Energy in general relativity:
A comparison between quasilocal definitions

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The most incomprehensible thing about the world is that it is comprehensible.

Albert Einstein

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Abstract

Using a 3+1 spacetime decomposition, we derive Brown-York's and Lynden-Bell-Katz's quasilocal energy definitions. Then, we analyse the properties of the two definitions in specific spacetimes and derive what laws of black hole mechanics come from each definition. Comparing the results in the Newtonian limit, we find a suitable interpretation for the localization of gravitational energy for each definition. Using reasonable arguments, we show which definition is more appropriate and consistent with some phenomena, like Mercury's perihelion precession. Finally, we suggest a way to unify the two definitions by modifying the expression for matter energy in special relativity.

Keywords

Quasilocal energy, Brown-York energy, Lynden-Bell-Katz energy, black hole thermodynamics, gravitational energy density, perihelion precession.

Resumo

Usando uma decomposição 3+1 do espaço-tempo, deduzem-se as definições de energia quase-locais de Brown-York e de Lynden-Bell e Katz. Analisam-se as propriedades das duas definições em espaços-tempo específicos e deduzem-se as leis da mecânica de buracos negros provenientes de cada definição. Ao comparar os resultados no limite Newtoniano, encontra-se uma interpretação adequada da localização da energia gravitacional para cada definição. Usam-se argumentos razoáveis para mostrar que definição é mais apropriada e consistente com alguns fenómenos, como a precessão do periélio de Mercúrio. Por fim, sugere-se um método para unificar as duas definições ao modificar a expressão da energia da matéria em relatividade restrita.

Palavras-chave

Energia quase-local, energia de Brown-York, energia de Lynden-Bell e Katz, termodinâmica de buracos negros, densidade de energia gravitacional, precessão do periélio.

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Preface

The research of this thesis has been done at Centro Multidisciplinar de Astrofísica (CENTRA) in the Physics Department of Instituto Superior Técnico. The work was partially supported by Fundação para a Ciência e Tecnologia (FCT), through project No. UID/FIS/00099/2013.

Below are listed the works included in this thesis in preparation and to be submitted soon in international journals:

- D. P. L. Bragança and J. P. S. Lemos, *A comparison of quasilocal energy definitions in general relativity* in preparation (2016), see Chapter 3 of this thesis
- D. P. L. Bragança and J. P. S. Lemos, *General relativistic perihelion precession: Isolating the three different effects from Newtonian gravitation, special relativity, and curved time*, submitted (2016), see Appendix B.

Chapter 1

Introduction

1.1 Motivation

A hundred years ago, Einstein's paper on the general theory of relativity [1] changed the paradigm in Physics. It modified forever the way we look at space and time. Special relativity had already shown that space and time were not two radically different concepts, but instead that they were intimately connected in a single entity: spacetime. In the special theory of relativity, spacetime is flat, meaning that the natural path of matter is always a straight line (like in Newton's theory of classical mechanics). However, in his general theory of relativity, Einstein realized that spacetime is malleable and can be deformed by matter.

Gravity is no longer viewed as a physical force acting at a distance, but really as the curvature of spacetime. This curvature changes the natural path, or geodesic, that matter follows, causing what we call gravitation. The cause of this curvature is, according to Einstein, energy and momentum of all matter and radiation. Curvature and matter energy-momentum are related by the Einstein equations [2]

$$R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi G}{c^4} T_{ab} \,, \tag{1.1}$$

where R_{ab} is Ricci's tensor, R its trace, g_{ab} is the spacetime metric, T_{ab} the matter energy-momentum tensor, G is Newton's gravitational constant and c the speed of light. Note that T_{ab} may include a cosmological constant part (also called dark energy). Although some approximate solutions were found [1], these equations were solved exactly for the first time by Schwarzschild in 1916 [3].

According to Einstein, any type of energy and momentum would cause space to curve, even the very energy and momentum of gravity [1]. However, finding a definition for this gravitational energy has proven to be a hard task. In fact, a consensual gravity's energy-momentum tensor was not yet found. Interesting reviews of the energy problem in general relativity are given in [4, 5].

This issue is of major importance when comparing gravity with other forces, which have a well-defined energy-momentum tensor given by Noether's canonical formalism. Furthermore, the absence of a gravity's energy-momentum tensor leads to the problem of quantization of the gravity field, since the energy of the field quanta should have definite values [6].

In his famous article, Einstein himself only derived a pseudotensor that does not obey the principle of general covariance [1], giving weird values in many situations [6]. For instance, Einstein's pseudotensor can be made non vanishing in Minkowski spacetime, where it should always vanish, and can vanish in Schwarzschild spacetime, where it should not.

Some physicists even say that a local gravitational energy-momentum tensor simply does not exist [4, 2], since the principle of equivalence always allows a change of variables where the metric becomes locally of Minkowsky type. The difficulties in finding such a tensor suggest that gravitational energy may not be localized [4]. In fact, it is possible to define gravity's global energy. For instance, ADM mass, Komar's mass or Bondi mass give reasonable results in asymptotically flat spacetimes [4, 7, 8, 9, 10]. In fact, Bondi mass is a major clue to find that gravitational waves are a physical phenomena in the sense that they carry energy. Moreover, some attempts have been made to calculate this energy in a bounded region. These are called quasilocal definitions. Physicists such as Hawking, Geroch or Penrose, among others, have made such quasilocal energy definitions [4], recovering some expected results in the appropriate limits.

1.2 Overview of gravitational energy definitions in general relativity

1.2.1 Global definitions of gravitational energy

As long as spacetime is asymptotically flat, it is possible to define its global energy and momentum. For instance, if we consider that spacetime is foliated by asymptotically flat spacelike hypersurfaces, we can follow the framework developed by Arnowitt, Deser and Misner (ADM) [7, 8, 4] in order to get the ADM energy and momentum of a particular spacelike hypersurface. This ADM energy-momentum transforms as a Lorentz 4-vector and gives some reasonable results for the invariant mass. For instance, in Minkowski spacetime ADM mass is zero, in Schwarzschild spacetime ADM mass is m and momentum is zero [8], and in the weak field limit ADM mass is the total mass of the system [9].

If we consider now the propagation of gravitational waves (that appear naturally with the linearized field equations), ADM formalism does not work because there are no asymptotically flat spacelike surfaces. In this situation, we must use Bondi's framework, valid in the null infinity [10, 4]. We find that the Bondi mass of the null hypersurface is not constant but decreases with time, as long as the so-called Bondi news function is not trivial. This is interpreted as the loss of mass of the source through the emission of gravitational waves. This framework gives another 4-vector, where the time component is called the Bondi-Sachs energy.

ADM mass and the Bondi-Sachs energy are generally accepted as giving the correct values for spatial and null infinity, respectively, for asymptotically flat spacetimes.

In order to define a global mass, and other objects such as gravitational waves, in an asymptotically de Sitter spacetime, we cannot use the same method used in asymptotically flat spacetimes. Instead, a new formalism has to be built. This was done by Balasubramanian, de Boer and Minic [11], Anninos, Seng Ng and Strominger [12], Dehghani and Khajehazad [13], Ghezelbash and Mann [14], and by Ashtekar, Bonga and Kesavan [15, 16, 17], among others.

1.2.2 Local definitions of gravitational energy

To introduce the concept of a local gravitational energy definition, we can consider the relativistically corrected Newtonian theory in such a way that the energy of the gravitational field (built in analogy with electrodynamics) contributes as a source term [4]. We then get the corrected Poisson equation,

$$\nabla^2\phi = 4\pi G\left(\rho + \frac{1}{c^2}(u+U)\right), \tag{1.2}$$

where ϕ is the Newtonian potential, ρ is the mass density, u is the internal energy density of the matter source and U is the gravitational field energy density, given by

$$U = -\frac{|\nabla\phi|^2}{8\pi G}. \tag{1.3}$$

Solving this equation in a vacuum leads to a perihelion precession equal to a twelfth of general relativity prediction, as is calculated in Section 4.3. This result suggests that this coupling of the field to itself is a relativistic correction of the Newtonian theory (there are other corrections to make up the full general relativity precession), revealing that gravitational energy density should exist in the Newtonian limit. In fact, we can build a relativistic scalar gravitational theory that takes the gravitational field energy also as a source of gravity, see for example [18, 19, 20, 21]. This suggests that there should be an object that describes this energy in general relativity.

In order to derive his field equations with source terms, Einstein justifies the *a priori* introduction of the matter energy-momentum tensor "from the requirement that the energy of the gravitational field shall act gravitationally in the same way as any other kind of energy" [1]. Indeed, he had previously found a pseudotensor t_b^a that described this gravity's energy, which was given by

$$t_b^a = \frac{1}{16\pi\sqrt{-g}}((g^{cd}\sqrt{-g})_{,b}(\Gamma^a_{cd} - \delta^a_d\Gamma^e_{ce}) - \delta^a_b g^{cd}(\Gamma^e_{cd}\Gamma^f_{ef} - \Gamma^f_{ce}\Gamma^e_{df})\sqrt{-g})\,, \tag{1.4}$$

g being the determinant of the metric g_{ab}, and Γ^a_{bc} the Christoffel symbols, given by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(g_{bd,c} + g_{cd,b} - g_{bc,d}\right)\,, \tag{1.5}$$

where $_{,a}$ represents differentiation with respect to the coordinate x^a. With this pseudotensor, Einstein's equations in vacuum can be rewritten as

$$\left(g^{ab}\Gamma^d_{cb}\right)_{,d} = \frac{1}{8\pi}\left(t^a_c - \frac{1}{2}\delta^a_c t\right)\,, \tag{1.6}$$

where t is the trace of t^b_a, and $\sqrt{-g} = 1$ is assumed. Einstein's guess that gravitational energy should act gravitationally as any other kind of energy leads to

$$\left(g^{ab}\Gamma^d_{cb}\right)_{,d} = 8\pi\left[(t^a_c + T^a_c) - \frac{1}{2}\delta^a_c\,(t+T)\right]\,, \tag{1.7}$$

where T is the trace of T^b_a. This is equivalent to Einstein's field equations in the standard form of Eq. (1.1). Einstein's energy-momentum pseudotensor also leads to a clear equation for the conservation of energy. To show this, let us take the standard equation of the vanishing of the matter energy-momentum divergence

$$\nabla_a T^{ab} = 0\,, \tag{1.8}$$

where ∇_a is the covariant derivative. This is not a conservation law, since it allows energy to be exchanged between matter and the gravitational field. In fact, Einstein realized that the vanishing of divergence of the energy-momentum tensor of matter showed that "laws of conservation of momentum and energy do not apply in the strict sense for matter alone, or else that they apply only [...] when the field intensities of gravitation vanish". Nevertheless, we can show that Eq. (1.8) is equivalent to

$$\left(\sqrt{-g}(t^{ab} + T^{ab})\right)_{,a} = 0\,, \tag{1.9}$$

which is an exact conservation law. Thus Einstein's pseudotensor has some interesting properties and gives some clues about the nature of gravitational energy.

However, this pseudotensor can always vanish locally if one makes a suitable coordinate transformation, and it can be non-zero in a Minkowski spacetime using non-Euclidean coordinates.

Taking Einstein's asymmetric pseudotensor as an example, Landau and Lifshitz derived another pseudotensor, which is symmetric. Other pseudotensors have been proposed [4]. However, no natural energy-momentum tensor has been found, even though the relation between the canonical and gravitational energy-momentum tensor has already been studied [22]. Some tensors that rely on some extra structure (a background metric, a preferred class of frame fields or a Killing symmetry) have been built, but there are always extra choices to be made [4].

It is also possible to define an energy density for gravitational waves. Isaacson [23, 24], within a shortwave approximation for gravitational waves, showed that a certain tensor, now called the Isaacson tensor, which gives the average over many wavelengths of a gravitational wave, actually yields the energy of the gravitational field in the wave. In this way the energy contained in a gravitational wave can be described in a coordinate invariant way and avoids the need resort to pseudotensors that have the problems already referred.

Other definitions of a gravitational energy density can be built from different approaches in specific spacetimes, see for example [4, 25, 26].

1.2.3 Quasilocal definitions of gravitational energy

There is some controversy on the non-tensorial nature of the gravitational energy-momentum density [4]. In fact, even if meaningful global energy definitions have been found, some physicists say that it is meaningless to consider such a non-tensorial quantity in general relativity [2]. However, an object in general relativity does not have to be a tensor in order to have meaning. For instance, the Christoffel symbols Γ^a_{bc} have very clear geometrical and physical content (they represent the gravitational force), but they are not tensorial, since they can always vanish locally in some coordinate system. Therefore, if the gravitational energy-momentum is a function of Γ^a_{bc}, it must also be non-tensorial. Thus, in order to have a solid quasilocal definition, we must consider an extended finite domain in which we define an expression for the gravitational field energy.

In the Newtonian framework, using the modified Poisson equation Eq. (1.2), it is possible to express the total mass-energy E_D(including gravitational energy) of a region D via a surface integral [4]. Indeed, we have

$$E_D = \frac{c^2}{4\pi G} \oint_S v^a (D_a \phi) dS \,, \tag{1.10}$$

where $S = \partial D$, D_a is the covariant derivative and v^a is the outward unit normal to S.

Similarly, in general relativity there are expressions of type (1.10) that define quasilocal gravitational energy in a spacelike hypersurface Σ in spacetime M. A list of reasonable criteria that should be satisfied by quasilocal definitions can be found in [4]. Hawking energy $E_H(S)$ follows from the idea that the bending of ingoing and outgoing light rays orthogonal to a 2-sphere S only depends on the mass-energy surrounded by S [4] (reminding Gauss's law in electrostatics).

Hawking's definition recovers the expected values at spatial infinity (ADM energy) and at null infinity (Bondi-Sachs energy) in asymptotically flat spacetimes. Hawking also defined a natural gravitational linear momentum that goes together with his energy definition. However, we do not know if this Hawking gravitational energy momentum is a 4-vector.

Hawking energy has a natural generalization by removing the contribution due to the extrinsic curvature of Σ in M. This energy is the Geroch energy, E_G, and it satisfies $E_G(S) \leq E_H(S)$. Geroch energy also tends to ADM energy in asymptotically flat spacetimes [4].

By studying the Einstein equations applied to a spherically symmetric distribution of fluid, Misner and Sharp derived an expression $m(r, t)$ for the total energy enclosed by a sphere of radial coordinate r at a given time t [27, 4]. Some properties of this energy were derived by Hayward [28], who got among other results the correct Newtonian limit, the ADM energy at spatial infinity and the Bondi-Sachs energy at null infinity.

Penrose suggested another definition of quasilocal energy-momentum, based on a "twistor" formalism [4]. This formalism is more specialized and will not be developed in this thesis. However, Penrose's energy-momentum gives the correct result in the limit of null infinity in asymptotically flat spacetimes (Bondi-Sachs energy-momentum) and it generates many interesting results (for instance the energy of a system of two black holes is less than the sum of the individual energies).

Bartnik had the idea of "quasilocalizing" ADM energy [4]. This means that the quasilocal energy of a compact, connected 3-manifold Σ is given by the infimum of all ADM masses describing asymptotically flat extensions of Σ.

Proceeding in analogy with the Hamilton-Jacobi formulation for classical mechanics, it is possible to define a canonical quasilocal energy [29]. This is called the Brown-York quasilocal energy definition and yields many interesting results. For instance, it allows for an elegant property of some black holes, established by Bose and Dadhich [30, 31] and developed but other authors such as Balart and Peña [32]. This energy also yields interesting results when calculated in the small sphere limit [33]. Furthermore, some authors [34, 35] calculated the Brown-York quasilocal energy for a spinning black hole, getting insightful results too. This energy definition can be extended for situations in which time becomes spacelike, for instance inside the event horizon [36]. However, it does not have any positivity property and needs a reference spacetime in order to be well-defined [4].

In stationary spacetimes, it is also possible to define the Lynden-Bell and Katz quasilocal energy [37, 38, 39, 40]. They begin by considering the total ADM spacetime energy and then they subtract to it the matter energy that they define. The result should be the gravitational energy. It is possible to put this idea in a quasilocal energy definition, and the result is similar to Brown-York's definition.

Most of these quasilocal energy definitions can be generalized to operators in loop quantum gravity [41].

There are other quasilocal energy definitions, but in this work we will mainly focus on Brown-York and on Lynden-Bell-Katz energy definitions.

1.3 Objectives

The main objectives of this thesis are to derive expressions for the gravitational energy and energy density from Brown-York's and Lynden-Bell-Katz's quasilocal energy definitions for specific spacetimes, to calculate what quasilocal black hole law of mechanics follow from each definition, to compare the results in appropriate limits and finally to choose which quasilocal definition is more appropriate to use.

A secondary objective of this thesis is to establish a relation between the gravitational mass m that appears in the Schwarzschild metric and the rest mass m^* of a star. This relation will be established when studying the interior Schwarzschild solution. The relation is independent from the energy definition that we use.

1.4 Notation

In this work, we use geometric units, which means that all the quantities have dimensions of $[\text{length}]^n$, where n can be any real number. We will also use $c = G = 4\pi\varepsilon_0 = 1$ to simplify the calculations. To obtain the physical quantity, we have to put back the c's and G's. The conversion factors from mass, time, charge and temperature to length are in the following table.

Mass	$\frac{G}{c^2}$
Time	c
Charge	$\frac{1}{c^2}\sqrt{\frac{G}{4\pi\varepsilon_0}}$
Temperature	$\frac{Gk_B}{c^4}$

TABLE 1.1: Conversion factors from physical quantities to length

We will also be using Einstein's summation convention [1], which means that whenever there are repeated covariant and contravariant indices, a sum between them is implied. For example

$$X^{ab}Y_{ac} \equiv \sum_a X^{ab}Y_{ac}\,,$$

for any tensors X^{ab} and Y_{ac}.

We shall also use a spacetime metric with signature $(-,+,+,+)$. Although, strictly speaking, the metric and the line element are different objects, we will sometimes refer to the line element as the metric, since there is a bijection between the two objects.

1.5 Thesis outline

This work is organized as follows. In Chapter 2, we introduce the 3+1 decomposition of the metric, and we derive Brown-York's and Lynden-Bell-Katz's quasilocal energy definitions. In Chapter 3, we start by defining an energy density from a quasilocal energy expression, and then we compare the two quasilocal energy expressions for specific spacetimes. In Chapter 4, we discuss the results and give some arguments to choose the most suitable definition.

Chapter 2

Derivation of Brown-York's and Lynden-Bell and Katz's energy definitions

There are different approaches to define quasilocal energy in general relativity. In this chapter, we derive the expressions for Brown-York and Lynden-Bell-Katz quasilocal energies that will be used throughout this work.

2.1 The 3+1 decomposition

In physics, energy is intimately related to time translations. In fact, it is usually defined as the conserved quantity associated with time translation invariance. However, in general relativity, space and time do not have independent natures. They are linked in a four dimensional curved manifold, spacetime. It is therefore comprehensible that, in order to clearly define energy in general relativity, we have to somehow separate time from space. This is achieved via a so-called 3+1 decomposition, where we use the available freedom for defining the metric to write it in a way that allow this separation. A clear approach to this 3+1 decomposition is given in [8].

Let us define our geometrical objects, following the same conventions as Brown and York [29]. We will consider a four dimensional spacetime region M, that is topologically identical to the product of a 3-space Σ and a real line interval (that represents the time coordinate). The boundary of Σ is B. The product of B with segments of timelike world lines orthogonal to Σ is called B_3, and is an element of the boundary of M. The end points of the world lines define two boundary hypersurfaces denoted Σ_i and Σ_f, that correspond respectively to $t = t_i$ and $t = t_f$. Note that the boundary of M is the sum of B_3, Σ_i, and Σ_f.

The spacetime metric is g_{ab}, the outward pointing spacelike unit normal to the three-boundary B_3 is n^a. The metric of B_3 is γ_{ab} and its extrinsic curvature is Θ_{ab}. These tensors are defined on B_3 and satisfy

$$n^a \gamma_{ab} = 0 \tag{2.1}$$

$$n^a \Theta_{ab} = 0 \,. \tag{2.2}$$

Note that γ^a_b is the projection tensor onto B_3. Indices i, j refer to intrinsic coordinates on B_3. Therefore, one can write γ_{ij} and Θ_{ij} when referring to tensors on B_3.

We write the future pointing timelike unit normal to the hypersurfaces Σ, that foliate spacetime, as u^a. The metric and extrinsic curvature for Σ are given by h_{ab} and K_{ab}, and h^a_b is the projection tensor onto Σ. When viewed as tensors on Σ, they will be written as h_{ij}, K_{ij}, and $h^i_j = \delta^i_j$.

The spacetime metric can be written, in adapted coordinates, following the usual ADM decomposition, as

$$\begin{aligned} ds^2 &= g_{ab} dx^a dx^b \\ &= -N^2\, dt^2 + h_{ij} \left(dx^i + V^i dt\right) \left(dx^j + V^j dt\right) \,, \end{aligned} \tag{2.3}$$

where N is called the lapse function and V^i is called the shift vector.

We shall assume that Σ is orthogonal to B_3, so that on B_3 we have $u^a n_a = 0$. This allows us to view n^a as a vector n^i in Σ with unit length. Because of this restriction, we can decompose the metric in B_3 as

$$\gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{\alpha\beta} (dx^\alpha + V^\alpha dt)(dx^\beta + V^\beta dt) \,, \tag{2.4}$$

where α, β refer to coordinates on B, $\sigma_{\alpha\beta}$ being the two-surface metric on B. The extrinsic curvature of B as embedded in Σ is denoted by $k_{\alpha\beta}$. These tensors can be viewed also as spacetime tensors σ_{ab} and k_{ab}, or as tensors on Σ by writing σ_{ij} and k_{ij}. The projection tensor onto B is σ^a_b.

We now establish some useful relations. We define time t as a scalar function that labels the hypersurfaces Σ. As the unit normal vector u^a must be proportional to the gradient of the function t, we find that

$$u_a = -N\, t_{,a} \,, \tag{2.5}$$

since $u_a u^a = -1$. We say that a vector field T^a is tangent to Σ, or spatial, if $u_a T^a = 0$. This generalizes naturally to higher-rank tensors. The metric g_{ab} can be decomposed as

$$g_{ab} = -u_a u_b + h_{ab} \,. \tag{2.6}$$

The spacetime covariant derivative is ∇_a, and the covariant derivative compatible with h_{ab} is D_a and is the projection of ∇_a onto Σ. This means that $D_a f = h_a^c \nabla_c f$ for a scalar function f, $D_a T^b = h_a^c h_d^b \nabla_c T^d$ for a spatial vector T^b. The extrinsic curvature of Σ as embedded in M is defined by

$$K_{ab} = -h_a^c \nabla_c u_b \,. \tag{2.7}$$

We will be using adapted coordinates such that t is the time coordinate and x^i, $i = 1, 2, 3$ lie in Σ. These coordinates have been used in Eq. (2.3). In these coordinates, the normal satisfies $u_a = -N\delta_a^0$, and spatial vectors have zero time component. Using Eqs. (2.3) and (2.6), we find that the shift vector is $V^i = h_0^i = -Nu^i$.

Define the momentum for the hypersurfaces Σ as

$$P^{ij} = \frac{1}{16\pi}\sqrt{h}\left(Kh^{ij} - K^{ij}\right) \,, \tag{2.8}$$

where $h = \det h_{ij}$, and $K = K_a^a$ is the trace of the extrinsic curvature.

Denoting n^a as the outward pointing spacelike normal to B_3, the metric γ_{ab} on B_3 can be written as

$$g_{ab} = \gamma_{ab} + n_a n_b \,, \tag{2.9}$$

and the extrinsic curvature of B_3 as

$$\Theta_{ab} = -\gamma_a^c \nabla_c n_b \,. \tag{2.10}$$

Introducing intrinsic coordinates x^i, $i = 1, 2, 3$ on B_3, the intrinsic metric becomes γ_{ij}. We define the boundary momentum by

$$\pi^{ij} = -\frac{1}{16\pi}\sqrt{-\gamma}\left(\Theta\gamma^{ij} - \Theta^{ij}\right) \,, \tag{2.11}$$

where $\gamma = \det \gamma_{ij}$.

We can now write an expression for the metric σ_{ab} on the two-boundaries B, as

$$g_{ab} = -u_a u_b + n_a n_b + \sigma_{ab} \,. \tag{2.12}$$

The extrinsic curvature of B as embedded in Σ is given by

$$k_{ab} = -\sigma_a^c \nabla_c n_b \,. \tag{2.13}$$

Introducing intrinsic coordinates x^α, $\alpha = 1, 2$ on B, the intrinsic metric becomes $\sigma_{\alpha\beta}$.

2.2 Derivation of Brown-York's energy definition

The Brown-York quasilocal energy was first derived in 1993, by employing a Hamilton-Jacobi formalism to the general relativity action in a 3+1 decomposition [29]. In this section we will first describe this formalism for classical mechanics and then apply it to general relativity.

2.2.1 Hamilton-Jacobi formalism for classical mechanics

The classical action S is defined by

$$S = \int L(q_i, \dot{q}_i, t)\, dt\,, \tag{2.14}$$

where $L(q_i, \dot{q}_i, t)$ is the classical Lagrangian, regarded as a function of the generalized coordinates q_i and their time derivative $\dot{q}_i$, and of time t. As L is related to the Hamiltonian H through

$$H = \sum_i p_i \dot{q}_i - L\,, \tag{2.15}$$

where the conjugate momenta p_i are given by

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\,, \tag{2.16}$$

we can write the action as

$$S = \int \left(\sum_i p_i \dot{q}_i - H(q_i, p_i, t) \right) dt\,. \tag{2.17}$$

Note that H is regarded as a function of q_i, p_i and t, that are considered independent.

We can now introduce a parameter λ that describes the system path in state space, which contains phase space and time. The action becomes

$$S = \int_{\lambda_i}^{\lambda_f} \left(\sum_i p_i\, q_i' - t' H \right) d\lambda\,, \tag{2.18}$$

where a prime means differentiation with respect to the parameter λ, λ_i describes the initial state of the system and λ_f describes the final state of the system. Let us now consider the variation of this action. Retaining the boundary terms that come from integration by parts, we get

$$\delta S = \sum_i \left\{ p_i\, \delta q_i|_{\lambda_i}^{\lambda_f} + \int_{\lambda_i}^{\lambda_f} d\lambda \left[\delta p_i \left(q_i' - t' \frac{\partial H}{\partial p_i} \right) - \delta q_i \left(p_i' + t' \frac{\partial H}{\partial q_i} \right) \right] \right\} - H\, \delta t|_{\lambda_i}^{\lambda_f}\,. \tag{2.19}$$

If we consider that the variations at λ_i and λ_f are zero, the requirement that the action has null variation leads to the Hamilton equations for the system configuration, that is

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{2.20}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} \,. \tag{2.21}$$

However, we will not consider that the variations vanish at the boundaries. Instead, we restrict the variations of the action among classical solutions, so that the integral in Eq. (2.19) vanishes trivially. Writing "cl" as the evaluation of a quantity at a classical solution, we get

$$\delta S^{\rm cl} = \sum_i p_i^{\rm cl}\, \delta q_i \Big|_{\lambda_i}^{\lambda_f} - H^{\rm cl}\, \delta t \Big|_{\lambda_i}^{\lambda_f} \,. \tag{2.22}$$

The Hamilton-Jacobi equations follow directly from Eq. (2.22). The classical momentum and Hamiltonian at the final state λ_f are

$$p_i^{\rm cl} = \frac{\partial S_{\rm cl}}{\partial q_i} \tag{2.23}$$

$$H^{\rm cl} = -\frac{\partial S_{\rm cl}}{\partial t} \,, \tag{2.24}$$

where q_i and t are also the values at the final state λ_f. There is therefore an intimate connection between the classical Hamiltonian (the energy) and the change in time of the classical action. Nevertheless, the choice of the action is not unique. In fact, we can add arbitrary functions of the fixed boundary states λ_i and λ_f without changing the classical equations of motion. This freedom allows us to choose the ground level for the energy if we use the Hamilton-Jacobi approach.

The Hamilton-Jacobi equation, Eq. (2.24), also shows that if we know the action, we can calculate the energy and the momentum. Conversely, if we know the momentum and the Hamiltonian, we can calculate the action. As in general relativity we can define an action, if we use the Hamilton-Jacobi formalism it should be possible to derive an expression for the total energy.

2.2.2 Applying the Hamilton-Jacobi formalism to general relativity

We begin by giving a suitable action for a region M of a 3+1 dimensional spacetime, with boundary B_3, Σ_i (in which $t = t_i$), and Σ_f (in which $t = t_i$). Such an action is given by [29]

$$S = \frac{1}{16\pi} \int_M d^4x \sqrt{-g}\, R + \frac{1}{8\pi} \int_\Sigma d^3x \sqrt{h}\, K \Big|_{t_i}^{t_f} - \frac{1}{8\pi} \int_{B_3} d^3x \sqrt{-\gamma}\, \Theta + S_m \,, \tag{2.25}$$

where $R = R^a_a$ is the Ricci scalar and S_m is the matter action, that may include a cosmological constant. The notation $\int_\Sigma d^3x \big|_{t_i}^{t_f}$ represents the integral evaluated at Σ_f minus the integral evaluated at Σ_i. The variation of S due to arbitrary variations of the metric and matter fields is

$$\begin{aligned}\delta S = &-\frac{1}{16\pi}\int_M d^4x\sqrt{-g}\,\delta g_{ab}G^{ab} + \frac{1}{2}\int_M d^4x\sqrt{-g}\,\delta g_{ab}T^{ab} \\ &+ \text{(boundary terms coming from the matter action)} \\ &+ \int_\Sigma d^3x\, P^{ij}\,\delta h_{ij}\Big|_{t_i}^{t_f} + \int_{B_3} d^3x\,\pi^{ij}\delta\gamma_{ij}\,,\end{aligned} \tag{2.26}$$

where $G^{ab} = R^{ab} - \frac{1}{2}g^{ab}R$ is the Einstein tensor, P^{ij} is the gravitational momentum conjugate to h_{ij}, and π^{ij} is the gravitational momentum conjugate to γ_{ij}. The calculation of this variation is detailed in Appendix A.

Proceeding in an analogous way to Hamilton's approach, the variations of the fields at the boundary are required to vanish, and in this case the solutions that extremize the action satisfy Einstein's equations

$$G^{ab} = 8\pi T^{ab}\,. \tag{2.27}$$

The freedom in S can be taken into account by subtracting an arbitrary function S_0 of the fixed boundary fields, just like in Section 2.2.1. Therefore, we can define the action

$$S' = S - S_0\,. \tag{2.28}$$

Note that S_0 depends on the the boundary metric γ_{ij} and on the initial and final metrics $h_{ij}(t_i)$ and $h_{ij}(t_f)$. The variation in S' is just the variation of S, given in Eq. (2.26) minus the variation of S_0, given by

$$\delta S_0 = \int_\Sigma d^3x\frac{\delta S_0}{\delta h_{ij}}\,\delta h_{ij}\Big|_{t_i}^{t_f} + \int_{B^3} d^3x\frac{\delta S_0}{\delta\gamma_{ij}}\delta\gamma_{ij} = \int_\Sigma d^3x P_0^{ij}\,\delta h_{ij}\Big|_{t_i}^{t_f} + \int_{B^3} d^3x\pi_0^{ij}\delta\gamma_{ij}\,, \tag{2.29}$$

where P_0^{ij} and π_0^{ij} are defined as the functional derivative of S_0 with respect to h_{ij} and γ_{ij}, respectively.

We now consider only classical actions S^{cl}, that is actions that satisfy Einstein's equations given in Eq. (2.27). The variation of classical actions depend only on the fixed boundary information, and, from Eq. (2.26), is given by

$$\begin{aligned}\delta S^{\text{cl}} = &\text{(boundary terms coming from the matter action)} \\ &+ \int_\Sigma d^3x\left(P^{ij} - P_0^{ij}\right)\delta h_{ij}\Big|_{t_i}^{t_f} + \int_{B^3} d^3x\left(\pi^{ij} - \pi_0^{ij}\right)\delta\gamma_{ij}\,,\end{aligned} \tag{2.30}$$

where P^{ij} and π^{ij} are evaluated at the classical solution. The analogues of Eqs. (2.23) and (2.24) are thus

$$P^{ij} - P_0^{ij} = \frac{\delta S_{\rm cl}}{\delta h_{ij}} \tag{2.31}$$

$$\pi^{ij} - \pi_0^{ij} = \frac{\delta S_{\rm cl}}{\delta \gamma_{ij}} \,. \tag{2.32}$$

Here, $P^{ij} - P_0^{ij}$ is the gravitational momentum at Σ_f and, since now the metric γ_{ij} takes the role of time in Eq. (2.24), the Hamiltonian is replaced by a surface stress-energy-momentum tensor, defined by

$$\tau^{ij} \equiv \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\rm cl}}{\delta \gamma_{ij}} = \frac{2}{\sqrt{-\gamma}} \left(\pi^{ij} - \pi_0^{ij} \right) \,. \tag{2.33}$$

It is important to note that τ^{ij} characterizes the entire system, and includes contributions from the gravitational field and from other matter fields.

Note also that the definition in Eq. (2.33) is analogous to the definition of the standard matter energy-momentum tensor

$$T^{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S^m}{\delta g_{ab}} \,. \tag{2.34}$$

From Eq. (2.33), we can define the proper energy surface density ϵ, the proper momentum surface density j_a, and the spatial stress s^{ab} as the normal and tangential projections of τ^{ij} on a two surface B

$$\epsilon \equiv u_i u_j \tau^{ij} \tag{2.35}$$

$$j_a \equiv -\sigma_{ai} u_j \tau^{ij} \tag{2.36}$$

$$s^{ab} \equiv \sigma^a_i \sigma^b_j \tau^{ij} \,. \tag{2.37}$$

Using the relationships

$$\frac{\partial \gamma_{ij}}{\partial N} = -\frac{2 u_i u_j}{N} \tag{2.38}$$

$$\frac{\partial \gamma_{ij}}{\partial V^a} = -\frac{2 \sigma_{a(i} u_{j)}}{N} \tag{2.39}$$

$$\frac{\partial \gamma_{ij}}{\partial \sigma_{ab}} = \sigma^a_{(i} \sigma^b_{j)} \,, \tag{2.40}$$

we can rewrite Eqs. (2.35)-(2.37) as

$$\epsilon = -\frac{1}{\sqrt{\sigma}}\frac{\delta S^{\rm cl}}{\delta N} \tag{2.41}$$

$$j_a = \frac{1}{\sqrt{\sigma}}\frac{\delta S^{\rm cl}}{\delta V^a} \tag{2.42}$$

$$s^{ab} = \frac{2}{\sqrt{-\gamma}}\frac{\delta S^{\rm cl}}{\delta \sigma_{ab}}\,. \tag{2.43}$$

These quantities are defined on a 2-surface B and, when integrated over B, represent the energy, momentum, and stress contained on the spacelike hypersurface Σ with boundary B. The quantity that interests us the most is the quasilocal energy in Σ, which is given by

$$E = \int_B d^2x\sqrt{\sigma}\epsilon\,. \tag{2.44}$$

We now simplify the expression of ϵ. In Appendix A, we show that

$$\frac{2}{\sqrt{-\gamma}}u_i u_j \pi^{ij} = \frac{k}{8\pi} \tag{2.45}$$

$$\frac{2}{\sqrt{-\gamma}}\sigma^a_i\sigma^b_j\pi^{ij} = \frac{1}{8\pi}\left[k^{ab} + (n\cdot a - k)\,\sigma^{ab}\right]\,, \tag{2.46}$$

where k is the trace of the extrinsic curvature k_{ij} of B, and $a^b = u^c\nabla_c u^b$ is the so-called acceleration of u^b. Therefore, ϵ and s^{ab} become

$$\epsilon = \frac{k}{8\pi} + \frac{1}{\sqrt{\sigma}}\frac{\delta S^{\rm cl}}{\delta N} \tag{2.47}$$

$$s^{ab} = \frac{1}{8\pi}\left[k^{ab} + (n\cdot a - k)\,\sigma^{ab}\right] - \frac{2}{\sqrt{-\gamma}}\frac{\delta S^{\rm cl}}{\delta\sigma_{ab}}\,. \tag{2.48}$$

Demanding that the energy surface density ϵ should depend only on h_{ij} and P^{ij} defined on Σ, and following Brown-York's method for specifying the ambiguity due to S_0, the reference action is

$$S_0 = -\int_{B_3} d^3x\left[N\sqrt{\sigma}\frac{k_0}{8\pi} + 2\sqrt{\sigma}V^a\left(\frac{\sigma_{ak}n_l P^{kl}}{\sqrt{h}}\right)_0\right]\,, \tag{2.49}$$

where k_0 and $\left(\frac{\sigma_{ak}n_l P^{kl}}{\sqrt{h}}\right)_0$ are chosen in a reference space where one considers a surface whose induced 2-metric is σ_{ab}. With this reference action, the energy surface density becomes

$$\epsilon = \frac{k - k_0}{8\pi}\,. \tag{2.50}$$

Note that, by definition, ϵ is zero when evaluated in the reference spacetime, so we can say that the freedom in choosing a reference spacetime is related to the freedom of choosing the ground level of the energy. With this expression, Eq. (2.44) becomes

$$E = \frac{1}{8\pi} \int_B d^2x \sqrt{\sigma} \left(k - k_0\right) \,. \tag{2.51}$$

We now consider the case of a static, spherically symmetric spacetime with metric given by

$$ds^2 = -N^2\, dt^2 + h^2\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \,, \tag{2.52}$$

where N and h are functions of r only. Here, Σ is the interior of a $t =$ constant slice with 2-boundary B determined by $r =$ constant. Calculating the extrinsic curvature k_{ab} yields

$$k_\theta^\theta = k_\phi^\phi = -\frac{1}{rh} \,. \tag{2.53}$$

The subtraction term k_0 is obtained using the results of Eq. (2.53) with $h = 1$ (that is, in Minkowski spacetime). Thus, we get for the proper energy density

$$\epsilon = \frac{1}{4\pi} \left(\frac{1}{r} - \frac{1}{rh}\right) \,, \tag{2.54}$$

and the quasilocal energy is given by

$$E_{\rm BY}(r) = r \left(1 - \frac{1}{h}\right) \,. \tag{2.55}$$

Eq. (2.55) will be used in Chapter 3 to find expressions for Brown-York's quasilocal energy in different static, spherically symmetric spacetimes.

Finally, we can calculate the surface pressure s, which is defined by

$$s = \frac{1}{2} \sigma_{ab} s^{ab} \,. \tag{2.56}$$

Using Eq. (2.43), the fact that the acceleration of the timelike unit normal u^a satisfies

$$n \cdot a = \frac{N'}{Nh} \,, \tag{2.57}$$

where a prime represents differentiation with respect to r, and that, according to Eq. (2.49), the reference action S_0 is given by

$$S_0 = \frac{1}{4\pi} \int dt\, d\theta\, d\phi N R \sin\theta \,, \tag{2.58}$$

we get that

$$s = \frac{1}{2} \sigma_{ab} s^{ab} = \frac{1}{8\pi} \left(\frac{N'}{Nh} + \frac{1}{rh} - \frac{1}{r}\right) \,. \tag{2.59}$$

This concludes our derivation of Brown-York's quasilocal energy definition. In the next section, we shall introduce Lynden-Bell and Katz's quasilocal energy, which, despite being very similar to Brown-York's, is derived in a completely different way.

2.3 Derivation of Lynden-Bell-Katz energy definition

Lynden-Bell and Katz's approach to find an expression for the gravitational energy is based in taking the total energy in a standard way, and then subtracting the matter energy [37]. Their first expression is not quasilocal in the sense that it is not calculated on the boundary, but is an integral through the slice Σ, and is only valid for stationary spacetimes. However, Israel's formalism (see [42, 43, 40])allowed Lynden-Bell-Katz's energy to be formulated in terms of surface integrals, and thus allowed a cleaner comparison between this definition and other quasilocal definition, namely Brown-York's.

2.3.1 Lynden-Bell and Katz definition of gravitational energy for static spacetimes

In this section, we will follow Lynden-Bell and Katz's approach to find an expression for the gravitational energy density in static, spherically symmetric spacetimes [37].

In special relativity, we can define the total energy of matter and radiation E_m as

$$E_m = \int_\Sigma T_a^b \xi^a u_b \sqrt{h} d\Sigma \,, \tag{2.60}$$

where ξ^a is the timelike Killing vector field of the Minkowski space, Σ is a spacelike hypersurface, T_a^b is the matter energy-momentum tensor, h_{ab} is the hypersurface induced metric, $\sqrt{h}d\Sigma$ is the volume element of the surface, and u_b is the future-pointing unit normal. Eq. (2.60) can be generalized for the curved space of general relativity, as long as there is a timelike Killing vector field.

Let us consider the Schwarzschild metric in Schwarzschild coordinates, which may describe an asymptotically flat spacetime at infinity, given by

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \,, \tag{2.61}$$

where m is the mass parameter. The ADM mass for this metric in the spacelike hypersurface Σ, that is the gravitational mass measured at infinity, is equal to m. However, the energy E_m is not always equal to m. Hence, we can write

$$m = E_m + E_{\text{grav}} \,, \tag{2.62}$$

where E_{grav} is the energy attributed to the gravitational field. This is valid for any stationary, asymptotically flat spacetime, provided that we replace m by the ADM mass for that spacetime.

We now consider a static spherically symmetric spacetime, in Schwarzschild coordinates. We perform a cut on the metric on a sphere with radial parameter r, so that the exterior spacetime is unchanged and the interior spacetime is flat. To accomplish this, there must be a surface matter distribution given by the discontinuities in the gradient of the metric tensor introduced by the cut. Since the exterior spacetime is unchanged, it is still described at infinity by the Schwarzschild metric, and the ADM mass is m. To calculate the gravitational field energy of the cut system, we take the difference between the total ADM mass and the matter energy, which has contributions from the matter exterior to the cut and from the surface distribution of the sphere. We write their sum as $E_m(r)$. By Eq. (2.62), the gravitational field energy of the cut system is

$$E_{\text{grav}}(r) = m - E_m(r) \,. \tag{2.63}$$

Assuming now that a flat spacetime has no gravitational field energy, it follows that all of it comes from the exterior spacetime. As outside the cut the spacetime of the cut system is identical to the spacetime of the original system, we also assume that the field energies are the same in the two cases.

To calculate the gravitational energy density, we change the position of the cut from the sphere of radial coordinate R to the sphere of radial coordinate $R + dR$. We write dE_{grav} as the field energy contained in the volume enclosed between the two spheres, in the original system. As this energy is uniformly distributed over spheres, we can divide dE_{grav} by the volume between the spheres to find the gravitational field energy density.

We evaluate the gravitational field energy density for Schwarzschild's spacetime as an example. Taking the isotropic radial coordinate of the sphere to be $r = R$, we get for the isotropic form of the metric

$$ds^2 = \begin{cases} -\left(\frac{1-\frac{m}{r}}{1+\frac{m}{r}}\right)^2 dt^2 + \left(1+\frac{m}{2r}\right)^4 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \,, & r > R \\ -\left(\frac{1-\frac{m}{R}}{1+\frac{m}{R}}\right)^2 dt^2 + \left(1+\frac{m}{2R}\right)^4 \left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \,, & r \leq R \end{cases} . \tag{2.64}$$

Note that we use isotropic coordinates because they are the only compatible with having a point massive source at the origin or with having a shell of matter at a fixed radius [44, 45], and therefore they are the most appropriate the work with matter shells. The energy density component of the appropriate energy-momentum tensor of the shell, which corresponds to the metric given in Eq. (2.64) is

$$T^0_0 = \frac{m}{4\pi R^2 \left(1+\frac{m}{2R}\right)^5}\delta(r-a)\,, \tag{2.65}$$

so we can calculate E_m directly, getting

$$E_m = \int T^0_0 \left(\frac{1-\frac{m}{R}}{1+\frac{m}{R}}\right)\left(1+\frac{m}{2R}\right)^6 r^2 \sin\theta dr\, d\theta\, d\phi = m - \frac{m^2}{2R}\,. \tag{2.66}$$

Therefore, the gravitational energy (outside the shell) is $E_{\text{grav}} = \frac{m^2}{2R}$ and so the gravitational energy between the spheres is $dE_m = \frac{m^2}{2R^2}\, dR$. Since the area of the sphere of the sphere with isotropic radius R is $4\pi R^2 \left(1+\frac{m}{2R}\right)^4$ and the radial distance corresponding to dR is $\left(1+\frac{m}{2R}\right)^2 dR$, the gravitational field energy density ρ_{grav} becomes

$$\rho_{\text{grav}} = \frac{1}{8\pi}\left[\frac{m}{R^2\left(1+\frac{m}{2R}\right)^3}\right]^2 . \tag{2.67}$$

This positive energy density is coordinate independent (as long as the change of coordinates keeps the Killing vector field).

We have obtained an expression for the gravitational field energy and energy density for stationary spacetimes. Now, we shall write it in a quasilocal form, so that the total energy inside a sphere can be calculated only from the information at the boundary.

2.3.2 Israel's expression for Lynden-Bell and Katz energy definition

Following Grøn's approach [38, 39], we now proceed to express Eq. (2.63) in terms of surface layers quantities. We consider the same spacetime cut as last section. However, now we call B the closed surface whose spacetime inside is flat, and that is defined by a constant radial coordinate. Considering the total energy inside B, E_B, as the sum of the matter and gravitational energies inside B. We can write

$$E_{\text{grav}} = E_B - E_m\,, \tag{2.68}$$

where E_{grav} is the gravitational energy inside B, and E_m is the matter energy inside B, which is given by

$$E_m = \int_V T^b_a \xi^a u_b\, dV\,, \tag{2.69}$$

where V is the volume inside B, and dV its invariant element. Note the similarity with Eq. (2.60).

Considering now the extrinsic curvature of B, which we write as k_{ij}, we can define [42, 46, 43] Lanczos' surface energy-momentum tensor S^i_j as

$$S^i_j = \frac{1}{8\pi}\left([k^i_j] - \delta^j_i\,[k]\right)\,, \tag{2.70}$$

where $k = k^i_i$ is the trace of the extrinsic curvature and $[\,]$ means discontinuity at B. Furthermore, the total energy inside the cut B is defined by

$$E_B = \int_B S^b_a \xi^a u_b \, dB\,. \tag{2.71}$$

Note that this is already a quasilocal quantity, as it is calculated exclusively with boundary information. The gravitational field energy is then given by

$$E_{\rm grav} = \int_B S^b_a \xi^a u_b \, dB - \int_V T^b_a \xi^a u_b \, dV\,. \tag{2.72}$$

If the matter energy-momentum tensor vanishes, then the gravitational energy is also given by Eq. (2.71).

In order to compare this formalism with Brown-York's, we only consider from now on the total energy. Using Eq. (2.70), the fact that the projection of the extrinsic curvature k_{ij} on u_a is zero, and that $\frac{\xi^a}{|\xi|} = u^a$, with $|\xi| \equiv \sqrt{-\xi^a\xi_a}$, we get that the total Lynden-Bell-Katz quasilocal energy can be written as

$$E_B = \frac{1}{8\pi}|\xi| \int_B d^2x \sqrt{\sigma}\,(k - k_0)\,. \tag{2.73}$$

Note the remarkable result that, apart from the factor $|\xi|$, this is the Brown-York energy found in Eq. (2.51). As we shall see, this factor makes all the difference in the interpretation and on the results of the two energy definitions.

Proceeding like in Section 2.2.2, and considering the metric for a static, spherically symmetric spacetime given in Eq. (2.52), we find that the Lynden-Bell and Katz quasilocal energy inside a sphere of radial coordinate r is

$$E_{\rm LBK} = N\,r\left(1 - \frac{1}{h}\right)\,, \tag{2.74}$$

where $N = |\xi|$.

Chapter 3

Comparison of Brown-York energy and Lynden-Bell-Katz energy in specific cases

We have derived two different quasilocal energy definitions. It would be interesting to understand in what they are different. As we shall see, there are differences in the interpretation of the localization of energy in the Newtonian limit, that are evident in the expressions for the energy density derived from the quasilocal energy expression. In this chapter, we compare the energies and energy densities in different spacetimes, and we derive the laws of black hole mechanics that follow from each definition. A calculation of the Lynden-Bell and Katz energy in some specific spacetimes is also presented in [47, 38]. For the Brown-York energy, the calculations for some exact solutions is presented in [36], where there is also a method to calculate the quasilocal energy inside the event horizon of a black hole. Nevertheless, the method of calculating the energy densities in order to compare two quasilocal energy definitions has never been done.

3.1 Definition of energy density for static, spherically symmetric spacetimes

In order to compare the two definitions of quasilocal energy, we shall compare the energy densities generated by each energy in the Newtonian limit. For that, we need a definition for energy density from a quasilocal energy expression. We consider static, spherically symmetric spacetimes because all the spacetimes that are going to be analyzed share this property.

Let us consider a sphere S with radial coordinate r. The quasilocal energy inside S is $E(r)$. Now consider a sphere S' with radial coordinate $r + dr$, which contains an energy $E(r + dr)$. The energy contained in the volume enclosed between S and S' is $dE = E(r+dr) - E(r)$. Because of spherically symmetry, this energy is uniformly distributed across the sphere.

To calculate the volume, we consider a metric in the form of Eq. (2.52). The volume is given by the product of the area of the sphere $4\pi r^2$ by the radial distance $h\,dr$. We can then define an energy density ρ as

$$\rho = \frac{1}{4\pi h\, r^2}\frac{dE}{dr}\,. \tag{3.1}$$

This expression is the same as the one obtained by Lynden-Bell and Katz [37].

3.2 Schwarzschild spacetime

The first spacetime that we are going to analyze is Schwarzschild's spacetime. It was the first exact solution of Einstein's equations, and corresponds to a vacuum solution with ADM mass m, no electric charge and no angular momentum. It has an intrinsic singularity at the origin and describes the simplest black hole.

The corresponding metric can be written, in Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{1}{1-\frac{2m}{r}}\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \tag{3.2}$$

where m is the standard mass parameter. We only study this solution outside the event horizon $r = 2m$, where there is a coordinate singularity that can be removed through a change of coordinates.

3.2.1 Brown-York energy in a Schwarzschild spacetime

The Brown-York quasilocal energy for a surface of constant r is, using Eq. (2.55),

$$E_{\mathrm{BY}}(r) = r\left(1 - \sqrt{1 - \frac{2m}{r}}\right)\,. \tag{3.3}$$

In the special cases $r = 2m$ (event horizon) and $r \to \infty$, we get

$$E_{\mathrm{BY}}(2m) = 2m\,, \tag{3.4}$$

$$\lim_{r\to\infty} E_{\mathrm{BY}}(r) = m\,. \tag{3.5}$$

This tells us that the total energy inside the event horizon (which includes gravitational energy and the energy of the singularity at $r = 0$) is twice the total spacetime energy m. Therefore, the total gravitational energy outside the horizon must be negative and equal to $-m$. This makes sense in a Newtonian perspective since gravity is exclusively an attractive force and thus the gravitational potential energy is always negative.

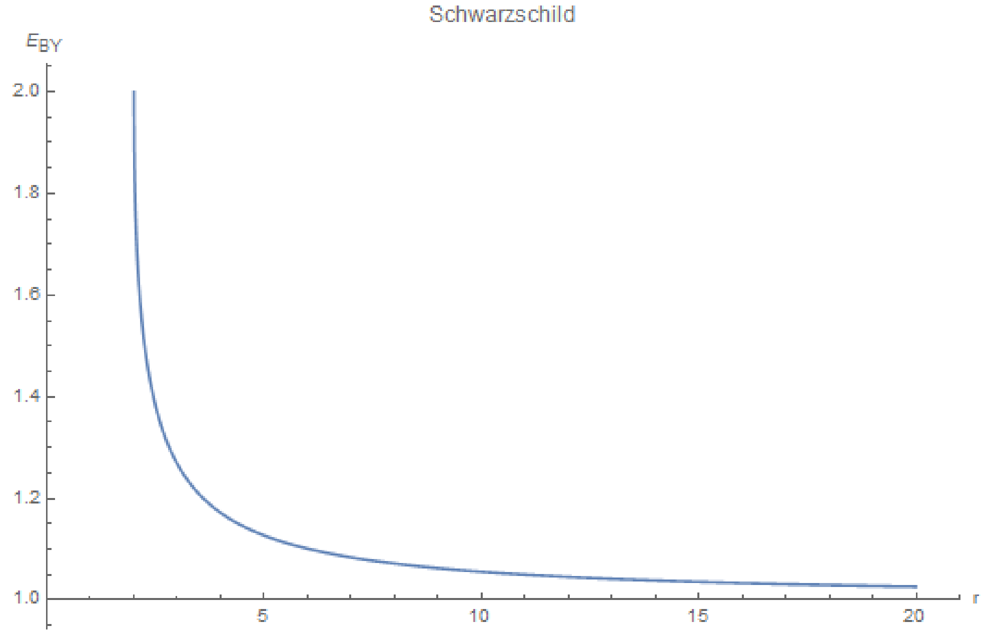

FIGURE 3.1: Brown-York energy for Schwarzschild spacetime, with m=1.

Calculating the density defined in Eq.(3.1) yields

$$\rho_{\mathrm{BY}} = \frac{\sqrt{1-\frac{2m}{r}}}{4\pi r^2}\frac{dE_{\mathrm{BY}}}{dr} = \frac{\sqrt{1-\frac{2m}{r}}}{4\pi r^2}\left(1-\frac{1-\frac{m}{r}}{\sqrt{1-\frac{2m}{r}}}\right) . \tag{3.6}$$

We can approximate the last expression of Eq. (3.6) to second order in $\frac{m}{r}$, getting

$$\rho_{\mathrm{BY}} \approx -\frac{m^2}{8\pi r^4} = -\frac{|\boldsymbol{g}|^2}{8\pi} , \tag{3.7}$$

where $\boldsymbol{g} = -\frac{m}{r^2}\,\boldsymbol{e_r}$ is the Newtonian gravitational field, $\boldsymbol{e_r}$ being the unit radial vector. It is interesting to note that the total Newtonian gravitational potential energy U can be calculated with

$$U = \int_{\mathrm{space}} -\frac{|\boldsymbol{g}|^2}{8\pi}\, dV . \tag{3.8}$$

This means that Brown-York's energy in Schwarzschild spacetime corresponds to the interpretation that the gravitational energy is all stored in the field.

3.2.2 Lynden-Bell-Katz energy in a Schwarzschild spacetime

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (2.74),

$$E_{\mathrm{LBK}}(r) = \sqrt{1-\frac{2m}{r}}\left(1-\sqrt{1-\frac{2m}{r}}\right) r\,. \tag{3.9}$$

In the special cases $r = 2m$ (event horizon) and $r \to \infty$, we get

$$E_{\mathrm{LBK}}(2m) = 0\,, \tag{3.10}$$

$$\lim_{r\to\infty} E_{\mathrm{LBK}}(r) = m\,. \tag{3.11}$$

This tells us that the total energy inside the event horizon is zero. Therefore, the total spacetime gravitational energy is contained outside the horizon, and is positive and equal to m. This result does not have an immediate interpretation within Newton's theory, since gravitational energy should be negative definite. However, the total energy at infinity coincides with Brown-York's.

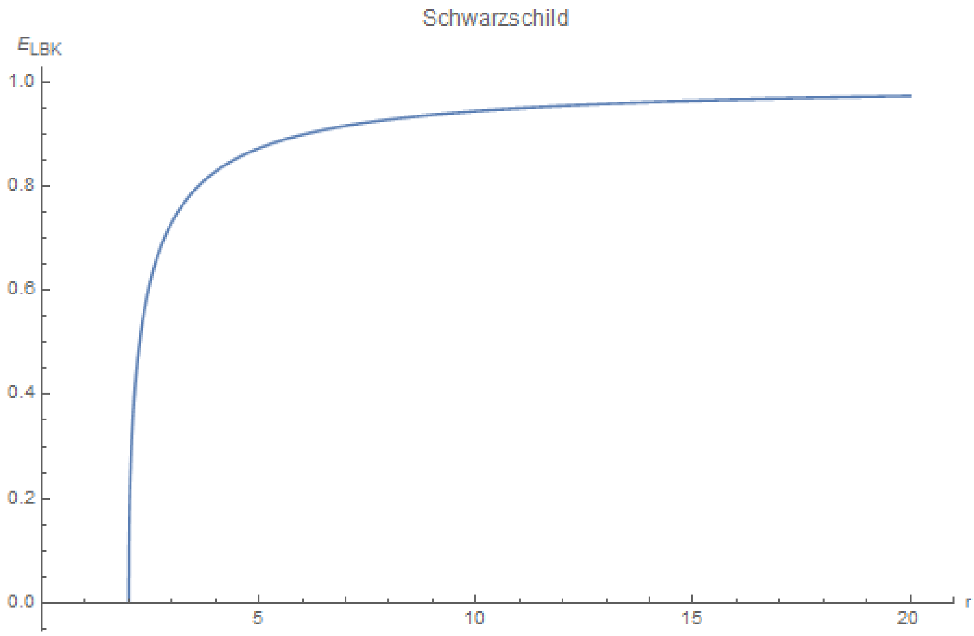

FIGURE 3.2: Lynden-Bell-Katz energy for Schwarzschild spacetime, with m=1.

Proceeding in analogy with Section 3.2.1, we can calculate the density ρ_{LBK} associated with this energy, getting

$$\begin{aligned}\rho_{\mathrm{LBK}} &= \frac{\sqrt{1-\frac{2m}{r}}}{4\pi r^2}\frac{dE_{\mathrm{LBK}}}{dr}\\ &= \frac{\sqrt{1-\frac{2m}{r}}}{4\pi r^2}\left(\left(1-\sqrt{1-\frac{2m}{r}}\right)\sqrt{1-\frac{2m}{r}} - \frac{m}{r} + \frac{m\left(1-\sqrt{1-\frac{2m}{r}}\right)}{\sqrt{1-\frac{2m}{r}}\,r}\right)\,.\end{aligned} \tag{3.12}$$

In order to better understand the differences between ρ_{LBK} and ρ_{BY}, we can approximate the last result of Eq. (3.12) to second order in $\frac{m}{r}$. This yields

$$\rho_{\mathrm{LBK}} \approx \frac{m^2}{8\pi r^4} = \frac{|\boldsymbol{g}|^2}{8\pi} \,. \tag{3.13}$$

This density is the opposite of the result of Eq. (3.7). However, this is less justifiable according to Newtonian theory. To solve this problem, Lynden-Bell and Katz interpret that gravitational energy is contained in the matter and in the field, and summing the two gives the total Newtonian gravitational potential energy. This will be more evident in Section 3.6.2.

3.3 Reissner-Nordström spacetime

The Reissner-Nordström spacetime is a solution of the Einstein-Maxwell equations in a vacuum [2], given by

$$G_{ab} = 8\pi T_{ab} \tag{3.14}$$

$$\nabla_b F^{ab} = 0 \,, \tag{3.15}$$

where F^{ab} is the electromagnetic field generated by a charge located in $r = 0$ and the energy-momentum tensor is given by

$$T_{ab} = \frac{1}{4\pi}\left(F^{ac}F^{b}_{c} - \frac{1}{4}g^{ab}F_{de}F^{de}\right) \,. \tag{3.16}$$

This solution describes a charged, non-rotating, black hole.

The corresponding metric can be written, in Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \frac{1}{1 - \frac{2m}{r} + \frac{q^2}{r^2}}\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) \,, \tag{3.17}$$

where m is the standard mass parameter and q is the electric charge of the singularity in geometric units.

3.3.1 Brown-York energy in a Reissner-Nordström spacetime

The Brown-York quasilocal energy for a surface of constant r , using Eq. (2.55),

$$E_{\mathrm{BY}}(r) = r\left(1 - \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}}\right) \,. \tag{3.18}$$

We now consider the case $|q| < m$. In the special cases $r = 0$ (singularity) $r = r_- = m - \sqrt{m^2 - q^2}$ (inner event horizon)$r = r_+ = m + \sqrt{m^2 - q^2}$ (outer event horizon) and $r \to \infty$, we get

$$\lim_{r \to 0} E_{\mathrm{BY}}(r) = -|q| \,, \tag{3.19}$$

$$E_{\mathrm{BY}}(r_-) = r_- \,, \tag{3.20}$$

$$E_{\mathrm{BY}}(r_+) = r_+ \,, \tag{3.21}$$

$$\lim_{r \to \infty} E_{\mathrm{BY}}(r) = m \,. \tag{3.22}$$

Note that Eq. (3.19) implies that the charged singularity has an intrinsic negative energy. That energy compensates the electromagnetic energy of the spacetime so that the total energy at infinity does not depend on the charge q. Note also that the energy at an event horizon is equal to the radius r of the horizon. Moreover, the total energy outside the outer horizon is negative and equal to $-\sqrt{m^2 - q^2}$. This means that, in that region, the positive definite electromagnetic energy does not compensate the negative gravitational energy.

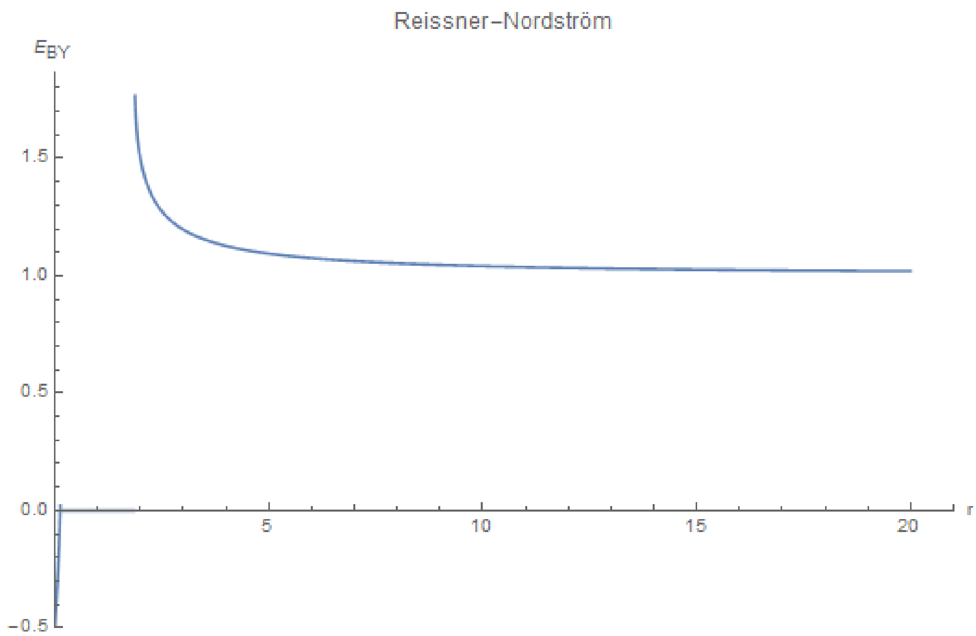

FIGURE 3.3: Brown-York energy for Reissner-Nordström spacetime, with m=1 and $q = \frac{1}{2}$.

We can now briefly consider the case where $|q| > m$. The only difference is that now there are no event horizons, so the line element Eq. (3.17) is everywhere regular (except for the intrinsic singularity at $r = 0$). The singularity becomes visible by a distant observer. However, the values of the quasilocal energy and energy density are not changed.

Proceeding in the same way as in Section 3.2.1, we can find the energy density $\rho_{\rm BY}$ outside the outer event horizon. Performing the calculations, we get

$$\rho_{\rm BY} = \frac{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{4\pi r^2}\frac{dE_{\rm BY}}{dr} = \frac{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{4\pi r^2}\left(1-\frac{1-\frac{m}{r}}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\right). \tag{3.23}$$

Approximating the last expression of Eq. (3.23) to second order in $\frac{m}{r}$ and in $\frac{q}{r}$, we get

$$\rho_{\rm BY} \approx -\frac{m^2}{8\pi r^4} + \frac{q^2}{8\pi r^4} = -\frac{|\boldsymbol{g}|^2}{8\pi} + \frac{|\boldsymbol{E}|^2}{8\pi}, \tag{3.24}$$

where $\boldsymbol{E} = \frac{q}{r^2}\,\boldsymbol{e_r}$ is the electric field produced by the charge q. Eq. (3.24) contains the result found in Eq. (3.7) and the electromagnetic energy density. This result is remarkable and very elegant since the gravitational and electromagnetic energy densities have exactly the same structure, differing only in the sign.

3.3.2 Lynden-Bell-Katz energy in a Reissner-Nordström spacetime

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (2.74),

$$E_{\rm LBK}(r) = \sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}\left(1-\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}\right) r\,. \tag{3.25}$$

As in Section 3.3.1, we consider the case $|q| < m$. In the special cases $r = 0$ (singularity) $r = r_- = m - \sqrt{m^2-q^2}$ (inner event horizon)$r = r_+ = m + \sqrt{m^2-q^2}$ (outer event horizon) and $r \to \infty$, we get

$$\lim_{r\to 0} E_{\rm BY}(r) = -\infty\,, \tag{3.26}$$

$$E_{\rm BY}(r_-) = 0\,, \tag{3.27}$$

$$E_{\rm BY}(r_+) = 0\,, \tag{3.28}$$

$$\lim_{r\to\infty} E_{\rm BY}(r) = m\,. \tag{3.29}$$

Note that Eq. (3.26) implies that the charged singularity has an intrinsic infinite energy, just like a point charge in classical electromagnetism. However, the energy inside r_- cancels exactly that infinite energy. Note also that the energy at any event horizon is zero. Moreover, the total energy outside the outer horizon is positive and equal to m, as in the case of the Schwarzschild spacetime.

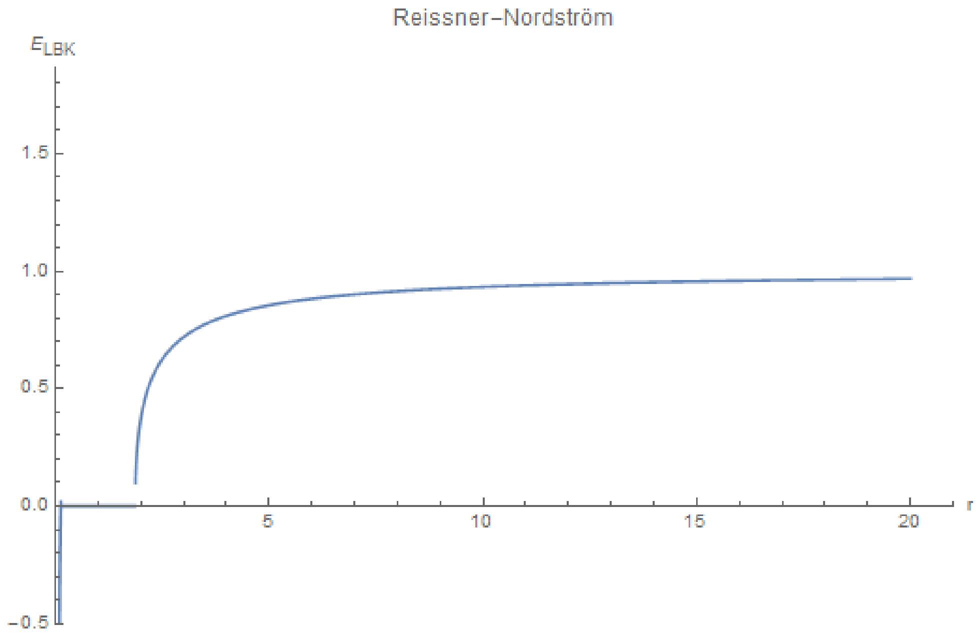

FIGURE 3.4: Lynden-Bell-Katz energy for Reissner-Nordström spacetime, with m=1 and $q = \frac{1}{2}$.

Proceeding in the same way as in Section 3.2.1, we can find the energy density ρ_{LBK} outside the outer event horizon. Performing the calculations, we get

$$\begin{aligned}\rho_{\text{LBK}} &= \frac{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{4\pi r^2}\frac{dE_{\text{LBK}}}{dr} \\ &= \frac{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{4\pi r^2}\frac{(q^2-r^2)\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}+r(r-m)}{r^2\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\,.\end{aligned} \tag{3.30}$$

Approximating the last expression of Eq. (3.30) to second order in $\frac{m}{r}$ and in $\frac{q}{r}$, we get

$$\rho_{\text{LBK}} \approx \frac{m^2}{8\pi r^4}+\frac{q^2}{8\pi r^4} = \frac{|\boldsymbol{g}|^2}{8\pi}+\frac{|\boldsymbol{E}|^2}{8\pi}\,. \tag{3.31}$$

where $\boldsymbol{E}$ is the electric field produced by the charge q, as in Eq. (3.24). Eq. (3.31) contains the result found in Eq. (3.13) and the electromagnetic energy density. The interpretation of the gravitational energy term is the same as the interpretation in Eq. (3.13). However, even though the gravitational energy term differs by a sign between Brown-York and Lynden-Bell-Katz energy densities, they both yield the expected result for the electromagnetic energy density.

3.4 De Sitter spacetime

The de Sitter spacetime is a solution of the Einstein equations Eq. (2.27) in a vacuum with a positive cosmological constant Λ, given by

$$G_{ab} + \Lambda g_{ab} = 0\,. \tag{3.32}$$

The corresponding spacetime metric can be written, in Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -\left(1 - \frac{r^2}{l^2}\right) dt^2 + \frac{1}{1 - \frac{r^2}{l^2}} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)\,, \tag{3.33}$$

where $l = \sqrt{\frac{3}{\Lambda}}$ is the cosmological radius, Λ being the cosmological constant that enters Einstein's equations.

3.4.1 Brown-York energy in a de Sitter spacetime

The Brown-York quasilocal energy for a surface of constant r is, using Eq. (2.55),

$$E_{\mathrm{BY}}(r) = r\left(1 - \sqrt{1 - \frac{r^2}{l^2}}\right)\,. \tag{3.34}$$

We now consider the case $r < l$. In the special cases $r = 0$ and $r = l$ (cosmological horizon), we get

$$E_{\mathrm{BY}}(0) = 0\,, \tag{3.35}$$

$$E_{\mathrm{BY}}(l) = l\,. \tag{3.36}$$

Note that, as found in Section 3.3.1, the energy at the horizon is equal to its radius r. Note also that, since now there is no charged singularity in $r = 0$, there is no energy associated with $r = 0$.

Analogously to Section 3.2.1, the energy density ρ_{BY} is given by

$$\rho_{\mathrm{BY}} = \frac{\sqrt{1 - \frac{r^2}{l^2}}}{4\pi r^2}\frac{dE_{\mathrm{BY}}}{dr} = \frac{\sqrt{1 - \frac{r^2}{l^2}}}{4\pi r^2}\left(1 - \frac{1 - \frac{2r^2}{l^2}}{\sqrt{1 - \frac{r^2}{l^2}}}\right)\,. \tag{3.37}$$

Approximating Eq. (3.37) to second order in $\left(\frac{r}{l}\right)^2$, we get

$$\rho_{\mathrm{BY}} \approx \frac{3}{8\pi l^2} - \frac{r^2}{32\pi l^4} = \frac{\Lambda}{8\pi} - \frac{1}{8\pi}\left(\frac{\Lambda r}{6}\right)^2\,. \tag{3.38}$$

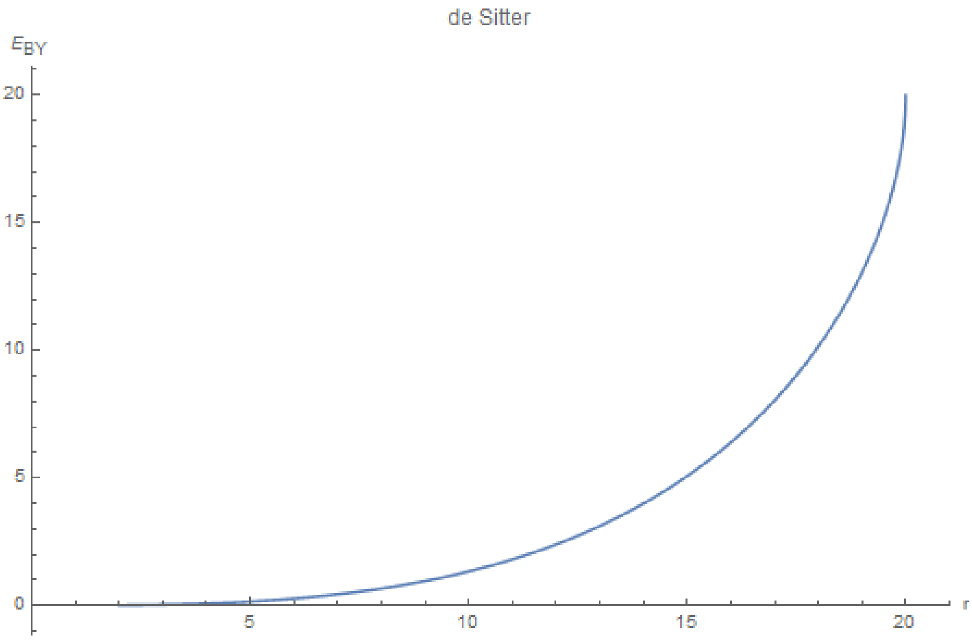

FIGURE 3.5: Brown-York energy for the de Sitter spacetime, with l=20.

We can interpret the right hand side of Eq. (3.38) in the following way. The first term, $\frac{\Lambda}{8\pi}$, is just the energy density associated with the cosmological constant, or dark energy, and the second term is associated with the gravitational field produced by the dark energy (since it has the dependence in Λ^2). This result is the expected Newtonian limit, since the gravitational field created by a continuous distribution of mass with density $\frac{\Lambda}{8\pi}$, as is the case in the De Sitter spacetime, is $\boldsymbol{g} = -\frac{\Lambda r}{6}\boldsymbol{e_r}$.

3.4.2 Lynden-Bell-Katz energy in a de Sitter spacetime

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (2.74),

$$E_{\text{LBK}}(r) = \sqrt{1 - \frac{r^2}{l^2}} \left(1 - \sqrt{1 - \frac{r^2}{l^2}}\right) r \,. \tag{3.39}$$

We consider the case $r < l$, as in Section 3.4.1. In the special cases $r = 0$ and $r = l$ (cosmological horizon), we get

$$E_{\text{LBK}}(0) = 0 \,, \tag{3.40}$$

$$E_{\text{LBK}}(l) = 0 \,. \tag{3.41}$$

Note that, as found in Section 3.3.2, the energy at the horizon is zero. Note also that, since now there is no charged singularity in $r = 0$, there is no energy associated with $r = 0$. The result that the energy inside the cosmological horizon is zero may be intriguing. It means that the gravitational energy and dark energy cancel exactly. We could use this result to support the idea that the total energy of the universe is zero, so that the universe could have come into existence from a quantum vacuum. However, this interpretation would only make sense when agreed that Lynden-Bell-Katz energy is the correct definition of energy in general relativity.

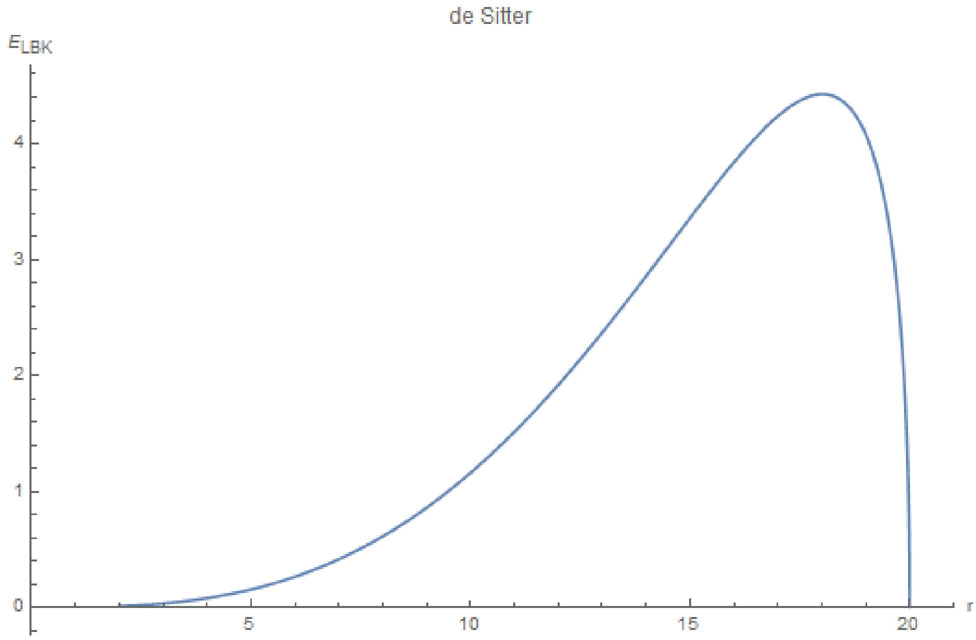

FIGURE 3.6: Lynden-Bell-Katz energy for the de Sitter spacetime, with l=20.

Analogously to Section 3.2.1, the energy density ρ_{LBK} is given by

$$\begin{aligned}\rho_{\mathrm{LBK}} &= \frac{\sqrt{1-\frac{r^2}{l^2}}}{4\pi r^2}\frac{dE_{\mathrm{LBK}}}{dr} \\ &= \frac{\sqrt{1-\frac{r^2}{l^2}}}{4\pi r^2}\frac{(3r^2-l^2)\sqrt{1-\frac{r^2}{l^2}}+l^2-2r^2}{l^2\sqrt{1-\frac{r^2}{l^2}}}\,. \end{aligned} \tag{3.42}$$

Approximating Eq. (3.42) to second order in $\left(\frac{r}{l}\right)^2$, we get

$$\rho_{\mathrm{LBK}} \approx \frac{3}{8\pi l^2} - \frac{11r^2}{32\pi l^4} = \frac{\Lambda}{8\pi} + \frac{1}{8\pi}\left(\frac{\Lambda r}{6}\right)^2 - \frac{19\Lambda^2 r^2}{288\pi}\,. \tag{3.43}$$

We can interpret the right hand side of Eq. (3.43) in the following way. The first term, $\frac{\Lambda}{8\pi}$, is just the energy density associated with the cosmological constant, or dark energy, as in Eq. (3.38). The second term is associated with the positive energy density stored in the gravitational field produced by the dark energy. The third term is the negative gravitational energy density stored in the dark energy. In order to be consistent with the Lynden-Bell and Katz interpretation for the localization of the gravitational energy, the sum of the last two terms should be equal to twice the gravitational potential energy in the matter, given by $\frac{1}{2}\frac{\Lambda}{8\pi}\Phi$, where Φ is the gravitational potential, plus the positive gravitational field energy density, given by $\frac{|\boldsymbol{g}|^2}{8\pi}$, where $\boldsymbol{g}$ is the gravitational field. We can make this calculation in a Newtonian framework, to see if we get the factor right.

The gravitational field is $\boldsymbol{g} = -\frac{\Lambda r}{6}\boldsymbol{e_r}$, as indicated in the previous section. Therefore, the gravitational potential is $\Phi = \frac{\Lambda r^2}{12} + \text{constant}$. Assuming homogeneity and isotropy, we assume that the constant vanishes. The matter gravitational potential energy density is then

$$\frac{1}{2}\frac{\Lambda r^2}{12}\frac{\Lambda}{8\pi} = \frac{\Lambda^2 r^2}{192\pi}\,. \tag{3.44}$$

The positive gravitational field energy density is the opposite of the energy density that appears in Eq. (3.38), that is

$$\frac{1}{8\pi}\left(\frac{\Lambda r}{6}\right)^2 . \tag{3.45}$$

Adding two times Eq. (3.44) and (3.45), we get

$$\frac{4\Lambda^2 r^2}{288\pi}\,, \tag{3.46}$$

which is different from the term found in Eq. (3.43), which is $-\frac{11\Lambda^2 r^2}{288\pi}$. This difference may be due to our expression for the gravitational potential, since it lead to a positive gravitational energy density, when we would expect a negative density. This could be fixed with a suitable constant of integration, but any choice for that constant would be an *ad hoc* artificial guess. It would be very interesting to find an elegant interpretation for the last term of Eq. (3.43).

3.5 Schwarzschild-de Sitter spacetime

Just like the de Sitter spacetime, the Schwarzschild-de Sitter spacetime is a solution of the Einstein equations Eq. (2.27) in a vacuum with a positive cosmological constant Λ. However, it has an intrinsic singularity at the origin and describes a black hole. The Einstein field equations are given by

$$G_{ab} + \Lambda g_{ab} = 0\,. \tag{3.47}$$

The spacetime metric can be written, in Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{r^2}{l^2}\right) dt^2 + \frac{1}{1 - \frac{2m}{r} - \frac{r^2}{l^2}}\, dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,, \tag{3.48}$$

where $l = \sqrt{\frac{3}{\Lambda}}$ is the cosmological radius, and m is the standard mass parameter of the black hole.

3.5.1 Brown-York energy in a Schwarzschild-de Sitter spacetime

The Brown-York quasilocal energy for a surface of constant r is, using Eq. (2.55),

$$E_{\rm BY}(r) = r\left(1 - \sqrt{1 - \frac{2m}{r} - \frac{r^2}{l^2}}\right) . \tag{3.49}$$

We now consider the case where r is greater than the radius of the event horizon r_{EH} but less than the cosmological horizon radius r_{CH}. Note that, as these two special cases cancel the square root in Eq. (3.49), the energies in those cases are equal to the respective radius. If we take the limit of Eq. (3.49) when $m \ll r \ll l$, and expand the square root to first order in $\frac{2m}{r} + \frac{r^2}{l^2}$, we get

$$E_{\rm BY}(r) \approx m + \frac{r^3}{2l^2} = m + \frac{\Lambda r^3}{6} . \tag{3.50}$$

The second term on the right hand side of Eq. (3.50) increases with r. This is a consequence of the presence of the cosmological constant Λ, that is equivalent to a constant energy density $\frac{\Lambda}{8\pi}$ over the whole space. In fact, $\frac{4}{3}\pi r^3 \frac{\Lambda}{8\pi} = \frac{\Lambda r^3}{6}$.

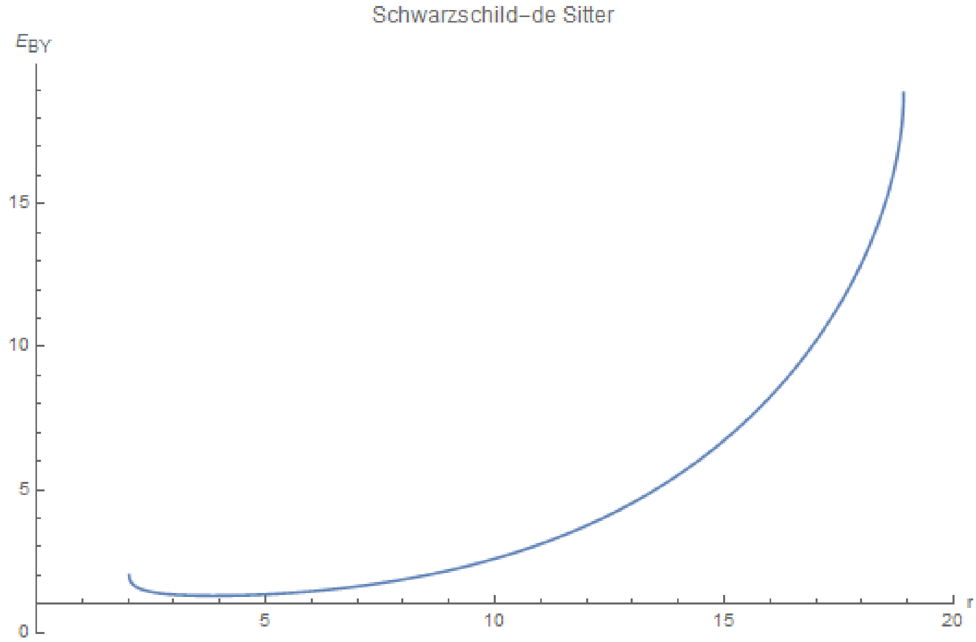

FIGURE 3.7: Brown-York energy for the Schwarzschild-de Sitter spacetime, with $m = 1$ and l=20.

Calculating the energy density ρ_{BY}, we get

$$\begin{aligned}\rho_{\text{BY}} &= \frac{\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}}{4\pi r^2}\frac{dE_{BY}}{dr}\\ &= \frac{\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}}{4\pi r^2}\left(1-\frac{1-\frac{m}{r}-\frac{2r^2}{l^2}}{\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}}\right)\,, \end{aligned}\tag{3.51}$$

and approximating to second order in $\frac{m}{r}$ and in $\left(\frac{r}{l}\right)^2$, we get

$$\rho_{\text{BY}} \approx \frac{3}{8\pi l^2}-\frac{m^2}{8\pi r^4}-\frac{m}{8\pi r l^2}-\frac{r^2}{32\pi l^4}=\frac{\Lambda}{8\pi}-\frac{1}{8\pi}\left(\frac{m}{r^2}+\frac{\Lambda r}{6}\right)^2\,. \tag{3.52}$$

With our previous results, we can interpret the terms of the right hand side of Eq. (3.52). The first term, $\frac{\Lambda}{8\pi}$, is the energy associated with dark energy, just as in Eq. (3.38). The second term is the energy associated with the gravitational field. Note that the gravitational field energy contribution is consistent with the value of gravitational field created by the point mass in the origin and by the continuous distribution of matter. Indeed, $\boldsymbol{g}=-\left(\frac{m}{r^2}+\frac{\Lambda r}{6}\right)\boldsymbol{e_r}$.

3.5.2 Lynden-Bell-Katz energy in a Schwarzschild-de Sitter spacetime

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (2.74),

$$E_{\text{LBK}}(r)=\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}\left(1-\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}\right)r\,. \tag{3.53}$$

We now consider the case where r is greater than the radius of the event horizon r_{EH} but less than the cosmological horizon radius r_{CH}. Note that, as these two special cases cancel the square root in Eq. (3.53), the energies in those cases are equal to zero. If we take the limit of Eq. (3.53) when $m \ll r \ll l$, and expand the square root to first order in $\frac{2m}{r}+\frac{r^2}{l^2}$, we get

$$E_{\text{LBK}}(r)\approx m+\frac{r^3}{2l^2}=m+\frac{\Lambda r^3}{6}\,, \tag{3.54}$$

which is the same result found in Eq. (3.50), and therein they have the same interpretation.

Calculating the energy density ρ_{LBK}, we get

$$\begin{aligned}\rho_{\text{LBK}} &= \frac{\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}}{4\pi r^2}\frac{dE_{\text{LBK}}}{dr}\\ &= \frac{\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}}{4\pi r^2}\,\frac{r(3r^2-l^2)\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}+l^2(r-m)-2r^3}{l^2 r\sqrt{1-\frac{2m}{r}-\frac{r^2}{l^2}}}\,. \end{aligned}\tag{3.55}$$

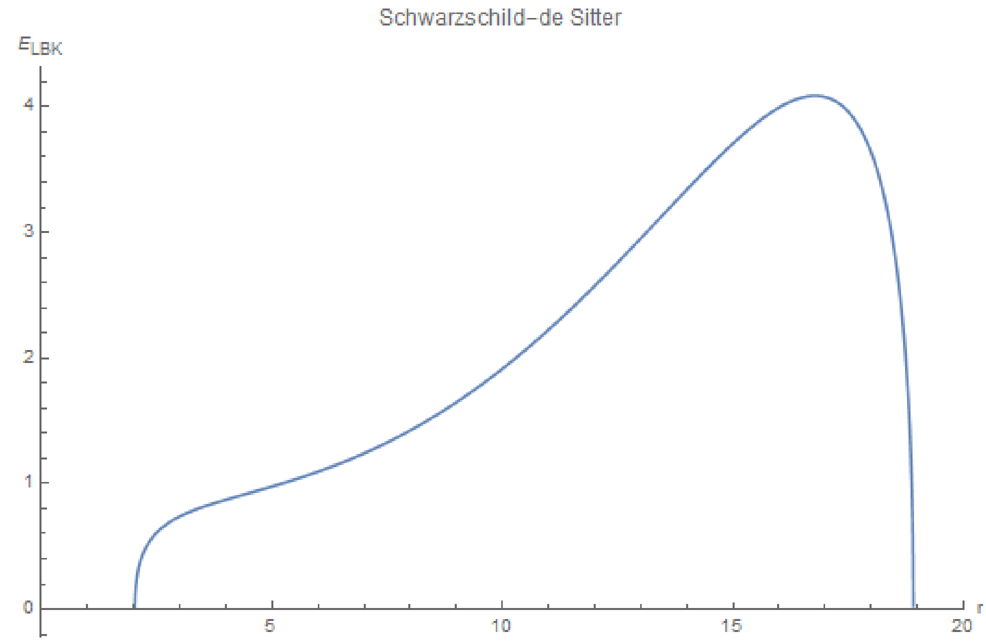

FIGURE 3.8: Lynden-Bell-Katz energy for the Schwarzschild-de Sitter spacetime, with $m = 1$ and l=20.

Approximating Eq. (3.55) to second order in $\frac{m}{r}$ and in $\left(\frac{r}{l}\right)^2$, we get

$$\rho_{\mathrm{LBK}} \approx \frac{3}{8\pi l^2} + \frac{m^2}{8\pi r^4} + \frac{m}{8\pi r l^2} - \frac{11 r^2}{32\pi l^4} = \frac{\Lambda}{8\pi} + \frac{1}{8\pi}\left(\frac{m}{r^2} + \frac{\Lambda r}{6}\right)^2 - \frac{19\Lambda^2 r^2}{288}\,. \tag{3.56}$$

The interpretation of the terms on the right hand side of Eq. (3.56) is the same as in Eq. (3.52), as long as we follow the Lynden-Bell-Katz interpretation for the gravitational energy. The first term, $\frac{\Lambda}{8\pi}$, is the energy associated with dark energy, just as in Eq. (3.43). The second term is the (positive) energy associated with the gravitational field produced by the mass m and by the continuous dark energy distribution. The third term represents the negative gravitational energy density stored in the dark energy, that also appeared in Eq. (3.43).

3.6 Interior Schwarzschild spacetime

Until now, we have only studied exterior solutions of the Einstein equations, that describe spacetime outside the source. In this section, we analyze an important interior solution: the Schwarzschild interior solution, named after its discoverer. Other important interior solutions were found by Tolman [48] and Whittaker [49], among others.

The Schwarzschild interior solution describes the interior of a star with constant energy density. It solves the Einstein equations Eq. (2.27) with an energy-momentum tensor given by

$$T_{ab} = (\rho_0 + p)u_a u_b + p\, g_{ab}\,, \tag{3.57}$$

where u^a is the velocity vector field of the fluid that constitutes the star, ρ_0 is the constant gravitational mass density, and p is the pressure (not constant).

The metric corresponding to this solution can be written, in Schwarzschild coordinates (t, r, θ, ϕ), as

$$ds^2 = -N^2\, dt^2 + \frac{1}{1 - \frac{8\pi\rho_0}{3} r^2}\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right) , \tag{3.58}$$

where N^2 is given by

$$N^2 = \left[\frac{3}{2}\sqrt{1 - \frac{8\pi\rho_0}{3} r_0^2} - \frac{1}{2}\sqrt{1 - \frac{8\pi\rho_0}{3} r^2} \right]^2 . \tag{3.59}$$

This metric describes the interior of a star of constant gravitational mass density ρ_0 and with radius r_0. Consistency with the outside Schwarzschild solution requires that

$$m = \frac{4}{3}\pi r_0^3\, \rho_0 . \tag{3.60}$$

3.6.1 Brown-York energy in an interior Schwarzschild spacetime

The Brown-York quasilocal energy for a surface of constant r is, using Eq. (2.55),

$$E_{\rm BY}(r) = r \left(1 - \sqrt{1 - \frac{8\pi\rho_0}{3} r^2} \right) = r \left(1 - \sqrt{1 - \frac{2m}{r_0^3} r^2} \right) . \tag{3.61}$$

In the special case $r = r_0$, we have

$$E_{\rm BY}(r_0) = r_0 \left(1 - \sqrt{1 - \frac{2m}{r_0}} \right) , \tag{3.62}$$

which agrees with the expression for the energy in the exterior Schwarzschild spacetime given in Eq. (3.3).

The weak field approximation of Eq. (3.62) is obtained by expanding the square root to second order in $\frac{m}{r_0}$. This yields

$$E_{\rm BY}(r_0) = m + \frac{1}{2}\frac{m^2}{r_0} . \tag{3.63}$$

We may now investigate if this result satisfies the correspondence principle, that is, if it is equivalent to the Newtonian result. Since the total energy measured at infinity is m, as calculated in Eq. (3.5), the gravitational energy stored outside the star is equal to $m - E_{\rm BY}(r_0) = -\frac{1}{2}\frac{m^2}{r_0}$. If we consider that, in the Newtonian theory of gravitation, the gravitational energy is stored in

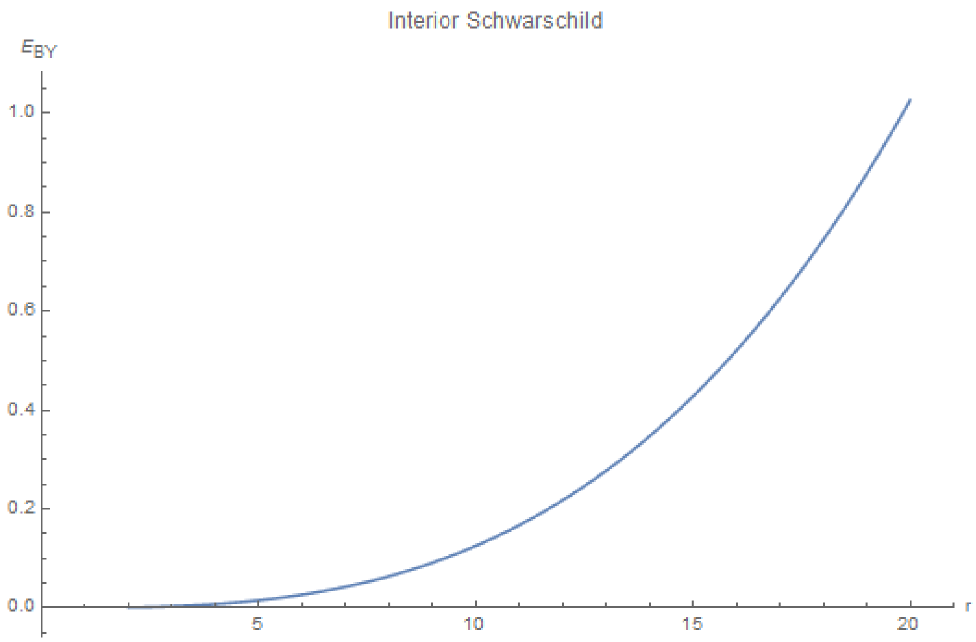

FIGURE 3.9: Brown-York energy for the interior Schwarzschild spacetime, with $m = 1$ and $r_0 = 20$.

the field with energy density given by Eq. (3.7), then it is very interesting to note that

$$\int_{r_0}^{\infty} -\frac{|\boldsymbol{g}|^2}{8\pi}\, 4\pi r^2\, dr = -\frac{1}{2}\frac{m^2}{r_0}\,, \tag{3.64}$$

where $\boldsymbol{g}$ is defined in Eq. (3.7). Therefore, the weak field limit of the Brown-York energy is consistent with the Newtonian result outside the star, provided that in the Newtonian theory we consider that gravitational energy is stored in the field.

Let us now analyze what happens inside the star. The Brown-York energy inside the star is given in Eq. (3.63). We can now calculate the energy inside the star in the Newtonian theory of gravitation. First, we consider that the star has a rest mass equal to m^*, that may be different from the gravitational mass m, which is the energy measured at infinity. The energy associated with matter is thus $E_{\text{matter}} = m^*$. Then, defining $\boldsymbol{g^*} = -\frac{m^* r}{r_0^3}\boldsymbol{e_r}$ as the gravitational field generated by the rest mass of the star, we can calculate the gravitational energy stored inside the star through

$$E_{\text{grav}} = \int_{0}^{r_0} -\frac{|\boldsymbol{g^*}|^2}{8\pi}\, 4\pi r^2\, dr = -\frac{1}{10}\frac{(m^*)^2}{r_0}\,. \tag{3.65}$$

Therefore, the total energy E_{Newton} inside the star is

$$E_{\text{Newton}} = E_{\text{matter}} + E_{\text{grav}} = m^* - \frac{1}{10}\frac{(m^*)^2}{r_0}\,. \tag{3.66}$$

For consistency in the weak field limit, E_{BY} must be equal to E_{Newton}. This constraint allows us to express the gravitational mass of the star m in terms of its rest mass m^*. In fact, equating the expressions of Eq. (3.63) and Eq. (3.66) yields

$$m + \frac{1}{2}\frac{m^2}{r_0} = m^* - \frac{1}{10}\frac{(m^*)^2}{r_0}\,, \tag{3.67}$$

and solving to second order in m^* gives

$$m = m^* - \frac{3}{5}\frac{(m^*)^2}{r_0}\,. \tag{3.68}$$

The second term of the right hand side of Eq. (3.68) is just the total gravitational potential energy of the star. Therefore, in this weak field limit, the gravitational mass measured at infinity is just the sum of the rest mass and the gravitational potential energy of the star. Thus, gravitational potential energy is itself a source of gravitational energy.

We can now proceed in the same way as in Section 3.2.1 to calculate the energy density from Eq. (3.61). We find that

$$\rho_{\mathrm{BY}} = \frac{\sqrt{1-\frac{2mr^2}{r_0^3}}}{4\pi r^2}\left(1 - \frac{1-\frac{4mr^2}{r_0^3}}{\sqrt{1-\frac{2mr^2}{r_0^3}}}\right)\,. \tag{3.69}$$

Approximating to second order in $\frac{mr^2}{r_0^3}$ yields

$$\rho_{\mathrm{BY}} \approx \frac{m}{\frac{4}{3}\pi r_0^3} - \frac{1}{8\pi}\frac{m^2 r^2}{r_0^6}\,. \tag{3.70}$$

The first term is just ρ_0, the density of the star. The second term is just the gravitational energy density stored in the gravitational field. Note that, in the Newtonian theory, the gravitational field inside a homogeneous sphere of mass m and radius r_0 is $\boldsymbol{g} = -\frac{mr}{r_0^3}\boldsymbol{e_r}$. Therefore, Eq. (3.70) is the expected expression for the energy density according to the Brown-York interpretation.

3.6.2 Lynden-Bell-Katz energy in an interior Schwarzschild spacetime

The Lynden-Bell-Katz quasilocal energy for a surface of constant r is, using Eq. (2.74),

$$E_{\mathrm{LBK}}(r) = \left(\frac{3}{2}\sqrt{1-\frac{8\pi\rho_0}{3}r_0^2} - \frac{1}{2}\sqrt{1-\frac{8\pi\rho_0}{3}r^2}\right)\left(1-\sqrt{1-\frac{8\pi\rho_0}{3}r^2}\right)r \tag{3.71}$$

$$= \left(\frac{3}{2}\sqrt{1-\frac{2m}{r_0}} - \frac{1}{2}\sqrt{1-\frac{2m}{r_0^3}r^2}\right)\left(1-\sqrt{1-\frac{2m}{r_0^3}r^2}\right)r\,. \tag{3.72}$$

In the special case $r = r_0$, we have

$$E_{\rm LBK}(r_0) = \sqrt{1 - \frac{2m}{r_0}} \left(1 - \sqrt{1 - \frac{2m}{r_0}}\right) r_0 \,, \tag{3.73}$$

which agrees with the expression for the energy in the exterior Schwarzschild spacetime given in Eq. (3.9).

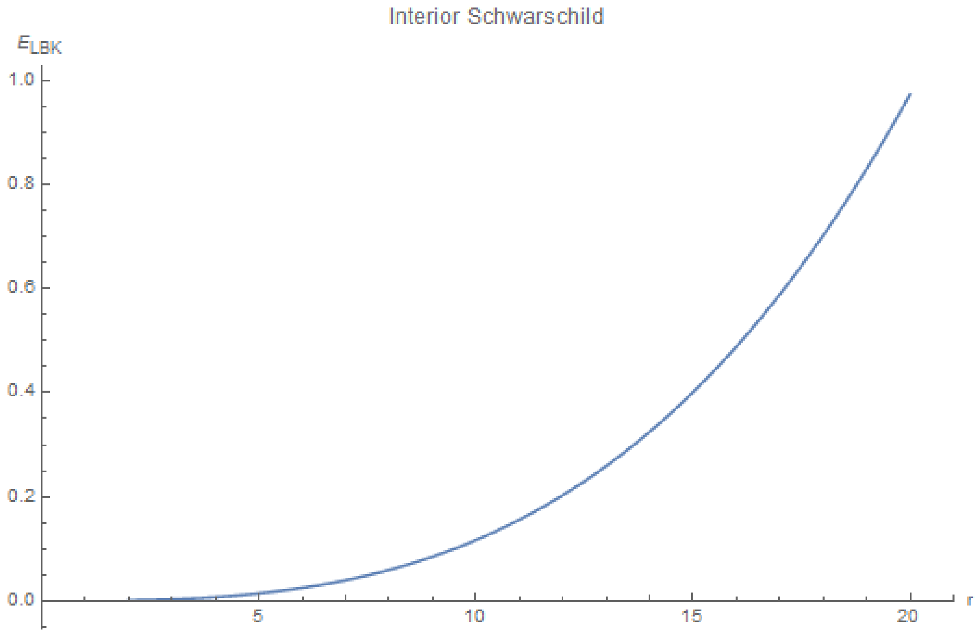

FIGURE 3.10: Lynden-Bell-Katz energy for the interior Schwarzschild spacetime, with $m = 1$ and $r_0 = 20$.

The weak field approximation of Eq. (3.73) is obtained by expanding the square root to second order in $\frac{m}{r_0}$. This yields

$$E_{\rm LBK}(r_0) = m - \frac{1}{2}\frac{m^2}{r_0} \,. \tag{3.74}$$

We may now investigate if this result is consistent with the Lynden-Bell and Katz interpretation of the localization of gravitational energy in Newton's theory. Since the total energy measured at infinity is m, as calculated in Eq. (3.5), the gravitational energy stored outside the star is equal to $m - E_{\rm LBK}(r_0) = \frac{1}{2}\frac{m^2}{r_0}$. Assuming that, in the Newtonian theory of gravitation, the gravitational energy stored in the field has an energy density given by Eq. (3.13), it is interesting to note that

$$\int\limits_{r_0}^{\infty} \frac{|\boldsymbol{g}|^2}{8\pi} 4\pi r^2 \, dr = \frac{1}{2}\frac{m^2}{r_0} \,, \tag{3.75}$$

where $\boldsymbol{g}$ is defined in Eq. (3.7). Therefore, the weak field limit of the Lynden-Bell-Katz energy is consistent with the Newtonian result outside the star, provided that in the Newtonian theory we consider that the gravitational energy stored in the field is positive and that the gravitational energy stored in matter in negative (and equal to twice the total gravitational energy).

Let us now analyze what happens inside the star. The Lynden-Bell-Katz energy inside the star is given in Eq. (3.74). We can now calculate the energy inside the star in the Newtonian theory of gravitation, considering the Lynden-Bell-Katz interpretation of the localization of gravitational energy. First, as in Section 3.6.1, we consider that the star has a rest mass equal to m^*. The energy associated with matter is thus $E_{\text{matter}} = m^*$. Then, as in Section 3.6.1, we define $\boldsymbol{g^*} = -\frac{m^* r}{r_0^3}\boldsymbol{e_r}$ as the gravitational field generated by the rest mass of the star. We can then calculate the gravitational energy stored in the field inside the star through

$$E_{\text{grav. field}} = \int_0^{r_0} \frac{|\boldsymbol{g^*}|^2}{8\pi}\, 4\pi r^2\, dr = \frac{1}{10}\frac{(m^*)^2}{r_0}\,. \tag{3.76}$$

Moreover, in this Lynden-Bell-Katz interpretation, there is also gravitational energy stored in the matter (being twice the total gravitational energy). As the gravitational potential energy U of a sphere of mass m^* and radius r_0 is $U = -\frac{3}{5}\frac{(m^*)^2}{r_0}$, the gravitational energy stored in the matter is

$$E_{\text{grav. matter}} = -\frac{6}{5}\frac{(m^*)^2}{r_0} \tag{3.77}$$

Therefore, the total energy E_{Newton} inside the star is

$$\begin{aligned} E_{\text{Newton}} &= E_{\text{matter}} + E_{\text{grav. field}} + E_{\text{grav. matter}} \\ &= m^* + \frac{1}{10}\frac{(m^*)^2}{r_0} - \frac{6}{5}\frac{(m^*)^2}{r_0} \\ &= m^* - \frac{11}{10}\frac{(m^*)^2}{r_0}\,. \end{aligned} \tag{3.78}$$

For consistency in the weak field limit, E_{LBK} must be equal to E_{Newton}. As in Section 3.6.1, this constraint allows us to express the gravitational mass of the star m in terms of its rest mass m^*. Equating the expressions of Eq. (3.74) and Eq. (3.78) yields

$$m - \frac{1}{2}\frac{m^2}{r_0} = m^* - \frac{11}{10}\frac{(m^*)^2}{r_0}\,, \tag{3.79}$$

and solving to second order in m^* gives

$$m = m^* - \frac{3}{5}\frac{(m^*)^2}{r_0}\,, \tag{3.80}$$

which is the same result obtained in Eq. (3.68), showing that this is the correct interpretation for the localization of energy in the Lynden-Bell-Katz energy definition.

We can now proceed in the same way as in Section 3.2.1 to calculate the energy density from Eq. (3.71). We find that

$$\rho_{\mathrm{LBK}} = \frac{\sqrt{1-\frac{2mr^2}{r_0^3}}}{4\pi r^2}\left[\frac{mr^2\left(1-2\sqrt{1-\frac{2mr^2}{r_0^3}}+3\sqrt{1-\frac{2m}{r_0}}\right)}{r_0^3\sqrt{1-\frac{2mr^2}{r_0^3}}}\right.$$
$$\left.+\left(1-\sqrt{1-\frac{2mr^2}{r_0^3}}\right)\left(\frac{3}{2}\sqrt{1-\frac{2m}{r_0}}-\frac{1}{2}\sqrt{1-\frac{2mr^2}{r_0^3}}\right)\right] \tag{3.81}$$

Approximating to second order in $\frac{mr^2}{r_0^3}$ yields

$$\rho_{\mathrm{LBK}} \approx \frac{m}{\frac{4}{3}\pi r_0^3}+\frac{1}{8\pi}\left(\frac{mr}{r_0^3}\right)^2+\left(\frac{3m^2r^2}{8\pi r_0^6}-\frac{9m^2}{8\pi r_0^4}\right). \tag{3.82}$$

As in Eq. (3.70), the first term is just ρ_0, the density of the star, the second term contains the positive gravitational field energy density, and the third term is the gravitational energy stored in the matter, which can be calculated with the Newtonian potential associated with a sphere of constant density. As expected, the sign of the gravitational field term is reversed in relation to Eq. (3.70), because of the difference in the interpretation of the localization of gravitational energy.

3.7 Black hole laws of mechanics and thermodynamics

In this section we consider a Reissner-Nordström spacetime, with metric given in Eq. (3.17). We derive the laws of mechanics and thermodynamics that follow directly from the different energy definitions. We follow the method used by Brown and York, which was applied to a Schwarzschild black hole [29].

3.7.1 Brown-York energy: Laws of mechanics and thermodynamics

The Brown-York quasilocal energy is given by Eq. (3.18). If the radius r, mass m and charge q are changed, the energy varies according to

$$dE_{\mathrm{BY}} = \left(1-\frac{1-\frac{m}{r}}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\right)dr+\frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}dm-\frac{q}{r\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}dq\,. \tag{3.83}$$

To simplify this expression, we use the fact that the electric potential is given by $\Phi = \frac{q}{r}$ and that the surface pressure s is given by Eq. (2.56), with $N = \sqrt{1 - \frac{2m}{r} + \frac{q^2}{r^2}}$ and $h = \frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}$, as

$$s = \frac{1}{2}\sigma_{ab}s^{ab} = \frac{1}{8\pi r}\left[\left(\frac{1-\frac{m}{r}}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\right) - 1\right] . \tag{3.84}$$

Therefore, Eq. (3.83) can be written as

$$dE_{\rm BY} = -s\, d(4\pi r^2) + \frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\left(dm - \Phi\, dq\right) . \tag{3.85}$$

We can now simplify Eq. (3.85) by replacing dm and dq by dA_+, where $A_+ = 4\pi r_+^2$ is the surface area of the outer event horizon of the black hole. Calculating, we find that

$$dA_+ = \frac{8\pi r_+^2}{\sqrt{m^2-q^2}}\left(dm - \Phi_+\, dq\right) , \tag{3.86}$$

where $\Phi_+ = \frac{q}{r_+}$ is the electric potential at the outer event horizon. Defining the quantity

$$\kappa = \frac{\sqrt{m^2-q^2}}{r_+^2} \tag{3.87}$$

(known as the surface gravity of the black hole), and using Eq. (3.86) in Eq. (3.85) yields

$$dE_{\rm BY} = -s\, d(4\pi r^2) + \frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\left(\frac{\kappa}{8\pi}dA_+ + \left(\Phi_+ - \Phi\right)\, dq\right) . \tag{3.88}$$

Note that, in the last term, only the change of electromagnetic energy outside the horizon and contained inside the sphere of radius r is taken into account, and not the total change in electromagnetic energy. Also, the factor $\frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}$ can be interpreted as the blue-shift of the energy from infinity to the sphere of radial parameter r. Letting $r \to \infty$, the first term disappears, $\Phi \to 0$, and $\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}} \to 1$. Therefore, we get the familiar first law of black hole mechanics

$$dE_{\rm BY} = \frac{\kappa}{8\pi}dA_+ + \Phi_+\, dq\, . \tag{3.89}$$

This equation is straightforwardly turned into the first law of thermodynamics for a black hole. We just have to replace in Eq. (3.89) the Bekenstein-Hawking entropy $S_{\rm BH} = \frac{A_+}{4}$ and the Hawking temperature $T_{\rm BH} = \frac{\kappa}{2\pi}$ to get

$$dE_{\rm BY} = T_{\rm BH}\, dS_{\rm BH} + \Phi_+\, dq\, . \tag{3.90}$$

Therefore, the Brown-York quasilocal energy allows a natural derivation of the first law of mechanics and thermodynamics for a Reissner-Nordström black hole.

3.7.2 Lynden-Bell-Katz energy: Laws of mechanics and thermodynamics

The Lynden-Bell-Katz quasilocal energy is given by Eq. (3.25). If the radius r, mass m and charge q are changed, the energy varies according to

$$\begin{aligned} dE_{\text{LBK}} =& \frac{(q^2-r^2)\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}+r(r-m)}{r^2\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}dr + \left(1-\frac{1-\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\right) dm \\ & -\frac{q}{r}\left(1-\frac{1-\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\right) dq\,. \end{aligned} \tag{3.91}$$

As in Section 3.7.1, we can simplify Eq. (3.91). Unlike Brown and York, Lynden-Bell and Katz did not provide a definition for the surface tension for their energy definition. However, we can assume that the first term in the right hand side represents the equivalent of the term $-s\,dA$ in Eq. (3.88). It is possible to find an expression for s that satisfies this criterion, but we will not write it here. Using also the fact that the electric potential is given by $\Phi = \frac{q}{r}$, Eq. (3.91) can be written as

$$dE_{\text{LBK}} = -s\,dA + \alpha\,(dm - \Phi\,dq)\ , \tag{3.92}$$

where α is defined by

$$\alpha = 1 - \frac{1-\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}\,. \tag{3.93}$$

We can further simplify Eq. (3.92) by replacing dm and dq by dA_+, as in Section 3.7.1. Using Eqs.(3.86) and (3.87), we get

$$dE_{\text{LBK}} = -s\,dA + \alpha\left(\frac{\kappa}{8\pi}dA_+ + (\Phi_+ - \Phi)\ dq\right)\,. \tag{3.94}$$

Note that, as in Eq. (3.88), in the last term, only the change of electromagnetic energy outside the horizon and contained inside the sphere of radius r is taken into account, and not the total change in electromagnetic energy. Note also that here α does not have the natural blue-shift interpretation that the factor $\frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}$ has in Eq. (3.88). Letting $r \to \infty$, the first term disappears, $\Phi \to 0$, and $\alpha \to 1$. Therefore, we get the first law of black hole mechanics that we had also find in (3.89)

$$dE_{\text{LBK}} = \frac{\kappa}{8\pi}dA_+ + \Phi_+\,dq\,. \tag{3.95}$$

Therefore, the Lynden-Bell-Katz quasilocal energy also allows a natural derivation of the first law of mechanics and thermodynamics for a Reissner-Nordström black hole.

Chapter 4

Discussion: picking the best definition

After analyzing many features of the Brown-York and Lynden-Bell-Katz energy definitions, we can now analyze the results and choose the most appropriate energy definition. We will first analyze the crucial differences of the two energies in the Newtonian limit and then check if there are ways to experimentally distinguish the two definitions. Finally, we propose a solution for a problem that appears when comparing the energy densities.

4.1 Differences between the energy densities in the Newtonian limit

The Newtonian gravitational potential energy is given by

$$U = \frac{1}{2} \int \rho \, \Phi \,, \tag{4.1}$$

where ρ is the mass density and Φ is the Newtonian gravitational potential. However, using Poisson's equation

$$\nabla^2 \Phi = 4\pi\rho \,, \tag{4.2}$$

we can integrate Eq. (4.1) by parts, yielding

$$U = \int_{\text{space}} -\frac{|\boldsymbol{g}|^2}{8\pi} \, dV \,. \tag{4.3}$$

Therefore, we could define a Newtonian gravitational energy density ρ_{grav} by

$$\rho_{\text{grav}} = -\frac{|\boldsymbol{g}|^2}{8\pi}\,. \tag{4.4}$$

This result was suggested in Eq. (3.8). However, formally speaking, we could also define the gravitational energy density by

$$\rho_{\text{grav}} = a\left(\frac{1}{2}\rho\,\Phi\right) + b\left(-\frac{|\boldsymbol{g}|^2}{8\pi}\right)\,, \tag{4.5}$$

for any real numbers a and b such that $a + b = 1$. In the pure Newtonian theory, there is no way to choose the correct pair (a, b).

The Brown-York energy density for a Schwarzschild spacetime in the weak field limit, calculated in Eq. (3.7), corresponds to $b = 1$, and therefore $a = 0$. On the other hand, the Lynden-Bell-Katz energy density for the same spacetime in the weak field limit corresponds to $b = -1$, and therefore $a = 2$. This difference is consistent through all spacetimes.

These results have a simple interpretation. They mean that, in the case of the Brown-York energy definition, all the gravitational energy is stored in the field, whereas in the Lynden-Bell-Katz definition, the energy is stored in the matter (twice the total energy) and in the field (the negative of the total energy). Therefore, on pure simplicity grounds, there is no doubt that the Brown-York energy in simpler than Lynden-Bell-Katz's. However, this criterion is very weak, and we need an experimental difference in order to choose undoubtedly the best energy.

4.2 The most suitable definition for black hole thermodynamics

In Section 3.7, we compared the laws of mechanics for black holes generated by the two energy definitions. We can now analyze which one is the most suitable for this situation.

In the far away limit, i.e. when $r \to \infty$, we obtain the same result, given in Eqs. (3.89) and (3.95). Therefore, we have to analyze and compare the results for any radial parameter r, that are given in Eqs. (3.88) and (3.94). The difference between the results is the factor in front of the term $\frac{\kappa}{8\pi}dA_+ + (\Phi_+ - \Phi)\,dq$. In the case of the Brown-York definition, the factor is $\frac{1}{\sqrt{1-\frac{2m}{r}+\frac{q^2}{r^2}}}$, which is the gravitational blue-shift from infinity to the surface of radius r. It is not so surprising to get this result, and it has this natural *a posteriori* explanation. In the case of the Lynden-Bell-Katz definition, the factor α is given in Eq. (3.93). Unlike the Brown-York case, it does not have such an intuitive interpretation. Therefore, even though the two definitions have the same far away limit, Brown-York's is more naturally interpreted and therefore more adapted to black-hole thermodynamics than Lynden-Bell-Katz's.

4.3 Mercury's perihelion precession: influence of a gravitational energy density

This argument is semi-classical and is related to the physical consequences of having a positive or negative definite gravitational energy density stored in the field. Since Brown-York's energy definition is related to having a negative gravitational energy density stored in the field and Lynden-Bell-Katz's is related to a positive gravitational energy density stored in the field, if we find such an observable distinction we can choose the most appropriate definition.

We consider the influence of a positive or negative gravitational energy density in the perihelion advance of the planets, assuming that gravitational energy acts gravitationally just like any other type of energy. To simplify our approach, we will not consider for now the influence of gravitational field pressure, even though it has an influence in the perihelion advance too.

Let us begin by considering a negative energy density given by

$$\rho_{\text{grav}} = -\frac{|\nabla\Phi|^2}{8\pi}\,. \tag{4.6}$$

In a vacuum, where there is only gravitational energy, Poisson's equation becomes

$$\nabla^2\Phi = -\frac{|\nabla\Phi|^2}{2}\,. \tag{4.7}$$

Assuming spherical symmetry around a massive sphere of gravitational mass M, Eq. (4.7) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g(r)\right) = \frac{g^2(r)}{2}\,, \tag{4.8}$$

where g, defined by

$$g(r)\boldsymbol{e_r} = -\nabla\Phi\,, \tag{4.9}$$

is the gravitational force per unit mass, and $\boldsymbol{e_r}$ is the unit radial vector. This differential equation can be solved to give

$$g(r) = \frac{1}{\frac{r}{2} - a\,r^2}\,, \tag{4.10}$$

where a is a constant of integration. We can find the value of a through the principle of correspondence, i.e., when the speed of light c goes to infinity, $c \to \infty$. In this way one gets $a = \frac{1}{M}$, where M is the gravitational mass of the sphere that generates the gravitational field.

We now solve the equation of motion for a test particle of mass m in polar coordinates (r, ϕ) in the gravitational field produced by the mass M. Denoting the position vector of the test particle by $\boldsymbol{r}$, Newton's second law

$$\ddot{\boldsymbol{r}} = -\nabla\Phi \tag{4.11}$$

yields in these coordinates

$$\ddot{r} - r\dot{\phi}^2 = g(r)\,, \tag{4.12}$$

where a dot means differentiation with respect to time t, $\dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}t}$. Since the force is radial, the angular momentum per unit mass h is conserved. Therefore, $h = r^2\dot{\phi}$ is constant. Performing the change of variable

$$u = \frac{1}{r}\,, \tag{4.13}$$

where r should be envisaged as $r(\phi)$, and so $u(\phi) = \frac{1}{r(\phi)}$, and rearranging, we obtain

$$u'' + u = \frac{1}{h^2 a - \frac{h^2 u}{2}}\,, \tag{4.14}$$

where a prime denotes differentiation with respect to ϕ, $u' \equiv \frac{\mathrm{d}u}{\mathrm{d}\phi}$. One can rewrite the right hand side of Eq. (4.14) as $\frac{M}{h^2}\frac{1}{1-Mu/2}$. Approximating to first order in $\frac{Mu}{2}$, Eq. (4.14) becomes

$$u'' + u = \frac{M}{h^2} + \frac{M^2 u}{2h^2}. \tag{4.15}$$

Solving for $u(\phi)$ yields

$$u(\phi) = \frac{M}{h^2} + b\cos\left(\sqrt{1 - \frac{M^2}{2h^2}}\phi\right). \tag{4.16}$$

Expanding the square root and retaining the first order term, we get an advance of the perihelion

$$\Delta\phi_{\text{grav}} = \frac{\pi M^2}{2h^2}\,. \tag{4.17}$$

Denoting the total general relativistic precession by $\Delta\phi_{\text{GR}}$ we thus get

$$\Delta\phi_{\text{grav}} = \frac{1}{12}\Delta\phi_{\text{GR}}\,. \tag{4.18}$$

that is, a negative self energy of the gravitational field accounts for $\frac{1}{12}$ of the advance predicted by general relativity.

Performing the same calculations using

$$\rho_{\text{grav}} = \frac{|\nabla\Phi|^2}{8\pi} \tag{4.19}$$

yields

$$\Delta\phi_{\text{grav}} = -\frac{\pi M^2}{2h^2}\,, \tag{4.20}$$

that is

$$\Delta\phi_{\text{grav}} = -\frac{1}{12}\Delta\phi_{\text{GR}}\,. \tag{4.21}$$

Therefore, a positive self energy of the gravitational field would cause a regression of the perihelion, instead of an observed advance of the perihelion.

This argument would work very well if the only effect brought by general relativity were this weight of gravitational energy itself. There are in fact other effects that explain the full general relativity prediction, namely the influence of gravitational field pressure, special relativistic dynamics and gravitational time dilation. In Appendix B, we calculate the contributions due to each of these effects. Nonetheless, it is only possible to explain the full general relativistic prediction if we consider that gravitational energy density is negative. This strongly suggests that the correct quasilocal energy definition should predict such a negative definite gravitational field energy density. Therefore, considering this argument, the most appropriate energy definition is Brown-York's.

4.4 A possible interpretation for the difference in the energies

We now recall the two quasilocal energy definitions in their most fundamental form, which are given in Eqs. (2.51) and (2.73). The Brown-York quasilocal energy E_{BY} and the Lynden-Bell and Katz quasilocal energy E_{LBK} inside the two-surface B are given by

$$E_{\mathrm{BY}} = \frac{1}{8\pi}\int_B d^2x\sqrt{\sigma}\,(k-k_0) \tag{4.22}$$

$$E_{\mathrm{LBK}} = \frac{1}{8\pi}|\xi|\int_B d^2x\sqrt{\sigma}\,(k-k_0)\ , \tag{4.23}$$

where $|\xi|$ is the positive (constant) norm of the timelike Killing vector at B, ξ^a, $\sigma = \det\sigma_{\alpha\beta}$ is the determinant of the two-surface metric $\sigma_{\alpha\beta}$, k is the extrinsic curvature of the surface as embedded in the spacetime M, and k_0 is the extrinsic curvature of the surface as embedded in a flat Minkowski spacetime. Note that

$$E_{\mathrm{LBK}} = |\xi|E_{\mathrm{BY}}\ , \tag{4.24}$$

which means that E_{LBK} can be interpreted as a red-shifted E_{BY} from B to infinity.

It is interesting to note that, in Eq. (2.60), we could also have defined E_m by multiplying the right hand side by $|\xi|^n$, since in special relativity $|\xi| = 1$. This modification would be still valid for general relativity. Taking $n = -1$, and proceeding in the same way, we would have got Brown-York's energy. Alternatively, in Eq. (2.60), we could have replaced ξ^a by u^a, keeping the result valid in special relativity. When generalizing to special relativity, we would also have got Brown-York's expression. Nevertheless, in the end of their original paper, Lynden-Bell and Katz discuss this possibility, arguing that they have the correct expression and giving a physical example to sustain their opinion. Despite this, they say that any expression with $n > -1$ would lead to the fact that all black hole's energy is contained outside the horizon, which is an important feature of their energy definition. However, in a more recent article, Lynden-Bell, Katz and Bičák argue that the correct total matter energy definition should take $n = -1$, and this naturally leads to Brown-York results [50]. For example, they find that the quasilocal energy at the horizon of a Schwarzschild black hole is $2m$. This interesting result suggests that in reality Brown-York and Lynden-Bell-Katz quasilocal energy definitions are equivalent and yield the same results, and therefore the same interpretation on the localization of the gravitational energy in the classical limit.

Chapter 5

Conclusion

In this work, we have derived Brown-York and Lynden-Bell-Katz quasilocal energy definitions. Then, we have compared them in Schwarzschild, Reissner-Nordström, de Sitter, Schwarzschild-de Sitter and interior Schwarzschild spacetimes. After defining an expression for energy density from quasilocal energy expressions, we calculated the energy density for each spacetime. We also compared the black hole law's of mechanics that followed from each definition, and arrived to the conclusion that Brown-York's black hole law of mechanics had a simpler interpretation than Lynden-Bell-Katz's, even though they both lead to the known black hole law of mechanics at infinity.

Analyzing the results in the Newtonian limit, we provided a different interpretation for the localization of gravitational energy for each energy definition. Namely, Brown-York's definition was consistent with having all the gravitational energy stored in the field, whereas Lynde-Bell-Katz's energy was consistent with having gravitational energy stored in the matter and in the field. Then, taking a semi-classical approach where we considered that gravitational energy should also act gravitationally as any other form of matter energy in Newton's theory, we calculated what would be Mercury's precession when considering each possibility of localization of gravitational energy. We found that Brown-York's interpretation lead to an advance of the perihelion of a twelfth of the full general relativistic advance, whereas Lynden-Bell-Katz's interpretation lead to a retrograde precession of a twelfth of the full general relativistic precession. Finally, analyzing carefully the expressions of the two energies, we gave an interpretation for the difference between them. We also suggested a way to unify the two energies, by reducing Lynden-Bell-Katz's expression to Brown-York's.

Many other quasilocal energy definitions exist in general relativity. They may yield meaningful results in some situations, and less meaningful results in other situations. Anyway, the problem of gravitational energy in general relativity is not solved by quasilocal definitions. Certainly, they may help to calculate the energy in many cases, but from a theoretical point of view these definitions will never be fully satisfying. In fact, since all the other forces have well-defined energy-momentum tensors, why should gravity be different? The recent direct detection of gravitational waves by LIGO [51] is a clear reminder that energy is transported by gravitational waves, and thus that there must be a gravitational energy-momentum tensor. Moreover, assuming that we will find a suitable quantum gravity theory, gravitons should carry energy and momentum, and this requires an energy-momentum tensor for gravity.

As is well-known, general relativity may be interpreted as a spin-2 gauge theory in Minkowski spacetime [52]. In such interpretation, the equivalence principle would be a consequence and not an "axiom" of the theory. However, this Minkowski spacetime would not be observable, since we would measure quantities exactly as if space were curved. This is interesting since, if space was not curved "in reality" but if its curvature was merely an illusion, it would be possible to define an energy-momentum tensor in that background Minkowski spacetime. However, today's consensual interpretation of general relativity is that space is really curved, so that the equivalence principle is in fact an axiom of the theory. It is then impossible to define a gravitational energy-momentum tensor, since the equivalence principle implies that gravity can always be made to vanish locally, through a suitable choice of coordinates.

To put it in a nutshell, in order to have an energy-momentum tensor for gravity while keep general relativity, we have to change the most accepted interpretation of the theory. As Einstein and Feynman said, the geometric approach is not essential for physics. Until then, we will have to deal with different pseudotensors and quasilocal energy definitions.

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Appendix A

Kinematics for the Brown-York quasi-local energy derivation

A.1 Variation of the gravitational action

In this section, we give a more detailed calculation of the variation of the gravitational action, given in Eq. (2.25).

We begin by calculating the variation of $\sqrt{-g}\,R$, which is given by

$$\begin{aligned}\delta(\sqrt{-g}\,R) &= (\delta\sqrt{-g})R + \sqrt{-g}(\delta R) \\ &= \frac{1}{2}\sqrt{-g}R\,g^{ab}\delta g_{ab} + \sqrt{-g}(-R^{ab}\delta g_{ab} + \nabla^a\left(\nabla^b\delta g_{ab} - g^{bc}\nabla_a\delta g_{bc}\right)).\end{aligned} \tag{A.1}$$

The variation of $\sqrt{h}K$ yields

$$\delta(\sqrt{h}K) = -\frac{1}{2}\sqrt{h}h^{ab}\delta g_{ab} + \sqrt{h}\delta K \tag{A.2}$$

$$\delta K = -\frac{1}{2}K^{ab}\delta g_{ab} - \frac{1}{2}u^a\left(\nabla^b\delta g_{ab} - g^{bc}\nabla_a\delta g_{bc}\right) + D_a c_1^a\,, \tag{A.3}$$

where $c_1^a = -\frac{1}{2}h^{ab}\delta g_{cb}u^c$. The variation of $\sqrt{-\gamma}\Theta$ is analogous

$$\delta(\sqrt{-\gamma}\Theta) = \frac{1}{2}\sqrt{-\gamma}\gamma^{ab}\delta g_{ab} + \sqrt{\gamma}\delta\Theta \tag{A.4}$$

$$\delta\Theta = -\frac{1}{2}\Theta^{ab}\delta g_{ab} + \frac{1}{2}n^a\left(\nabla^b\delta g_{ab} - g^{bc}\nabla_a\delta g_{bc}\right) + D_a c_2^a\,, \tag{A.5}$$

where $c_2^a = -\frac{1}{2}\gamma^{ab}\delta g_{cb}n^c$. After integration, the total derivative of Eq. (A.1) cancels with other boundary terms, and we discard the boundary divergences $D_a c_1^a$ and $D_a c_2^a$. Thus, we get

$$\begin{aligned}\delta S = \frac{1}{16\pi}\int_M d^4x\sqrt{-g}\left(\frac{1}{2}g^{ab}R - R^{ab}\right)\delta g_{ab} + \frac{1}{8\pi}\int_\Sigma d^3x\sqrt{h}\,\frac{1}{2}(h^{ij}K - K^{ij})\delta h_{ij}\Big|_{t_i}^{t_f} \\ - \frac{1}{8\pi}\int_{B_3} d^3x\sqrt{-\gamma}\,\frac{1}{2}(\gamma^{ij}\Theta - \Theta^{ij})\delta\gamma_{ij}\,,\end{aligned} \tag{A.6}$$

and, using the definitions of the gravitational momenta P^{ij} and π^{ij}, and using the Einstein tensor $G^{ab} = R^{ab} - \frac{1}{2}g^{ab}R$, we get

$$\begin{aligned}\delta S = &-\frac{1}{16\pi}\int_M d^4x\sqrt{-g}\,\delta g_{ab}G^{ab} \\ &+ \int_\Sigma d^3x\,P^{ij}\,\delta h_{ij}\Big|_{t_i}^{t_f} + \int_{B_3} d^3x\,\pi^{ij}\delta\gamma_{ij}\,.\end{aligned} \tag{A.7}$$

This is just Eq. (2.26) without the terms coming from S_m.

A.2 Calculation of other expressions

In this section we calculate the expressions in Eqs. (2.45) and (2.46), following the approach of Brown and York [29].

We begin by writing π^{ij} in terms of the variables h_{ij}, P_{ij}, N and V^i. The extrinsic curvature Θ_{ab} can be split by $\delta_a^c = h_a^c - u_a u^c$, which results in

$$\Theta_{ab} = -h_a^c h_b^d \gamma_c^e \nabla_e n_d - u_a u_b u^c u^d \gamma_c^e \nabla_e n_d + 2h_{(a}^c u_{b)} u^d \gamma_c^e \nabla_e n_d\,. \tag{A.8}$$

As $u^a n_a = 0$ on B_3, the projections onto B_3 and Σ commute, and we get for the first term

$$\begin{aligned}-h_a^c h_b^d \gamma_c^e \nabla_e n_d &= -\gamma_a^c h_b^d h_c^e \nabla_e n_d \\ &= -\sigma_a^c D_c n_b \\ &= k_{ab}\,.\end{aligned} \tag{A.9}$$

For the second term, we get

$$-u_a u_b u^c u^d \gamma_c^e \nabla_e n_d = u_a u_b n_d a^d\,, \tag{A.10}$$

where $a^d = u^c \nabla_c u^d$ is the acceleration of u^a. The third term on the right hand side of Eq. (A.8) can also be simplified, giving

$$\begin{aligned} 2h^c_{(a} u_{b)} u^d \gamma^e_c \nabla_e n_d &= -2\sigma^c_{(a} u_{b)} n^d h^e_c \nabla_e u_d \\ &= 2\sigma^c_{(a} u_{b)} n^d K_{cd} \,. \end{aligned} \tag{A.11}$$

Collecting all simplifications, Θ_{ab} can be expressed as

$$\Theta_{ab} = k_{ab} + u_a u_b n_d a^d + 2\sigma^c_{(a} u_{b)} n^d K_{cd} \,. \tag{A.12}$$

It follows that

$$\Theta = k - n_d a^d \,. \tag{A.13}$$

Finally, we can express the momentum π^{ij} as

$$\pi^{ij} = \frac{N\sqrt{\sigma}}{16\pi}[k^{ij} + n_c a^c \sigma^{ij} - k\gamma^{ij}] - \frac{2N\sqrt{\sigma}}{\sqrt{h}} \sigma^{(i}_k u^{j)} P^{kl} n_l \,. \tag{A.14}$$

Note that, in this equation, indices i, j refer to coordinates on B_3 and indices k, l refer to coordinates on Σ.

Using Eq. (A.14), it follows that

$$\frac{2}{\sqrt{-\gamma}} u_i u_j \pi^{ij} = \frac{k}{8\pi} \tag{A.15}$$

$$\frac{2}{\sqrt{-\gamma}} \sigma^a_i \sigma^b_j \pi^{ij} = \frac{1}{8\pi} \left[k^{ab} + (n \cdot a - k)\, \sigma^{ab}\right] , \tag{A.16}$$

which were the results we wanted to prove.

Appendix B

All the contributions to Mercury's perihelion precession

In this appendix, we calculate the contributions of three different effects to the precession of Mercury's perihelion. The three effects are the gravitational field self-energy as source of gravity, special relativistic dynamics and gravitational time dilation. We will write explicitly the G's and c's.

B.1 Perihelion precession: Contribution from the gravitational field self-energy and pressure

B.1.1 Gravitational field energy density and pressure

In electrodynamics, one can consider that the electromagnetic energy is stored in the field, with a positive energy density given by $\rho_{\rm EM} = \frac{1}{2}\left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2\right)$, where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively.

A similar expression can be obtained for the gravitational field in Newtonian gravitation. Indeed, using Poisson's equation, one can show that the total gravitational potential energy $E_{\rm grav}$ in a given volume V can be written as [25, 53, 20, 4]

$$\begin{aligned} E_{\rm grav} &= \frac{1}{2}\int \rho_{\rm m}\Phi \, \mathrm{d}V \\ &= \int -\frac{|\nabla\Phi|^2}{8\pi G}\mathrm{d}V, \end{aligned} \tag{B.1}$$

where $\rho_{\rm m}$ is the matter density, Φ is the gravitational potential and G is Newton's gravitational constant, and ∇ is the gradient operator. Therefore, in the Newtonian theory of gravitation, one may define a gravitational field energy density as

$$\rho_{\rm grav} \equiv -\frac{|\nabla\Phi|^2}{8\pi G}\,. \tag{B.2}$$

It is interesting to note that this is a negative definite energy density. This stems from the fact that gravity in the Newtonian theory is exclusively an attractive force. This approach is consistent to our level of approximation [20].

Moreover, also in analogy to electrodynamics, one can define a stress tensor for the Newtonian gravitational field given by [4]

$$\sigma_{ab} \equiv \frac{1}{4\pi G}\left(\Phi_{,a}\Phi_{,b} - \frac{1}{2}h_{ab}|\nabla\Phi|^2\right), \tag{B.3}$$

where h_{ab} is the Euclidean metric of the three-space and $\Phi_{,a}$ denotes derivative of Φ with relation to the coordinate with index a.

B.1.2 Improved field equation for static spacetimes

For static and asymptotically flat spacetimes [4], one can identify the Newton scalar potential Φ with the logarithm of the length of the corresponding timelike Killing vector. Then, Einstein's equations yield the following field equation

$$\nabla^2\Phi = \frac{4\pi G}{c^2}\left(\rho + 3p\right) - \frac{1}{c^2}|\nabla\Phi|^2\,, \tag{B.4}$$

where ∇^2 is the Laplacian, c is the velocity of light, ρ is the total energy density of matter and radiation, and p is its pressure. Note that the gravitational field energy density enters in Eq. (B.4) through $8\pi G\rho_{\rm grav} = -|\nabla\Phi|^2$, see Eq. (B.2). Note also that the average spatial stress P of the field Φ, defined by $P = \frac{\sigma^a_a}{3}$, where σ_{ab} is given in Eq. (B.3), appears naturally in Eq. (B.4). Thus, we can identify the second term on the right hand side of Eq. (B.4) as $\frac{4\pi G}{c^2}(\rho_{\rm grav} + 3P)$.

In vacuum, $\rho = 0$ and $p = 0$, so Eq. (B.4) turns into

$$\nabla^2\Phi = -\frac{1}{c^2}|\nabla\Phi|^2\,. \tag{B.5}$$

Assuming spherical symmetry around a massive sphere, writing

$$g(r)\boldsymbol{e_r} = -\nabla\Phi\,, \tag{B.6}$$

where g is the gravitational force per unit mass, and $\boldsymbol{e_r}$ is the unit radial vector, Eq. (B.4) becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 g(r)\right) = \frac{g^2(r)}{c^2}, \tag{B.7}$$

This differential equation can be solved to give

$$g(r) = \frac{1}{\frac{r}{c^2} - a\,r^2}\,, \tag{B.8}$$

where a is a constant of integration. We can find the value of a through the principle of correspondence, i.e., when the speed of light c goes to infinity, $c \to \infty$. In this way one gets $a = \frac{1}{GM}$, where M is the gravitational mass of the sphere that generates the gravitational field.

B.1.3 Solving the equation of motion

We now solve the equation of motion for a test particle of mass m in polar coordinates (r, ϕ) in the gravitational field produced by the mass M. Denoting the position vector of the test particle by $\boldsymbol{r}$, Newton's second law

$$\ddot{\boldsymbol{r}} = -\nabla\Phi \tag{B.9}$$

yields in these coordinates

$$\ddot{r} - r\dot{\phi}^2 = g(r)\,, \tag{B.10}$$

where a dot means differentiation with respect to time t, $\dot{r} \equiv \frac{\mathrm{d}r}{\mathrm{d}t}$. Since the force is radial, the angular momentum per unit mass h is conserved. Therefore, $h = r^2\dot{\phi}$ is constant. Performing the change of variable

$$u = \frac{1}{r}\,, \tag{B.11}$$

where r should be envisage as $r(\phi)$, and so $u(\phi) = \frac{1}{r(\phi)}$, and rearranging, we obtain

$$u'' + u = \frac{1}{h^2 a - \frac{h^2 u}{c^2}}\,, \tag{B.12}$$

where a prime denotes differentiation with respect to ϕ, $u' \equiv \frac{\mathrm{d}u}{\mathrm{d}\phi}$. One can rewrite the right hand side of Eq. (B.12) as $\frac{GM}{h^2}\frac{1}{1-GMu/c^2}$. Approximating to first order in $\frac{GMu}{c^2}$, Eq. (B.12) becomes

$$u'' + u = \frac{GM}{h^2} + \frac{(GM)^2 u}{c^2 h^2}. \tag{B.13}$$

Solving for $u(\phi)$ yields

$$u(\phi) = \frac{GM}{h^2} + b\cos\left(\sqrt{1 - \frac{(GM)^2}{c^2h^2}}\,\phi\right). \tag{B.14}$$

Expanding the square root and retaining the first order term, we get an advance of the perihelion

$$\Delta\phi_{\mathrm{grav}} = \frac{\pi (GM)^2}{c^2 h^2}\,. \tag{B.15}$$

Denoting the total general relativistic precession by $\Delta\phi_{\rm GR}$ we thus get

$$\Delta\phi_{\rm grav} = \frac{1}{6}\Delta\phi_{\rm GR}\,. \tag{B.16}$$

that is the self energy of the gravitational field and its pressure account for $\frac{1}{6}$ of the advance predicted by general relativity. It is interesting to see that if we had considered a positive gravitational energy density and pressure, we would have got a precession in the other direction.

Note that if one considers only the contribution of gravitational field energy density (excluding pressure), one gets $\frac{1}{12}$ of the general relativistic perihelion precession [25].

B.2 Perihelion precession: Contribution from special relativistic dynamics

Special relativity also has a contribution for the advance of the perihelion. In fact, in some standard physics textbooks, the calculation of this contribution is left as an exercise [54, 55], see also [56, 57, 58]. Here, we calculate it explicitly.

Newton's second law for a particle of mass m can be generalized to be compatible with special relativity. The correct equation is

$$\boldsymbol{a} = \frac{1}{m\gamma}\left(\boldsymbol{F} - \frac{(\boldsymbol{v}\cdot\boldsymbol{F})\boldsymbol{v}}{c^2}\right) \tag{B.17}$$

where $\boldsymbol{a}$ is the particle's acceleration, $\boldsymbol{v}$ its velocity, $\boldsymbol{F}$ the force exerted on the particle, and γ the Lorentz-Fitzgerald contraction factor $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, where $v^2 \equiv |\boldsymbol{v}|^2$. Using $\boldsymbol{F} = -\frac{GMm}{r^2}\boldsymbol{e_r}$, and using the conservation of angular momentum per unit mass $h = \gamma r^2\dot{\phi}$, the radial component of Eq. (B.17) becomes

$$\ddot{r} - \frac{h^2}{\gamma^2 r^3} = -\frac{GM}{\gamma r^2}\left(1 - \frac{\dot{r}^2}{c^2}\right) \tag{B.18}$$

Define again the variable u by $u = r^{-1}$. Then we have the following identities,

$$\dot{r} = -\frac{h}{\gamma}u' \tag{B.19}$$

$$\begin{aligned}\ddot{r} = &-\frac{u^2h^2}{\gamma}\left\{\frac{1}{\gamma}u'' + \right.\\ &\left.+\frac{GM}{c^2}u'^2\left[\frac{h^2}{\gamma^2c^2}\left(u^2+u'^2\right) - 1\right]\right\}\,,\end{aligned} \tag{B.20}$$

where a prime denotes differentiation with respect to ϕ and we have used $\left(\gamma^{-1}\right)' = \frac{GM}{c^2}u'\left(\frac{h^2}{\gamma^2c^2}\left(u^2+u'^2\right)-1\right)$. Then, using Eqs. (B.19) and (B.20) in Eq. (B.18) we find, after some algebra,

$$u''+u=\gamma\frac{GM}{h^2}+\frac{GM}{c^2}u'^2\left(\gamma-\gamma^{-1}\right) \\ -\frac{GMu'^2h^2}{c^4\gamma}\left(u^2+u'^2\right). \quad \text{(B.21)}$$

Neglecting the last term which is of fourth order, and approximating $\gamma \approx 1+\frac{1}{2}\frac{v^2}{c^2}$ and $\gamma^{-1} \approx 1-\frac{1}{2}\frac{v^2}{c^2}$, we get

$$u''+u=\frac{GM}{h^2}+\frac{GMv^2}{2c^2h^2}+\frac{GM(u')^2v^2}{c^6}. \quad \text{(B.22)}$$

We can also work through that

$$v^2=\frac{h^2}{\gamma^2}\left(u'^2+u^2\right). \quad \text{(B.23)}$$

Thus the last term of Eq. (B.22) is fourth order and can be neglected. Write now

$$u=u_0+u_1, \quad \text{(B.24)}$$

where u_0 is the Newtonian solution given by (see, e.g., [2])

$$u_0=\frac{GM}{h^2}(1+e\cos\phi), \quad \text{(B.25)}$$

with e being the eccentricity of the orbit, and u_1 is a first order correction to the solution. Then substitution Eqs. (B.23)-(B.25) into Eq. (B.22) yields

$$u_1''+u_1=\frac{GM}{2c^2\gamma^2}\left(u'^2+u^2\right). \quad \text{(B.26)}$$

Keeping only the first order corrections we get

$$u_1''+u_1=\frac{(GM)^3}{2c^2h^4}(1+e^2+2e\cos\phi). \quad \text{(B.27)}$$

Making the guess $u_1=A\phi\sin\phi$, we get $u_1''+u_1=2A\cos\phi$. Therefore, $A=\frac{em^3c^4}{2h^4}$. Neglecting the constant correction, since it does not have any effect in the perihelion precession, one finally obtains

$$\begin{aligned} u &= u_0+u_1 \\ &= \frac{GM}{h^2}\left(1+e\cos\phi+\frac{(GM)^2}{c^2h^2}\frac{e}{2}\phi\sin\phi\right). \end{aligned} \quad \text{(B.28)}$$

Defining a small parameter $\varepsilon \equiv 3\frac{(GM)^2}{c^2h^2}$, Eq. (B.28) becomes $u = \frac{GM}{h^2}\left(1 + e\cos\phi + \varepsilon\frac{e}{6}\phi\sin\phi\right)$, i.e.,

$$u = \frac{GM}{h^2}\left[1 + e\cos\left(\phi\left(1 - \frac{\varepsilon}{6}\right)\right)\right] . \tag{B.29}$$

This gives a special relativistic advance of $\Delta\phi_{\mathrm{SR}} = \frac{\pi\varepsilon}{3}$, i.e.,

$$\Delta\phi_{\mathrm{SR}} = \frac{\pi(GM)^2}{c^2h^2} , \tag{B.30}$$

i.e.,

$$\Delta\phi_{\mathrm{SR}} = \frac{1}{6}\Delta\phi_{\mathrm{GR}} \tag{B.31}$$

This is the same advance as calculated in [57], see the text books [54, 55] for exercises.

Therefore, using special relativistic dynamics in flat spacetime, one may explain some of the advance predicted by general relativity. This makes sense since general relativity should include special relativistic effects.

B.3 Perihelion precession: Contribution from the curved time in the Newtonian approximation

Using the method of Cheng [59], Roseveare [60] managed to isolate the effect of gravitational time dilation, a form of time curvature. We will follow a similar approach to isolate the effect of curved time, or gravitational time dilation, in the Newtonian limit. A static spacetime metric $g_{\alpha\beta}$ can be written

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{00}dt^2 + g_{ab}\, dx^a dx^b , \tag{B.32}$$

where Greek indices α, β run from 0 to 3 and Latin indices a, b correspond to the spatial part of the metric and run from 1 to 3. In a Newtonian approximation in spherical coordinates [2] the metric can be written as

$$ds^2 = -(c^2 + 2\Phi)dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right) , \tag{B.33}$$

where Φ is the Newtonian gravitational potential, which far away from the source is

$$\Phi = -\frac{GM}{r} . \tag{B.34}$$

In order to find the particle's motion in such a field we search for the corresponding particle's Lagrangian. The classical Lagrangian L of a particle of mass m moving with kinetic energy K in a gravitational potential energy U is is given by

$$L = K - U\,, \tag{B.35}$$

Now Φ is the gravitational potential given in Eq. (B.34), so the potential energy of the particle of mass m is

$$U = -\frac{GmM}{r}\,, \tag{B.36}$$

To find the appropriate expression for the kinetic energy K of the particle measured by a static observer in the metric given by Eq. (B.32), we begin by calculating the observed total energy E of the particle. The four-velocity v^α and four-momentum p_α of the particle are given by $v^\alpha = \frac{dx^\alpha}{d\tau}$ and $p_\alpha = m\, g_{\alpha\beta}v^\beta$, respectively, where τ is the proper time of the particle. If an observer has four-velocity u^β, then the total observed energy of the particle measured in the local observer's frame, supposed inertial, is $E = -p_\beta u^\beta$. The kinetic energy K can then be expressed by subtracting the rest energy mc^2 from the total energy E, i.e., $K = -p_\beta u^\beta - mc^2$. Considering now the metric given in Eq. (B.32), which is diagonal in our chosen spherical coordinates, we have that the particle's four velocity can be written as $v^\alpha = \frac{dx^\alpha}{dt}\frac{dt}{d\tau} = \frac{dx^\alpha}{dt}v^0$, where $v^0 = \frac{dt}{d\tau}$. Then, since $g_{\alpha\beta}v^\alpha v^\beta = -c^2$, we get $g_{\alpha\beta}v^\alpha v^\beta = g_{00}\left(v^0\right)^2 + \left(v^0\right)^2 v^2 = -c^2$, where $v^2 = g_{ab}v^a v^b$ is the square of the spatial velocity of the particle, with $v^a = \frac{dx^a}{dt}$ being the spatial components of the velocity of the particle with respect to time t. We can then solve the equation above for v^0, getting $v^0 = c\sqrt{\frac{-1}{g_{00}+v^2}}$. We want the kinetic energy measured by a static observer. An observer u^α can be envisaged, dynamically, as a test particle, and if the observer is static it has spatial velocity zero, $u^a = 0$. Thus, the derivation above holds, but now the particle observer has $u^a = 0$ and $u^2 = 0$. So, $u^0 = c\sqrt{\frac{-1}{g_{00}}}$. Since the metric is diagonal and $u^a = 0$, the kinetic energy expression simplifies to $K = mg_{00}\, v^0 u^0 - mc^2$. Then using the expressions for v^0 and u^0 just found we find for the kinetic energy the expression $K = mc^2\left(\sqrt{\frac{g_{00}}{g_{00}+v^2}} - 1\right)$. Using the g_{00} given in Eq. (B.33), expanding the fraction and the square root, and keeping only the first order correction in the Newtonian potential Φ, gives $K = \frac{1}{2}m\left(1 - \frac{2\Phi}{c^2}\right)v^2$. Substituting $v^2 = \dot{r}^2 + r^2\dot{\phi}^2$ for a particle moving in the equatorial plane $\theta = \frac{\pi}{2}$ (without loss of generality) and Φ given in Eq. (B.34) one finally obtains

$$K = \frac{1}{2}m\left(1 + \frac{2GM}{c^2 r}\right)\left(\dot{r}^2 + r^2\dot{\phi}^2\right)\,. \tag{B.37}$$

Since $L = K - U$ as displayed in Eq. (B.35), and U and K are given in Eqs. (B.36) and (B.37), respectivey, we have

$$L = \frac{1}{2}m\left(1 + \frac{2GM}{c^2 r}\right)\left(\dot{r}^2 + r^2\dot{\phi}^2\right) + \frac{GmM}{r}\,. \tag{B.38}$$

This is the same Lagrangian used in [60], we have derived it by a different method. Note that we are in the approximation of weak field and low speeds.

Then using the Euler-Lagrange equations derived from the Lagrangian (B.38) gives an advance of the perihelion due to the curved time (see [60], p. 174)

$$\Delta\phi_{\rm t} = \frac{4\pi(GM)^2}{c^2h^2}\,, \tag{B.39}$$

i.e.,

$$\Delta\phi_{\rm t} = \frac{2}{3}\Delta\phi_{\rm GR} \tag{B.40}$$

which is 2/3 of general relativity.

We considered in this section a Newtonian type metric and not the exact Schwarzschild solution. This is because we wanted to isolate the most important effect, namely, of time curvature, in this weak field approximation. Of course, had we taken the exact solution, we would account for the full general relativistic prediction.

B.4 Perihelion precession: overall effect

Taking all the effects into account, we can sum all the contributions, getting

$$\frac{\Delta\phi_{\rm grav} + \Delta\phi_{\rm SR} + \Delta\phi_{\rm t}}{\Delta\phi_{\rm GR}} = \frac{1}{6} + \frac{1}{6} + \frac{2}{3} = 1\,. \tag{B.41}$$

B.5 Conclusion

Adding all the effects that were considered, namely the presence of gravitational field energy density and pressure as sources of gravity, special relativistic dynamics, and the curvature of time in a Newtonian approximation, we get the full general relativity prediction for the advance of Mercury's perihelion. Our result gives an interesting and plausible explanation for the effects that are hidden in general relativity in the perihelion precession calculation