Reliability and Quality Control
LMAC, MMAC

2nd. Semester – 2023/2024
2024/04/16 — 17:10
MAP30 #2

Duration: **30** minutes

- Write your number and name below.
- Add your answers to this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and TWO QUESTIONS. The total of points is 5.0.

Number: **Name**:

1. Suppose that a product is designed to have a time to failure (in hours) with an exponential distribution with parameter λ_1. A proportion p of the units of the product comes out with an exponential distribution with parameter λ_2 ($\lambda_2 > \lambda_1$). Suppose that all units of the product go through a burn-in for $t^\star$ hours.

The time to failure of the units which did not fail during this burn-in, $T^\star$, has reliability function

$$R^\star(t) = \begin{cases} 1, & t < t^\star \\ \frac{1}{C^\star} \times \left[(1-p) \times e^{-\lambda_1 t} + p \times e^{-\lambda_2 t}\right], & t \geq t^\star, \end{cases}$$

where $C^\star = (1-p) \times e^{-\lambda_1 t^\star} + p \times e^{-\lambda_2 t^\star}$.

(a) Derive an expression for $E(T^\star)$. **(1.5)**

- **Relevant failure time**
  $T^\star$ = time of units which did not fail during the burn-in
- **Requested expected value**

$$\begin{aligned} E(T^\star) &\overset{T^\star \geq 0}{=} \int_0^{+\infty} R^\star(t)\,dt = \int_0^{t^\star} dt + \frac{1}{C^\star} \times \int_{t^\star}^{+\infty} \left[(1-p) \times e^{-\lambda_1 t} + p \times e^{-\lambda_2 t}\right] dt \\ &= t^\star - \frac{1}{C^\star} \times \left(\frac{1-p}{\lambda_1} \times e^{-\lambda_1 t} + \frac{p}{\lambda_2} \times e^{-\lambda_2 t} \Big|_{t^\star}^{+\infty} \right) \\ &= t^\star + \frac{1}{C^\star} \times \left(\frac{1-p}{\lambda_1} e^{-\lambda_1 t^\star} + \frac{p}{\lambda_2} \times e^{-\lambda_2 t^\star} \right) \\ &= t^\star + \frac{1}{\lambda_1 \lambda_2 C^\star} \times \left[(1-p)\lambda_2 \times e^{-\lambda_1 t^\star} + p\lambda_1 \times e^{-\lambda_2 t^\star} \right]. \end{aligned}$$

(b) Assume from now on that $\lambda_1^{-1} = 10000$, $\lambda_2^{-1} = 100$, $p = 0.05$, and $t^\star = 200$. Suppose these values led to: $E(T^\star) \simeq 10128.6$; a decreasing plot of $[R^\star(t)]^{1/t}$, for $t \in (0, +\infty)$. Comment on these results. **(0.5)**

- **Comment on** $E(T^\star) = 10128.6$
  A unit surviving $t^\star$ hours of burn-in is expected to operate an additional $E(T^\star) - t^\star$ hours in the field. If $\lambda_1^{-1} = 10000$, $1/\lambda_2^{-1} = 100$, and $p = 0.05$, then a unit surviving $t^\star = 200$ hours of burn-in, is expected to operate an additional $E(T^\star) - t^\star \simeq 10128.6 - 200 = 9928.6$ hours in the field.
- **Comment on the plot of** $[R^\star(t)]^{1/t}$
  According to Definition 3.32, a decreasing plot of $[R^\star(t)]^{1/t}$, for $\in (0, +\infty)$, means that the r.v. $T^\star$ is IHRA (increasing hazard rate in average).

(c) Capitalize on the stochastic ageing behaviour of $T^\star$ to provide: an upper bound to $\sqrt{V(T^\star)}$; a lower bound for the expected failure time of a 3-out-of-5 system with units with failure times i.i.d. to $T^\star$. **(1.5)**

- **Requested upper bound to** $SD(T^\star)$

  Since $T^\star \in IHRA$, we can invoke Corollary 3.54 and write

  $$CV(T^\star) = \frac{\sqrt{V(T^\star)}}{E(T^\star)} \le 1 \quad \Leftrightarrow \quad SD(T^\star) = \sqrt{V(T^\star)} \le E(T^\star) \simeq 10128.6.$$

- **Individual failure times**

  $T_i^\star$ = failure time of valve $i \overset{i.i.d.}{\sim} T^\star, \quad i = 1,\dots,5$

  $E(T_i^\star) = \mu_i = \mu^\star \simeq 10128.6, \quad i = 1,\dots,5$

- **System**

  3-out-of-5 system

  TT = failure time of the 3-out-of-5 system

- **Lower bound for** $E(TT)$

  Since $T_i^\star \overset{i.i.d.}{\sim} T^\star \in IHRA$, we can apply Theorem 3.69 and provide a lower bound to $\mu = E(TT)$ depending on the $p^* = \binom{5}{3} = 10$ minimal path sets, all with cardinality 3:

  $$\mu \quad \ge \quad \max_{j=1,\dots,p^*} \left\{ \left(\sum_{i \in \mathscr{P}_j} \mu_i^{-1} \right)^{-1} \right\} \overset{\mu_i=\mu^\star}{=} \max_{j=1,\dots,p^\star} \left\{ \left(\frac{\#\mathscr{P}_j}{\mu^\star} \right)^{-1} \right\} = \frac{\mu^\star}{\min_{j=1,\dots,p^\star}\{\#\mathscr{P}_j\}} = \frac{\mu^\star}{3} \simeq 3376.2.$$

  [The subsets with cardinality 3 from the set $\{1,2,3,4,5\}$ are: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2,5\}$, $\{1,3,4\}$, $\{1,3,5\}$, $\{1,4,5\}$, $\{2,3,4\}$, $\{2,3,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.]

  [Alternatively, we could have capitalize on the fact that $T_i^\star \overset{i.i.d.}{\sim} IHRA \overset{Prop.3.36}{\Rightarrow} T_i^\star \overset{i.i.d.}{\sim} NBUE$ and invoke instead Theorem 3.65, which leads to the same lower bound to $E(TT)$.]

2. A mechanical engineer concluded that the exponential model fits 20 failure times quite well. The complete data set led to a ML estimate of the expected time to failure equal to $\hat{\lambda}^{-1} = 3543.37$ hours.

Identify the pivotal quantity you would use to obtain a 95% confidence interval (CI) for the expected time to failure. Determine this CI (BUT DO NOT DERIVE IT). **(1.5)**

- **Distribution assumption**

  It is fairly reasonable to admit that the failure times T_i are such that:

  $T_i \overset{i.i.d.}{\sim} T \sim \text{exponential}(\lambda), \quad i = 1,\dots,n \ (n = 20).$

- **Life test**

  Complete data is available!

  $\hat{E}(T) = \hat{\lambda}^{-1} = \frac{\sum_{i=1}^n t_i}{n} = 3543.37$

  $\hat{\lambda} = [\hat{E}(T)]^{-1} = \frac{1}{3543.37}$

- 95% **confidence interval (CI) for** $E(T) = \frac{1}{\lambda}$

  - **Pivotal quantity**

    $Z = 2\lambda \sum_{i=1}^n T_i \sim \chi^2_{(2n)}$ (check lecture note just after Exercise 5.25)

  - **Requested CI**

    Since

    $$CI_{95\%}(\lambda) \overset{(5.22)}{=} \left[\frac{\hat{\lambda} \times F^{-1}_{\chi^2_{(2n)}}(\alpha/2)}{2n}, \frac{\hat{\lambda} \times F^{-1}_{\chi^2_{(2n)}}(1-\alpha/2)}{2n} \right] = [\lambda_L, \lambda_U]$$

    we get the requested CI:

    $$\begin{aligned} CI_{95\%}(1/\lambda) &= \left[\frac{1}{\lambda_U}, \frac{1}{\lambda_L} \right] \\ &= \left[\frac{\hat{\lambda}^{-1} \times 2n}{F^{-1}_{\chi^2_{(2n)}}(1-\alpha/2)}, \frac{\hat{\lambda}^{-1} \times 2n}{F^{-1}_{\chi^2_{(2n)}}(\alpha/2)} \right] \\ &= \left[\frac{3543.37 \times 2 \times 20}{59.34}, \frac{3543.37 \times 2 \times 20}{24.43} \right] \simeq [2388.52, 5801.67]. \end{aligned}$$