firm should exit the market right away, and thus the optimal strategy is to choose $\tau = 0$. In order to avoid such trivial cases we choose $a \geq 0$ and $b \geq 0$ such that the function $ax - b$ takes both positive and negative values. With this choice the function $ax - b$ is increasing function with respect to the demand level x.

Simple calculations prove that we may rewrite the total expected pay-off functional as follows:

$$E_x \left[\int_0^\tau e^{-\gamma s}(aX_s - b)ds \right] = E_x \left[\int_0^\tau e^{-\gamma s} aX_s ds + (e^{-\gamma\tau}\frac{b}{\gamma}) \right] - \frac{b}{\gamma}$$

Therefore, $\mathcal{V}(x) = V(x) - \frac{b}{\gamma}$ with

$$V(x) := \sup_{\tau \in S} E_x \left[\int_0^\tau e^{-\gamma s} aX_s ds + e^{-\gamma\tau}\frac{b}{\gamma} \right]. \tag{2.1}$$

Thus we have an exit problem with profit function $\Pi(x) = ax$ and $C = \frac{b}{\gamma}$ the salvage value. In order to solve this optimization problem, we start by studying the corresponding HJB equation:

$$\max\left(ax + \mu x V'(x) + \frac{1}{2}\sigma^2 x^2 V''(x) - \gamma V(x), \frac{b}{\gamma} - V(x) \right) = 0. \tag{2.2}$$

As discussed in the previous chapter we assume that the condition (1.9) holds true. Indeed, if we are considering an affine profit function, condition (1.9) holds true if and only if $\gamma - \mu \geq 0$. In order to see this, we note that

$$E_x \left[\int_0^\infty e^{-\gamma s} a X_s ds \right] = \int_0^\infty e^{-\gamma s} E(aX_s) ds = \int_0^\infty axe^{(\mu-\gamma)s} ds$$

which is convergent if and only if $\gamma - \mu \geq 0$. In the previous calculation we used the formula for the expected value of GBM [5, 7].

In order to find an optimal stopping strategy we first guess the continuation region. Since our goal is to maximize profit, it seems reasonable that, when the initial demand is low, we stop and gain the value $\frac{b}{\gamma}$. Otherwise, if the demand is high, we remain earning $\Pi(x)$, until time τ. Therefore, the continuation region that we propose is

$$\mathcal{C} = \{x : x \geq x^\star\}.$$

Thus the value function $V(x)$ should be such that $V(x) = \frac{b}{\gamma}$ if $x < x^\star$, and if $x \geq x^\star$ then $V(x)$ is solution of

$$\mu x V^{'}(x) + \frac{1}{2}\sigma^2 x^2 V^{''}(x) - \gamma V(x) + ax = 0. \tag{2.3}$$

In order to solve (2.3), we simply note that the solution to the corresponding homogeneous equation is $A_1 x^{\beta_1} + A_2 x^{\beta_2}$, with β_1 and β_2 given by (1.13) and (1.14) respectively. A particular solution of (2.3) is

$$V_0(x) = \alpha x + \beta.$$

Thus $A_1 x^{\beta_1} + A_2 x^{\beta_2} + \alpha x + \beta$ is solution to (2.3) if and only if α and β are such that:

$$\mu x \alpha + 0 - \gamma(\alpha x + \beta) + ax = 0$$

which implies that $\alpha = \frac{a}{\gamma - \mu}$ and $\beta = 0$.

Consequently all the functions of the form

$$V(x) = \frac{a}{\gamma - \mu} x + A_1 x^{\beta_1} + A_2 x^{\beta_2}$$

are solutions of the ODE (2.3), where A_1 and A_2 are constants that depend on boundary values. However when the demand is high we do not expect to exercise the exit option. Thus, $\lim_{x\to\infty} \frac{V(x)}{V_0(x)} < \infty$. This holds true if and only if $A_2 = 0$ because $\beta_2 \geq 0$ as we can see in the appendix. Therefore

$$V(x) = \frac{a}{\gamma - \mu} x + A_1 x^{\beta_1}. \tag{2.4}$$

As we expect to have, if $x < x^\star$ then

$$V(x) = \frac{b}{\gamma}. \tag{2.5}$$

We now wish to paste (2.4) and (2.5) together in such a way that the resulting function is the value function.
To obtain the value function all that is left, is to determine values for A_1 and $x^\star$. The fit and smooth conditions ensure that A_1 and $x^\star$ are unique. By the fit condition

$$\frac{b}{\gamma} = \frac{a}{\gamma-\mu}x^\star + A_1 x^{\star\beta_1} \Rightarrow A_1 = \frac{\frac{b}{\gamma} - \frac{a}{\gamma-\mu}x^\star}{x^{\star\beta_1}}. \tag{2.6}$$

By the smooth condition

$$\begin{aligned}
(\frac{b}{\gamma})' &= (\frac{a}{\gamma-\mu}x^\star + A_1 x^{\star\beta_1})' \\
\Leftrightarrow 0 &= \frac{a}{\gamma-\mu} + A_1\beta_1 x^{\star\beta-1} \\
\Leftrightarrow 0 &= \frac{a}{\gamma-\mu} + \beta_1(\frac{b}{\gamma x^\star} - \frac{a}{\gamma-\mu}) \\
\Leftrightarrow x^\star &= \frac{b}{a}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right) \qquad (2.7)
\end{aligned}$$

Since $A_1 = \frac{\frac{b}{\gamma} - \frac{a}{\gamma-\mu}x^\star}{x^{\star\beta_1}}$ then, replacing $x^\star$ by $\frac{b}{a}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right)$, we have

$$A_1 = \frac{\frac{b}{\gamma}\frac{-1}{(\beta_1-1)}}{(x^\star)^{\beta_1}} \geq 0.$$

Additionally, $x^\star \leq \frac{b}{a}$, meaning that exit is optimal for values of demand where the profit flow $ax - b$ is negative. To prove $x^\star \leq \frac{b}{a}$, consider the ODE (2.3) at $x = x^\star$:

$$-\gamma V(x^\star) + \underbrace{\mu x^\star V'(x^\star)}_{=0} + \frac{1}{2}\sigma^2 x^{\star 2} V''(x^\star) + ax^\star = 0 \Leftrightarrow$$

$$ax^\star - b = -\frac{1}{2}\sigma^2 x^{\star 2} V''(x^\star) \tag{2.8}$$

To prove that $\frac{1}{2}\sigma^2 x^{\star 2} V''(x^\star) \geq 0$ note that

$$0 = V'(x^\star) = \frac{a}{\gamma-\mu} + A_1\beta_1 x^{\star\beta_1-1} \Rightarrow A_1\beta_1 x^{\star\beta_1-1} = -\frac{a}{\gamma-\mu}$$

therefore

$$V''(x^\star) = A_1\beta_1(\beta_1-1)x^{\star\beta_1-2} = A_1\beta_1 x^{\star\beta_1-1}\frac{\beta_1-1}{x^\star} = -\frac{a}{\gamma-\mu}\frac{\beta_1-1}{x^\star}$$

Since $a \geq 0$ and $\beta_1 \leq 0$ then $V''(x^\star) = -\frac{a}{\gamma-\mu}\frac{\beta_1-1}{x^\star} \geq 0$, and in view of (2.8), it holds that $x^\star \leq \frac{b}{a}$.

In the next proposition, we provide the value of the firm

Proposition. *The solution of the optimal stopping problem defined in (**??**) is given by:*

$$\mathcal{V}(x) = \begin{cases} 0 & : x < x^\star \\ \frac{ax}{\gamma-\mu} - \frac{b}{\gamma} + A_1 x^{\beta_1} & : x \geq x^\star \end{cases} \tag{2.9}$$

where $A_1 \geq 0$ is given by (2.6) and $x^\star \leq \frac{b}{a}$, is given by (2.7).

Proof. By construction, $V(x)$ is continuous in $\mathbb{R}^+$ and with continuous derivative. As $\mathcal{V}(x) = V(x) - \frac{b}{\gamma}$, $\forall x > 0$, we need to prove only that $V(x)$ is solution of the HJB equation (2.2). In order to prove that we follow the following steps

- $ax + \mu x V'(x) + \frac{1}{2}\sigma^2 x^2 V''(x) - \gamma V(x) \leq 0$

  for $x \leq x^\star$, with $V(x)$ given by: $V(x) = \frac{b}{\gamma}$.

  To see this note that

  $$ax + \mu x V'(x) + \frac{1}{2}\sigma^2 x^2 V''(x) - \gamma V(x) = ax - b$$

  As $x \leq x^\star \leq \frac{b}{a}$ then $ax - b \leq 0$ for $x \leq x^\star$. The result holds.

- $\frac{b}{\gamma} - V(x) \leq 0$ for $x \geq x^\star$

  For $x = x^\star$, by the fit condition it follows that $\frac{b}{\gamma} - \frac{a}{\gamma-\mu} x^\star - A_1 x^{\star\beta_1} = 0$, and thus the result holds for $x = x^\star$. The function $\left(\frac{b}{\gamma} - V(x)\right)$ is decreasing for $x \geq x^\star$. In fact, by the smooth condition, the derivative computed at $x = x^\star$ is equal to 0. As $-A_1\beta_1 \geq 0$ and $\beta_1 - 1 \leq 0$, then it follows that $\left(\frac{b}{\gamma} - V(x)\right)' \leq 0$ for $x \geq x^\star$.

  We conclude that $\left(\frac{b}{\gamma} - V(x)\right) \leq 0$ for $x \geq x^\star$.

□

With this analysis we are also studying the situation of a monomial case. As we can see in Guerra, Nunes and Oliveira [8] that the exit problem with $\Pi(X_s) = aX_s^\theta - b$ can be reduced to this problem. To see this we use Ito's Lemma and conclude that X_s^θ is a GBM with volatility $\sigma_\theta = \sigma\theta$ and drift $\mu_\theta = \mu\theta + \frac{1}{2}\sigma^2\theta(\theta-1)$.

2.1.1 Comparative statics for the exit problem

In this section we study the influence of the market expectations, μ and σ, and the parameters a and b on the decision to exit. Simple calculations show that:

$$\begin{aligned}\frac{\partial x^\star}{\partial \sigma} &= \frac{b}{a}\frac{\gamma-\mu}{\gamma}\frac{-\frac{\partial \beta_1}{\partial \sigma}}{(\beta_1-1)^2} \leq 0\\ \frac{\partial x^\star}{\partial \mu} &= \frac{b}{a}\frac{1}{\gamma}\frac{\frac{\partial \beta_1}{\partial \mu}}{(\beta_1-1)^2} \leq 0\end{aligned} \tag{2.10}$$

where in the first inequality we have used the fact that $\frac{\partial \beta_1}{\partial \sigma} \geq 0$ and in the second that $\frac{\partial \beta_1}{\partial \mu} \leq 0$ (see appendix for such proofs). Therefore if the drift or the volatility of the uncertainty process X increases, the decision to exit the market is postponed.

Regarding the behaviour of $x^\star$ with the parameters of the gain function, we have

$$\begin{aligned}\frac{\partial x^\star}{\partial b} &= \frac{1}{a}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right) \geq 0\\ \frac{\partial x^\star}{\partial a} &= -\frac{b}{a^2}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right) \leq 0\end{aligned}$$

it follows that $x^\star$ increases with b (i.e. the decision to exit the market is postponed if the profit flow shifts upward) and decreases with the slope.

2.2 Investment problem

In the previous sections we have addressed the problem of exiting the market, and here we solve the investment problem as described in (1.3) considering an affine profit function. After some calculations we prove that we may rewrite the functional $\tilde{J}(x,\tau)$ as follows:

$$\begin{aligned}\tilde{J}(x,\tau) =& E_x\left[\int_\tau^\infty e^{-\gamma s}(aX_s-b)ds\right]\\ =& \frac{ax}{\gamma-\mu} + E_x\left[\int_0^\tau e^{-\gamma s}(-a)X_s ds + e^{-\gamma\tau}\frac{(-b)}{\gamma}\right]\end{aligned} \tag{2.11}$$

therefore $\tilde{\mathcal{V}}(x) = \frac{ax}{\gamma-\mu} + V(x)$ where

$$V(x) := \sup_{\tau\in S} E_x\left[\int_0^\tau e^{-\gamma s}(-a)X_s ds + e^{-\gamma\tau}\frac{(-b)}{\gamma}\right]. \tag{2.12}$$

Similarly to the exit problem, we need to assume $a \geq 0$ and $b \geq 0$ in order to avoid trivial problems (i.e. problems where (2.11) is maximized with $\tau = 0$ or with $\tau = \infty$). Comparing

(2.12) with (2.1), we conclude that the problem of investment is similar to the exit problem, where only the signs of a and b need to be changed. But this remark is misleading, as we cannot use the solution of the exit problem derived in the previous section because the profit function $ax - b$, for the investment case, is no longer an increasing function but it is a decreasing function.
Some of the results presented for the exit problem are still valid in the investment problem. Notably, the HJB equation is similar, just replacing a and b by $-a$ and $-b$ we have

$$\max\{\mu x V^{'}(x) + \frac{1}{2}\sigma^2 x^2 V^{''}(x) - \gamma V(x) - ax, -\frac{b}{\gamma} - V(x)\} = 0 \tag{2.13}$$

and also the homogeneous and non-homogeneous solutions, which now is:

$$V(x) = -\frac{a}{\gamma - \mu}x + A_1 x^{\beta_1} + A_2 x^{\beta_2}$$

where A_1 and A_2 are still to be derived. The major difference are precisely the conditions that we use to derive A_1 and A_2. Here, for the investment decision, we need to guess another continuation and stopping regions. Intuitively, to maximize the expected profit one invest when the initial demand is low, and one exits the market when the initial demand is above a certain level $x^\star$. Thus, the continuation region is $\mathcal{C} = \{x : x \leq x^\star\}$. Therefore in the stopping region, the value of the firm is $-\frac{b}{\gamma}$ and in the continuation region $V(x)$ is solution of the ODE:

$$\mu x V^{'}(x) + \frac{1}{2}\sigma^2 x^2 V^{''}(x) - \gamma V(x) + (-a)x = 0 \tag{2.14}$$

so our guess is that the solution for the value of the firm is:

$$V(x) = \begin{cases} -\frac{a}{\gamma-\mu}x + A_1 x^{\beta_1} + A_2 x^{\beta_2} & : x < x^\star \\ -\frac{b}{\gamma} & : x \geq x^\star \end{cases} \tag{2.15}$$

Recalling that $\beta_1 \leq 0$, and since $V(0) = 0$ then $A_1 = 0$. Furthermore, using a similar approach to the one used for the exit problem, the fit and smooth pasting conditions imply that:

$$A_2 = \frac{-\frac{b}{\gamma} + \frac{a}{\gamma-\mu}x^\star}{x^{\star\beta_2}} \tag{2.16}$$

$$x^\star = \frac{b}{a}\left(\frac{\gamma - \mu}{\gamma}\frac{\beta_2}{\beta_2 - 1}\right) \tag{2.17}$$

Similarly to the exit problem, $x^\star$ is positive as $\beta_2 > 1$. To prove $A_2 \geq 0$ we use the expression for $x^\star$ in the numerator of A_2, obtaining

$$A_2 = \frac{\frac{b}{\gamma}\frac{1}{(\beta_2 - 1)}}{x^{\star\beta_2}} \geq 0.$$

Additionally, $x^\star \geq \frac{b}{a}$, meaning that to invest is optimal for values of demand where the profit flow $ax - b$ is positive. To see this, consider (2.14) at $x = x^\star$:

$$-\gamma V(x^\star) + \mu x V'(x^\star) + \frac{1}{2}\sigma^2 x^2 V''(x^\star) + (-a)x^\star = 0$$

$$ax^\star - b = \frac{1}{2}\sigma^2 x^{\star 2} V''(x^\star) \tag{2.18}$$

To prove that $\frac{1}{2}\sigma^2 x^{\star 2} V''(x^\star) \geq 0$ note that

$$0 = V'(x^\star) = -\frac{a}{\gamma - \mu} + A_2\beta_2 x^{\star\beta_2 - 1} \Rightarrow A_2\beta_2 x^{\star\beta_2-1} = \frac{a}{\gamma - \mu}$$

therefore

$$V''(x^\star) = A_2\beta_2(\beta_2 - 1)x^{\star\beta_2-2} = A_2\beta_2 x^{\star\beta_2-1}\frac{\beta_2 - 1}{x^\star} = \frac{a}{\gamma-\mu}\frac{\beta_2 - 1}{x^\star}$$

Since $a \geq 0$ then $V''(x^\star) = \frac{a}{\gamma-\mu}\frac{\beta_2-1}{x^\star} \geq 0$, and, in view of (2.18), it holds that $x^\star \geq \frac{b}{a}$.

In the next proposition, we provide the value of the firm

Proposition. *$\tilde{\mathcal{V}}(x)$ is given by:*

$$\tilde{\mathcal{V}}(x) = \begin{cases} A_2 x^{\beta_2} & : x < x^\star \\ \frac{ax}{\gamma-\mu} - \frac{b}{\gamma} & : x \geq x^\star \end{cases}$$

where $A_2 \geq 0$ and $x^\star \geq \frac{b}{a}$ are given by (2.16) and (2.17) respectively.

Proof. By construction, $V(x)$ is continuous in $\mathbb{R}^+$ and with continuous derivative. As $\tilde{\mathcal{V}}(x) = V(x) + \frac{ax}{\gamma-\mu}$, $\forall x > 0$, we need to prove only that $V(x)$ is solution of the HJB equation (2.13). We omit the proof that $V(x)$ is the solution of the HJB equation (2.13), because it is analogous to the exit problem. □

2.2.1 Comparative statics for the investment problem

As for the exit problem for the investment problem we study the influence of the market expectations, μ and σ, and the parameters a and b on the decision to invest. Simple calculations show that:

$$\frac{\partial x^\star}{\partial \sigma} = \frac{b}{a}\frac{\gamma - \mu}{\gamma}\frac{-\frac{\partial\beta_2}{\partial\sigma}}{(\beta_2 - 1)^2} \geq 0$$

$$\frac{\partial x^\star}{\partial \mu} = \frac{b}{a}\frac{1}{\gamma}\frac{\frac{\partial\beta_2}{\partial\mu}}{(\beta_2 - 1)^2} \leq 0 \tag{2.19}$$

where in the first inequality we have used the fact $\frac{\partial\beta_2}{\partial\sigma} \leq 0$ and in the second that $\frac{\partial\beta_2}{\partial\mu} \leq 0$ (see appendix for such proofs). Therefore if σ increases, the decision to invest is anticipated whereas if μ increases, it is postponed.

Regarding the behaviour of $x^\star$ with the parameters of the profit function, we have

$$\frac{\partial x^\star}{\partial b} = \frac{1}{a}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_2}{\beta_2-1}\right) \geq 0$$
$$\frac{\partial x^\star}{\partial a} = -\frac{b}{a^2}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_2}{\beta_2-1}\right) \leq 0$$

it follows that $x^\star$ increases with b (i.e. the decision to invest in the market is postponed if the profit flow shifts upward) and decreases with the slope.

2.3 Changing market problem

We now consider that the firm has the option to switch from one market characterized by the profit function $\Pi_1(x) = a_1 x - b_1$, to another one characterized by the profit function $\Pi_2(x) = a_2 x - b_2$ at time τ. We are assuming that $a_1, a_2 \geq 0$, for the same reasons as before. Simple calculations prove that we may rewrite the functional $I(x,\tau)$ as follows:

$$I(x,\tau) = E_x\left[\int_0^\tau e^{-\gamma s}(a_1 X_s - b_1)ds + \int_\tau^\infty e^{-\gamma s}(a_2 X_s - b_2)ds\right] \tag{2.20}$$

$$= E_x\left[\int_0^\tau e^{-\gamma s}(\tilde{a} X_s - \tilde{b})ds\right] + \underbrace{E_x\left[\int_0^\infty e^{-\gamma s}(a_2 X_s - b_2)ds\right]}_{=\frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma}} \tag{2.21}$$

with $\tilde{a} = (a_1 - a_2)$ and $\tilde{b} = (b_1 - b_2)$.

We will say that Π_1 is more risky than Π_2, if $a_1 \geq a_2$. Moreover, if $b_2 \geq b_1$ then $\Pi_1(x) \geq \Pi_2(x)$, and consequently it is never optimal to switch i.e.($\tau = \infty$). In order to avoid such trivial case we choose $b_1 \geq b_2$. Thus $\tilde{a} = (a_1 - a_2) \geq 0$ and $\tilde{b} = (b_1 - b_2) \geq 0$.
This problem can be solved using the results derived in the exit problem, with a and b given by $\tilde{a}$ and $\tilde{b}$. Thus the value function for such case is given by:

$$\mathcal{G}(x) = \begin{cases} \frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma} + 0 & : x < x^\star \\ \frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma} + \frac{\tilde{a}}{\gamma-\mu}x - \frac{\tilde{b}}{\gamma} + A_1 x^{\beta_1} = \frac{a_1}{\gamma-\mu}x - \frac{b_1}{\gamma} + A_1 x^{\beta_1} & : x \geq x^\star \end{cases}$$

where $A_1 = \frac{\frac{\tilde{b}}{\gamma} - \frac{\tilde{a}}{\gamma-\mu}x^\star}{x^{\star\beta_1}} \geq 0$ and $x^\star = \frac{\tilde{b}}{\tilde{a}}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_1}{\beta_1-1}\right) \leq \frac{\tilde{b}}{\tilde{a}} = c$. Note that $c = \frac{\tilde{b}}{\tilde{a}}$ is precisely the point where Π_1 and Π_2 intersect, therefore the value x where the firm should optimally change from the first market to the second one is smaller or equal to the point where both markets are equally profitable.
The other relevant case occurs when $a_1 \leq a_2$, in that case we say that Π_2 is more risky than Π_1. Furthermore, if $b_2 \leq b_1$ then $\Pi_1(x) \leq \Pi_2(x)$, therefore we switch immediately, i.e.$\tau = 0$. In order to avoid such trivial case we choose $b_2 \geq b_1$. Thus $\tilde{a} = (a_1 - a_2) \leq 0$ and $\tilde{b} = (b_1 - b_2) \leq 0$.
This problem can be solved using the results derived in the investment problem, with a and b given by $\tilde{a}$ and $\tilde{b}$. The value function for such case is given by:

$$\mathcal{G}(x) = \begin{cases} A_2 x^{\beta_2} + \frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma} & : x < x^\star \\ \frac{a_1}{\gamma-\mu}x - \frac{b_1}{\gamma} & : x \geq x^\star \end{cases}$$

where $x^\star = \frac{\tilde{b}}{\tilde{a}}\left(\frac{\gamma-\mu}{\gamma}\frac{\beta_2}{\beta_2-1}\right) \geq \frac{\tilde{b}}{\tilde{a}} = c$ and $A_2 = \frac{\frac{\tilde{b}}{\gamma} - \frac{\tilde{a}}{\gamma-\mu}x^\star}{x^{\star\beta_2}} \geq 0$. Therefore the value x where the firm should optimally change from the first market to the second one is larger or equal to the point where both markets are equally profitable.

When one faces the decision to change from a first market to a second market, besides the influence of $\tilde{a} = a_1 - a_2$ and $\tilde{b} = b_1 - b_2$ (which follows the same pattern as the dependency of $x^\star$ with respect to a and b in the exit and investment problems), it is also interesting to study how $x^\star$ varies when we rotate the function Π_1 around Π_2, keeping the intersection point fixed.

The point where Π_1 and Π_2 intersects is $c = \frac{b_1 - b_1}{a_1 - a_2} = \frac{\tilde{b}}{\tilde{a}}$. Then since $x^\star = \frac{\tilde{b}}{\tilde{a}} \left(\frac{\gamma - \mu}{\gamma} \frac{\beta_1}{\beta_1 - 1} \right) = Kc$, where $K = \frac{\gamma - \mu}{\gamma} \frac{\beta_1}{\beta_1 - 1}$, we conclude that $x^\star$ remains the same when we rotate the function Π_1 around Π_2, keeping the intersection point fixed.

Chapter 3

Polynomial profit functions

In this section we solve the exit and investment problems for a class of profit functions of the form $a_1x^\theta - a_2x - b$. Mathematically, this kind of functions increase the difficulty of the problem because sometimes we cannot obtain explicitly the threshold $x^\star$.
Financially, this kind of function allow us to consider more realistic scenarios as well as changes of the market with different types of profit functions. We consider $a_1, a_2 \geq 0$, and to avoid $a_1x^\theta - a_2x - b$ changing sign more than once, thus giving us a disconnected continuation region, or $a_1x^\theta - a_2x - b \gtreqless 0$, thus giving trivial solutions to our problem, we consider $b \geq 0$ and $\theta \geq 1$.

3.1 Exit problem

As for the exit problem with affine profit function, simple calculations prove that we may rewrite the total expected pay-off functional as follows:

$$E_x\left[\int_0^\tau e^{-\gamma s}\left(a_1X_s^\theta - a_2X_s - b\right)ds\right] = E_x\left[\int_0^\tau e^{-\gamma s}(a_1{X_s}^\theta - a_2X_s)ds + e^{-\gamma\tau}\frac{b}{\gamma}\right] - \frac{b}{\gamma}$$

Therefore $\mathcal{V}(x) = V(x) - \frac{b}{\gamma}$, with

$$V(x) := \sup_{\tau\in S} E_x\left[\int_0^\tau e^{-\gamma s}(a_1{X_s}^\theta - a_2X_s)ds + e^{-\gamma\tau}\frac{b}{\gamma}\right]. \tag{3.1}$$

Then we may rewrite (3.1) as an exit problem with profit function $\Pi(x) = a_1x^\theta - a_2x$ and salvage value $C = \frac{b}{\gamma}$. In order to solve the optimization problem, we start by studying the corresponding HJB equation:

$$\max\{\mu xV'(x) + \frac{1}{2}\sigma^2x^2V''(x) - \gamma V(x) + a_1x^\theta - a_2x, \frac{b}{\gamma} - V(x)\} = 0 \tag{3.2}$$

As discussed in section (1.3) we assume that $E_x\left[\int_0^\infty e^{-\gamma s}\Pi(X_s)ds\right] < \infty$. The next result guarantees that $J(x,\tau)$ is well defined, and finite for all x and τ.

Proposition. *The condition* $E_x\left[\int_0^\infty e^{-\gamma s}(a_1X_s^\theta - a_2X_s)ds\right] < \infty$ *holds true if and only if* $\theta \in [1, \beta_2[$.

Proof. Using Fubbini's theorem we have

$$E_x\left[\int_0^\infty e^{-\gamma s}(a_1X_s^\theta - a_2X_s)ds\right] = E_x\left[\int_0^\infty e^{-\gamma s}a_1X_s^\theta ds\right] - E_x\left[\int_0^\infty e^{-\gamma s}a_2X_s ds\right]$$
$$= \int_0^\infty e^{-\gamma s}E_x(a_1X_s^\theta)ds - \int_0^\infty e^{-\gamma s}E_x(a_2X_s)ds$$

Using Ito's Lemma, $E_x(X_s^\theta) = x^\theta e^{(\mu_\theta-\gamma)s}$ where $\mu_\theta = \mu\theta + \frac{1}{2}\sigma^2\theta(\theta-1)$.
Therefore $E_x[\int_0^\infty e^{-\gamma s}\Pi(X_s)ds]$ is finite as long as $\gamma - \mu > 0$ and $\gamma - \mu_\theta > 0 \Leftrightarrow -\frac{1}{2}\sigma^2\theta^2 + (\frac{1}{2}\sigma^2 - \mu)\theta + \gamma \geq 0$, which is equivalent to have $\theta \in]\beta_1, \beta_2[$. For $\theta \in]\beta_1, \beta_2[$, if $\mu \geq 0$, then $\gamma - \mu \geq \gamma - \mu_\theta \geq 0$. And if $\mu \leq 0$, then $\gamma - \mu \geq 0$ and $\gamma - \mu_\theta \geq 0$. Since $\beta_1 \leq 0 \leq 1$ then $E_x[\int_0^\infty e^{-\gamma s}\Pi(X_s)ds]$ is finite as long $\theta \in [1, \beta_2[$. □

As above, for the linear problems, we first guess the continuation region.
Since $\Pi(x) = a_1x^\theta - a_2x \leq 0$ for $0 \leq x \leq \sqrt[\theta-1]{\frac{a_2}{a_1}}$, then when the initial demand is low, we exit the market and gain the salvage cost $\frac{b}{\gamma}$. Otherwise, if the demand is high, we remain in production, earning $\Pi(x)$ per unit time, until time τ. Therefore, the continuation region for $V(x)$ is $\mathcal{C} = \{x : x \geq x^\star\}$. We guess that if $x \leq x^\star$ then $V(x) = \frac{b}{\gamma}$ and if $x \geq x^\star$ then $V(x)$ is solution of

$$\mu x V'(x) + \frac{1}{2}\sigma^2x^2V''(x) - \gamma V(x) + a_1x^\theta - a_2x = 0 \tag{3.3}$$

Equation (3.3) is similar to equation (2.3). In particular the homogeneous part is the same. A particular solution to (3.3) is $V_0(x) = \alpha x^\theta - \beta x$, therefore

$$V(x) = \alpha x^\theta - \beta x + A_1x^{\beta_1} + A_2x^{\beta_2}$$

is solution to (3.3), with

$$\alpha = \frac{a_1}{\gamma - \mu\theta - \frac{1}{2}\sigma^2\theta(\theta-1)}$$
$$\beta = \frac{a_2}{\gamma - \mu}$$

Therefore we guess that if $x > x^\star$

$$V(x) = \frac{a_1x^\theta}{\gamma - \mu_\theta} - \frac{a_2x}{\gamma - \mu} + A_1x^{\beta_1} + A_2x^{\beta_2}.$$

With an argument such as the one used in section (2), we can show that $A_2 = 0$.

So, we propose $V(x)$ defined by

$$V(x) = \begin{cases} \frac{b}{\gamma} & : x < x^\star \\ \frac{a_1 x^\theta}{\gamma-\mu_\theta} - \frac{a_2 x}{\gamma-\mu} + A_1 x^{\beta_1} & : x \geq x^\star \end{cases} \tag{3.4}$$

We now determine values for A_1 and $x^\star$, as done before for the linear problems. By the fit condition

$$A_1 = \frac{\frac{a_2 x^\star}{\gamma-\mu} - \frac{a_1 (x^\star)^\theta}{\gamma-\mu_\theta} + \frac{b}{\gamma}}{(x^\star)^{\beta_1}} \tag{3.5}$$

and by the smooth condition

$$\frac{a_1\theta(x^\star)^{\theta-1}}{\gamma-\mu_\theta} - \frac{a_2}{\gamma-\mu} + \beta_1 A_1 x^{\star\beta_1-1} = 0 \Leftrightarrow \frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2(1-\beta_1)}{\gamma-\mu}x^\star + \frac{\beta_1 b}{\gamma} = 0$$

The value of $x^\star$ is a zero of the polynomial

$$f(x) := \frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}(x)^\theta - \frac{a_2(1-\beta_1)}{\gamma-\mu}x + \frac{\beta_1 b}{\gamma}. \tag{3.6}$$

Unfortunately, by Abel Ruffini theorem, $f(x)$ does not have an algebraic solution for all $\theta \in (1, \beta_2)$. Even though we cannot find explicitly the threshold $x^\star$ we can still find lower and upper bounds for $x^\star$, as we will see next.

Also, we are able to prove that there is a unique positive solution $x^\star$ to the equation $f(x) = 0$. Since the derivative of $f(x)$ is

$$f'(x) = \frac{a_1\theta(\theta-\beta_1)}{\gamma-\mu_\theta}(x)^{\theta-1} - \frac{a_2(1-\beta_1)}{\gamma-\mu}$$

then $f'(x) \geq 0$ for $x \in [x_1, \infty[$, where x_1 is the zero of $f'(x)$. Furthermore, as $f(0) = \frac{\beta_1 b}{\gamma} \leq 0$ and $\lim_{x\to\infty} f(x) = \infty$, we conclude there is an unique positive solution to the equation $f(x) = 0$, which we denote by $x^\star$.

We now show that $y^\star := \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}}$ is a lower bound of $x^\star$.

$$\begin{aligned} f(y^\star) &= \frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}(y^\star)^\theta - \frac{a_2(1-\beta_1)}{\gamma-\mu}y^\star + \frac{\beta_1 b}{\gamma} \\ &= y^\star\left(\frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}(y^\star)^{\theta-1} - \frac{a_2(1-\beta_1)}{\gamma-\mu}\right) + \frac{\beta_1 b}{\gamma} \\ &= \beta_1\left(\sqrt[\theta-1]{\frac{a_2}{a_1}\frac{\gamma-\mu_\theta}{\gamma-\mu}}\frac{a_2}{\gamma-\mu}(1-\frac{1}{\theta}) + \frac{b}{\gamma}\right) \end{aligned}$$

Given that $\beta_1 \leq 0$ and $(1 - \frac{1}{\theta}) \geq 0$, then $f(y^\star) \leq 0$. As $f(x) \leq 0$ for $x \in (0, x^\star)$ then $x^\star \geq y^\star = \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}}$.

To find an upper bound of $x^\star$, first we consider c such that $a_1c^\theta - a_2c - b = 0$.
As for $f(x)$, the previous equation does not have an algebraic solution $\forall\theta \in (1, \beta_2)$. However, we can study the function $a_1x^\theta - a_2x - b$. After analysing the derivative of $a_1x^\theta - a_2x - b$ we conclude $a_1x^\theta - a_2x - b \leq 0$ if $x \in [0, c]$. Therefore the following result holds:

Proposition. *$x^\star \leq c$, where c is such that $a_1c^\theta - a_2c - b = 0$.*

Proof. Using the fact that $V(x^\star) = \frac{b}{\gamma}$ and $V'(x^\star) = 0$, one has

$$\mu x^\star V^{'}(x^\star) + \frac{1}{2}\sigma^2(x^\star)^2V^{''}(x^\star) - \gamma V(x^\star) + a_1x^{\star\theta} - a_2x^\star = 0$$

$$\Leftrightarrow a_1x^{\star\theta} - a_2x^\star - b = -\frac{1}{2}\sigma^2(x^\star)^2V^{''}(x^\star). \tag{3.7}$$

To prove that $-\frac{1}{2}\sigma^2(x^\star)^2V^{''}(x^\star) \leq 0$ first note that

$$0 = V^{'}(x^\star) = \frac{a_1\theta}{\gamma - \mu_\theta}x^{\star\theta-1} - \frac{a_2}{\gamma - \mu} + \beta_1A_1x^{\star\beta_1-1}$$

$$\Leftrightarrow \beta_1A_1x^{\star\beta_1-1} = -\left(\frac{a_1\theta}{\gamma - \mu_\theta}x^{\star\theta-1} - \frac{a_2}{\gamma - \mu}\right). \tag{3.8}$$

Therefore

$$-\tfrac{1}{2}\sigma^2(x^\star)^2V^{''}(x^\star) = -\tfrac{1}{2}\sigma^2(x^\star)^2\left(\tfrac{a_1\theta(\theta-1)}{\gamma-\mu_\theta}x^{\star\theta-2} + \beta_1(\beta_1 - 1)A_1x^{\star\beta_1-2}\right)$$

Considering (3.8) we obtain the following upper limit to $-\frac{1}{2}\sigma^2(x^\star)^2V^{''}(x^\star)$

$$\begin{aligned}
-\frac{1}{2}\sigma^2(x^\star)^2V^{''}(x^\star) &= -\frac{1}{2}\sigma^2(x^\star)^2\left[\frac{a_1\theta(\theta-1)}{\gamma-\mu_\theta}x^{\star\theta-2} - \frac{(\beta_1-1)}{x^\star}\left(\frac{a_1\theta}{\gamma-\mu_\theta}x^{\star\theta-1} - \frac{a_2}{\gamma-\mu}\right)\right] \\
&= -\frac{1}{2}\sigma^2\left[\frac{a_1\theta(\theta-\beta_1)}{\gamma-\mu_\theta}x^{\star\theta} - \frac{a_2(1-\beta_1)}{\gamma-\mu}x^\star\right] \\
&\leq -\frac{1}{2}\sigma^2\left[\frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}x^{\star\theta} - \frac{a_2(1-\beta_1)}{\gamma-\mu}x^\star\right]
\end{aligned}$$

Given that $0 = f(x^\star)$, then

$$\frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2(1-\beta_1)}{\gamma-\mu}x^\star = -\frac{\beta_1 b}{\gamma} \tag{3.9}$$

Therefore, $a_1 x^{\star\theta} - a_2 x^\star - b = -\frac{1}{2}\sigma^2 (x^\star)^2 V''(x^\star) \leq \frac{1}{2}\sigma^2 \frac{\beta_1 b}{\gamma} \leq 0$. Since $a_1 x^{\star\theta} - a_2 x^\star - b \leq 0$, and $a_1 x^\theta - a_2 x - b \leq 0$ for $x \in (0, c)$, then $x^\star \leq c$ meaning that exit is optimal for values of demand where the profit flow $a_1 x^\theta - a_2 x - b$ is negative. □

We now prove the following lemma.

Lemma 1. $\frac{a_2}{\gamma-\mu} x^\star - \frac{a_1}{\gamma-\mu_\theta} (x^\star)^\theta + \frac{b}{\gamma} = -\frac{1}{\beta_1} \left(\frac{a_1 \theta}{\gamma-\mu_\theta} (x^\star)^\theta - \frac{a_2}{\gamma-\mu} x^\star \right)$

Proof.

$$
\begin{aligned}
f(x^\star) &= \frac{a_1(\theta - \beta_1)}{\gamma - \mu_\theta} (x^\star)^\theta - \frac{a_2(1-\beta_1)}{\gamma - \mu} x^\star + \frac{\beta_1 b}{\gamma} = 0 \Leftrightarrow \\
&= \left(\frac{a_2}{\gamma - \mu} x^\star - \frac{a_1}{\gamma - \mu_\theta} (x^\star)^\theta + \frac{b}{\gamma} \right) = -\frac{1}{\beta_1} \left(\frac{a_1 \theta}{\gamma - \mu_\theta} (x^\star)^\theta - \frac{a_2}{\gamma - \mu} x^\star \right)
\end{aligned}
$$

□

Using the previous lemma we have

$$
A_1 = \frac{\frac{a_2 x^\star}{\gamma-\mu} - \frac{a_1 (x^\star)^\theta}{\gamma-\mu_\theta} + \frac{b}{\gamma}}{(x^\star)^{\beta_1}} = \frac{-\frac{1}{\beta_1}\left(\frac{a_1\theta}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2}{\gamma-\mu} x^\star \right)}{(x^\star)^{\beta_1}}
$$

To prove that $A_1 \geq 0$ we study the function $g(x) := -\frac{1}{\beta_1}\left(\frac{a_1\theta}{\gamma-\mu_\theta}(x)^\theta - \frac{a_2}{\gamma-\mu} x \right)$. Since the derivative of $g(x)$ is $-\frac{1}{\beta_1}\left(\frac{a_1\theta^2}{\gamma-\mu_\theta}(x)^{\theta-1} - \frac{a_2}{\gamma-\mu} \right)$ then $g'(x) \geq 0$ for $x \in [x_2, \infty]$, where x_2 is the positive zero of $g^{'}(x)$. We now show that $g(y^\star) = 0$.

$$
\begin{aligned}
g(y^\star) &= -\frac{1}{\beta_1}\left(\frac{a_1\theta}{\gamma - \mu_\theta}(y^\star)^\theta - \frac{a_2}{\gamma - \mu} y^\star \right) \\
&= -\frac{y^*}{\beta_1}\left(\frac{a_1\theta}{\gamma - \mu_\theta}(y^\star)^{\theta-1} - \frac{a_2}{\gamma - \mu} \right) = 0.
\end{aligned}
$$

Since $g(y^\star) = 0$, $x_2 \leq y^\star \leq x^\star$ and $g(x)$ increases for $x \geq x_2$ then $g(x^\star) \geq 0$. We conclude that $A_1 \geq 0$.

In the next proposition, we provide the value of the firm

Proposition. *The solution of the optimal stopping problem defined on (3.1) is given by:*

$$
\mathcal{V}(x) = \begin{cases} 0 & : x < x^\star \\ \frac{a_1 x^\theta}{\gamma - \mu\theta - \frac{1}{2}\sigma^2\theta(\theta-1)} - \frac{a_2 x}{\gamma-\mu} - \frac{b}{\gamma} + A_1 x^{\beta_1} & : x \geq x^\star \end{cases}
$$

where $A_1 \geq 0$ is given by (3.5) and $x^\star \in [y^\star, c]$ is such that $f(x^\star) = 0$, with $f(x)$ given by (3.6), $y^\star = \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}}$ and c such that $a_1c^\theta - a_2c - b = 0$.

Proof. By construction, $V(x)$ is continuous in $\mathbb{R}^+$ with continuous derivative. As $\mathcal{V}(x) = V(x) - \frac{b}{\gamma}$, $\forall x > 0$, we need to prove only that $V(x)$ is solution of the HJB equation (3.2). In order to prove such result we follow the following steps

- $a_1x^\theta - a_2x + \mu xV'(x) + \frac{1}{2}\sigma^2x^2V''(x) - \gamma V(x) \leq 0$ for $x \leq x^\star$, with $V(x) = \frac{b}{\gamma}$.

  To see this note that

  $$a_1x^\theta - a_2x + \mu xV'(x) + \frac{1}{2}\sigma^2x^2V''(x) - \gamma V(x) = a_1x^\theta - a_2x - b$$

  As $x \leq x^\star \leq c$ and $f(x) \leq 0$ for $x \in [0, c]$ then $a_1x^\theta - a_2x - b \leq 0$ for $x \leq x^\star$. Therefore the result holds.

- $\left(\frac{b}{\gamma} - V(x)\right) \leq 0$ for $x \geq x^\star$, with $V(x) = \frac{a_1x^\theta}{\gamma-\mu_\theta} - \frac{a_2x}{\gamma-\mu} + A_1x^{\beta_1}$. By the fit condition, the result for $x = x^\star$ is trivially verified. In order to prove this for $x > x^\star$, then we note that the function $\frac{b}{\gamma} - V(x)$ is decreasing for $x > x^\star$. In fact, by the smooth condition, the derivative computed at $x = x^\star$ is equal to 0. As $-A_1\beta_1 \geq 0$ and $\beta_1 - 1 \leq 0$, then it follows that $\left(\frac{b}{\gamma} - V(x)\right)' \leq 0$ for $x \geq x^\star$.

  Thus we conclude that $\frac{b}{\gamma} - V(x) \leq 0$ for $x \geq x^\star$.

$\square$

3.1.1 Comparative statics for the exit problem

In this section we study the influence of the market expectation of the parameters μ, σ and b on the decision to exit.

First we prove that $\frac{\partial f}{\partial x^\star} \geq 0$. Using (3.9) we have

$$\begin{aligned}\frac{\partial f}{\partial x^\star} =& \frac{a_1\theta(\theta-\beta_1)}{\gamma-\mu_\theta}(x^\star)^{\theta-1} - \frac{a_2(1-\beta_1)}{\gamma-\mu} \\ \geq& \frac{a_1(\theta-\beta_1)}{\gamma-\mu_\theta}(x^\star)^{\theta-1} - \frac{a_2(1-\beta_1)}{\gamma-\mu} = -\frac{\beta_1 b}{\gamma x^\star} \geq 0.\end{aligned}$$

We now study how $x^\star$ varies if μ increases. By the implicit derivative theorem, we have

$$\frac{\partial x^\star}{\partial \mu} = -\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial x^\star}}.$$

Using Lemma 1 we obtain

$$\begin{aligned}\frac{\partial f}{\partial \mu} &= \frac{a_1\theta(\theta-\beta_1)}{(\gamma-\mu_\theta)^2}(x^\star)^\theta - \frac{\partial \beta_1}{\partial \mu}\frac{a_1}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2(1-\beta_1)}{(\gamma-\mu)^2}x^\star + \frac{\partial \beta_1}{\partial \mu}\frac{a_2}{\gamma-\mu}x^\star + \frac{\partial \beta_1}{\partial \mu}\frac{b}{\gamma}\\
&= \frac{\partial \beta_1}{\partial \mu}\left(\frac{a_2}{\gamma-\mu}x^\star - \frac{a_1}{\gamma-\mu_\theta}(x^\star)^\theta + \frac{b}{\gamma}\right) + \frac{a_1\theta(\theta-\beta_1)}{(\gamma-\mu_\theta)^2}(x^\star)^\theta - \frac{a_2(1-\beta_1)}{(\gamma-\mu)^2}x^\star\\
&= \frac{\partial \beta_1}{\partial \mu}\left(-\frac{1}{\beta_1}\left[\frac{a_1\theta}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2}{\gamma-\mu}x^\star\right]\right) + \frac{a_1\theta(\theta-\beta_1)}{(\gamma-\mu_\theta)^2}(x^\star)^\theta - \frac{a_2(1-\beta_1)}{(\gamma-\mu)^2}x^\star\\
&= \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{\theta-\beta_1}{\gamma-\mu_\theta}\right)(x^\star)^\theta + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} - \frac{1-\beta_1}{\gamma-\mu}\right)x^\star\end{aligned}$$

To show that $\frac{\partial f}{\partial \mu} \geq 0$, first we analyse the polynomial

$$p_\theta(x) := \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{\theta-\beta_1}{\gamma-\mu_\theta}\right)x^\theta + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} - \frac{1-\beta_1}{\gamma-\mu}\right)x$$

For the moment let's suppose that $a_1 \geq a_2$. If $\theta = 1$ then using (2.10) we have that

$$-\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial x^\star}} = \frac{\partial x^\star}{\partial \mu} = \frac{b}{a_1-a_2}\frac{1}{\gamma}\frac{\frac{\partial \beta_1}{\partial \mu}}{(\beta_1-1)^2} \leq 0.$$

Since $\frac{\partial f}{\partial x^\star} \geq 0$, then for $\theta = 1$, $\frac{\partial f}{\partial \mu} \geq 0$. Substituting θ by 1, in $p_\theta(x^\star)$ we have

$$0 \leq \left.\frac{\partial f}{\partial \mu}\right|_{\theta=1} = p_1(x^\star) = -\frac{a_1-a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} - \frac{1-\beta_1}{\gamma-\mu}\right)x^\star$$

As $x^\star > 0$, then $\frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} - \frac{1-\beta_1}{\gamma-\mu}\right) \leq 0$.

To prove that $\frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{\theta-\beta_1}{\gamma-\mu_\theta}\right) \geq 0$ note that $\frac{\theta-\beta_1}{\gamma-\mu_\theta} = \frac{\theta-\beta_1}{-\frac{1}{2}\sigma^2(\theta-\beta_1)(\theta-\beta_2)} = \frac{2}{\sigma^2}\frac{1}{(\beta_2-\theta)}$. Thus the derivative of $\frac{\theta-\beta_1}{\gamma-\mu_\theta}$ with respect to θ is $\frac{2}{\sigma^2}\frac{1}{(\beta_2-\theta)^2} > 0$. Therefore,

$$\begin{aligned}&\frac{(\theta-\beta_1)}{\gamma-\mu_\theta} \geq \frac{(1-\beta_1)}{\gamma-\mu} \Rightarrow -\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{(\theta-\beta_1)}{\gamma-\mu_\theta} \geq -\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{(1-\beta_1)}{\gamma-\mu} \geq 0\\
\Rightarrow &\frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \mu} + \frac{(\theta-\beta_1)}{\gamma-\mu_\theta}\right) \geq 0\end{aligned}$$

We are now in the position to study the polynomial $p_\theta(x)$, for any $\theta \in [1, \beta_2[$. Let $a^\star$ such that $p_\theta(a^\star) = 0$, then

$$a^\star = \sqrt[\theta-1]{\frac{\frac{a_2}{\gamma-\mu}\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{1-\beta_1}{\gamma-\mu}\right)}{\frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{\theta-\beta_1}{\gamma-\mu_\theta}\right)}} \Leftrightarrow a^\star = \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}\frac{\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{1-\beta_1}{\gamma-\mu}\right)}{\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{\theta-\beta_1}{\gamma-\mu_\theta}\right)}}$$

Since the derivative of $p_\theta(x)$ with respect to x is

$$p_\theta^{'}(x) = \frac{a_1\theta^2}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{(\theta-\beta_1)}{\gamma-\mu_\theta}\right)x^{\theta-1}+\frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}-\frac{(1-\beta_1)}{\gamma-\mu}\right)$$

then $p_\theta(x)$ is increasing in x, for a fixed θ, if $x \geq a_0$, where a_0 is such that $p_\theta{}^{'}(a_0) = 0$.
To prove that $a^\star \leq x^\star$, we show that $a^\star \leq y^\star$. This is because $y^\star$ is a lower bound of $x^\star$. Hence, by transitivity, $a^\star \leq x^\star$. Recalling that

$$y^\star = \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}},$$

then, multiplying the above equality by $\sqrt[\theta-1]{\frac{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{1-\beta_1}{\gamma-\mu}}{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{\theta-\beta_1}{\gamma-\mu_\theta}}}$, we have

$$(y^\star)\sqrt[\theta-1]{\frac{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{1-\beta_1}{\gamma-\mu}}{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{\theta-\beta_1}{\gamma-\mu_\theta}}} = \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}\frac{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{1-\beta_1}{\gamma-\mu}}{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{\theta-\beta_1}{\gamma-\mu_\theta}}} = a^\star.$$

To show that $\sqrt[\theta-1]{\frac{\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{1-\beta_1}{\gamma-\mu}\right)}{\left(-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{\theta-\beta_1}{\gamma-\mu_\theta}\right)}} \leq 1$, note that

$$\frac{\theta-\beta_1}{\gamma-\mu_\theta} \geq \frac{1-\beta_1}{\gamma-\mu} \Rightarrow -\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{\theta-\beta_1}{\gamma-\mu_\theta} \geq -\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{1-\beta_1}{\gamma-\mu}$$
$$\Rightarrow \sqrt[\theta-1]{\frac{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{1-\beta_1}{\gamma-\mu}}{-\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu}+\frac{\theta-\beta_1}{\gamma-\mu_\theta}}} \leq 1.$$

Therefore $a^\star \leq \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}} = y^\star \leq x^\star$. Since $a_0 \leq a^\star \leq x^\star$ and $p_\theta(x)$ is increasing in x for a fixed θ, if $x \geq a_0$, then

$$0 = p_\theta(a^\star) \leq p_\theta(x^\star) = \frac{\partial f}{\partial\mu}.$$

Given that $\frac{\partial f}{\partial x^\star} \geq 0$ and $\frac{\partial f}{\partial\mu} \geq 0$, then by the implicit derivative theorem, $\frac{\partial x^\star}{\partial\mu} \leq 0$. Thus, we conclude that $x^\star$ decreases if μ increases.
Now since $\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} - \frac{1-\beta_1}{\gamma-\mu}$ does not depend neither in a_1 nor in a_2 then, if $a_2 \geq a_1$, $\frac{1}{\beta_1}\frac{\partial\beta_1}{\partial\mu} - \frac{1-\beta_1}{\gamma-\mu} \leq 0$. Therefore $\frac{\partial x^*}{\partial\mu} \leq 0$ for $a_2 \geq a_1$.

We now study how $x^\star$ varies if σ increases. First note that, by the implicit derivative theorem

$$\frac{\partial x^\star}{\partial \sigma} = -\frac{\frac{\partial f}{\partial \sigma}}{\frac{\partial f}{\partial x^\star}} \tag{3.10}$$

Again using Lemma 1, we have

$$\begin{aligned}
\frac{\partial f}{\partial \sigma} &= \frac{a_1(\theta-\beta_1)\sigma(\theta(\theta-1))}{(\gamma-\mu_\theta)^2}(x^\star)^\theta - \frac{\partial \beta_1}{\partial \sigma}\frac{a_1}{\gamma-\mu_\theta}(x^\star)^\theta + \frac{\partial \beta_1}{\partial \sigma}\frac{a_2}{\gamma-\mu}x^\star + \frac{\partial \beta_1}{\partial \sigma}\frac{b}{\gamma} \\
&= \frac{\partial \beta_1}{\partial \sigma}\left(\frac{a_2}{\gamma-\mu}x^\star - \frac{a_1}{\gamma-\mu_\theta}(x^\star)^\theta + \frac{b}{\gamma}\right) + \frac{a_1(\theta-\beta_1)(\sigma\theta(\theta-1))}{(\gamma-\mu_\theta)^2}(x^\star)^\theta \\
&= \frac{\partial \beta_1}{\partial \sigma}\left(-\frac{1}{\beta_1}\left[\frac{a_1\theta}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2}{\gamma-\mu}x^\star\right]\right) + \frac{a_1(\theta-\beta_1)(\sigma(\theta(\theta-1))}{(\gamma-\mu_\theta)^2}(x^\star)^\theta \\
&= \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma} + \frac{(\theta-\beta_1)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)(x^\star)^\theta + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma}\right)x^\star.
\end{aligned}$$

As above, to show that $\frac{\partial f}{\partial \sigma} \geq 0$, we study the polynomial

$$q_\theta(x) := \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma} + \frac{(\theta-\beta_1)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)x^\theta + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma}\right)x.$$

Since $\frac{\partial \beta_1}{\partial \sigma} \geq 0$ then $-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma} \geq 0$. Therefore $\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma} + \frac{(\theta-\beta_1)\sigma(\theta(\theta-1))}{\gamma-\mu_\theta}\right) \geq 0$.

Let $b^\star$ be a zero of $q_\theta(x)$. Then

$$b^\star = \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}\frac{-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma}}{-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma} + \frac{(\theta-\beta_1)\sigma(\theta(\theta-1))}{\gamma-\mu_\theta}}} \leq \sqrt[\theta-1]{\frac{a_2}{a_1\theta}\frac{\gamma-\mu_\theta}{\gamma-\mu}} = y^\star \leq x^\star$$

Since the derivative of $q_\theta(x)$ with respect to x is

$$q_\theta{}'(x) = \frac{a_1\theta}{\gamma-\mu_\theta}\left(-\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma} + \frac{(\theta-\beta_1)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)x^\theta + \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_1}\frac{\partial \beta_1}{\partial \sigma}\right)$$

then $q_\theta(x)$ is increasing in x, for a fixed θ, if $x \geq b_0$, where b_0 is such that $q_\theta{}'(b_0) = 0$. Given that $b_0 \leq b^\star \leq x^\star$ and $q_\theta{}'(x) \geq 0$, if $x \geq b_0$, then

$$0 = q_\theta(b^\star) \leq q_\theta(x^\star) = \frac{\partial f}{\partial \sigma}.$$

Since $\frac{\partial f}{\partial x^\star} \geq 0$ and $\frac{\partial f}{\partial \sigma} \geq 0$, then by the implicit derivative theorem, $\frac{\partial x^\star}{\partial \sigma} \leq 0$. We conclude that $x^\star$ decreases if σ increases.

Note that since $\frac{\partial f}{\partial b} = \frac{\beta_1}{\gamma} \leq 0$ then $\frac{\partial x^\star}{\partial b} \geq 0$.

3.2 Investment problem

As for the investment problem with affine profit function, simple calculations prove that we may rewrite functional $\tilde{J}(x,\tau)$ as follows:

$$\begin{aligned}\tilde{J}(x,\tau) =& E_x\left[\int_\tau^\infty e^{-\gamma s}\left(a_1 X_s^\theta - a_2 X_s - b\right) ds\right]\\ =& E_x\left[\int_0^\tau e^{-\gamma s}\left(-a_1 X_s^\theta + a_2 X_s\right) ds - e^{-\gamma\tau}\frac{b}{\gamma}\right] + \frac{a_1 x^\theta}{\gamma-\mu_\theta} - \frac{a_2 x}{\gamma-\mu}.\end{aligned}$$

Therefore $\tilde{\mathcal{V}}(x) = V(x) + \frac{a_1 x^\theta}{\gamma-\mu_\theta} - \frac{a_2 x}{\gamma-\mu}$, where

$$V(x) := \sup_{\tau\in S} E_x\left[\int_0^\tau e^{-\gamma s}(-a_1 X_s^\theta + a_2 X_s) ds - e^{-\gamma\tau}\frac{b}{\gamma}\right]. \tag{3.11}$$

Then the HJB equation for $V(x)$ is

$$\max\{\mu x V^{'}(x) + \frac{1}{2}\sigma^2 x^2 V^{''}(x) - \gamma V(x) - a_1 x^\theta + a_2 x, -\frac{b}{\gamma} - V(x)\} = 0, \tag{3.12}$$

where $\Pi(x) = -a_1 x^\theta + a_2 x$ is the running cost and $C = -\frac{b}{\gamma}$ is the terminal cost.
As in the case of affine profit functions, comparing (3.1) with (3.11), we conclude that the problem of investment is similar to the exit problem, where only the signs of a_1, a_2 and b need to be changed. The main difference comes from the fact that the continuation region for the investment problem is not the same as the continuation region for the exit problem. To see this note that since $\Pi(x) \geq 0$ for $0 \leq x \leq \sqrt[\theta-1]{\frac{a_2}{a_1}}$, then one invests in the market when the initial demand is low, and one exits the market when the initial demand is above a certain level $x^\star$. Therefore, the continuation region for $V(x)$ is $\mathcal{C} = \{x : x \leq x^\star\}$. We conclude that for the stopping region $V(x) = -\frac{b}{\gamma}$ and in the continuation region the value function is solution of the ODE:

$$\mu x V^{'}(x) + \frac{1}{2}\sigma^2 x^2 V^{''}(x) - \gamma V(x) - a_1 x^\theta + a_2 x = 0 \tag{3.13}$$

After some calculations we propose $V(x)$, for $\theta \in [1, \beta_2[$, to be

$$V(x) = \begin{cases} \frac{a_2 x}{\gamma-\mu} - \frac{a_1 x^\theta}{\gamma-\mu_\theta} + A_2 x^{\beta_2} & : x < x^\star \\ -\frac{b}{\gamma} & : x \geq x^\star \end{cases} \tag{3.14}$$

We now determine values for A_2 and $x^\star$. By the fit condition

$$-\frac{b}{\gamma} = \frac{a_2 x^\star}{\gamma-\mu} - \frac{a_1 (x^\star)^\theta}{\gamma-\mu_\theta} + A_2 x^{\star\beta_2} \Rightarrow A_2 = \frac{\left(\frac{a_1 (x^\star)^\theta}{\gamma-\mu_\theta} - \frac{a_2 x^\star}{\gamma-\mu} - \frac{b}{\gamma}\right)}{x^{\star\beta_2}} \tag{3.15}$$

and by the smooth condition

$$\begin{aligned}
&\frac{a_2}{\gamma-\mu}-\frac{a_1\theta(x^\star)^{\theta-1}}{\gamma-\mu_\theta}+\beta_2 A_2 x^{\star\beta_2-1}=0\\
\Leftrightarrow&\frac{a_2}{\gamma-\mu}-\frac{a_1\theta(x^\star)^{\theta-1}}{\gamma-\mu_\theta}+\beta_2\left(\frac{a_1(x^\star)^{\theta-1}}{\gamma-\mu_\theta}-\frac{a_2}{\gamma-\mu}-\frac{b}{\gamma x^\star}\right)=0\\
\Leftrightarrow&\frac{a_1(\beta_2-\theta)}{\gamma-\mu_\theta}(x^\star)^\theta-\frac{a_2(\beta_2-1)}{\gamma-\mu}x^\star-\frac{\beta_2 b}{\gamma}=0.
\end{aligned}$$

The value of $x^\star$ is a zero of the polynomial

$$f(x):=\frac{a_1(\beta_2-\theta)}{\gamma-\mu_\theta}(x)^\theta-\frac{a_2(\beta_2-1)}{\gamma-\mu}x-\frac{\beta_2 b}{\gamma}. \tag{3.16}$$

Thus the investment threshold $x^\star$ is the zero of $f(x)$ which cannot be found analytically. Following the same approach as for the exit problem, we first prove that there is a unique positive solution $x^\star$ to the equation $f(x)=0$. Since the derivative of $f(x)$ is

$$f'(x)=\frac{a_1\theta(\beta_2-\theta)}{\gamma-\mu_\theta}(x)^{\theta-1}-\frac{a_2(\beta_2-1)}{\gamma-\mu},$$

then $f'(x)\geq 0$ for $x\in[x_1,\infty[$, where x_1 is the zero of $f'(x)$. Furthermore $f(0)=-\frac{\beta_2 b}{\gamma}\leq 0$ and $\lim_{x\to\infty} f(x)=\infty$, we conclude that there is a unique positive solution to the equation $f(x)=0$, which we denote by $x^\star$.

We now show that $y^\star := \sqrt[\theta-1]{\frac{a_2}{a_1}\frac{\gamma-\mu_\theta}{\gamma-\mu}\frac{\beta_2-1}{\beta_2-\theta}}$ is a lower bound of $x^\star$. First note that

$$\begin{aligned}
f(y^\star)&=\frac{a_1(\beta_2-\theta)}{\gamma-\mu_\theta}(y^\star)^\theta-\frac{a_2(\beta_2-1)}{\gamma-\mu}y^\star-\frac{\beta_2 b}{\gamma}\\
&=y^\star\left[\frac{a_1(\beta_2-\theta)}{\gamma-\mu_\theta}(y^\star)^{\theta-1}-\frac{a_2(\beta_2-1)}{\gamma-\mu}\right]-\frac{\beta_2 b}{\gamma}=-\frac{\beta_2 b}{\gamma}\leq 0.
\end{aligned}$$

Given that $f(y^\star)\leq 0$ and $f(x)\leq 0$ if $x\in(0,x^\star)$ then $x^\star\geq y^\star=\sqrt[\theta-1]{\frac{a_2}{a_1}\frac{\gamma-\mu_\theta}{\gamma-\mu}\frac{\beta_2-1}{\beta_2-\theta}}$. Since $\gamma-\mu_\theta=-\frac{1}{2}\sigma^2(\theta-\beta_1)(\theta-\beta_2)$ then $y^\star=\sqrt[\theta-1]{\frac{a_2}{a_1}\frac{\theta-\beta_1}{1-\beta_1}}$.

We note that the lower bound for the investment problem is greater or equal than the lower bound for the exit problem.

Proposition. *$x^\star\geq c$ where c is such that $a_1c^\theta-a_2c-b=0$. Meaning that investment is optimal for values of demand where the profit flow $a_1x^\theta-a_2x-b$ is positive.*

Proof. We omit the proof because it is analogous to the proof for the exit problem. □

We now prove the following lemma

Lemma 2. $\frac{a_1}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2}{(\gamma-\mu)}x^\star + \frac{b}{\gamma} = -\frac{1}{\beta_2}\left(\frac{a_2}{\gamma-\mu}x^\star - \frac{a_1\theta}{\gamma-\mu_\theta}(x^\star)^\theta\right)$

Proof.

$$\begin{aligned} f(x^\star) &= \frac{a_1(\beta_2-\theta)}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2(\beta_2-1)}{\gamma-\mu}x^\star - \frac{\beta_2 b}{\gamma} = 0 \\ &\Rightarrow \left(\frac{a_1}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2}{\gamma-\mu}x^\star - \frac{b}{\gamma}\right) = -\frac{1}{\beta_2}\left[\frac{a_2}{\gamma-\mu}x^\star - \frac{a_1\theta}{\gamma-\mu_\theta}(x^\star)^\theta\right] \end{aligned}$$

□

Using the previous lemma, we can now prove that $A_2 \geq 0$.

$$A_2 = \frac{\frac{a_1(x^\star)^\theta}{\gamma-\mu_\theta} - \frac{a_2x^\star}{\gamma-\mu} - \frac{b}{\gamma}}{(x^\star)^{\beta_2}} = \frac{-\frac{1}{\beta_2}\left[\frac{a_2}{\gamma-\mu}x^\star - \frac{a_1\theta}{\gamma-\mu_\theta}(x^\star)^\theta\right]}{(x^\star)^{\beta_2}}.$$

Studying the function $g(x) := -\frac{1}{\beta_2}\left(\frac{a_2}{\gamma-\mu}x - \frac{a_1\theta}{\gamma-\mu_\theta}x^\theta\right)$, we can show that $g(x^\star) \geq 0$. Since the derivative of $g(x)$ is $-\frac{1}{\beta_2}\left(\frac{a_2}{\gamma-\mu} - \frac{a_1\theta^2}{\gamma-\mu_\theta}x^{\theta-1}\right)$ then $g'(x) \geq 0$ for $x \in [x_2, \infty[$ where x_2 is the positive zero of $g'(x)$. As $g(y^\star) = \frac{a_2y^\star}{\gamma-\mu}\frac{\theta-1}{\beta_2-\theta} \geq 0$, $x_2 \leq y^\star \leq x^\star$, then $g(x^\star) \geq g(y^\star) \geq 0$. We thus conclude that $A_2 \geq 0$.

In the next proposition, we provide the value of the firm.

Proposition. *$\tilde{\mathcal{V}}(x)$is given by:*

$$\tilde{\mathcal{V}}(x) = \begin{cases} A_2x^{\beta_2} & : x < x^\star \\ \frac{a_1x^\theta}{\gamma-\mu\theta-\frac{1}{2}\sigma^2\theta(\theta-1)} - \frac{a_2x}{\gamma-\mu} - \frac{b}{\gamma} & : x \geq x^\star, \end{cases}$$

where $A_2 \geq 0$ is given by (3.15) and $x^\star \geq y^\star$ is such that $f(x^\star) = 0$, with $f(x)$ given by (3.16), $y^\star = \sqrt[\theta-1]{\frac{a_2}{a_1}\frac{\theta-\beta_1}{1-\beta_1}}$.

Proof. By construction, $V(x)$ is continuous in $\mathbb{R}^+$ with continuous derivative. As $\tilde{\mathcal{V}}(x) = V(x) + \frac{a_1x^\theta}{\gamma-\mu\theta-\frac{1}{2}\sigma^2\theta(\theta-1)} - \frac{a_2x}{\gamma-\mu}$, $\forall x > 0$, we need to prove only that $V(x)$ is solution of the HJB equation (3.12). We omit that $V(x)$ is the solution of the HJB equation (3.12), because the proof is analogous to the exit problem. □

3.2.1 Comparative statics for the investment problem

In this section we study the influence of the market expectations, μ and σ, and the parameters b on the decision to invest.

We begin by studying how $x^\star$ varies if μ and σ increase, when $b = 0$. If $b = 0$ then $x^\star = c \sqrt[\theta-1]{\frac{\theta-\beta_1}{1-\beta_1}} = y^\star$ where $c = \sqrt[\theta-1]{\frac{a_2}{a_1}}$. In this case we can solve $x^\star$ analytically, and therefore the comparative statics follows easily. Since $\frac{\partial \beta_1}{\partial \mu} \leq 0$, then

$$\frac{\partial x^\star}{\partial \mu} = c\frac{1}{\theta-1}\left(\frac{\theta-\beta_1}{1-\beta_1}\right)^{\frac{1}{\theta-1}-1}\frac{\partial \beta_1}{\partial \mu}\frac{\theta-1}{(1-\beta_1)^2} \leq 0$$

and, given that $\frac{\partial \beta_1}{\partial \sigma} \geq, 0$ then

$$\frac{\partial x^\star}{\partial \sigma} = c\frac{1}{\theta-1}\left(\frac{\theta-\beta_1}{1-\beta_1}\right)^{\frac{1}{\theta-1}-1}\frac{\partial \beta_1}{\partial \sigma}\frac{\theta-1}{(1-\beta_1)^2} \geq 0.$$

Now if $b > 0$, using the implicit derivative theorem, we have

$$\frac{\partial x^\star}{\partial \mu} = -\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial x^\star}}.$$

To prove that $\frac{\partial f}{\partial x^\star} \geq 0$, note that

$$\begin{aligned}\frac{\partial f}{\partial x^\star} =& \frac{a_1\theta(\beta_2-\theta)}{\gamma-\mu_\theta}(x^\star)^{\theta-1} - \frac{a_2(\beta_2-1)}{\gamma-\mu}\\ \geq& \frac{a_1(\beta_2-\theta)}{\gamma-\mu_\theta}(x^\star)^{\theta-1} - \frac{a_2(\beta_2-1)}{\gamma-\mu} = \frac{\beta_2 b}{\gamma x^\star} \geq 0.\end{aligned}$$

By Lemma 2, we have

$$\begin{aligned}\frac{\partial f}{\partial \mu} &= \frac{a_1\theta(\beta_2-\theta)}{(\gamma-\mu_\theta)^2}(x^\star)^\theta + \frac{\partial \beta_2}{\partial \mu}\frac{a_1}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{a_2(\beta_2-1)}{(\gamma-\mu)^2}x^\star - \frac{\partial \beta_2}{\partial \mu}\frac{a_2}{\gamma-\mu}x^\star - \frac{\partial \beta_2}{\partial \mu}\frac{b}{\gamma}\\ &= \frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2-\theta}{\gamma-\mu_\theta}\right)(x^\star)^\theta - \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2-1}{\gamma-\mu}\right)x^\star\end{aligned}$$

To show that $\frac{\partial f}{\partial \mu} \geq 0$, first we analyse the polynomial

$$p_\theta(x) := \frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2-\theta}{\gamma-\mu_\theta}\right)x^\theta - \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2-1}{\gamma-\mu}\right)x$$

For the moment let's suppose that $a_1 \geq a_2$. We saw, by (2.19), that if $\theta = 1$ then

$$-\frac{\frac{\partial f}{\partial \mu}}{\frac{\partial f}{\partial x^\star}} = \frac{\partial x^\star}{\partial \mu} = \frac{b}{a_1 - a_2}\frac{1}{\gamma}\frac{\frac{\partial \beta_2}{\partial \mu}}{(\beta_2 - 1)^2} \leq 0.$$

Since $\frac{\partial f}{\partial x^\star} \geq 0$, then for $\theta = 1$, $\frac{\partial f}{\partial \mu} \geq 0$. Substituting θ by 1, in $p_\theta(x^\star)$ we have

$$0 \leq \left.\frac{\partial f}{\partial \mu}\right|_{\theta=1} = p_1(x^\star) = \frac{a_1 - a_2}{\gamma - \mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - 1}{\gamma - \mu}\right) x^\star.$$

Therefore, since $x^\star > 0$, then

$$\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - 1}{\gamma - \mu} \geq 0 \Rightarrow -\frac{a_2}{\gamma - \mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - 1}{\gamma - \mu}\right) \leq 0.$$

If $\frac{a_1\theta}{\gamma - \mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - \theta}{\gamma - \mu_\theta}\right)$ were negative then $p_\theta(x) < 0, \forall x$. But since, for $b = 0$, $y^\star = x^\star$ and $\frac{\partial x^\star}{\partial \mu} \leq 0$, then $0 \leq \left.\frac{\partial f}{\partial \mu}\right|_{b=0} = p_\theta(y^\star)$, thus

$$\frac{a_1\theta}{\gamma - \mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - \theta}{\gamma - \mu_\theta}\right) \geq 0,$$

which contradicts the statement that $\frac{a_1\theta}{\gamma - \mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - \theta}{\gamma - \mu_\theta}\right) \leq 0$.
Therefore one needs to study the behaviour of $p_\theta(x)$, in order to prove that $\frac{\partial f}{\partial \mu} \geq 0$. The derivative of $p_\theta(x)$ is given by:

$$p_\theta^{'}(x) = \frac{a_1\theta^2}{\gamma - \mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - \theta}{\gamma - \mu_\theta}\right) x^{\theta - 1} - \frac{a_2}{\gamma - \mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - 1}{\gamma - \mu}\right)$$

Thus ${p_\theta}'(x) \geq 0$ if $x \geq a_0$ where a_0 such that ${p_\theta}'(a_0) = 0$. Therefore $0 \leq p_\theta(y^\star) \leq p_\theta(x^\star) = \frac{\partial f}{\partial \mu}$.
Given that $\frac{\partial f}{\partial x^\star} \geq 0$ and $\frac{\partial f}{\partial \mu} \geq 0$, then by the implicit derivative theorem, $\frac{\partial x^\star}{\partial \mu} \leq 0$. We conclude that $x^\star$ decreases if μ increases. Now since $-\frac{a_2}{\gamma - \mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - 1}{\gamma - \mu}\right)$ does not depend on a_1 and a_2 then, if $a_2 \geq a_1$, $-\frac{a_2}{\gamma - \mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \mu} + \frac{\beta_2 - 1}{\gamma - \mu}\right) \leq 0$. Therefore, $\frac{\partial x^\star}{\partial \mu} \leq 0$ for $a_2 \geq a_1$.

Again using the implicit derivative theorem we have

$$\frac{\partial x^\star}{\partial \sigma} = -\frac{\frac{\partial f}{\partial \sigma}}{\frac{\partial f}{\partial x^\star}} \tag{3.17}$$

Using Lemma 2 we have

$$\frac{\partial f}{\partial \sigma} = \frac{a_1(\beta_2-\theta)\sigma(\theta(\theta-1))}{(\gamma-\mu_\theta)^2}(x^\star)^\theta + \frac{\partial \beta_2}{\partial \sigma}\frac{a_1}{\gamma-\mu_\theta}(x^\star)^\theta - \frac{\partial \beta_2}{\partial \sigma}\frac{a_2}{\gamma-\mu}x^\star - \frac{\partial \beta_2}{\partial \sigma}\frac{b}{\gamma}$$
$$= \frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} + \frac{(\beta_2-\theta)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)(x^\star)^\theta - \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma}\right)x^\star.$$

As above, to show that $\frac{\partial f}{\partial \sigma} \leq 0$, we study the polynomial

$$q_\theta(x) := \frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} + \frac{(\beta_2-\theta)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)x^\theta - \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma}\right)x.$$

Since $\beta_2 \geq 0$ and $\frac{\partial \beta_2}{\partial \sigma} \leq 0$, then $-\frac{a_2}{\gamma-\mu}\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} \geq 0$.

If $\frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} + \frac{(\beta_2-\theta)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)$ were positive then $q_\theta(x) > 0$, $\forall x$. But since for $b = 0$, $y^\star = x^\star$ and $\frac{\partial x^\star}{\partial \sigma} \geq 0$ then $0 \geq \frac{\partial f}{\partial \sigma}\Big|_{b=0} = q_\theta(y^\star)$, thus $\frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} + \frac{(\beta_2-\theta)\sigma(\theta-1)}{\gamma-\mu_\theta}\right) \leq 0$, which contradicts the statement that $\frac{a_1\theta}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} + \frac{(\beta_2-\theta)\sigma(\theta-1)}{\gamma-\mu_\theta}\right) \geq 0$.

Therefore one needs to study the behaviour of $q_\theta(x)$, in order to prove that $\frac{\partial f}{\partial \sigma} \leq 0$. The derivative of $q_\theta(x)$ is given by:

$$q_\theta{}'(x) = \frac{a_1\theta^2}{\gamma-\mu_\theta}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma} + \frac{(\beta_2-\theta)\sigma(\theta-1)}{\gamma-\mu_\theta}\right)x^{\theta-1} - \frac{a_2}{\gamma-\mu}\left(\frac{1}{\beta_2}\frac{\partial \beta_2}{\partial \sigma}\right).$$

Thus $q_\theta{}'(x) \leq 0$ if $x \geq b_0$, where b_0 is such that $q_\theta{}'(b_0) = 0$. Therefore $0 \geq q_\theta(y^\star) \geq q_\theta(x^\star) = \frac{\partial f}{\partial \sigma}$.
Since $\frac{\partial f}{\partial x^\star} \geq 0$ and $\frac{\partial f}{\partial \sigma} \leq 0$, then by the implicit derivative theorem, $\frac{\partial x^\star}{\partial \sigma} \geq 0$. We conclude that $x^\star$ increases if σ increases.

Note that since $\frac{\partial f}{\partial b} = -\frac{\beta_2}{\gamma} \leq 0$ then $\frac{\partial x^\star}{\partial b} \geq 0$.

3.3 Changing market problem

In this section first we consider that the firm has the option to switch from one market characterized by a monomial profit function $\Pi_1(x) = a_1 x^\theta - b_1$, to another one characterized by a linear profit function $\Pi_2(x) = a_2 x - b_2$, at time τ. We are assuming that $a_1, a_2 \geq 0$. To avoid $\Pi_1(x)$ intersects more than once or intersect $\Pi_2(x)$, we assume $b_1 \geq b_2$.

Simple calculations prove that we may rewrite the functional $I(x, \tau)$ as follows:

$$I(x,\tau) = E_x\left[\int_0^\tau e^{-\gamma s}(a_1 X_s^\theta - b_1)ds + \int_\tau^\infty e^{-\gamma s}(a_2 X_s - b_2)ds\right] \tag{3.18}$$

$$= E_x\left[\int_0^\tau e^{-\gamma s}(a_1 X_s^\theta - a_2 X_s - (b_1 - b_2))ds\right] + \underbrace{E_x\left[\int_0^\infty e^{-\gamma s}(a_2 X_s - b_2)ds\right]}_{=\frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma}} \tag{3.19}$$

This problem can be solved using the results derived in the exit problem with polynomial profit functions, with $b = b_1 - b_2$. Thus the value function for such case is given by:

$$\mathcal{G}(x) = \begin{cases} \frac{a_2}{\gamma-\mu}x - \frac{b_2}{\gamma} & : x < x^\star \\ \frac{a_1}{\gamma-\mu_\theta}x^\theta - \frac{b_1}{\gamma} + A_1 x^{\beta_1} & : x \geq x^\star \end{cases}$$

where $A_1 = \frac{\frac{a_2 x^\star}{\gamma-\mu} - \frac{a_1 (x^\star)^\theta}{\gamma-\mu_\theta} + \frac{b_1 - b_2}{\gamma}}{x^{\star\beta_1}} \geq 0$ and $x^\star \leq c$ is such that $f(x^\star) = 0$, with $f(x)$ given by (3.6), and c such that $\Pi_1(c) = \Pi_2(c)$. Note that c is precisely the point where Π_1 and Π_2 intersect, therefore the value x where the firm should optimally change from the first market to the second one is smaller or equal to the point where both markets are equally profitable.

Now we consider that the firm has the option to switch from one market characterized by a linear profit function $\Pi_1(x) = a_1 x - b_1$, to another one characterized by a monomial profit function $\Pi_2(x) = a_2 x^\theta - b_2$ at time τ. Again we are assuming that $a_1, a_2 \geq 0$. To avoid $\Pi_1(x)$ intersects more than once or does not intersect $\Pi_2(x)$ we assume $b_1 \leq b_2$.
Simple calculations prove that we may rewrite the functional $I(x, \tau)$ as follows:

$$I(x,\tau) = E_x\left[\int_0^\tau e^{-\gamma s}(a_1 X_s - a_2 X_s^\theta + (b_2 - b_1))ds\right] + \underbrace{E\left[\int_0^\infty e^{-\gamma s}(a_2 X_s^\theta - b_2)ds\right]}_{=\frac{a_2}{\gamma-\mu_\theta}x^\theta - \frac{b_2}{\gamma}}$$

Therefore this problem can be solved using the results derived in the investment problem with polynomial profit function, with $b = b_2 - b_1$. Thus the value function for such case is given by:

$$\mathcal{G}(x) = \begin{cases} A_2 x^{\beta_2} + \frac{a_1}{\gamma-\mu}x - \frac{b_1}{\gamma} & : x < x^\star \\ \frac{a_2}{\gamma-\mu_\theta}x^\theta - \frac{b_2}{\gamma} & : x \geq x^\star \end{cases}$$

where $A_2 = \frac{\frac{a_2 (x^\star)^\theta}{\gamma-\mu_\theta} - \frac{a_1 x^\star}{\gamma-\mu} - \frac{(b_2-b_1)}{\gamma}}{x^{\star\beta_2}} \geq 0$ and $x^\star \geq c$ is such that $f(x^\star) = 0$, with $f(x)$ given by (3.16), and c such that $\Pi_1(c) = \Pi_2(c)$. Therefore the value x where the firm should optimally change from the first market to the second one is larger or equal to the point where both markets are equally profitable.

Chapter 4

Conclusion

In this work we analysed three profit maximization problems: the "exit problem", the "investment problem" and the "changing market problem". We assumed that the market demand followed a Geometric Brownian Motion. We solved the exit and investment problems for a class of profit functions of the form $\Pi(x) = a_1 x^\theta - a_2 x - b$. Using the Hamilton-Jacobi-Bellman equation, first we found the value function for the exit problem. Even though the exit threshold could not be calculated analytically, we determined an upper and lower bound. We then presented comparative statistics with respect to the drift and the volatility and concluded that if the drift or the volatility of the uncertainty process X increases, then the decision to exit the market is postponed. We then determined the value function for the investment problem. In this case we could only calculate a lower bound for the investment threshold. Again we presented comparative statistics with respect to the drift and the volatility and concluded that if the volatility increases, the decision to invest is anticipated whereas if the drift increases, it is postponed. Using the results derived in the exit problem we calculated the value function for the changing market problem of a firm that has the option to switch from one market characterised by a monomial profit function to a market characterised by a linear profit function. We then presented comparative statistics with respect to the drift (denoted by μ) and the volatility (denoted by σ) and concluded that the influence of μ and σ of the uncertainty process X, follows the same pattern as the dependency of $x^\star$ with respect to μ and σ in the exit problem. Conversely, we calculated the value function for the changing market problem of a firm that has the option to switch from one market characterised by a linear profit function to a market characterised by a monomial profit function. Again we presented comparative statistics with respect to μ and σ and concluded that the influence of μ and σ of the uncertainty process X, follows the same pattern as the dependency of $x^\star$ with respect to μ and σ in the investment problem.

Bibliography

[1] Michael D Intriligator. *Mathematical optimization and economic theory*, volume 39. Siam, 1971.

[2] Thomas S Ferguson. Who solved the secretary problem? *Statistical science*, pages 282–289, 1989.

[3] Johnathan Mun. *Real options analysis: Tools and techniques for valuing strategic investments and decisions*, volume 137. John Wiley & Sons, 2002.

[4] Avinash K Dixit and Robert S Pindyck. *Investment under uncertainty.* Princeton University Press, 1994.

[5] Tomas Björk. *Arbitrage theory in continuous time.* Oxford University Press, 2009.

[6] Kevin Ross. Stochastic control in continuous time. *Lecture Notes on Continuous Time Stochastic Control, Springer*, 2008.

[7] Bernt Øksendal. Stochastic differential equations. In *Stochastic differential equations*, pages 65–84. Springer, 2003.

[8] Manuel Guerra, Claudia Nunes, and Carlos Oliveira. Exit option for a class of profit functions. *International Journal of Computer Mathematics*, pages 1–16, 2016.

Appendix

Properties of β_1

In this last section we study some properties of β_1 and β_2.
Let $P(t) = \frac{1}{2}\sigma^2 t^2 + (\mu - \frac{1}{2}\sigma^2)t - \gamma$. Then β_1 and β_2 are zeros of $P(t)$.
First we study how β_1 varies if we increase σ. Using the implicit derivative theorem we have:

$$\frac{\partial \beta_1}{\partial \sigma} = -\frac{\frac{\partial P}{\partial \sigma}}{\frac{\partial P}{\partial \beta_1}} = -\frac{\sigma\beta_1^2 - \sigma\beta_1}{\sigma^2\beta_1 + (\mu - \frac{1}{2}\sigma^2)} = \frac{\sigma\beta_1(\beta_1 - 1)}{\sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}} \geq 0.$$

We now study how β_1 varies if we increase μ. Using the implicit derivative theorem we have:

$$\frac{\partial \beta_1}{\partial \mu} = -\frac{\frac{\partial P}{\partial \mu}}{\frac{\partial P}{\partial \beta_1}} = -\frac{\beta_1}{\sigma^2\beta_1 + (\mu - \frac{1}{2}\sigma^2)} = \frac{\beta_1}{\sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}} \leq 0.$$

Properties of β_2

Lemma 3. *$\beta_2 \geq 1$ for all σ and μ.*

Proof.

$$\begin{aligned}
\gamma \geq \mu \Rightarrow \beta_2 &= \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}}{\sigma^2} \\
&\geq \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\mu}}{\sigma^2} \\
&= \frac{-(\mu - \frac{1}{2}\sigma^2) + \sqrt{(\mu + \frac{1}{2}\sigma^2)^2}}{\sigma^2} \\
&= \frac{-(\mu - \frac{1}{2}\sigma^2) + (\mu + \frac{1}{2}\sigma^2)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1
\end{aligned}$$

□

First we study how β_2 varies if we increase σ. Using the implicit derivative theorem we have:

$$\frac{\partial \beta_2}{\partial \sigma} = -\frac{\frac{\partial P}{\partial \sigma}}{\frac{\partial P}{\partial \beta_2}} = -\frac{\sigma\beta_2^2 - \sigma\beta_2}{\sigma^2\beta_2 + (\mu - \frac{1}{2}\sigma^2)} = -\frac{\sigma\beta_2(\beta_2 - 1)}{\sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}} \leq 0.$$

We now study how β_2 varies if we increase μ. Using the implicit derivative theorem we have:

$$\frac{\partial \beta_2}{\partial \mu} = -\frac{\frac{\partial P}{\partial \mu}}{\frac{\partial P}{\partial \beta_2}} = -\frac{\beta_2}{\sigma^2\beta_2 + (\mu - \frac{1}{2}\sigma^2)} = -\frac{\beta_2}{\sqrt{(\mu - \frac{1}{2}\sigma^2)^2 + 2\sigma^2\gamma}} \leq 0.$$