Reliability and Quality Control
LMAC, MMA

2nd. Semester – 2022/2023
2023/04/11 — 17:10
MAP30 #2

Duration: **30** minutes

- Write your number and name below.
- Add your answers to this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and TWO QUESTIONS. The total of points is 4.0.

Number: **Name**:

1. The failure time T of a variable choke valve is assumed to have a Weibull distribution with shape and scale parameters $\alpha^\star = 2$ and $\delta^\star = 1$ (respectively).

(a) Given that the valve has survived the first t ($t > 0$) time units, what is the probability that it will survive the next u ($u > 0$) time units? **(1.0)**
How does this probability compare with the one that the valve is still operating at time u? Comment.

- **Failure time**
$T \sim \text{Weibull}(\delta^\star = 1, \alpha^\star = 2)$
- **Reliability function of T**
$R(t) \overset{(4.22)}{=} \exp\left[-\left(\frac{t}{\delta^\star}\right)^{\alpha^\star}\right] = e^{-t^2}, \quad t \geq 0$
- **Requested probability**

$$\begin{aligned} P(T > t+u \mid T > t) &= \frac{P(T > t+u, T > t)}{P(T > t)} = \frac{P(T > t+u)}{P(T > t)} = \frac{R(t+u)}{R(t)} = \frac{e^{-(t+u)^2}}{e^{-t^2}} \\ &= e^{-2tu-u^2} \end{aligned}$$

- **Requested comparison**

$$\begin{aligned} \frac{P(T > t+u \mid T > t)}{P(T > u)} &= \frac{e^{-2tu-u^2}}{e^{-u^2}} \\ &= e^{-2tu} \\ &< 1. \end{aligned}$$

- **Comment**
This is an expected result because T has a Weibull distribution with a shape parameter $\alpha > 1$, hence $T \in IHR$. This in turn implies that $T \in NBU$ (see Prop. 3.36), i.e., $R(u) \geq R(t+u)/R(t)$ $\Leftrightarrow R_T(u) \geq R_{T-t|T>t}(u) \Leftrightarrow T \geq_{st} (T-t|T>t)$, according to Def. 3.14.

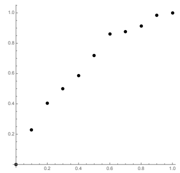
(b) Consider a 3-out-of-4 system with variable choke valves with failure times that are i.i.d. to T. **(2.0)**
Determine an upper bound (as sharp as reasonably possible) for the expected failure time of this 3-out-of-4 system.

- **Individual failure times**
T_i = failure time of valve $i \overset{i.i.d.}{\sim} T, \quad i = 1,\ldots,4$
$E(T_i) = \mu_i = \mu^\star \overset{form.}{=} \delta^\star \times \Gamma(1+1/\alpha^\star) = \Gamma(3/2) = 1/2 \times \Gamma(1/2) = \frac{\sqrt{\pi}}{2}, \quad i = 1,\ldots,4$
- **System**
3-out-of-4 system
TT = failure time of the 3-out-of-4 system
- **Minimal cut sets**
$\mathcal{K}_1 = \{1,2\}, \mathcal{K}_2 = \{1,3\}, \mathcal{K}_3 = \{1,4\}, \mathcal{K}_4 = \{2,3\}, \mathcal{K}_5 = \{2,4\}, \mathcal{K}_6 = \{3,4\}$
$q = \binom{4}{2} = 6$ minimal cut sets

- **Important**
We have already mentioned that the individual times to failure $T_i \overset{i.i.d.}{\sim} T \in IHR$; hence $T \in IHRA$ (see Prop. 3.36).
Under these circumstances, we can apply Th. 3.69 and provide an upper bound to $\mu = E(TT)$.
- **Upper bound for $\mu = E(TT)$**

$$\begin{aligned} \mu &\overset{Th.3.69}{\leq} \min_{j=1,\ldots,q} \int_0^{+\infty} \left[1 - \prod_{i \in \mathcal{K}_j} (1 - e^{-t/\mu_i})\right] dt \\ &\overset{\mu_i = \mu^\star}{=} \min_{j=1,\ldots,q} \int_0^{+\infty} \left[1 - (1 - e^{-t/\mu^\star})^{\#\mathcal{K}_j}\right] dt \\ &= \int_0^{+\infty} \left[1 - (1 - e^{-t/\mu^\star})^{\min_{j=1,\ldots,q} \#\mathcal{K}_j}\right] dt \\ &\overset{\#\mathcal{K}_j = 2, \forall j}{=} \int_0^{+\infty} \left[1 - (1 - e^{-t/\mu^\star})^2\right] dt \\ &= \int_0^{+\infty} \left(2e^{-t/\mu^\star} - e^{-2t/\mu^\star}\right) dt \\ &= 2\mu^\star \int_0^{+\infty} f_{exp(1/\mu^\star)}(t)\, dt - \frac{\mu^\star}{2} \int_0^{+\infty} f_{exp(2/\mu^\star)}(t)\, dt \\ &= \frac{3\mu^\star}{2} = \frac{3\sqrt{\pi}}{4}. \end{aligned}$$

2. An engineer collected ten failure times of that same type of variable choke valve.
She used Mathematica: to obtain the TTT plot on the right; to perform a Kolmogorov-Smirnov test with null hypothesis $H_0 : T \sim \text{Weibull}(\delta = 1, \alpha = 2)$; to obtain the ML estimates $\hat{\alpha} = 2.059$ and $\hat{\delta} = 0.962$.



(a) What conclusion can the engineer draw from the TTT plot? **(0.5)**
Comment on the p-value of the goodness-of-fit test ($p - value = 0.986$), namely in light of the TTT plot above.

- **Comment on the TTT plot**
The TTT plot suggests a concave curve above the 45^o line, thus the data can be fitted by an IHR distribution,[1] according to Note 5.5 of the lecture notes.
- **Comment on the p-value of the Kolmogorov-Smirnov test**
[Recall that the $p-$value is the largest significance level leading to the non rejection of the null hypothesis. Thus,] for these particular data set and null hypothesis $H_0 : T \sim \text{Weibull}(\delta = 1, \alpha = 2)$: should not reject H_0 for any significance level $\alpha_0 \leq p-$value $= 0.986$, specifically the usual significance levels $(1\%, 5\%, 10\%)$.
This decision is consistent with the TTT plot because the conjectured Weibull distribution in H_0 is IHR after all it has a shape parameter larger than 1.

(b) Determine the ML estimate of the median failure time of a single variable choke value. **(0.5)**

- **Failure time**
$T \sim \text{Weibull}(\delta, \alpha), \quad \delta, \alpha > 0$ (UNKNOWN)
- **ML estimate of the median failure time**
Since
$me : P(T \leq me) = 0.5 \Leftrightarrow R(me) = 0.5 \Leftrightarrow \exp\left[-\left(\frac{me}{\delta}\right)^\alpha\right] = 0.5 \Leftrightarrow me = \delta\,[-\ln(0.5)]^{1/\alpha} = h(\delta, \alpha),$
we can invoke the invariance property of the ML estimators and get the requested ML estimate:
$$\widehat{me} = \hat{h}(\delta, \alpha) = h(\hat{\delta}, \hat{\alpha}) = \hat{\delta}\,[-\ln(0.5)]^{1/\hat{\alpha}} = 0.962 \times [-\ln(0.5)]^{1/2.059} \simeq 0.805134.$$