An introduction to special relativity focused on the geometric construction of the hyperbolic plane

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Abstract — Einstein’s special relativity has introduced a radical transformation in our interpretation of space and time. In contemporaneous physics, namely in quantum field theory, special relativity is a fundamental tool. Both classical and quantum electrodynamics strongly depend on this new vision.

The main goal of this dissertation is to develop a new vector approach, based on a direct geometric construction, of the hyperbolic plane as a bi-dimensional simplification of the whole four-dimensional Minkowskian spacetime. It is intended to create a visual and intuitive interpretation of the relativity of simultaneity, thereby allowing to directly read off both length contraction and time dilation. This geometric algebra, therefore, makes almost irrelevant the usual emphasis on passive coordinate transformations. Obviously, then, those effects that strictly depend on the whole four-dimensional construction – such as Thomas rotation – are beyond the present approach.

Several typical examples of classical paradoxes are considered: the pole-in-barn paradox; the twin (or clock) paradox. The previously developed method is used, thereby clearly showing its soundness and applicability. Also, this alternative approach has another usefulness: more than exploring what is relative (like space and time, when considered separately), it stresses what is absolute and invariant – like the spacetime interval invariance underneath the new metric which prohibits a Euclidean interpretation of spacetime plots. Finally, the twin paradox is herein addressed with parabolic and hyperbolic paths, to stress that acceleration can (and should) be included in special relativity.

Keywords — Minkowskian spacetime, Hyperbolic plane, Boost / Active Lorentz transformation, Doppler effect, Hyperbolic motion, Twin paradox, Geometric algebra of the hyperbolic plane.

1. INTRODUCTION

In 1905, Einstein published two articles that had a great impact on the theory of relativity [1]. One of them was titled On the electrodynamics of moving bodies and contains essentially what is known now by the theory of special relativity. In this article Einstein gets essentially the same results that had been obtained by other authors, but in a much more simple and clear. In another article, as a supplement to the above, he deduces in less than three pages the probably most famous formula in Physics: $E = mc^2$.

From the point of view of new scientific results, the Einstein work didn't bring many contributions. It can be said that almost all the equations derived by Einstein had already been obtained before. However, the jobs that have been developed over here were based on the existence of ether. The ether was considered a useful concept, able to provide an understanding of the phenomena, although they never managed to prove its existence.

For Einstein, the ether was a useless concept, since it was not possible to be detected. In the article published in 1905, he adopted the concept that it wasn't detectable should be deleted.

Einstein's theory of relativity is generally famous for being extremely complex and difficult to understand. But, in fact, your exposure by Einstein was quite clear and succinct. The same didn’t happen in the work carried out by Lorentz and Poincaré. These authors did not present a final version and teaching of his ideas. Showed a gradual construction, through several attempts of a theory that suffered several changes constantly. They were not clear about what should be taken as a starting point. They didn't have a set of principles and a method of deduction of results. Each case was studied by a new method.

In contrast, Einstein presents its assumptions with more clarity. He starts of two postulates making more simple and clear deductions, both under the conceptual point of view as a mathematician. There is a general method which is across all cases. This was an aspect in which Einstein was impressed and appreciated by the entire scientific community. Even for those who did not agree with the conceptual and epistemological aspects of the theory of Einstein realized that its methodology was quite attractive.

In 1908, Hermann Minkowski (1864-1909) gave his famous lecture space and time at a conference of German scientists from various fields. He found himself faced with the following problem: how to explain to those present, many of them weren’t physicists or mathematicians, the theory of special relativity introduced by Einstein in 1905 whose mathematical treatment involved mathematical concepts (tensors, non-euclidean geometry). Minkowski made it speaking only of the change that the theory of special relativity introduced on the concepts of space and time, but without resorting to sophisticated math.

In order to explain the meaning of relativity and spacetime, Minkowski had the idea to represent the movement of objects throughout of Minkowski diagrams.

Minkowski also took their diagrams to represent, for each observer, two cones: the cone of future events related to this observer and the cone of the past events. Also introduced the distinction, now classic, between timelike vectors and spacelike vectors.

About this subject, the theory of special relativity, there is a huge bibliography can be consulted. From introductory level bibliography [2-10], as well as intermediate level bibliography [11-18], until the bibliography of advanced level [19-23].
2. **SIMULTANEITY, MINKOWSKI DIAGRAMS AND INvariance of the interval**

The essence of the theory of special relativity, formulated by Albert Einstein in 1905, lies in the revised concept of simultaneity. According to the Galilean transformation, the time is universal and absolute, so, does not depend on the reference frame in which it is measured. The theory of special relativity of Einstein starts with two postulates that according to Newtonian mechanics are purely contradictory to each other.

- **First postulate** (principle of relativity): the laws of physics are the same in all inertial reference frames.

- **Second postulate** (c invariance): the speed of light in a vacuum does not depend on the speed of the source.

According to the physics pre-relativistic, there is an incompatibility between the two previously described postulates. In Newtonian mechanics, there is no limit speed for a velocity of a particle. Considering a particle with a velocity \( c \) in a wagon with a speed \( v \), an observer at the station sees the particle with a velocity \( w = c + v > c \).

### 2.1 Relativity of simultaneity

The concept of absolute simultaneity is incompatible with the second postulate. Then we will proceed to review the concept of simultaneity, using various geometric problems.

Consider an observer \( O' \) traveling on a train in motion, with speed \( \beta \) described by coordinate system \( S' \) and a stationary observer \( O \) at the station described by coordinate system \( S \) as shown in the following figure.

![Figure 1 - Simultaneity.](image)

From Figure 1 it is concluded that from the point of view of the observer \( O' \) the events A and B are simultaneous, but from the point of view of the observer \( O \) they are not. It is concluded that the simultaneity of events is a relative concept, depends on the inertial frame of reference measured. This implies then that the time is not absolute, so, the time does not flow in the same way to different reference. A consequence of this is that the equitemps of different references in a spacetime diagram no longer coincide.

The next figure shows the equilocs and equitemps of the reference frame \( S \).

![Figure 2 – Equilocs and equitemps of \( S' \).](image)

The axes of \( S \) and \( S' \) are described by the following equations:

\[
\begin{align*}
\text{axis } x \ (\text{equitemp of } S) & \quad \mapsto t = 0 \quad \mapsto t' = -\beta x' \\
\text{axis } t \ (\text{equiloc of } S) & \quad \mapsto x = 0 \quad \mapsto x' = -\beta t' \\
\text{axis } x' \ (\text{equitemp of } S') & \quad \mapsto t' = 0 \quad \mapsto t = \beta x \\
\text{axis } t' \ (\text{equiloc of } S') & \quad \mapsto x' = 0 \quad \mapsto x = \beta t
\end{align*}
\]

(1)

### 2.2 The passive Lorentz transformation

With defined equations in (1) it is easy to get the passive Lorentz transformation. The passive Lorentz transformation aims to derive a relation between a transformation of coordinates \((x, t)\) and a coordinate system \((x', t')\). This means that, knowing the coordinates of a given event in a certain frame of reference, with the Lorentz transformation is possible to obtain the coordinates of the same event in another frame of reference.

So the passive Lorentz transformation is

\[
\begin{align*}
t' &= \gamma(t - \beta x) \\
x' &= \gamma(x - \beta t)
\end{align*}
\]

(2)

and in matrix form, we have

\[
\begin{pmatrix}
t' \\
x'
\end{pmatrix} = \gamma \begin{pmatrix}
1 & -\beta \\
-\beta & 1
\end{pmatrix} \begin{pmatrix}
t \\
x
\end{pmatrix}
\]

(3)

With

\[
\gamma = \frac{1}{\sqrt{1 - \beta^2}}
\]

(4)

where \(-1 \leq \beta \leq 1\) and \(\gamma \geq 1\).

### 2.3 Invariance of the interval

In special relativity is not all relative, in fact there are an absolute. This absolute is the invariance of the interval. Is given by

\[
(t')^2 - (x')^2 = t^2 - x^2
\]

(5)

In equation (5) is over to obtain a result of particular relevance in relativity is the invariance of the spacetime interval. On the basis of this result we can say that the geometry of Minkowski...
spacetime is a non-euclidean geometry: the plan \((x, t)\) has a hyperbolic geometry. The concept of "distance" is thus revised in the theory of special relativity. While in the euclidean plane the locus of points which are a fixed distance from a given point (center) is a circle. In the hyperbolic plane the locus of the events a "distance" (spacetime interval) holds a given event is a hyperbola.

By equation 5 results the following hyperbolas

\[ t^2 - x^2 = \pm s^2 \]

\[ t^2 - x^2 = \tau^2 \]

\[ t^2 - x^2 = 0 \]

\[ x^2 - t^2 = \chi^2 \]

So there are two hyperbolas and two straight lines (asymptotes). The timelike hyperbola and the spacelike hyperbola are described by \( t^2 - x^2 = \tau^2 \) and \( x^2 - t^2 = \chi^2 \), respectively. And the lines (asymptotes) of light are described by \( t = \pm x \). In Figure 3 follows the representation of these hyperbolas.

The hyperbola \( \mathcal{H}_t \) with equation \( x^2 - t^2 = \chi^2 \) corresponds to the locus of events \( B(x, t) \) that the interval between the events \( B \) and \( O(0, 0) \) is \( \mathcal{F} = -\chi^2 \). The degenerate hyperbola with equation \( t^2 - x^2 = 0 \) corresponds to the locus of events \( C(x, t) \) that the interval between the events \( C \) and \( O(0, 0) \) is \( \mathcal{F} = 0 \). The hyperbolas identified represent the hyperbolic geometry. Is here where the special relativity works. For comparison, the equivalent in euclidean geometry is a circle where the locus of points \( P(x, t) \) whose the distance (euclidean) between points \( P \) and \( O(0, 0) \) is \( \mathcal{D} = r \).

Considering two events \( E_1(x_1, t_1) \) and \( E_2(x_2, t_2) \). The interval between them is defined by

\[ \mathcal{F}(E_1, E_2) = (t_2 - t_1)^2 - (x_2 - x_1)^2 \]

and the measure of this interval is

\[ \mu(E_1, E_2) = \sqrt{[(t_2 - t_1)^2 - (x_2 - x_1)^2]} \]

Thus in Figure 4 the measure corresponding to the hyperbola \( \mathcal{H}_t \) is \( \mu = \tau \) and the measure corresponding to the hyperbola \( \mathcal{H}_t \) is \( \mu = \chi \). The concept of measurement is, therefore, a concept of the hyperbolic plane that corresponds to the concept of distance of the euclidean plane. The euclidean distance between \( O(0, 0) \) and \( A(x, t) \) is given by

\[ \mathcal{D} = \sqrt{t_A^2 + x_A^2} = r \]

From the point of view of \( O \) we have:

\[ A \rightarrow O \rightarrow (x_A, t_A) = (\beta t_A, t_A) \]

In this figure are present two observers: the observer \( O \rightarrow (x, t) \) in what is called the axis \( x \) by \( O \) and the axis \( x \) by \( O \) and the observer \( \mathcal{P} \rightarrow (x', t') \) in what is called the axis \( t \) by \( \mathcal{P} \) and the axis \( t' \) by \( \mathcal{P} \). The hyperbola \( \mathcal{H}_t \) with equation \( t^2 - x^2 = \tau^2 \) corresponds to the locus of events \( A(x, t) \) that the interval between the events \( A \) and \( O(0, 0) \) is \( \mathcal{F} = \tau^2 \).

In order to clarify the difference between euclidean geometry and hyperbolic geometry shows the case in Figure 4.
in that \( \mu \) is the measure of the interval (Lorentzian metric) and \( r \) the distance (euclidean metric). Note that, only when we have \( \beta = 0 \), the euclidean distance coincides with the measure of the interval, thus, \( \mu = \tau = \chi = r \). For \( \beta = 1 \) we got \( \mu = 0 \). This is the case of degenerate hyperbola.

2.4 The active Lorentz transformation

In section 2.2 the passive Lorentz transformation has been derived. We will now derive the active Lorentz transformation. Taking into account the event vector

\[
\mathbf{r} = t \mathbf{e}_0 + x \mathbf{e}_1 = t' \mathbf{f}_0 + x' \mathbf{f}_1
\]

The active Lorentz transformation is

\[
\begin{align*}
\mathbf{f}_0 &= \gamma (\mathbf{e}_0 + \beta \mathbf{e}_1) \\
\mathbf{f}_1 &= \gamma (\mathbf{e}_1 + \beta \mathbf{e}_0)
\end{align*}
\]  

and in matrix form, we have

\[
\begin{pmatrix}
\mathbf{f}_0 \\
\mathbf{f}_1
\end{pmatrix} = \gamma
\begin{pmatrix}
1 & \beta \\
\beta & 1
\end{pmatrix}
\begin{pmatrix}
\mathbf{e}_0 \\
\mathbf{e}_1
\end{pmatrix} \iff
\begin{pmatrix}
\mathbf{e}_0 \\
\mathbf{e}_1
\end{pmatrix} = \gamma
\begin{pmatrix}
1 & -\beta \\
-\beta & 1
\end{pmatrix}
\begin{pmatrix}
\mathbf{f}_0 \\
\mathbf{f}_1
\end{pmatrix} \tag{15}
\]

The active Lorentz transformation transforms \( \{\mathbf{e}_0, \mathbf{e}_1\} \) in \( \{\mathbf{f}_0, \mathbf{f}_1\} \).

Consider that in this way comes

\[
\mathbf{G} = \begin{pmatrix}
\mathbf{f}_0 & \mathbf{f}_0 & \mathbf{f}_1 \\
\mathbf{f}_1 & \mathbf{f}_0 & \mathbf{f}_1
\end{pmatrix} = \begin{pmatrix}
\mathbf{e}_0 & \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_1
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
0 & -1
\end{pmatrix} \tag{17}
\]

With the inner product of vectors crossed gets

\[
\mathbf{M} = \begin{pmatrix}
\mathbf{e}_0 \cdot \mathbf{f}_0 & \mathbf{e}_0 \cdot \mathbf{f}_1 \\
\mathbf{e}_1 \cdot \mathbf{f}_0 & \mathbf{e}_1 \cdot \mathbf{f}_1
\end{pmatrix} = \begin{pmatrix}
\gamma & \gamma \beta \\
-\gamma \beta & -\gamma
\end{pmatrix} \tag{18}
\]

In Figure 4 we can see these vectors represented in a Minkowski diagram.

The most important in Figure 5 is the nature of non-euclidean geometry. The four vectors \( \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{f}_0, \mathbf{f}_1\} \) are unit vectors, therefore they have the same "length". Note that a timelike vector (\( \mathbf{e}_0 \) or \( \mathbf{f}_0 \)) defines a family of equilocus and a spacelike vector (\( \mathbf{e}_1 \) or \( \mathbf{f}_1 \)) defines a family of equitemps.

3. Time dilation, length contraction and causality

This chapter begins by addressing formally two of the most well-known consequences of the theory of special relativity. It is time dilation and length contraction. These are two consequences of relativity of simultaneity.

3.1 Time dilation

From the Figure 6 it is possible to derive the time dilation.

\[
\overline{OA} = Te_0, \quad \overline{OB} = T_0 \mathbf{f}_0 \tag{19}
\]

From the Figure 6, we have

\[
\overline{OB} = \overline{OA} + \overline{AB} \implies T_0 \mathbf{f}_0 = T \mathbf{e}_0 + L \mathbf{e}_1 \tag{20}
\]

It follows, therefore, the time dilation

\[
T = \gamma T_0 \geq T_0 \tag{21}
\]

where \( T_0 \) is the proper time of the observer \( O \), and \( T \) is the time measured by the observer \( O \).

So the observer \( O \) verifies that the clock of \( O' \) (that is on the move, from their point of view, marking a time \( T_0 \)) delays in relation to its own clock (marking, for the same interval of spacetime, a time \( T \)).

3.2 Length contraction

From the Figure 7 it is possible to derive the length contraction.
relativity correspond to a deviation from the common sense. An example of this type of "paradoxes" is known in the literature as the pole-barn paradox.

The pole-barn paradox is a thought experiment in special relativity. It involves a pole, parallel to the ground, travelling horizontally and therefore undergoing a Lorentz length contraction. As a result, the pole fits inside a barn which would normally be too small to contain it. On the other hand, from the point of view of an observer moving with the pole, it is the barn that is moving, so it is the barn which will be contracted to an even smaller size, thus being unable to contain the pole. This apparent paradox results from the mistaken assumption of absolute simultaneity. The pole fits into the barn only if both of its ends are simultaneously inside the barn. In relativity, simultaneity is relative to each observer, and so the question of whether the pole fits inside the barn is relative to each observer, and the paradox is resolved. The situation can be further illustrated by the Minkowski diagram below.

![Minkowski diagram](image)

The solution to the apparent paradox lies in the relativity of simultaneity: what one observer (e.g. with the barn) considers to be two simultaneous events may not in fact be simultaneous to another observer (e.g. with the pole). When we say the pole "fits" inside the barn, what we mean precisely is that, at some specific time, the position of the back of the pole and the position of the front of the pole were both inside the barn; in other words, the front and back of the pole were inside the barn simultaneously. As simultaneity is relative, then, two observers can disagree without contradiction on whether the pole fits. To the observer with the barn, the back end of the pole was in the barn at the same time that the front end of the pole was, and so the pole fit; but to the observer with the pole, these two events were not simultaneous, and the pole did not fit.

3.3 Reciprocity

Both time dilation and length contraction are real and reciprocal effects. Both points of view (the $\mathcal{O}$ and the $\mathcal{P}$) are correct. In fact, the conclusion is always the same: the clocks in motion delay in relation to the stationary clock. The principle of relativity (first postulate) states that the laws of physics are the same in all inertial reference frames. If time dilation is observed from the point of view of $\mathcal{O}$ the same effect has to be present from the point of view of $\mathcal{P}$. The same goes for the length contraction: the ruler that is in motion is greater in relation to ruler that is at rest. This effect is reciprocal, if one observes the length contraction from the point of view of $\mathcal{O}$ the same effect has to be present from the point of view of $\mathcal{P}$.

3.4 The pole-barn paradox

One of the practical manifestations of relativity of the concept of simultaneity is the length contraction. But, as time dilation, this is a reciprocal effect. However, if misinterpreted, can give rise to (apparent) paradox. The term paradox is associated with situations where logical contradictions. But, as will be noted, there is nothing contradictory in its logic. What happens is, quite simply, a shock in relation to common sense or intuition. Therefore, almost all of the most important results of special
This figure shows that with superluminous signals there is a violation of causality. It is concluded that it is not possible to send information with supraluminous speed and, therefore, all physical actions only propagate with light speed or subluminous speed.

4. Bondi Calculus

This chapter begins to derive the Bondi k-calculus by Bondi’s radar method. And it is given by

$$k = \frac{1 + \beta}{1 - \beta}$$

(25)

For discover how the composition of velocities works it is uses again the Bondi’s radar method. So the composition of velocities is given by

$$\beta = \beta_1 \oplus \beta_2 = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$$

(26)

4.1 Doppler effect

From the Figure 10 it is possible to derive the Doppler effect.

According to Bob, Alice's journey lasted a time interval $T$. Once Alice traveled to a distance $L$, so

$$L = \beta \frac{T}{2}$$

(29)

But from a point of view of Alice, her journey lasted a time interval $T'$ and it was

$$T' = \frac{T}{\gamma}$$

(30)

During the journey of Alice there is a period of time lived by Bob that Alice unknown. Bob has already watched all the way through the journey of Alice. This is where a breach of reciprocity between the twins. This happens due to the fact that Alice in reverse, event A, instantly change the referential frame. We can see it in the next figure.

And for the case where the references frames approach the Doppler effect expression is given by

$$f' = kf \quad \Rightarrow \quad f' = \sqrt{\frac{1 + \beta}{1 - \beta}} f$$

(28)

4.2 The twin paradox

In this chapter is also studied the twin paradox. The twin paradox consists in two twins, Alice and Bob, which in the beginning have synchronized clocks, and are separated when one of them (Alice) starts a journey. At some point, that same twin reverses the direction of motion, returning to the starting point where the brother awaits. As Bob sees his sister moving away due time dilation he says that time passes slower for her sister. However since speed is relative, and assuming that Alice does not undergo any acceleration, she will also see his brother Bob moving away and will assert that it is in Bob that time slows down. The statement is followed by the appropriate question: when they find each other which one will actually be younger?

The universe line of Alice is represented like in the following figure.
In order to agree with each other, both twins agree to send to other electromagnetic signals with the same frequency $f$. As is known by the Doppler effect, the signals sent with a certain frequency are received with another frequency. We can see it in the figure below.

In Figure 13 the left figure the signals are sent by Bob and are received by Alice. On right figure the signals are sent by Alice and are received by Bob. Note that, when they are moving away the signals are received with a frequency $f' = f/k < f$ and when they are approaching the signals are received with a frequency $f' = kf > f$. Therefore in Figure 13 it can conclude that Alice receives ten signals that were sent by Bob and Bob receives eight signals that were sent by Alice. Wherefore both agree that when they find each other the Alice is younger than Bob.

4.3 The twin paradox with a parabolic section

However, in that present case remains a problem. As we can see in Figure 11, Alice suffers an infinite acceleration in the event A. As a result of this infinite acceleration, there is an abrupt change of *equitemps* of Alice. What the practical point of view makes the impossibility of her survival. In order to avoid this problem it is considered a new version for the universe line of Alice. It is showed in the next figure.

Now the universe line of Alice is characterized by a parabolic section that avoids the infinite acceleration on event A. So, the universe line of Alice is, from the point of view of Bob, described by the following equations:

$$x(t) = \begin{cases} \beta t, & 0 \leq t \leq t_1 \\ \left( \frac{L-\delta}{2} + \frac{\beta^2}{\delta} \left( t - \frac{L}{\beta} \right)^2 \right), & t_1 \leq t \leq t_2 \\ \beta (T-t), & t_2 \leq t \leq T \end{cases}$$

(31)

And the proper time of Alice is described by:

$$\tau(T) = 2\tau_0 = T' + \frac{\delta}{\beta^2} \left[ \sin^{-1}\left( \frac{\beta}{\gamma} \right) - \frac{\beta}{\gamma} \right]$$

(32)

When $\delta = 0$ the result obtained in the previous section: $\tau(T) = T' = (2L)/(\gamma \beta)$. The parameter $\delta$ define the family of parabolas, where $0 \leq \delta \leq L$. Note that when $\delta \neq 0$ (when there is a parabolic motion) gets a $\tau(T) > T'$, so, this new universe line of Alice (Figure 14) corresponds to a greater proper time interval for Alice in relation to the universe line considered in Figure 11.

5. Hyperbolic Motion and Applications

In this chapter the hyperbolic motion is studied. Then is held (again) the study of the twin paradox, but now the universe line of Alice is made up of sections of hyperbola.

In the general case, we have

$$(x-x_0)^2 - (t-t_0)^2 = X^2$$

(33)

These hyperbolas can be seen in the following figure.
The corresponding parametric equations are:

\[
\begin{align*}
  x(\phi) &= x_0 + X \cosh(\phi) \\
  t(\phi) &= t_0 + X \sinh(\phi) \\
  x(\phi) &= x_0 - X \cosh(\phi) \\
  t(\phi) &= t_0 + X \sinh(\phi)
\end{align*}
\]

In this case, we obtain

\[
x(t) = -X + \sqrt{t^2 + X^2}
\]

(35)

5.1 The twin paradox with hyperbolic sections

Now Alice is moving with hyperbolic motion. The universe line of Alice will have three sections characterized by a hyperbola. We can see it in Figure 16.

The second section is:

\[
x(t) = \frac{T}{\zeta} \left[ \sqrt{1 + \zeta^2 \left( \frac{t}{T} - \frac{1}{2} \right)^2} + 2 \sqrt{1 + \frac{\zeta^2}{16} - 1} \right], \quad \frac{T}{4} \leq t \leq \frac{3T}{4}
\]

(37)

Finally, the third section is:

\[
x(t) = \frac{T}{\zeta} \sqrt{1 + \zeta^2 \left( \frac{t}{T} - 1 \right)^2}, \quad \frac{3T}{4} \leq t \leq T
\]

(38)

The maximum distance between Alice and Bob corresponds to

\[
L = x \left( \frac{T}{2} \right) = \frac{2T}{\zeta} \left( \sqrt{1 + \frac{\zeta^2}{16} - 1} \right)
\]

(39)

The Alice universe line tends to "null path" \( O \rightarrow \text{P}_1 \rightarrow \text{P} \rightarrow \text{P}_2 \rightarrow \text{B} \) when \( \zeta \rightarrow \infty \). This "null path" corresponds to the path of an electromagnetic signal and therefore impossible to Alice, with

\[
x(t) = \begin{cases} 
  t, & 0 \leq t < \frac{T}{2} \\
  T - t, & \frac{T}{2} < t \leq T
\end{cases} \quad \Rightarrow \quad \beta(t) = 1
\]

(40)

The relative velocity is given by \( \beta(t) = \frac{dx}{dt} \), such that

\[
\beta(t) = \begin{cases} 
  \sqrt{1 + \zeta^2 \left( \frac{t}{T} - \frac{1}{2} \right)^2}, & 0 \leq t \leq \frac{T}{4} \\
  \sqrt{1 + \zeta^2 \left( \frac{t}{T} - 1 \right)^2}, & \frac{T}{4} \leq t \leq \frac{3T}{4} \\
  \sqrt{1 + \zeta^2 \left( \frac{t}{T} - 1 \right)^2}, & \frac{3T}{4} \leq t \leq T 
\end{cases}
\]

(41)

In turn, the time dilation coefficient

\[
\gamma(t) = \frac{1}{\sqrt{1 - \beta^2(t)}}
\]

(42)
is given by

\[
\gamma(t) = \begin{cases} \\
\sqrt{1 + \zeta^2 \left( \frac{t}{T} \right)^2}, & 0 \leq t \leq \frac{T}{4} \\
\sqrt{1 + \zeta^2 \left( \frac{t-1}{T} \right)^2}, & \frac{T}{4} \leq t \leq \frac{3T}{4} \\
\sqrt{1 + \zeta^2 \left( \frac{t}{T} - 1 \right)^2}, & \frac{3T}{4} \leq t \leq T
\end{cases}
\]  

(43)

The maximum value of \( \gamma(t) \) occurs in \( t = T/4 \) and \( t = (3T)/4 \), being

\[
\gamma_{\text{max}} = \sqrt{1 + \frac{\zeta^2}{16}} = 1 + \frac{\zeta}{2} \frac{L}{T}
\]

(44)

For relative acceleration \( a(t) = d\beta / dt \), comes

\[
a(t) = \begin{cases} \\
\frac{\zeta}{T} \left[ 1 + \zeta^2 \left( \frac{t}{T} \right)^2 \right]^{3/2}, & 0 \leq t \leq \frac{T}{4} \\
\frac{-\zeta}{T} \left[ 1 + \zeta^2 \left( \frac{t-1}{T} \right)^2 \right]^{3/2}, & \frac{T}{4} \leq t \leq \frac{3T}{4} \\
\frac{\zeta}{T} \left[ 1 + \zeta^2 \left( \frac{t}{T} - 1 \right)^2 \right]^{3/2}, & \frac{3T}{4} \leq t \leq T
\end{cases}
\]

(45)

Therefor

\[
a(t) = \begin{cases} \\
\frac{\zeta}{T}, & 0 \leq t \leq \frac{T}{4} \\
\frac{-\zeta}{T}, & \frac{T}{4} \leq t \leq \frac{3T}{4} \\
\frac{\zeta}{T}, & \frac{3T}{4} \leq t \leq T
\end{cases}
\]

(46)

And the proper time of Alice is described by

\[
\tau(t) = \begin{cases} \\
\frac{T}{\zeta} \sinh^{-1} \left( \frac{\zeta}{T} \frac{t}{T} \right), & 0 \leq t \leq \frac{T}{4} \\
\frac{T}{\zeta} \sinh^{-1} \left[ \zeta \left( \frac{t}{T} - \frac{1}{2} \right) \right] + \frac{2T}{\zeta} \sinh^{-1} \left( \frac{\zeta}{4} \right), & \frac{T}{4} \leq t \leq \frac{3T}{4} \\
\frac{T}{\zeta} \sinh^{-1} \left[ \zeta \left( \frac{t}{T} - 1 \right) \right] + \frac{4T}{\zeta} \sinh^{-1} \left( \frac{\zeta}{4} \right), & \frac{3T}{4} \leq t \leq T
\end{cases}
\]

(47)

In particular, the total time that Alice’s journey is

\[
T' = \tau(T) = \frac{4T}{\zeta} \sinh^{-1} \left( \frac{\zeta}{4} \right)
\]

(48)

The Figure 17 shows the variation of \( \tau / T \) in function of \( t / T \) for different values of the parameter \( \zeta \). With that figure pretends to show the evolution along the journey, the proper time of Alice as a function of time of Bob (considering various values of acceleration of Alice).

Figure 17 - Variation of \( \tau(t) \).

Note that how higher is the parameter \( \zeta \) more is the difference of the ages between the twins.

The next figure shows the variation of the ratio between the duration of the journey from the point of view of Alice \( T' \), and the duration of the journey from the point of view of Bob \( T \) with the parameter \( \zeta \).

Figure 18 - Variation of \( T'/T \) with \( \zeta \).
6. CONCLUSIONS

The subject of this thesis is a broad and comprehensive approach by the scientific community, it is easily found in various articles and specialty books. However, the most majority of these works focuses on the passive Lorentz transformation, therefore, determine how the coordinates of an event become between different frames of inertia. It results in a study of special relativity more analytical.

The importance of this thesis is introduce the special relativity based on a geometric, graphic and intuitive interpretation of all the effects of relativity without recourse to the coordinates, based essentially on a vector analysis. The developed approach is essentially a geometric algebra of the hyperbolic plane. So focuses on the active Lorentz transformation that transforms the vectors that characterize a given reference frame in other vectors that characterize another reference frame.

Note also the analysis of the twin paradox under the classical perspective of reference with relative motion uniform but also considering uniformly accelerated reference. However, the latter analysis is rarely considered, even in classical literature.

REFERENCES


