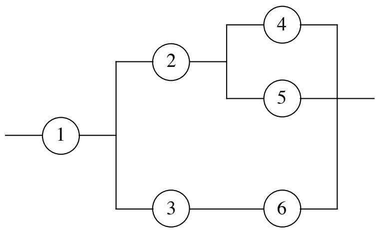
Reliability and Quality Control
LMAC, MMA

2nd. Semester – 2023/2024
02/07/2024 — 08:00
Exame Época de Recurso

- Duration: **2 hours**
- **Please justify your answers.**
- This test has **two pages** and **seven questions**. The total of points is **20.0**.

1. A lifting station for wastewater containing sewage depends on six components set according to the following system block diagram:



(a) Identify the minimal path sets and minimal cut sets of this system, and provide an expression (do not simplify it!) for its structure function. **(1.5)**

- **Minimal path sets**
  $\mathscr{P}_1 = \{1,2,4\}$, $\mathscr{P}_2 = \{1,2,5\}$, $\mathscr{P}_3 = \{1,3,6\}$,
  $p^* = 3$ minimal path sets
- **Minimal cut sets**
  $\mathscr{K}_1 = \{1\}$, $\mathscr{K}_2 = \{2,3\}$, $\mathscr{K}_3 = \{2,6\}$, $\mathscr{K}_4 = \{4,5,3\} = \{3,4,5\}$, $\mathscr{K}_5 = \{4,5,6\}$,
  $q = 5$ minimal cut sets
- **Structure function**

$$\phi(\underline{X}) \stackrel{Th.1.30}{=} 1 - \prod_{j=1}^{p^*}\left(1 - \prod_{i\in\mathscr{P}_j} X_i\right)$$
$$= 1 - (1 - X_1X_2X_4) \times (1 - X_1X_2X_5) \times (1 - X_1X_3X_6)$$

Obs. — Equivalently,

$$\phi(\underline{X}) \stackrel{Th.1.30}{=} \prod_{j=1}^{q}\left[1 - \prod_{i\in\mathscr{K}_j}(1 - X_i)\right]$$
$$= [1-(1-X_1)] \times [1-(1-X_2)(1-X_3)] \times [1-(1-X_2)(1-X_6)]$$
$$\times [1-(1-X_3)(1-X_4)(1-X_5)] \times [1-(1-X_4)(1-X_5)(1-X_6)].$$

(b) Suppose now that the components operate independently and $p_i = p = 0.99$, $i = 1,\ldots,6$. **(1.5)**
Provide two non trivial lower bounds for the reliability of the system. Which bound is stricter?

- **Reliability of the components**
  $p_i = p = 0.99$, $i = 1,\ldots,6$
- **First lower bound for $r(\underline{p})$**
  The components form a coherent system and operate independently, therefore

$$r(\underline{p}) \stackrel{Th.1.68}{\geq} \prod_{j=1}^{q}\left[1 - \prod_{i\in\mathscr{K}_j}(1-p_i)\right] \stackrel{p_i=p}{=} \prod_{j=1}^{q}\left[1-(1-p)^{\#\mathscr{K}_j}\right]$$
$$= [1-(1-p)] \times [1-(1-p)^2]^2 \times [1-(1-p)^3]^2$$
$$\stackrel{p=0.99}{=} 0.989800.$$

- **Second lower bound for $r(\underline{p})$**
  Since the components form a coherent system and operate in an independent fashion — hence, in a positively associated manner —, we can also apply Theorem 1.70 (min-max bounds!).

$$r(\underline{p}) \overset{(1.42)}{\geq} \max_{j=1,\ldots,p^*} \prod_{i\in\mathscr{P}_j} p_i \overset{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathscr{P}_j} = p^{\min_{j=1,\ldots,p^*}\#\mathscr{P}_j} \overset{\#\mathscr{P}_j=3,\forall j}{=} p^3 = 0.970299.$$

- **Which bound is stricter?**
  Since $0.989800 > 0.970299$ the lower bound given by Theorem 1.68 is stricter than the one obtain by invoking Theorem 1.70.

2. A leaf spring for a truck is assumed to have a time to failure (X, in years) with hazard rate function equal to $\lambda(t) = \left(\frac{t}{4}\right)^3, \quad t \geq 0$.

(a) What is the probability that this leaf spring fails after age $E(X) = 4 \times \Gamma(1/4+1)$? **(1.5)**
Obs.: It might be useful to know that $\Gamma(1/4) \simeq 3.625610$.

- **R.v.**
  X = time to failure of a leaf spring for a truck
- **Hazard rate function**
  $\lambda(t) = \left(\frac{t}{4}\right)^3$, $t \geq 0$
- **Survival function**
  Using Prop. 3.3, we get

$$\begin{aligned} R(t) &= \exp\left[-\int_0^t \lambda(u)\,du\right] \\ &= \exp\left[-\int_0^t \left(\frac{u}{4}\right)^3 du\right] \\ &= \exp\left[-\left(\frac{u}{4}\right)^4\Big|_0^t\right] \\ &= \exp\left[-\left(\frac{t}{4}\right)^4\right],\ t \geq 0. \end{aligned}$$

- **Requested probability**
  Since $E(X) = 4 \times \Gamma(1/4+1) = 4 \times \frac{1}{4}\Gamma(1/4) \simeq 3.625610$, we get

$$\begin{aligned} R[E(X)] &= \exp\left\{-\left[\frac{E(X)}{4}\right]^4\right\} \\ &\simeq \exp\left(-\frac{3.625610^4}{256}\right) \\ &\simeq 0.509172. \end{aligned}$$

(b) Assume that: T_i represents the time to failure of the leaf spring i ($i = 1,\ldots,4$) and $T_i \sim_{i.i.d.} X$; 3 out of 4 leaf springs are crucial from a safety standpoint of the truck. **(1.5)**
Derive the reliability function of the associated time to failure?

- **Individual times to failure, common reliability function**
  T_i = time to failure of leaf spring i, $i = 1,\ldots,4$
  $T_i \overset{i.i.d.}{\sim} X$, $i = 1,\ldots,4$
  $R_i(t) = R(t) = \exp\left[-\left(\frac{t}{4}\right)^4\right]$, $t \geq 0$
- **Duration of the system**
  We are dealing with a 3-out-of-4 system, thus

$$T = T_{(n-k+1)} = T_{(4-3+1)} = T_{(2)}.$$

- **Reliability function of T**

$$\begin{aligned} R_T(t) &\stackrel{(2.8)}{=} F_{binomial(4,1-R(t))}(4-3) \\ &= \sum_{j=0}^{1} \binom{4}{j} [1-R(t)]^j \, [R(t)]^{4-j} \\ &= [R(t)]^4 + 4\,[1-R(t)]\,[R(t)]^3 \\ &= 4\,[R(t)]^3 - 3\,[R(t)]^4 \\ &= 4\,e^{-\frac{3t^4}{256}} - 3\,e^{-\frac{t^4}{64}}, \quad t \geq 0. \end{aligned}$$

(c) What can be said about the stochastic ageing character of this system with four leaf springs? **(0.5)**

- **Devising the stochastic ageing character of T**

  Since $\lambda(t) = \left(\frac{t}{4}\right)^3$, $t \geq 0$, is an increasing function in t, we can conclude that $T_i \stackrel{i.i.d.}{\sim} IHR$, $i = 1, \dots, 4$. By applying Prop. 3.25, we can add that any order statistic is also IHR, namely

$$T = T_{(2)} \in IHR.$$

(d) Now, suppose the four leaf springs operate in a positively associated manner. **(2.0)**
Obtain an appropriate lower bound for the expected time to failure of this *new* system.

- **Important**

  We are dealing with positively associated r.v. therefore we can resort to Theorem 2.22 to provide bounds for R_T.

- **Lower bound for $R_T(t)$**

  Note that the minimal path sets associated with a 3-out-of-4 system are

$$\mathscr{P}_1 = \{1,2,3\}, \quad \mathscr{P}_2 = \{1,2,4\}, \quad \mathscr{P}_3 = \{1,3,4\}, \quad \mathscr{P}_4 = \{2,3,4\}, \quad p^* = 4 \text{ minimal path sets.}$$

  Consequently, by invoking Theorem 2.22, we get

$$R_T(t) \;\geq\; \max_{j=1,\dots,p^*} \left[\prod_{i \in \mathscr{P}_j} R_i(t) \right] \stackrel{R_i(t)=R(t)}{=} \max_{j=1,\dots,p^*} [R(t)]^{\#\mathscr{P}_j} = [R(t)]^{\min_{j=1,\dots,p^*} \#\mathscr{P}_j} \stackrel{\#\mathscr{P}_j=3,\forall j}{=} [R(t)]^3.$$

- **Lower bound for μ**

$$\begin{aligned} E(T) &\stackrel{(2.10)}{=} \int_0^{+\infty} R_T(t)\,dt \\ &\geq \int_0^{+\infty} [R(t)]^3\,dt \\ &\stackrel{(b)}{=} \int_0^{+\infty} \exp\left[-3 \times \left(\frac{t}{4}\right)^4\right] dt \\ &\stackrel{(b)}{=} \int_0^{+\infty} R_{Weibull\left(\frac{4}{3^{1/4}},4\right)}(t)\,dt \\ &= \frac{4}{3^{1/4}} \times \Gamma(1/4+1) \\ &= \frac{1}{3^{1/4}} \times \Gamma(1/4) \\ &\simeq 2.754868. \end{aligned}$$

3. Twenty-four vehicles have completed a 74 000 kilometer scheduled run at an automobile company's proving grounds. During the test eight radiator hoses failed and were replaced at kilometers: 2 760, 3 700, 7 100, 17 220, 29 500, 48 400, 52 600, 65 000. **(1.5)**

Calculate the ML estimate of the expected number of replacements of the original radiator hoses during the 24 000 kilometer warranty period supposing that: two million vehicles have been sold; proving ground kilometers are assumed to be equivalent to customer kilometers; the times to replacement are independent and exponentially distributed.

- **Failure times**

  T_i = failure time of a radiator hose i

  $T_i \overset{i.i.d.}{\sim} \text{exponential}(\lambda), \quad i = 1,\dots,n \quad (n = 20)$

  $\lambda > 0$ (UNKNOWN)

- **Censored data**

  Since the test lasted for $t_0 = 74\,000$ hours and none of the $r = 8$ failed radiator hoses was replaced, we are dealing with a Type I/item censored testing with replacement.

- **Total time on test**

  According to Definition 5.17, the cumulative total time in test is in this particular case:

$$\begin{aligned} \tilde{t} &= n \times t_0 \\ &= 24 \times 74\,000 \\ &= 1\,776\,000 \end{aligned}$$

- **ML estimate of λ**

  According to Table 5.11, it is equal to

$$\begin{aligned} \hat{\lambda} &= \frac{r}{\tilde{t}} \\ &= \frac{8}{1\,776\,000}. \end{aligned}$$

- **Another unknown parameter**

  If two million vehicles are sold, then the r.v. Y, representing number of replacements of the original radiator hoses during the 24 000 kilometer warranty period, has a binomial distribution with parameters $m = 2\,000\,000$ and $p = P(T \le 24\,000) = 1 - e^{-24\,000\times\lambda}$.

  Its expected value is equal to

$$h(\lambda) = 2\,000\,000 \times \left(1 - e^{-24\,000\times\lambda}\right).$$

- **Requested ML estimate**

  By invoking the invariance property of the ML estimators, we can add that

$$\begin{aligned} \widehat{h(\lambda)} &= h(\hat{\lambda}) \\ &= 2\,000\,000 \times \left(1 - e^{-24\,000\times\hat{\lambda}}\right) \\ &= 2\,000\,000 \times \left(1 - e^{-24\,000\times\frac{8}{1776000}}\right) \\ &\simeq 204\,938.878. \end{aligned}$$

4. The target fraction of live births from c-sections is $p_0 = 0.15$.

(a) A sample of 125 live births led to 25 c-sections performed in the sample. When plotted on a $p-$chart with 3-sigma control limits, what does the observed value of the control statistic suggest? **(0.5)**

- **Answer**

  The observed value of the control statistic is equal to $t = \frac{25}{125} = 0.2$. Since it is above the center line $p_0 = 0.15$, we only need to calculate the

$$UCL = p_0 + 3\sqrt{\frac{p_0(1-p_0)}{n}} = 0.15 + 3\sqrt{\frac{0.15(1-0.15)}{125}} \simeq 0.245812,$$

  and conclude that $t < UCL$, thus the observed value of the control statistic suggests that the fraction of c-sections is in-control.

(b) Obtain an approximate value of the in-control ARL of the $np-$chart used to control the fraction of **(1.5)** c-sections, with $n = 20$, $p_0 = 0.15$, and 3–sigma control limits.

- **Control statistic of the $np-$chart and its distribution**
  Y_N = number of nonconforming items in the N^{th} batch, $N \in \mathbb{N}$
  $Y_N \sim \text{binomial}(n, p)$
- **3-sigma control limits**

$$\begin{aligned} LCL &= \left\lceil \max\{0, np_0 - 3 \times \sqrt{np_0(1-p_0}\} \right\rceil = \left\lceil \max\left\{0, 20 \times 0.15 - 3 \times \sqrt{20 \times 0.15 \times (1-0.15}\right\} \right\rceil \\ &= \left\lceil \max\left\{0, 3 - 3\sqrt{2.55}\right\} \right\rceil = \lceil \max\{0, -1.791\} \rceil = 0 \\ UCL &= \left\lfloor np_0 + 3 \times \sqrt{np_0(1-p_0\}} \right\rfloor = \left\lfloor 20 \times 0.15 - 3 \times \sqrt{20 \times 0.15 \times (1-0.15} \right\rfloor \\ &= \left\lfloor 3 + 3\sqrt{2.55} \right\rfloor = \lfloor 7.791 \rfloor = 7. \end{aligned}$$

- **Probability of triggering a signal**

$$\begin{aligned} \xi(\theta) &= 1 \times P(Y_N \not\in [LCL, UCL] \mid p = p_0 + \theta) \\ &= 1 - \left[F_{binomial(n,p_0+\theta)}(UCL) - F_{binomial(p_0+\theta)}(LCL-1)\right]. \end{aligned}$$

- **Requested in-control ARL**
  Since $RL(0) \sim \text{geometric}(\xi(0))$,

$$\begin{aligned} ARL(0) &= \frac{1}{\xi(0)} = \frac{1}{1 - \left[F_{binomial(20,0.15+0)}(7) - F_{binomial(20,0.15+0)}(0-1)\right]} \\ &\overset{tables}{=} \frac{1}{1 - 0.9941 + 0} \\ &\simeq 169.492. \end{aligned}$$

- **[Obs.:** If we use a calculator instead of the tables then $ARL(0) \simeq 168.886$.]

5. An engineer is monitoring the proportion p of incompletely filled low voltage liquid crystal display units in a high-yield process. The associated control statistic X is the cumulative count of conforming display units between two nonconforming ones. The p.f. and c.d.f. of X is given by $P_p(X = x) = (1-p)^x\, p$, $x \in \mathbb{N}_0$, and $F_p(x) = 1-(1-p)^{1+x}$, $x \in \mathbb{N}_0$, where the target value of p is known and denoted by $p_0 \in (0,1)$ and the true value of p is equal to $\rho \times p_0$, with $0 < \rho < 1/p_0$.

(a) The engineer is planning to use a chart with conventional $k-$sigma control limits ($k \in \mathbb{R}^+$): $LCL = (1-p_0)/p_0 - k\sqrt{1-p_0}/p_0$ and $UCL = (1-p_0)/p_0 + k\sqrt{1-p_0}/p_0$. **(1.5)**
For which values of k is the LCL positive?
Elaborate on the importance of a positive LCL in this particular context.

- **Control statistic**
  X = number of nonconforming items between two consecutive conforming items
  $X \sim \text{geometric}^\star(p)$
- **P.f., c.d.f., expected value and variance of the control statistic**
  $P_p(X = x) = (1-p)^x\, p$, $x \in \mathbb{N}_0$
  $F_p(x) = 1-(1-p)^{1+x}$, $x \in \mathbb{N}_0$
  $E_p(X) = \frac{1-p}{p}$
  $V_p(X) = \frac{1-p}{p^2}$
- **[On the control limits**
  They are equal to:

$$E_{p_0}(X) \pm k\sqrt{V_{p_0}(X)} = \frac{1-p_0}{p_0} \pm k\sqrt{\frac{1-p_0}{p_0^2}}, \quad k > 0.]$$

- **Requested values of k**

$$k \in \mathbb{R}^+ \quad : \quad LCL > 0 \quad \Leftrightarrow \quad (1-p_0)/p_0 - k\sqrt{(1-p_0)/p_0^2} > 0 \quad \Leftrightarrow \quad k < \sqrt{1-p_0}.$$

- **Importance of a positive LCL**
  It is essential that the chart has a positive LCL so the chart is able to detect increases in p (i.e., to detect a quality deterioration) in a fairly quick fashion.
  If $LCL \not> 0$, we are bound to deal with an upper one-sided chart, whose out-of-control ARL in the presence of some increases in p is surely (and unreasonably) larger than the in-control ARL.

(b) A statistician anticipated both downward and upward shifts and decided to adopt an alternative chart with control limits $LCL^\star = 4$ and $UCL^\star = 7428$. Moreover, it triggers a signal with probability: **(2.0)**

- one if the control statistic X is below $LCL^\star$ or above $UCL^\star$;
- $\gamma_{LCL^\star} = 0.415872$ (resp. $\gamma_{UCL^\star} = 0.349557$) if the control statistic is equal to $LCL^\star$ (resp. $UCL^\star$).

The in-control ARL of this new chart is equal to $1/\alpha = 200$.

Verify that, when p shifts from its target value $p_0 = 0.001$ to $p = 1.1 \times p_0$, the out-of-control ARL is approximately equal to 194.950. Comment on this ARL value.

- **Probability of a signal**
  Judging by the description above, when $p = \rho \times p_0$ ($\rho \in (0, 1/p_0)$), this alternative chart triggers a signal with probability

$$\begin{aligned}
\xi^\star(\rho) &= 1 \times \left(1 - \left\{F_{\rho\times p_0}(UCL^\star) - F_{\rho\times p_0}[(LCL^\star)^-]\right\}\right) \\
&\quad +\gamma_{LCL^\star} \times P_{\rho\times p_0}(X = LCL^\star) \\
&\quad\quad +\gamma_{UCL^\star} \times P_{\rho\times p_0}(X = UCL^\star) \\
&= \left[1 - (1-\rho\times 0.001)^{1+(4-1)} + (1-\rho\times 0.001)^{1+7428}\right] \\
&\quad +0.415872 \times (1-\rho\times 0.001)^4\,\rho\times 0.001 \\
&\quad\quad +0.349557 \times (1-\rho\times 0.001)^{7428}\,\rho\times 0.001.
\end{aligned}$$

  Thus, the probability of a valid signal, when $\rho = 1.1$, is given by

$$\begin{aligned}
\xi^\star(1.1) &= \left[1 + (1-1.1\times 0.001)^{1+7428} - (1-1.1\times 0.001)^{1+4}\right] \\
&\quad +0.415872 \times (1-1.1\times 0.001)^4\,1.1\times 0.001 \\
&\quad\quad +0.349557 \times (1-1.1\times 0.001)^{7428}\,1.1\times 0.001 \\
&\simeq 0.005130.
\end{aligned}$$

- **ARL function and requested ARL value**
  Since we are still dealing with a Shewhart chart with independent control statistics, we have

$$\begin{aligned}
ARL^\star(\rho) &= \frac{1}{\xi^\star(\rho)} \\
ARL^\star(1.1) &= \frac{1}{0.005130} \\
&\simeq 194.950.
\end{aligned}$$

- **Comment**
  This chart is able to signal an increase in p sooner (in average) than to trigger a false alarm — $ARL^\star(1) > ARL^\star(1.1)$ — a very desirable property.

6. The high-voltage output of a certain power supply used in a copy machine is assumed to have a normal distribution with nominal mean and standard deviation equal to $\mu_0 = 350$ and $\sigma_0 = 2.0$ (V dc at 20 milliamps). Samples of $n = 9$ power supply units have been inspected every half-hour. The process mean and standard deviation have increased and the magnitudes of the associated shifts are $\delta = \sqrt{n}(\mu - \mu_0)/\sigma_0 = 0.25$ and $\theta = \sigma/\sigma_0 = \sqrt{23.574603/21.95}$, respectively. **(2.0)**

Obtain and interpret the first quartile of the out-of-control RL of a $\bar{X}-$chart with 3−sigma limits.

- **Quality characteristic**
  X = high-voltage output of a certain power supply used in a copy machine
  $X \sim \text{normal}(\mu, \sigma^2)$

- **Control statistic of the standard $\bar{X}-$chart and its distribution**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
  $\bar{X}_N \sim \text{normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$, where $\delta \in \mathbb{R}$ (resp. $\theta \geq 1$) represents the magnitude of a shift in μ (resp. an upward shift in σ)

- **Control limits**
  $LCL_\mu = \mu_0 - \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$,
  where $\gamma_\mu = 3$, $\mu_0 = 350$ and $\sigma_0 = 2.0$.

- **Shifts in the process mean and standard deviation**
  $\delta = \sqrt{n}\,(\mu - \mu_0)/\sigma_0 = 0.25$
  $\theta = \frac{\sigma}{\sigma_0} = \sqrt{\frac{23.574603}{21.95}} \simeq 1.036346$

- **Probability of a signal**

$$\begin{aligned}
\xi_\mu(\delta,\theta) &= P(\bar{X}_N \notin [LCL_\mu, UCL_\mu] \mid \delta,\theta) \\
&= \ldots \\
&= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right] \\
&= 1 - \left[\Phi\left(\frac{3 - 0.25}{1.036346}\right) - \Phi\left(\frac{-3 - 0.25}{1.036346}\right)\right] \\
&\simeq 1 - [\Phi(2.65) - \Phi(-3.14)] \\
&\overset{tables}{=} 1 - [0.9960 - (1 - 0.999155)] \\
&= 0.004845
\end{aligned}$$

- **Requested** 25% **percentage point of the out-of-control** RL_μ
  Given that $RL_\mu(\delta,\theta) \sim \text{geometric}\left(\xi_\mu(\delta,\theta)\right)$, we have

$$\begin{aligned}
F^{-1}_{RL_\mu(\delta,\theta)}(p) &\overset{Table\ 9.1}{=} \inf\left\{m \in \mathbb{N} : F_{RL_\mu(\delta,\theta)}(m) \geq p\right\} \\
&= \left\lceil \frac{\ln(1-p)}{\ln\left[1 - \xi_\mu(\delta,\theta)\right]} \right\rceil \\
&= \left\lceil \frac{\ln(1-0.25)}{\ln(1-0.004845)} \right\rceil \\
&= \lceil 59.233 \rceil \\
&= 60.
\end{aligned}$$

- **Interpretation of the** $F^{-1}_{RL_\sigma(\delta,\theta)}(0.25)$
  When there is a shift in the location and spread of this quality characteristic with magnitude $(\delta,\theta) = (0.25, \sqrt{23.574603/21.95})$, the probability that the $\bar{X}-$chart triggers a valid signal within the first 60 samples is of at least 25%.

7. The drained weight after filling of contents of a can of tomatoes is required to be at most 22.8 oz.

(a) Consider a sampling plan by variables with UNKNOWN STANDARD DEVIATION, $(n_s, k_s) = (59, 1.937713)$, and risk points $(p_1, 1-\alpha) = (1\%, 0.95)$ and $(p_2, \beta) = (5\%, 0.1)$. **(2.0)**
Confirm that this plan is indeed related to these risk points and compare (n_s, k_s) to $(n_\sigma, k_\sigma) = (19, 1.943305)$.

- **Single sampling plan by variables with** UNKNOWN STANDARD DEVIATION
  n_s (sample size)
  k_s (acceptance constant)

- **Producer's and consumer's risk points**
  $(p_1, 1-\alpha) = (1\%, 0.95)$
  $(p_2, \beta) = (5\%, 0.1)$

- **Requested verification**
  If we consider $n_s = 59$ and $k_s = 1.937713$ then

$$
\begin{aligned}
P_a(p_1) &\overset{(13.39)}{\simeq} \Phi(\theta_{p_1}) \\
&\overset{(13.41)}{\simeq} \Phi\left[\frac{\Phi^{-1}(1-p_1) - k_s\sqrt{\frac{3n_s-4}{3n_s-3}}}{\sqrt{\frac{1+\frac{3n_s k_s^2}{6n_s-8}}{n_s}}}\right] \\
&= \Phi\left[\frac{2.3263 - 1.937713\sqrt{\frac{3\times59-4}{3\times59-3}}}{\sqrt{\frac{1+\frac{3\times59\times1.937713^2}{6\times59-8}}{59}}}\right] \\
&\simeq \Phi(1.77) \\
&\overset{table}{=} 0.9616 \\
&\geq 1-\alpha = 0.95 \\
P_a(p_2) &\simeq \Phi(\theta_{p_2}) = \Phi\left[\frac{1.6449 - 1.937713\sqrt{\frac{3\times59-4}{3\times59-3}}}{\sqrt{\frac{1+\frac{3\times59\times1.937713^2}{6\times59-8}}{59}}}\right] \\
&\simeq \Phi(-1.29) \\
&\overset{table}{=} 1 - 0.9015 \\
&= 0.0985 \\
&\leq \beta = 0.1.
\end{aligned}
$$

  Hence, the sampling plan complies with the two given risk points.

- **Comment**
  Admitting that the standard deviation is unknown is more realistic:
  - it does not change the acceptance constant significantly (in this exercise $k_s \simeq k_\sigma$ down to the first decimal place);
  - but it requires the collection of a much larger sample (in this case $n_s = 3.1 \times n_\sigma$).

(b) Suppose a sample of size n_s was taken from a lot, and $\bar{x} = 22.876942$, $s = 1.116276$. **(0.5)**
Should the lot be accepted or rejected?

- **Checking whether or not the lot should be accepted**
  The lot should be accepted iff

$$Q = \frac{U - \bar{x}}{s} \geq k_s,$$

  where Q is the quality index, U is the upper specification limit, $\bar{x}$ and s represent the mean of a sample with size n_s, and k_s the acceptance constant. For this sample, we have

$$
\begin{aligned}
Q &= \frac{22.8 - 22.876942}{1.116276} \\
&= -0.068927 \\
&\ngeq 1.937713,
\end{aligned}
$$

  therefore we should reject the lot.