DM
DEPARTAMENTO
DE MATEMÁTICA
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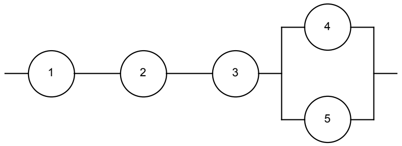
Reliability and Quality Control
LMAC, MMA

2nd. Semester – 2022/2023
2023/03/07 — 17:10
MAP30 #1

Duration: **30** minutes
- Write your number and name below.
- Add your answers to this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and TWO QUESTIONS. The total of points is 4.0.

Number: **Name**:

1. A car needs wheels (component 1), an engine (component 2), a transmission (component 3), and one out of two brake systems (components 4 and 5) to function.



Admit that the components of this car system operate independently and their reliabilities are equal to $p_i = p$ $(i = 1,\ldots,5)$.

(a) Identify the minimal path sets and the minimal cut sets of this system. **(1.5)**

Provide expressions for its: i) structure function (in terms of the minimal cut sets); ii) reliability.

- **Minimal path sets**
  $\mathcal{P}_1 = \{1,2,3,4\}$, $\mathcal{P}_2 = \{1,2,3,5\}$, $p^* = 2$ minimal path sets
- **Minimal cut sets**
  $\mathcal{K}_1 = \{1\}$, $\mathcal{K}_2 = \{2\}$, $\mathcal{K}_3 = \{3\}$, $\mathcal{K}_4 = \{4,5\}$, $q = 4$ minimal path sets
- **Structure function** (in terms of the minimal cut sets)

$$\begin{aligned}\phi(\underline{X}) &\overset{Th.1.30}{=} \prod_{j=1}^{q}\left[1-\prod_{i\in\mathcal{K}_j}(1-X_i)\right]\\ &= [1-(1-X_1)]\times[1-(1-X_2)]\times[1-(1-X_3)]\times[1-(1-X_4)(1-X_5)]\\ &= X_1\times X_2\times X_3\times(X_4+X_5-X_4X_5)\end{aligned}$$

- **Reliability**

$$\begin{aligned}r(\underline{p}) &= E[\phi(\underline{X})]\\ &= E[X_1\times X_2\times X_3\times(X_4+X_5-X_4X_5)]\\ &\overset{X_i\sim_{i.i.d.}Ber(p_i)}{=} p_1p_2p_3(p_4+p_5-p_4p_5)\\ &\overset{p_i=p}{=} p^3(2p-p^2) \qquad [\equiv p^4(2-p)].\end{aligned}$$

(b) Admit now that the components of the car system operate in a positively associated fashion. **(1.0)**

Determine the min-max lower and upper bounds to the reliability of this system.

- **Min-max upper bound**
  [Since we are dealing with a coherent system with components operating in a positively associated fashion, we can apply Theorem 1.70 and obtain]

$$r(\underline{p}) \overset{Th.1.70}{\leq} \min_{j=1,\ldots,q}\left[1-\prod_{i\in\mathcal{K}_j}(1-p_i)\right]$$

$$\begin{aligned}r(\underline{p}) &\overset{p_i=p}{=} \min_{j=1,\ldots,q}\left[1-(1-p)^{\#\mathcal{K}_j}\right]\\ &= 1-(1-p)^{\min_{j=1,\ldots,q}\#\mathcal{K}_j}\\ &= p\\ r(\underline{p}) &\overset{Th.1.70}{\geq} \max_{j=1,\ldots,p^*}\left[\prod_{i\in\mathcal{P}_j}p_i\right]\\ &\overset{p_i=p}{=} \max_{j=1,\ldots,p^*} p^{\#\mathcal{P}_j}\\ &= p^{\min_{j=1,\ldots,p^*}\#\mathcal{P}_j}\\ &= p^4.\end{aligned}$$

2. Admit that the times to failure of the components of the car system are independent and exponentially distributed with parameter λ. **(1.5)**

Write down the time to failure of the car system (T) in terms of the times to failure of its components. Derive expressions for $R_T(t)$ and $E(T)$.

- **Time to failure** (components)
  T_i = time to failure of component i
  $T_i \overset{i.i.d.}{\sim} \exp(\lambda)$, $i = 1,\ldots,5$

$$R_i(t) = P(T_i > t) = R(t) = \begin{cases} e^{-\lambda t}, & t\geq 0\\ 1, & t<0\end{cases}$$

- **Time to failure** (car system)
  $T = \min\{T_1, T_2, T_3, \max\{T_4, T_5\}\}$
- **Requested reliability**

$$\begin{aligned}R_T(t) &= P(T>t)\\ &= R_1(t)\times R_2(t)\times R_3(t)\times\{1-[1-R_4(t)]\times[1-R_5(t)]\}\\ &\overset{R_i(t)=R(t)}{=} [R(t)]^3\times\{1-[1-R(t)]^2\}\\ &= [R(t)]^3\times\{[2R(t)-R(t)]^2\}\\ &= [R(t)]^4\times[2-R(t)]\\ &= e^{-4\lambda t}\times\left(2-e^{-\lambda t}\right), \quad t>0\\ [&\equiv r(R(t),\ldots,R(t)) = r(e^{-\lambda t},\ldots,e^{-\lambda t}).]\end{aligned}$$

- **Expected value of** T

$$\begin{aligned}E(T) &\overset{(2.10)}{=} \int_0^{+\infty} R(t)\,dt\\ &= \int_0^{+\infty} e^{-4\lambda t}\times\left(2-e^{-\lambda t}\right)dt\\ &= \frac{1}{2\lambda}\int_0^{+\infty}4\lambda e^{-4\lambda t}\,dt-\frac{1}{5\lambda}\int_0^{+\infty}5\lambda e^{-\lambda t}\,dt\\ &= \frac{1}{2\lambda}\int_0^{+\infty}f_{exp(4\lambda)}(t)\,dt-\frac{1}{5\lambda}\int_0^{+\infty}f_{exp(5\lambda)}(t)\,dt\\ &= \frac{1}{2\lambda}\times 1-\frac{1}{5\lambda}\times 1\\ &= \frac{3}{10\lambda}.\end{aligned}$$