4.Optimal Control

Optimal Control Problem – Fixed final time; Free final state

Let x be the state of a plant with input u defined by

$$\dot{x} = f(x,u) \qquad x(0) = x_0 \qquad t \in [0,T] \quad u(t) \in U$$

T *given*

Find the function u, defined in the time interval $[0,T]$ that maximizes

$$J(u) = \Psi(x(T)) + \int_0^T L(x,u)dt$$

Pontriagyn Principle

Along an optimal trajectory of x, u, and λ, the following necessary conditions for the maximization of J are verified:

$$\dot{x} = f(x,u) \qquad x(0) = x_0 \qquad t \in [0, T] \quad u(t) \in U$$

$$-\dot{\lambda}'(t) = \lambda'(t) f_x\big(x(t), u(t)\big) + L_x\big(x(t), u(t)\big)$$

$$\lambda'(T) = \Psi_x(x) \,\Big|_{x=x(T)} \longleftarrow$$

Terminal condition of the co-state

At each t, the Hamiltonian defined by

$$H(\lambda, x, u) = \lambda' f(x,u) + L(x,u)$$

is maximum, as a function of u, for the optimal value of u.

$$\Psi_x(x)\Big|_{x=x(T)} = \begin{bmatrix} \left.\dfrac{\partial \Psi(x)}{\partial x_1}\right|_{x=x(T)} & \cdots & \left.\dfrac{\partial \Psi(x)}{\partial x_n}\right|_{x=x(T)} \end{bmatrix}$$

$$L_x(x,u) = \begin{bmatrix} \dfrac{\partial L}{\partial x_1} & \cdots & \dfrac{\partial L}{\partial x_n} \end{bmatrix} \qquad f_x = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial f_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

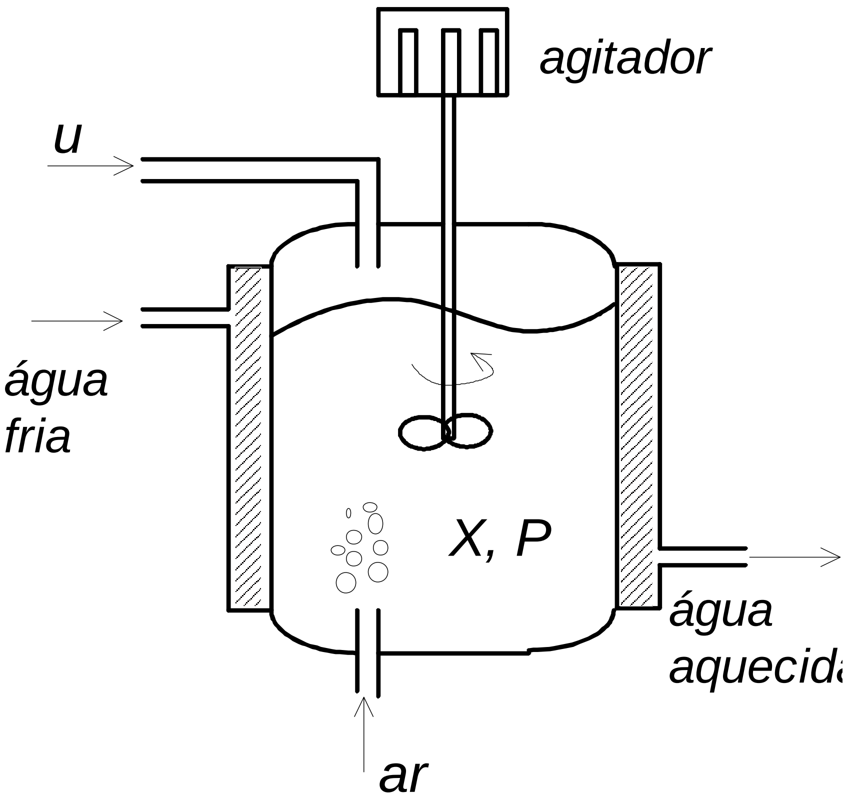
The vector λ is called co-state, and the corresponding differential equation is called the adjoint equation.

Penicillin Fermentation reactor

X – Quantity of fungi per unit volume

P – Quantity of penicilin per unit volume

u – Manipulated variable, substract rate (sugar)



Fungi produce penicillin.

A very simplified model of the fermentation

Growth due to "food

Mortality

$$\dot{X} = buX - \mu X$$

$$\dot{P} = c(1-u)X$$

Fungi production

Production inibition due to substract

Fermentation Optimal Control Problem

Model and initial conditions:

$$\dot{X} = uX - 0{,}5X$$

$$\dot{P} = (1-u)X$$

Initial conditions:

$$X(0) = 1$$

$$P(0) = 0$$

Objective:

Find $u(t)$ $0 \le t \le T$, T fixed, so that $J = P(T)$ is maximum given the constraint

$$0 \le u \le 1$$

Write the adjoint equation

The cost functional is

$$J = \psi\big(x(T)\big) + \int_0^T L(x,u)dt$$

In this case

$$J_{fermenter} = P(T)$$

Therefore $L(x,u) = 0$

and $\psi(x(T)) = P(T)$, and thus

$$\psi_x(x(T)) = \left.\begin{bmatrix} \dfrac{\partial\psi}{\partial x_1} & \dfrac{\partial\psi}{\partial x_2} \end{bmatrix}\right|_{x=x(T)} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The co-state has two components

$$\lambda'(t) = \begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix}$$

Since the Lagrangian is zero:

$$L_x(x,u) = [0 \quad 0]$$

Since $f(x,u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} = \begin{bmatrix} (u-0.5)x_1 \\ (1-u)x_2 \end{bmatrix}$ it is $f_x(x,u) = \begin{bmatrix} u-0.5 & 0 \\ 1-u & 0 \end{bmatrix}$

Adjoint equation

$$-\dot{\lambda}' = \lambda' f_x(x,u) + L_x(x,u)$$

$$f_x(x,u) = \begin{bmatrix} u-0.5 & 0 \\ 1-u & 0 \end{bmatrix} \qquad L_x(x,u) = 0$$

In this case the adjoint equation is

$$-\dot{\lambda}_1 = (u-0.5)\lambda_1 + (1-u)\lambda_2$$

$$-\dot{\lambda}_2 = 0$$

With terminal condition

$$\lambda_1(T) = 0 \quad \lambda_2(T) = 1$$

$$-\dot{\lambda}_1 = (u-0.5)\lambda_1 + (1-u)\lambda_2 \qquad\qquad -\dot{\lambda}_2 = 0$$

$$\lambda_1(T) = 0 \quad \lambda_2(T) = 1$$

Considering the terminal conditions

$$\lambda_2(t) = 1 \qquad 0 \le t \le T$$

And the equation for the 1st component of the co-state becomes

e a equação para a primeira componente do co-estado reduz-se a

$$-\dot{\lambda}_1 = (u-0.5)\lambda_1 + 1 - u$$

Dificulty: The equation depends on $u(t)$ and $u(t)$ depends on $\lambda(t)$...

Hints

a) Write the Hamiltonian for this special case. Remember that

$$H(\lambda, x, u) = \lambda' f + L$$

b) Assume that you know $\lambda(t)$. Find $u(t)$ that maximizes H for each t.

Remember the constraint $0 \le u \le 1$ and assume that $X > 0$

c) From b) you know the shape of $u(t)$ as a function of t. In particular, what is the value of $u(t)$ for t close to T? And the corresponding equation for $\lambda_1(t)$ during thgis time period?

d) Go “backwards” in time. What happens to $\lambda_1(t)$? And $u_{optim}(t)$?

$$H = \lambda' f + L$$

$$H = \lambda_1 f_1(X, P) + \lambda_2 f_2(X, P) + 0$$

$$H = \lambda_1 (u - 0.5) X + (1 - u) X$$

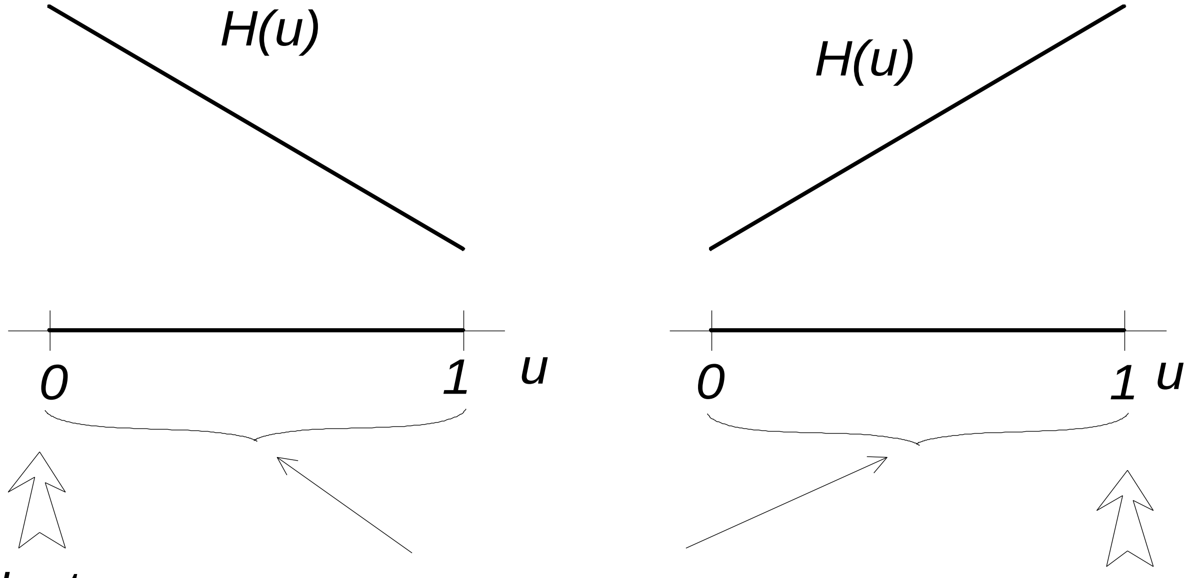
Can be written as

$$H = \left[(\lambda_1 - 1) u + (1 - 0.5 \lambda_1) \right] X$$

The Hamiltonian H is a linear function of u.

Assuming $X > 0$, H growing or decreasing depends just on $\lambda_1 - 1$.

$$H = \left[(\lambda_1 - 1)u + (1 - 0.5\lambda_1) \right] X$$



Neste caso
u_{opt}=0

Intervalo de valores admissíveis para u

Neste cas
u_{opt}=1

$\lambda_1 < 1$

$\lambda_1 > 1$

Since

$$\lambda_1(T) = 0$$

for t close to T, $\lambda_1(t) = 0$. Thus, since $\lambda_1(T) < 1$, the corresponding optimal control is

$$u_{opt}(t) = 0$$

Close to T, the adjoint equation becomes

$$-\dot{\lambda}_1 = (\underbrace{u}_{=0} - 0.5)\lambda_1 + 1 - \underbrace{u}_{=0}$$

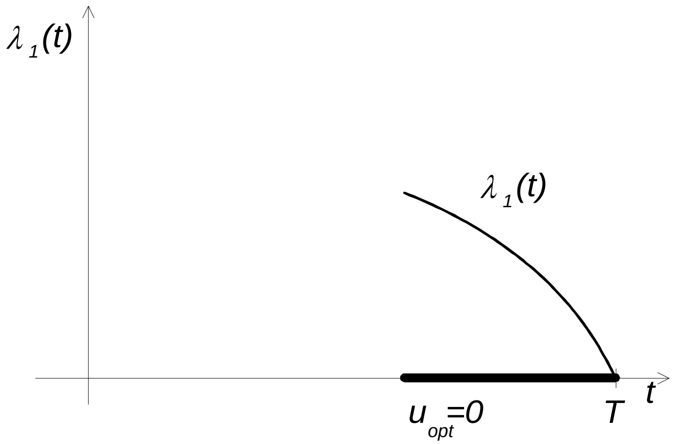
$$\dot{\lambda}_1(t) = 0.5\lambda_1 - 1$$

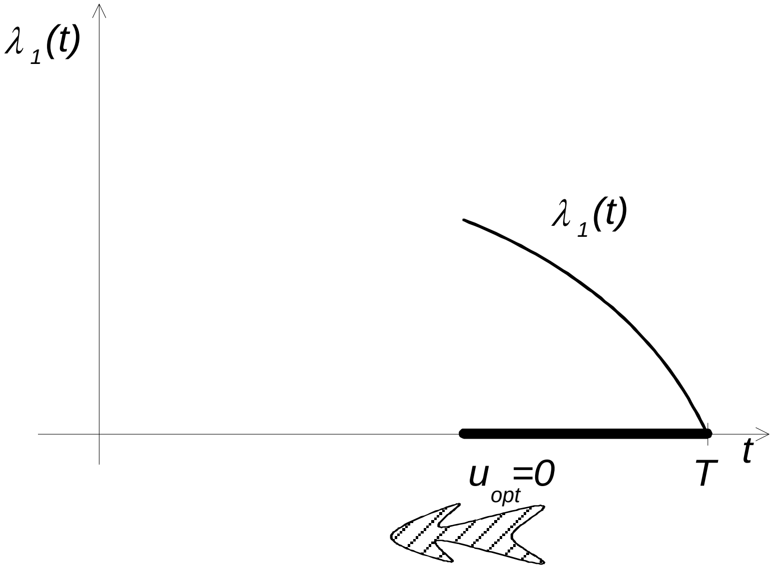
Near the end of the optimization interval the adjoint equation becomes

$$\dot{\lambda}_1(t) = 0.5\lambda_1(t) - 1 \qquad \lambda_1(T) = 0$$

It has the solution

$$\lambda_1(t) = \frac{1}{0.5}\left(1 - e^{0.5(t-T)}\right)$$





"Moving" in this sense *u*

becomes 1 at instant t_s in which

$$\lambda_1(t_s) = 1$$

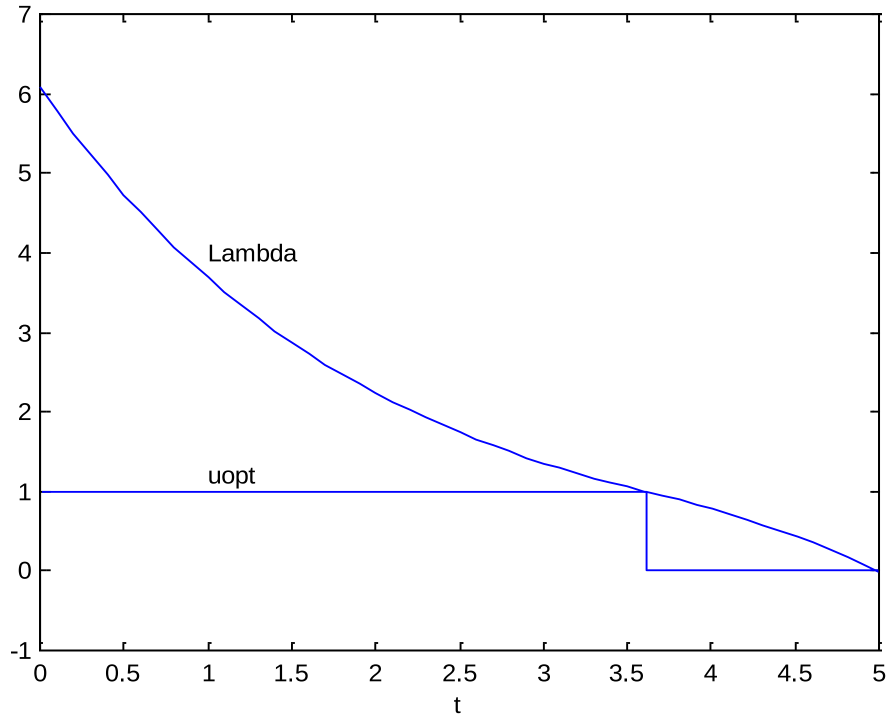
$$\frac{1}{0.5}\left(1 - e^{0.5(t_s - T)}\right) = 1$$

$$e^{0.5(t_s - T)} = 0.5$$

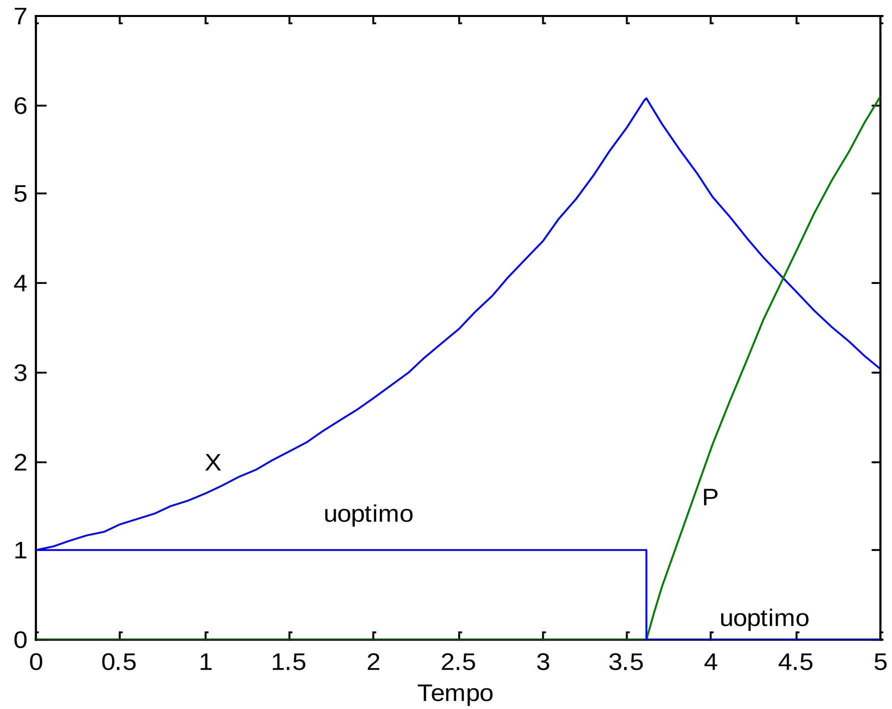
$$\log 0.5 = 0.5(t_s - T)$$

$$t_s = T + 2\log o.5 \cong T - 1.39$$

Example for the situation in which *T=5*

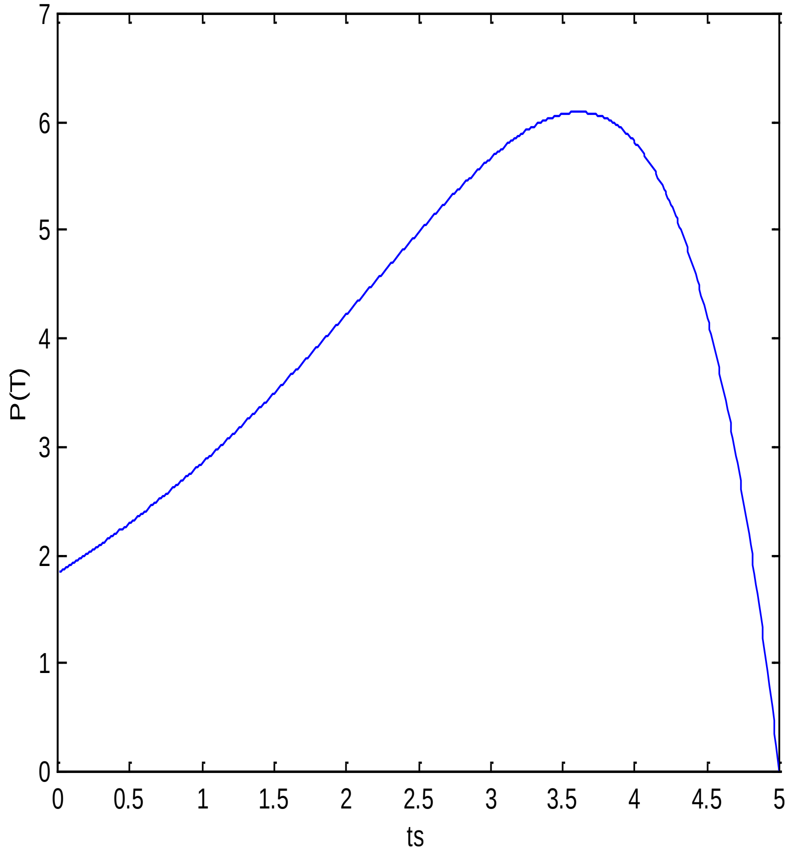


The optimal solution admits the following interpretation: Initially, all the effort is to make the fungi colony to grow. Due to the inhibition effect of the substrate there is no penicillin production. After the switching instant, the control variable is adjusted to maximize the penicillin production.



Assuming a bang-bang shape for the control function, the switching instant corresponds to the maximum.

It is remarked that Pontryagin's Principle yields not only the switching instant but also the shape of the control function.



Proof of Pontryagin's Principle

Objective:

Proof using a variational technique.

Basic optoimal control problem

Let x be the state of a plant with manipulated input u, satisfying the state equation

$$\dot{x} = f(x,u) \qquad x(0) = x_0 \qquad t \in [0,T] \quad T \text{ constant} \quad u(t) \in U$$

Optimize the function u, defined in $[0,T]$ that maximizes

$$J(u) = \Psi(x(T)) + \int_0^T L(x,u)dt$$

Proof strategy

If u_{opt} maximizes $J(u)$ any small variation δu causes a decrese in $J(u)$:

$$\delta J = J(u_{opt} + \delta u) - J(u_{opt}) < 0$$

Cost modification

$$\bar{J} = J - \int_0^T \lambda'(t)\left[\dot{x}(t) - f(x(t), u(t))\right]dt$$

Since thye term between square brackets vanishes along the plant state trajectories, then $\bar{J} = J$ and the u that optimizes $\bar{J}$ is the same that optimizes J.

Therefore,. We may select λ such as to simplify the problem.

This quantity is named *co-state*.

Hamiltonian function

The Hamiltonian is defined by

$$H(\lambda, x, u) = \lambda' f(x, u) + L(x, u)$$

With this definition, write the cost as

$$\bar{J} = J - \int_0^T \lambda'(t)\left[\dot{x}(t) - f(x(t), u(t))\right]dt = \Psi(x(T) + \int_0^T \left[L(x, u) + \lambda' f(x, u) - \lambda' \dot{x}\right]dt$$

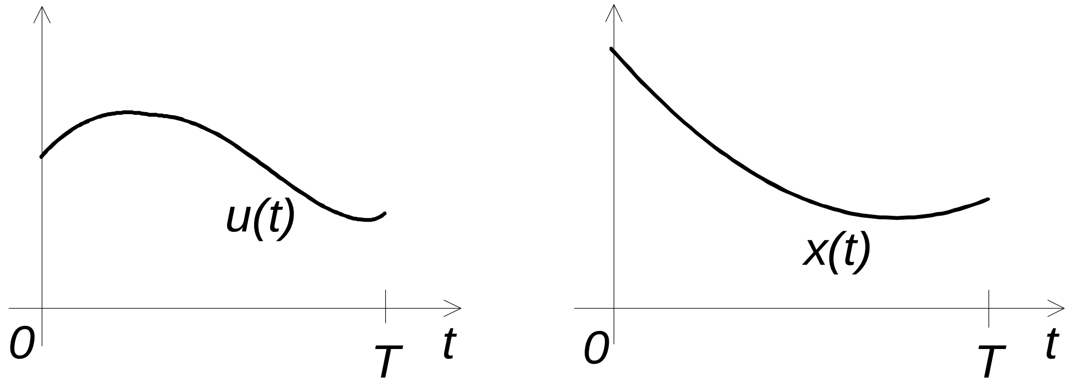
or

$$\bar{J} = \Psi(x(T)) + \int_0^T \left[H(\lambda(t), x(t), u(t)) - \lambda'(t)\dot{x}(t)\right]dt$$

Perturbation of the optimal control

Let $\{u(t), 0 \le t \le T\}$ be the optimal control

Together with the initial condition, it defines the optimal state trajectory $\{x(t), 0 \le t \le T\}$.



Add a small perturbation to *u* that defines the optimal control, to obtain a perturbed control function *v*.

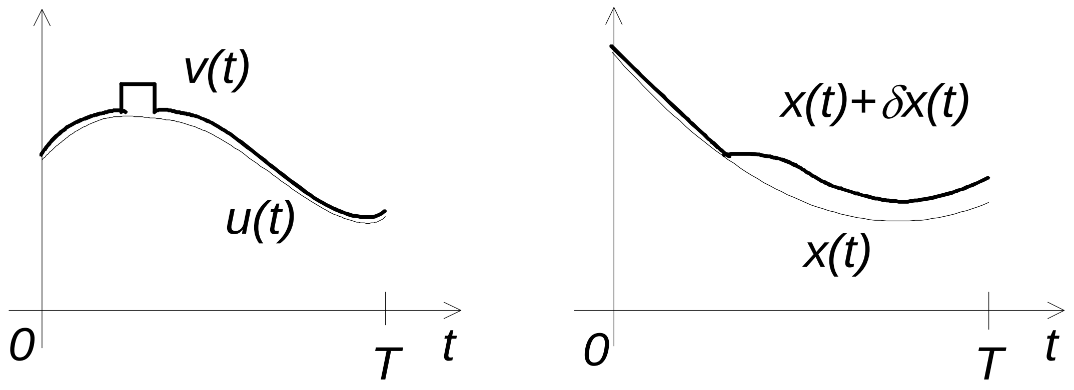
The variation is small in the sense that

$$\int_0^T |u(t) - v(t)| dt < \varepsilon$$

with ε a small number.

The state trajectory that corresponds to *v* deviates little from the optimal state, that corresponds to *u*.

Let $\delta x(t)$ be this state deviation.



Let $\delta\bar{J}$ be the corresponding deviation of the objective function

$$\delta\bar{J} = \bar{J}(v) - \bar{J}(u)$$

Since *u* is optimal, this deviation is **negative**.

Variation of the objective function

Recall that

$$\bar{J} = \Psi\big(x(T)\big) + \int_0^T \Big[H\big(\lambda(t), x(t), u(t)\big) - \lambda'(t)\dot{x}(t)\Big]dt$$

The variation is thus

$$\delta\bar{J} = \Psi\big(x(T) + \delta x(T)\big) - \Psi\big(x(T)\big) + \int_0^T \Big[H\big(\lambda, x + \delta x, v\big) - H\big(\lambda, x, u\big) - \lambda'\delta\ddot{x}\Big]dt$$

Recall the rule of integration by parts

Since $\frac{d}{dt}(ab) = \dot{a}b + a\dot{b}$ it is $$\int_0^T (\dot{a}b)dt = (ab)\Big|_0^T - \int_0^T (a\dot{b})dt$$

Apply this rule with

$$a = \delta x \qquad b = \lambda'$$

$$\int_0^T \lambda' \delta\dot{x} dt = \lambda'(T)\delta x(T) - \lambda'(0)\delta x(0) - \int_0^T \dot{\lambda}' \delta x dt$$

Remark that $\delta x(0) = 0$ because the variation in the optimal contyrol does not cause a variation in the initial condition.

$$\int_0^T \lambda' \delta\dot{x} dt = \lambda'(T)\delta x(T) - \int_0^T \dot{\lambda}' \delta x dt$$

We concluded that

$$\delta\bar{J} = \Psi\big(x(T) + \delta x(T)\big) - \Psi\big(x(T)\big) + \int_0^T \Big[H\big(\lambda, x + \delta x, v\big) - H\big(\lambda, x, u\big) - \lambda' \delta\dot{x}\Big] dt$$

Therefore:

$$\delta\bar{J} = \Psi\big(x(T) + \delta x(T)\big) - \Psi\big(x(T)\big) - \lambda'(T)\delta x(T) + \int_0^T \Big[H\big(\lambda, x + \delta x, v\big) - H\big(\lambda, x, u\big) + \dot{\lambda}' \delta x\Big] dt$$

By integration by parts we could express the variation of the state derivative in variations of the state and the co-state λ .

$$\delta \bar{J} = \Psi\big(x(T) + \delta x(T)\big) - \Psi\big(x(T)\big) - \lambda'(T)\delta x(T) + \int_0^T \Big[H\big(\lambda, x + \delta x, v\big) - H\big(\lambda, x, u\big) + \dot{\lambda}'\delta x\Big]dt$$

Approximete by 1st order Taylor expansions:

$$\Psi\big(x(T) + \delta x(T)\big) \approx \Psi\big(x(T)\big) + \Psi_x\big(x(T)\big)\delta x(T)$$

$$H\big(\lambda, x + \delta x, v\big) \approx H\big(\lambda, x, v\big) + H_x\big(\lambda, x, v\big)\delta x$$

Therefore, up to terms of 2nd order or higher:

$$\delta \bar{J} = \left[\Psi_x\left(x(T)\right) - \lambda'(T)\right]\delta x(T) + \int_0^T \left[H_x\left(\lambda, x, u\right) + \dot{\lambda}'\right]\delta x dt + \int_0^T \left[H\left(\lambda, x, v\right) - H\left(\lambda, x, u\right)\right]dt$$

Selecting λ as the solution of the adjoint equation

$$-\dot{\lambda}'(t) = H_x\left(\lambda(t), x(t), u(t)\right)$$

With terminal condition

$$\lambda'(T) = \Psi_x\left(x(T)\right)$$

The variation of the functional is reduced to

$$\delta \bar{J} = \int_0^T \left[H\left(\lambda(t), x(t), v(t)\right) - H\left(\lambda(t), x(t), u(t)\right)\right]dt$$

$$\delta \bar{J} = \int_0^T \left[H\big(\lambda(t), x(t), \underset{\substack{\uparrow \\ \text{Perturbed}}}{v(t)}\big) - H\big(\lambda(t), x(t), \underset{\substack{\uparrow \\ \text{Optimal}}}{u(t)}\big) \right] dt$$

$$\delta\bar{J} = \int_0^T \left[H\big(\lambda(t), x(t), v(t)\big) - H\big(\lambda(t), x(t), u(t)\big) \right] dt$$

If u is optimal, Then, at each time ***t***:

$$H\big(\lambda(t), x(t), v\big) \leq H\big(\lambda(t), x(t), u(t)\big)$$

$$\forall v \in U$$

This statement must be proved.

$$\delta \bar{J} = \int_0^T \left[H\left(\lambda(t), x(t), v(t)\right) - H\left(\lambda(t), x(t), u(t)\right) \right] dt$$

Assume by contradiction that there is t_1 and a function φ such that

$$H\left(\lambda(t_1), x(t_1), \varphi(t_1)\right) > H\left(\lambda(t_1), x(t_1), u(t_1)\right)$$

Since H is continuous, there exists an interval $\left[t_1 - \sigma, t_1 + \sigma\right]$ in which this property holds. Select $v(t) = u(t)$ except in this interval where we do $v(t) = \varphi(t)$. With this choice,

$$\delta \bar{J} = \int_{t_1 - \sigma}^{t_1 + \sigma} \left[H\left(\lambda(t), x(t), v(t)\right) - H\left(\lambda(t), x(t), u(t)\right) \right] dt > 0$$

This contradicts the assumption that u is the optimal control.

Problems with equality constraints on the terminal state

Let x be the state of a plant with input u defined by

$$\dot{x} = f(x, u) \qquad x(0) = x_0 \qquad t \in [0, T] \quad u(t) \in U$$

T *given*

Find the function u, defined in the time interval $[0, T]$ that maximizes

$$J(u) = \Psi(x(T)) + \int_0^T L(x, u) dt$$

Subject to the equality constraints in the terminal state

$$x_i(T) = \bar{x}_i \qquad i = 1, 2, \ldots, r \leq n$$

Maximum Principle (Equality constraints on the terminal state)

Along the optimal trajectory for x, u and λ the following necessary conditions for the maximization of J are satisfied

$$\dot{x} = f(x, u) \qquad x(0) = x_0 \qquad t \in [0, T] \;\; u(t) \in U$$

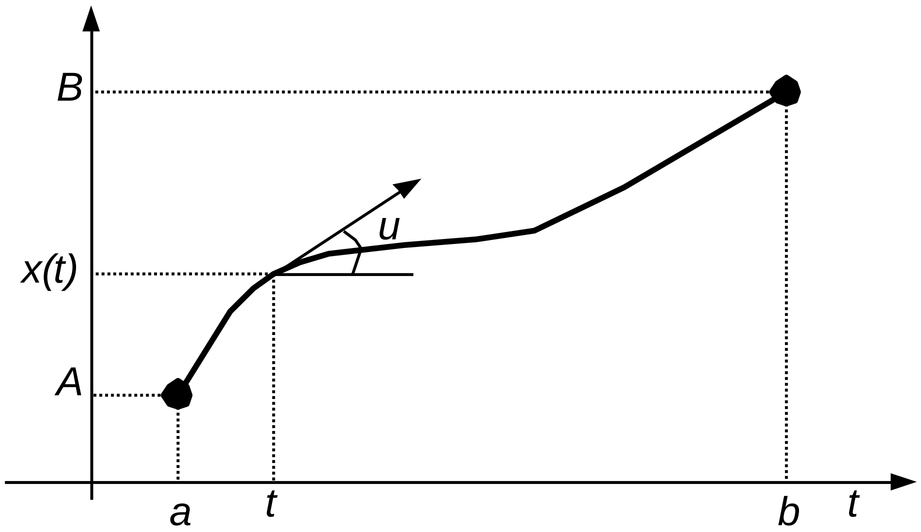
$$x_i(T) = \bar{x}_i \qquad i = 1, 2, \ldots, r \le n$$

$$-\dot{\lambda}'(t) = \lambda'(t) f_x\big(x(t), u(t)\big) + L_x\big(x(t), u(t)\big)$$

$$\lambda_i'(T) = \Psi_x\big(x(T)\big)_i \qquad i = r+1, r+2, \ldots, n$$

For each t, the Hamiltonian $H(\lambda, x, u) = \lambda' f(x, u) + L(x, u)$ is maximum for the optimal value of $u(t)$.

Exercise: Minimum length path between 2 points



What is the path with minimum length between the extreme points (a, A) and (b, B)? The length of the curve x connecting the two points is $J = \int_a^b \sqrt{1 + (\dot{x}(t))^2} dt$. Formulate this as an OCP and solve it using PMP.

Solution: Define the dynamics by

$\frac{dx}{dt} = u$ with initial and terminal conditions $x(a) = A$, $x(b) = B$

$$J(u) = \int_a^b \sqrt{1 + u(t)^2}\, dt$$

The OCP is

$$\max_u J(u)$$

$$s.\ t.\ \dot{x} = u$$

$$x(a) = A$$

$$x(b) = B$$

$$J(u) = \int_a^b \sqrt{1 + u(t)^2}\, dt \quad \rightarrow \quad L(x,u) = \sqrt{1+u^2} \quad \rightarrow \quad L_x = 0$$

$$f(x,u) = u \quad \rightarrow \quad f_x = 0$$

Adjoint equation: $\dot{\lambda} = 0$

Since there is a terminal condition on the state, there is no terminal condition on the co-state, but from the adjoint equation we know it is a constant.

Hamiltonian: $H(\lambda, x, u) = \lambda u + \sqrt{1+u^2}$

Maximum condition: $\frac{\partial H}{\partial u} = 0 \quad \rightarrow \quad \lambda + \frac{u}{\sqrt{1+u^2}} = 0$

Since λ is a constant, the optimal control will also be a constant. The slope u is constant and hence the optimal curve is a straight line.

Find the constant that defines the optimal control from the initial and terminal conditions.

Solve the dynamics equation $\dot{x} = u$ to get $x(t) = x(a) + \int_a^t u \, d\sigma$

$$x(t) = A + u(t - a)$$

Apply the terminal condition to get $B = A + u(b - a)$ or

$$u = \frac{B - A}{b - a}$$

End of exercise

General procedure to solve OCP with terminal state equality constraints

1. Solve the OCP. Since there are no terminal conditions for the co-state, the solution is obtained up to a constant.
2. Solve the state equation with the optimal control. This solution is parameterized by a constant in the control.
3. Compute the constant using the terminal condition on the state.

Exercise: Mobile robot with specified terminal state

A mobile robot moves along a line with coordinate x_1, with velocity x_2 and is modelled by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u$$

Find the control law that minimizes the energy consumed

$$J(u) = \frac{1}{2}\int_0^1 u^2(t)dt$$

when the robot moves between the initial and terminal states given by

$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (return to the same place but stop).

Solution: $f(x,u)=\begin{bmatrix} x_2 \\ u \end{bmatrix}$, $\quad f_x=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad L(x,u)=-\frac{1}{2}u^2$, $\quad L_x=\begin{bmatrix} 0 & 0 \end{bmatrix}$

Adjoint equation: $\quad [-\dot{\lambda}_1 \quad -\dot{\lambda}_2]=[\lambda_1 \quad \lambda_2]\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$\dot{\lambda}_1=0, \qquad \dot{\lambda}_2=-\lambda_1$$

$$\lambda_1=C_1, \qquad \lambda_2=C_2-C_1t$$

Hamiltonian: $\quad H(\lambda,x,u)=\lambda_1x_2+\lambda_2u-\frac{1}{2}u^2$

Maximum condition: $\quad \frac{\partial H}{\partial u}=\lambda_2-u=0 \qquad u^*(t)=\lambda_2(t)=C_2-C_1t$

To compute the constants C_1 and C_2, solve the state equations with $u = u^*$ and impose the initial and terminal conditions on the state.

$$\dot{x}_2 = C_2 - C_1 t$$

$$x_2(t) = x_2(0) + \int_0^t (C_2 - C_1 \sigma) d\sigma = 1 + C_2 t - \frac{1}{2} C_1 t^2$$

$$\dot{x}_1 = x_2 = 1 + C_2 t - \frac{1}{2} C_1 t^2$$

$$x_1(t) = x_1(0) + \int_0^t x_2(\sigma) d\sigma = 1 + \int_0^t \left[1 + C_2 \sigma - \frac{1}{2} C_1 \sigma^2\right] d\sigma$$

$$x_1(t) = 1 + t + \frac{1}{2} C_2 t^2 - \frac{1}{6} C_1 t^3$$

$$x_1(t) = 1 + t + \frac{1}{2} C_2 t^2 - \frac{1}{6} C_1 t^3$$

$$x_2(t) = 1 + C_2 t - \frac{1}{2} C_1 t^2$$

To compute C_1 and C_2, make $t = 1$ (final instant) and use the terminal conditions. To get

$$\begin{cases} 1 + \frac{1}{2} C_2 - \frac{1}{6} C_1 = 0 \\ 1 + C_2 - \frac{1}{2} C_1 = 0 \end{cases}, \qquad C_1 = -6, \qquad C_2 = -4$$

$$u^*(t) = 6t - 4$$

End of exercise

Example: Optimal velocity transfer – minimum energy

Solve the following OCP:

$$\min_u J(u) := \frac{1}{2}\int_0^T u^2(t)dt$$

s. t. $\dot{v} = -av + bu, \quad a, b > 0$

$$v(0) = V_1$$

$$v(T) = V_2$$

Dado que se pretende minimizar J, a lagrangiana é

$$L = -\frac{1}{2}u^2$$

e a hamiltoniana é

$$H = \lambda(-av + bu) - \frac{1}{2}u^2.$$

A condição de máximo é

$$\frac{\partial H}{\partial u} = \lambda b - u = 0,$$

pelo que o controlo ótimo é

$$u^*(t) = \lambda(t)b.$$

$$u^*(t) = \lambda(t)b.$$

Por outro lado, a equação é

$$\dot{\lambda} = \lambda a,$$

que tem por solução

$$\lambda(t) = C - 1e^{-a(T-t)},$$

em que C_1 é uma constante. Repare-se que com esta maneira de escrever a solução é $C_1 = \lambda(T)$. No entanto, como o estado (ou seja, neste caso, a velocidade) terminal é imposto, não há uma condição terminal explícita no valor terminal do coestado.

O controlo ótimo é, pois,

$$u^*(t) = bC_1e^{-a(T-t)},$$

O controlo ótimo é, pois,

$$u^*(t) = bC_1 e^{-a(T-t)},$$

pelo que a velocidade ótima satisfaz a equação, parametrizada por C_1,

$$\dot{v} = -av + b^2 C_1 e^{-a(T-t)}.$$

Para resolver esta equação, toma-se a transformada de Laplace

$$sV(s) - v(0) = -aV(s) + b^2 C_1 e^{-aT} \mathcal{L}(e^{at}),$$

$$V(s) = \frac{1}{s+a}v(0) + b_2 C_1 e^{-aT} \frac{1}{(s+a)(s-a)}$$

ou, através de uma expansão em frações simples,

$$V(s) = \frac{1}{s+a}v(0) - \frac{b^2}{a} C_1 e^{-aT} \frac{1}{2} \left(\frac{1}{s-a} - \frac{1}{s+a} \right).$$

Invertendo a transformada de Laplace e usando a condição inicial, vem

$$v(t) = V_1 e^{-at} - -\frac{b^2}{a} C_1 e^{-aT} \frac{1}{2} \left(e^{at} - e^{-at} \right),$$

$$V(s) = \frac{1}{s+a}v(0) - \frac{b^2}{a}C_1 e^{-aT}\frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right).$$

Invertendo a transformada de Laplace e usando a condição inicial, vem

$$v(t) = V_1 e^{-at} - -\frac{b^2}{a}C_1 e^{-aT}\frac{1}{2}\left(e^{at} - e^{-at}\right),$$

ou, ainda, como

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2},$$

é

$$v(t) = V_1 e^{-at} - \frac{b^2}{a}C_1 e^{-aT}\sinh(at).$$

A constante C_1 é calculada aplicando a condição terminal

$$V_2 = V1e^{-at} - \frac{b^2}{a} C_1 e^{-aT} \sinh(aT),$$

de onde

$$C_1 = \frac{a}{b^2} \cdot \frac{e^{aT}}{\sinh(aT)} \left(V_2 - V_1 e^{-aT}\right)$$

e, substituindo na expressão de λ,

$$\lambda(t) = \frac{a}{b^2} \cdot \frac{V_2 - V_1 e^{-aT}}{\sinh(aT)} e^{at}.$$

Finalmente, o controlo ótimo é

$$u^*(t) = \frac{a}{b} \cdot \frac{V_2 - V_1 e^{-aT}}{\sinh(aT)} e^{at}.$$

Fim do exemplo

Example: Optimal velocity transfer – minimum fuel

Solve the following OCP:

$$\min_u J(u) := \int_0^T u(t)dt$$

s. t. $\dot{v} = -av + bu, \quad a, b > 0$

$$0 \leq u(t) \leq \bar{u}$$

$$v(0) = V_1$$

$$v(T) = V_2$$

Assume that T is large enough so that there is one control switch.

Free terminal time problems

In addition to the conditions of the Maximum Principle, the following condition must hold:

$$H\big(\lambda(T), x(T), u(T)\big) = 0$$

Example

$$\min_{u,T} J(u) = \frac{1}{2}\int_0^T u^2(t)dt$$

sujeito a

$$\dot{v} = -v + u,$$

$$v(0) = V_1, \quad v(T) = V_2$$

com T livre.

Resolva o problema para um T genérico e mostre que a trajetória ótima do estado se pode escrever na forma

$$v^*(t) = \alpha e^{-t} + \beta e^t,$$

em que α e β são constantes que dependem de T. Em seguida, determine o valor ótimo de T impondo a condição de que a hamiltoniana se anula, que é válida para problemas de tempo mínimo.

Solution

$$H = -\lambda v + \lambda u - \frac{1}{2}u^2.$$

A equação adjunta é

$$\dot{\lambda} = \lambda$$

e tem por solução

$$\lambda(t) = C_1 e^t.$$

A condição de máximo é

$$\frac{\partial H}{\partial u} = \lambda - u = 0,$$

pelo que o controlo ótimo é

$$u^*(t) = \lambda(t),$$

ou seja,

$$u^*(t) = C_1 e^t.$$

A trajetória ótima da velocidade satisfaz, pois, a equação

$$\dot{v} = -v + C_1 e^t,$$

que pode ser resolvida com a transformada de Laplace. Sendo $V(s) = \mathcal{L}(v)$ a transformada de Laplace de v,

$$sV(s) - V_1 = -V(s) + C_1 \frac{1}{s-1}.$$

$$sV(s) - V_1 = -V(s) + C_1 \frac{1}{s-1}.$$

Resolvendo em ordem a $V(s)$

$$V(s) = V_1 \frac{1}{s+1} + C_1 \frac{1}{(s+1)(s-1)},$$

ou, como

$$\frac{1}{(s+1)(s-1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right),$$

vem

$$V(s) = \left(V_1 - \frac{C_1}{2}\right)\frac{1}{s+1} + \frac{C_1}{2} \cdot \frac{1}{s-1}.$$

Invertendo a transformada

$$v(t) = \left(a - \frac{C_1}{2}\right) e^{-1} + \frac{C_1}{2} e^t. \tag{12.33}$$

Fazendo $t = T$ e usando a condição terminal para a velocidade, obtém-se uma equação algébrica que resolvida em ordem a $\frac{C_1}{2}$ dá

$$\frac{C_1}{2} = \frac{b - ae^{-T}}{e^T - e^{-T}} =: \beta \tag{12.34}$$

e, ainda,

$$a - \frac{C_1}{2} = \frac{ae^T - b}{e^T - e^{-T}} := \alpha. \tag{12.35}$$

Assim, por (12.33), a trajetória para a velocidade é

$$v(t) = \frac{ae^T - b}{e^T - e^{-T}} e^{-t} + \frac{b - ae^{-T}}{e^T - e^{-T}} e^t$$

ou, usando as definições de α e β em (12.34 e (12.35),

$$x(t) = \alpha e^{-t} + \beta e^t,$$

$$\lambda(t) = 2\beta e^t$$

e

$$u^*(t) = 2\beta e^t.$$

Assim, por (12.33), a trajetória para a velocidade é

$$v(t) = \frac{ae^T - b}{e^T - e^{-T}} e^{-t} + \frac{b - ae^{-T}}{e^T - e^{-T}} e^t$$

ou, usando as definições de α e β em (12.34 e (12.35),

$$x(t) = \alpha e^{-t} + \beta e^t,$$

$$\lambda(t) = 2\beta e^t$$

e

$$u^*(t) = 2\beta e^t.$$

Para calcular o valor ótimo de T, observe-se que, ao longo de uma trajetória ótima, a hamiltoniana, dada por (12.32), é por conseguinte

$$H = -2\beta e^t \left(\alpha e^{-t} + \beta e^t\right) + 4\beta^2 e^{2t} - \frac{1}{2} \cdot 4\beta^2 e^{2t}$$

ou seja, simplificando,

$$H = -2\alpha\beta = -2\frac{(ae^T - b)(b - ae^{-T})}{\left(e^T - e^{-t}\right)^2},$$

que, como se esperava, é constante. A condição de otimalidade em relação a T consiste em $H = 0$. Para H se anular tem de ser

$$e^T = \frac{b}{a}, \quad \text{ou seja} \quad T = \ln\frac{b}{a},$$

para $b > a$ (por forma a que $T > 0$), ou

$$e^{-T} = \frac{a}{b}, \quad \text{ou seja} \quad T = \ln\frac{a}{b},$$

Example: Push cart

Problem: Given the car with dynamic equations

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = u(t)$$

Find the optimal control that satisfies the constraint $|u(t)| \leq 1$ and brings the car from the initial condition $[x_1(0) \quad x_2(0)]'$ to the origin $[0 \quad 0]'$ in minimum time.

The cost function is written

$$J = \int_0^T 1 \, dt \qquad T \text{ free}$$

Since the terminal state is completely fixed, there are no constraints on the final co-state. The co-state is thus known up to constants. The co-state equation is thus:

$$-\dot{\lambda}'(t) = \lambda'(t) f_x\big(x(t), u(t)\big) + L_x\big(x(t), u(t)\big)$$

Since $L = 1$, it follows that $L_x = 0$

Since $f(x,u) = \begin{bmatrix} x_2 \\ u \end{bmatrix}$ it is $f_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$-\begin{bmatrix} \dot{\lambda}_1 & \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-\begin{bmatrix} \dot{\lambda}_1 & \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The co-state equations are thus:

$$\dot{\lambda}_1 = 0$$
$$\dot{\lambda}_2 = -\lambda_1$$

These equations have the solution

$$\lambda_1(t) = \pi_1$$
$$\lambda_2(t) = \pi_2 - \pi_1(t)$$

π_1, π_2 unknown constants

The Hamiltonian $H = \lambda' f + L$ is

$$H(\lambda, x, u) = 1 + \lambda_1 x_2 + \lambda_2 u$$

Since the Hamiltonian is linear on u, the optimal control is attained at the maximum and minimum of the admissible values for u that are -1 and +1.

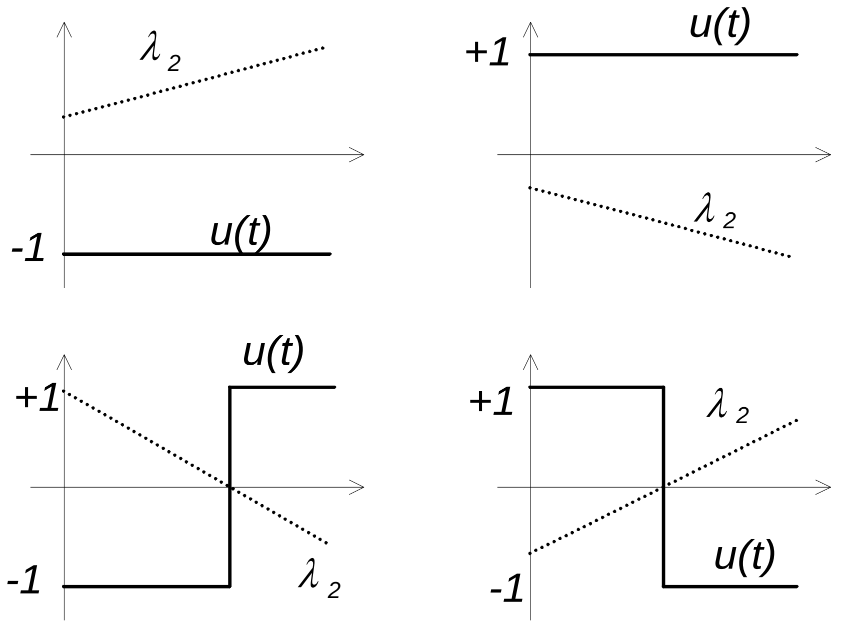
In this case we want to minimize the Hamiltonian.

For the Hamiltonian to be minimum:

* When $\lambda_2 > 0$ the optimal control is $u_{opt} = -1$

* When $\lambda_2 < 0$ the optimal control is $u_{opt} = +1$

There are the following possibilities:



Since $\lambda_2(t)$ is a straight line, $\lambda_2(t) = \pi_2 - \pi_1 t$, the optimal control has at most one switch.

How to find the switching instants?

Solve the state equations in a period of time in which u is constant:

$$x_2(t) = x_2(0) + ut$$

$$x_1(t) = x_1(0) + x_2(0)t + \frac{1}{2}ut^2$$

To obtain the corresponding orbits on the state plane, eliminate t between these equations. From the first:

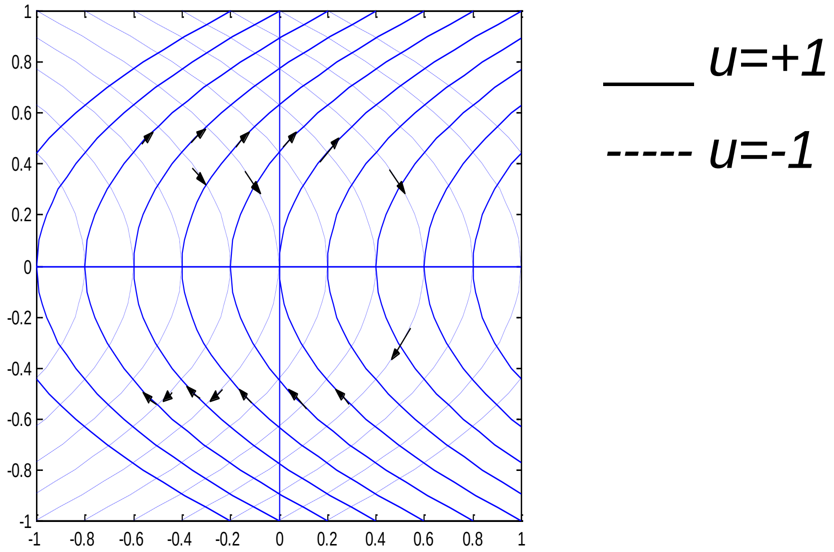
$$t = u\left(x_2(t) - x_2(0)\right)$$

Replace on the second:

$$x_1(t) = x_1(0) + \frac{1}{2}ux_2^2(t) - \frac{1}{2}ux_2^2(0)$$

$$x_1(t) = x_1(0) + \frac{1}{2} u x_2^2(t) - \frac{1}{2} u x_2^2(0)$$

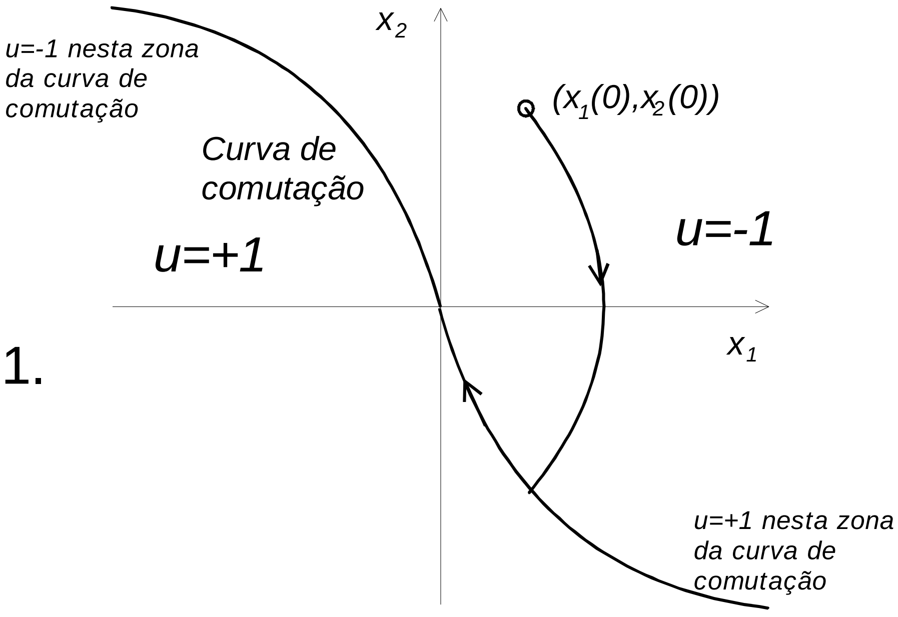
The orbits are parabolas with an horizontal axis, with the concavity turned to the left if $u = -1$ and turned to the right if $u = 1$.



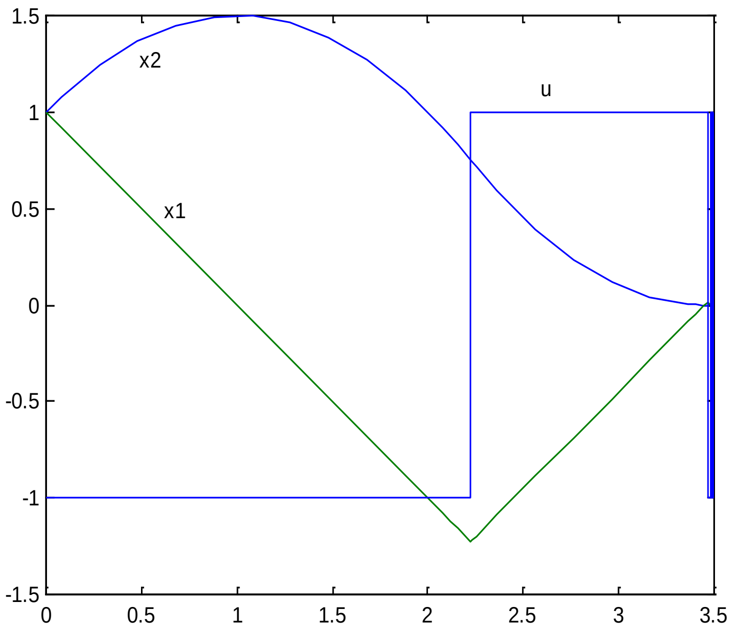
Since there can be only one switch in the optimal control, this fact leads to a simple rule to select the control depending on the region of the state space where we are:

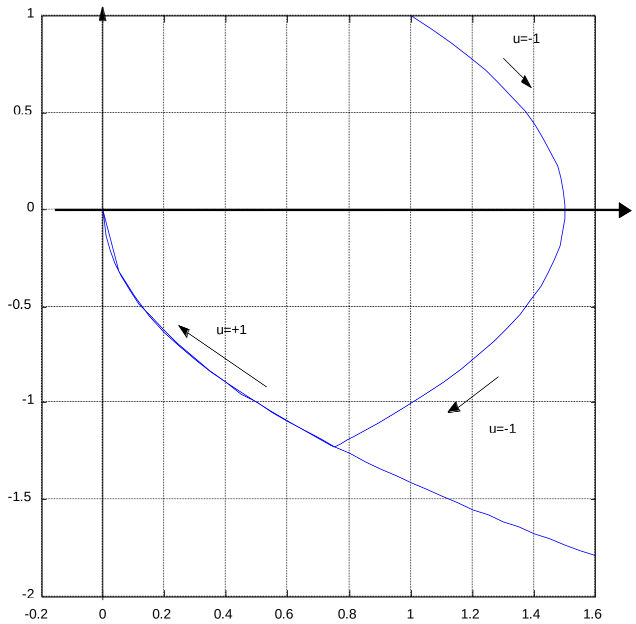
Above the switching curve the control is -1.

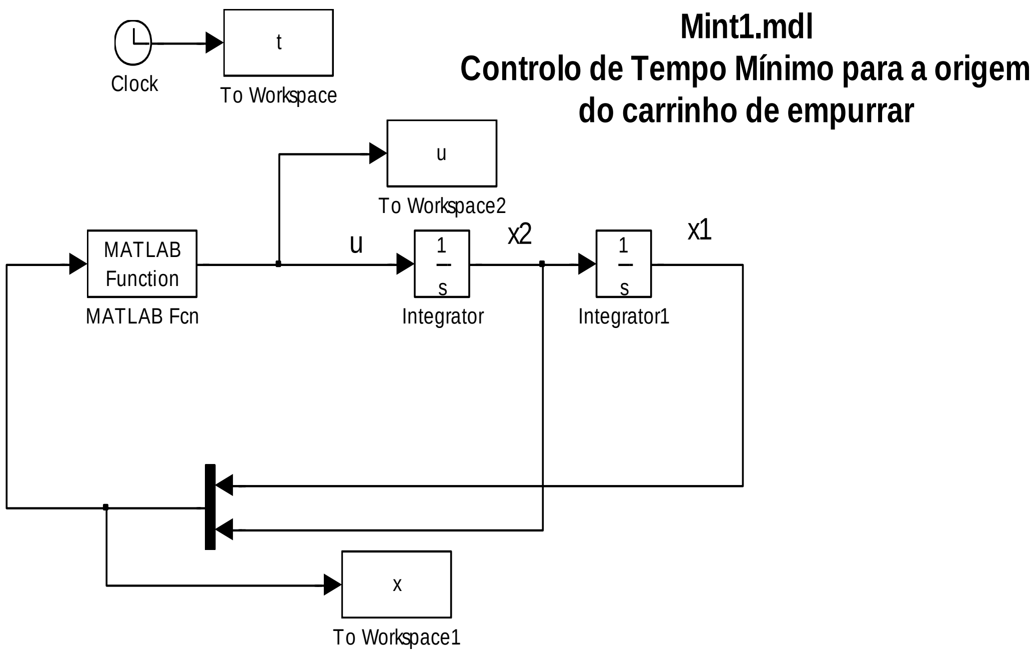
Below, it is +1.



When we are over the switching curve in the upper branch the control is -1 and in the lower branch is +1.


x2
u
x1
1.5
1
0.5
0
-0.5
-1
-1.5
0 0.5 1 1.5 2 2.5 3 3.5


u=-1
u=+1
u=-1
1
0.5
0
-0.5
-1
-1.5
-2
-0.2 0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6




```
function out=comuta(u)
% Calcula o controlo óptimo para o
% problema de tempo
% mínimo para a origem do
% carrinho de empurrar
if u(1)<0
  if u(2)>sqrt(-2*u(1))
    out=-1;
  else
    out=+1;
  end;
else
  if u(2)>-sqrt(2*u(1))
    out=-1;
  else
    out=+1;
  end;
end;
```

The Hamiltonian is constant for time invariant problems

Consider the case in which both L and f do not explicitly depend on the time t.

For the class of problems in which the optimality condition is $\frac{\partial H}{\partial u(t)} = 0$ and u is smooth, prove that the Hamiltonian is constant in time, i. e., that $\frac{dH}{dt} = 0$.

Help:

$$\dot{x} = f(x,u), \qquad -\dot{\lambda}^T = \lambda^T f_x + L_x, \qquad H(\lambda,x,u) = \lambda^T f(x,u) + L(x,u)$$

$$\frac{dH}{dt} = \dot{\lambda}^T f + \lambda^T f_x \dot{x} + \lambda f_u u + L_x \dot{x} + L_u \dot{u}$$

$$\frac{dH}{dt} = -\lambda^T f_x f - L_x f + \lambda^T f_x f + \lambda f_u \dot{u} + L_x f + L_u \dot{u}$$

$$\frac{dH}{dt} = (\lambda f_u + L_u)\dot{u} = \frac{\partial H}{\partial u}\dot{u} = 0\dot{u} = 0$$

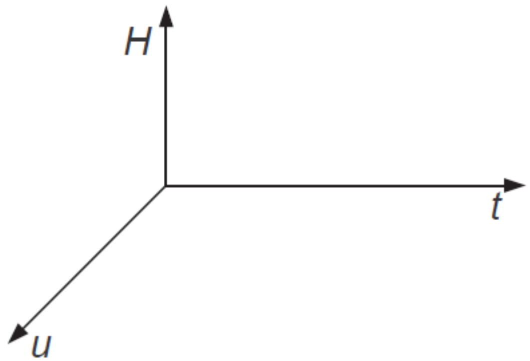
Exercício

$$J(u) = x_1(T) - \int_0^T u(t)dt,$$

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = u, \qquad 0 \le u \le \bar{u},$$

Resolva este problema de controlo ótimo (maximizar J, suponha $T > 1$) e calcule a Hamiltoniana ao longo do tempo, sobre uma trajetória ótima para λ, x e u. Desenhe a evolução da Hamiltoniana como função de u ao longo do tempo.



Solução

$$f_1(x,u) = x_2,$$

$$f_2(x,u) = u,$$

pelo que

$$f_x(x,u) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Como

$$\psi(x(T)) = x_1(T),$$

é

$$\psi_x(x(T)) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

A equação adjunta (11.6) é, pois,

$$\begin{bmatrix} -\dot{\lambda}_1 & -\dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

ou seja,

$$\dot{\lambda}_1 = 0,$$

$$\dot{\lambda}_2 = -\lambda_1.$$

Como as condições terminais do coestado, dadas por (11.7), são

$$\begin{bmatrix} \lambda_1(T) & \lambda_2(T) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

conclui-se que (repare que a derivada de λ_1 em ordem a t é zero, pelo que λ_1 é constante)

$$\lambda_1(t) = 1,$$

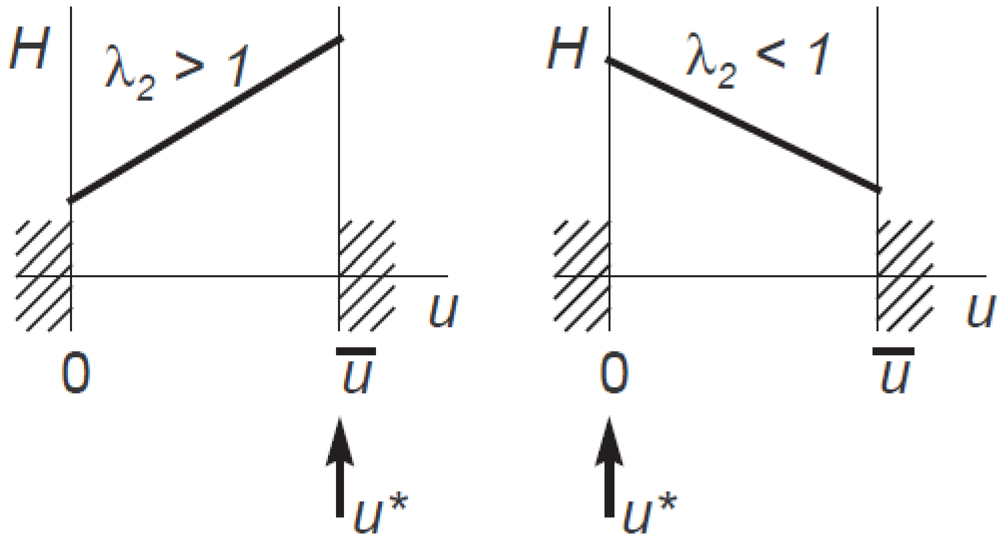
pelo que a equação para λ_2 é

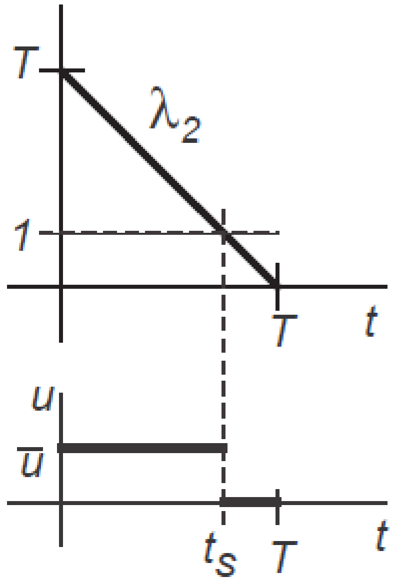
$$\dot{\lambda}_2(t) = -1.$$

Tendo em conta a condição inicial $\lambda_2(T) = 0$, a solução desta equação diferencial é

$$\lambda_2(t) = T - t.$$

$$H = \lambda_1 x_2 + (\lambda_2 - 1)u.$$





$$t_s = T - 1.$$

No intervalo $0 \leq t \leq t_s$ o estado vem dado por

$$x_2(t) = x_2(0) + \int_0^t \bar{u} d\sigma = \bar{u}t,$$

$$x_1(t) = x_1(0) + \int_0^t \bar{u}\sigma d\sigma = \frac{\bar{u}}{2}t^2.$$

O coestado é, para $0 \leq t \leq T$

$$\lambda_1(t) = 1, \quad \lambda_2(t) = T - t.$$

No de tempo $0 \leq t \leq t_s$ a hamiltoniana é, por conseguinte,

$$H = (T - 1)\bar{u}.$$

No intervalo $t_s \leq t \leq T$ o estado é dado pela integração de

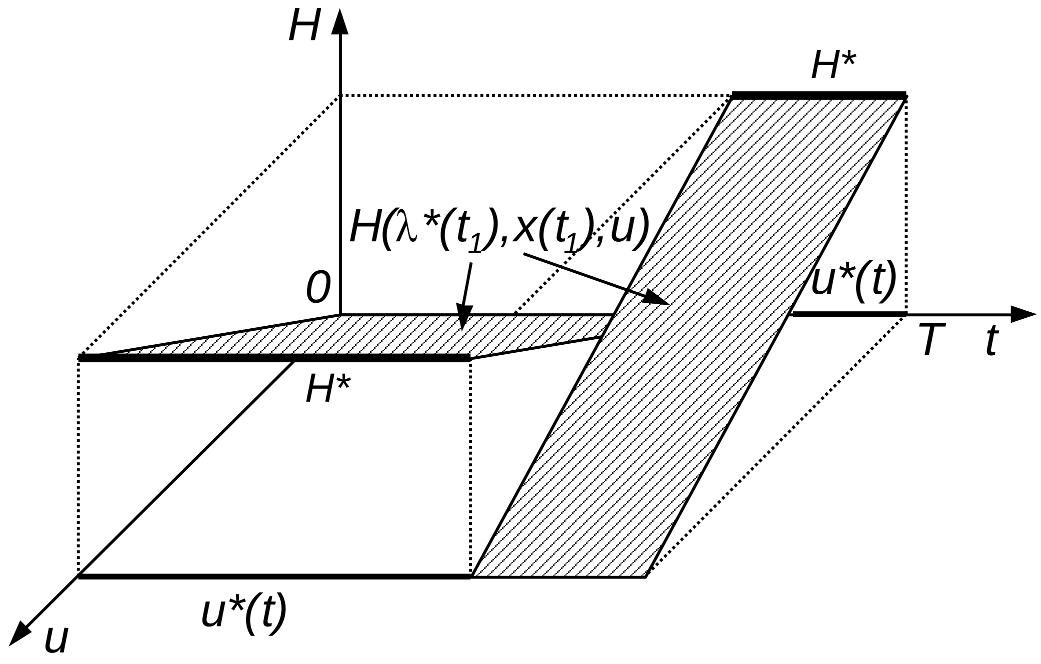
$$\dot{x}_2 = 0$$

donde

$$x_2(t) = x_2(t_s),$$

em que $x_2(t_s) = \bar{u}t_s = \bar{u}(T-1)$ é o valor final do intervalo anterior. Neste segundo intervalo, a hamiltoniana é pois $H = x_2(t) = \bar{u}(T-1)$, que é igual ao valor obtido no intervalo $[0, t_s]$. Ao longo de uma trajetória ótima, a hamiltoniana é, por conseguinte, constante em todo o intervalo $[0, T]$ e igual a $\bar{u}(T-1)$.

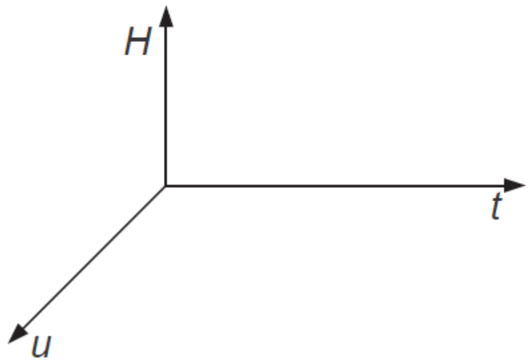
b) A figura 11.12 mostra a hamiltoniana para $x = x^*$ e $\lambda = \lambda^*$. O seu gráfico em função de u, para cada t, é constituído por uma rampa que comuta de inclinação para $t = t_s$, mas mantendo um valor constante para $u = u^*$.

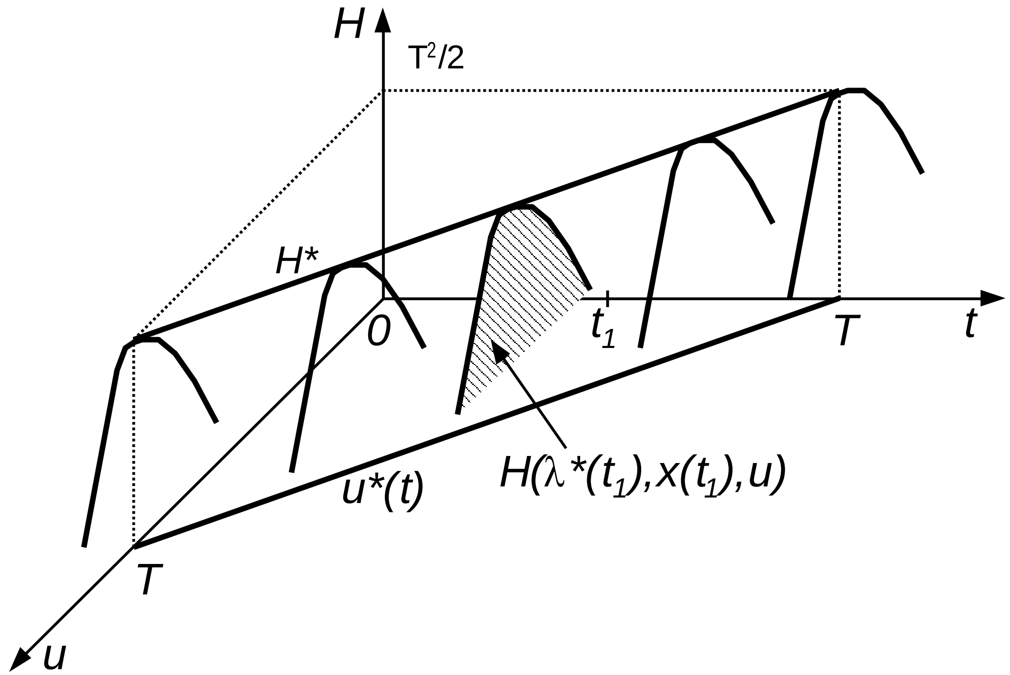

H
H*
$H(\lambda^*(t_1),x(t_1),u)$
0
u*(t)
T
t
H*
u
u*(t)

Exercício

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u, \qquad J(u) = x_1(T) - \frac{1}{2}\int_0^T u^2(t)dt.$$

Resolva este problema de controlo ótimo (maximizar J, suponha $T > 1$) e calcule a Hamiltoniana ao longo do tempo, sobre uma trajetória ótima para λ, x e u. Desenhe a evolução da Hamiltoniana como função de u ao longo do tempo.




H
T2/2
H*
0
t1
T
t
u*(t)
H(λ*(t1),x(t1),u)
T
u

The Linear Quadratic Problem

Dynamics:

$$\dot{x}(t) = Ax(t) + bu(t)$$

$$x(0) = x_0 \qquad u(t) \in R^m$$

Cost functional:

T fixo

$$J = \frac{1}{2}\int_0^T \left[x'(t)Qx(t) + u'Ru\right]dt \qquad Q = Q' \geq 0 \qquad R = R' > 0$$

Since we want to minimize J the Lagrangian is

$$L(x,u) = -\frac{1}{2}(x'Qx + u'Ru)$$

Adjoint equation

$$-\dot{\lambda}' = \lambda' f_x + L_x$$

$-\dot{\lambda}'(t) = \lambda'(t)A - x'(t)Q$ subject to the terminal condition $\lambda(T) = 0$

Hamiltonian

$$H(\lambda, x, u) = \lambda' f(x, u) + L(x, u)$$

$$H(\lambda, x, u) = \lambda'(t)Ax(t) + \lambda'(t)bu(t) - \frac{1}{2}x'(t)Qx(t) - \frac{1}{2}u'(t)Ru(t)$$

Minimum condition on the Hamiltoniana

The Hamiltonian

$$H(\lambda, x, u) = \lambda'(t)Ax(t) + \lambda'(t)bu(t) - \frac{1}{2}x'(t)Qx(t) - \frac{1}{2}u'(t)Ru(t)$$

Is a quadratic function. A necessary condition of minimum is therefore

$$\frac{\partial H}{\partial u} = 0$$

or

$$\lambda'(t)b - u'(t)R = 0$$

Thus, the optimal control verifies

$$u(t) = R^{-1}b'\lambda(t)$$

Thus, the optimal trajectory verifies

$$u_{opt}(t)$$

$$\dot{x}(t) = Ax(t) + bR^{-1}b'\lambda(t)$$

$$\dot{\lambda}(t) = Qx(t) - A'\lambda(t)$$

Subject to

$$x(0) = x_0 \qquad \lambda(T) = 0$$

This is a problem in which the unknowns (x and λ) are specified at two points (0 and T). It is said to be a *Two point boundary value problem*.

How to solve it?

State and co-state equations with optimal control

$$\dot{x} = Ax + bR^{-1}b'\lambda$$

$$\dot{\lambda} = Qx - A'\lambda$$

Assume that there is a matrix $P(t)$ such that

$$\lambda = -Px$$

Under this assumption, the state and co-state equations can be written as

$$\dot{x} = \left[A - bR^{-1}b'P\right]x$$

$$\dot{\lambda} = \left[Q + A'P\right]x$$

Let's try to get an equation for $P(t)$. We have

$$\lambda = -Px$$

Differentiate

$$\dot{\lambda} = -\dot{P}x - P\dot{x}$$

Use the state and co-state equations

$$\left(Q + A'P\right)x = -\dot{P}x - P\left(A - bR^{-1}b'P\right)x$$

Factorize x

$$\left[\dot{P} + PA + A'P - PbR^{-1}b'P + Q\right]x = 0$$

$$\left[\dot{P} + PA + A'P - PbR^{-1}b'P + Q\right]x = 0$$

In order that this identity holds for all x, the term between brackets must vanish.

In this way, we arrive at the **Riccati differential equation:**

$$-\dot{P} = PA + A'P - PbR^{-1}b'P + Q$$

$$P(T) = 0 \qquad \text{(why?)}$$

Linear Quadratic (LQ) Problem

Given a system with linear dynamics

$$\dot{x}(t) = Ax(t) + bu(t) \qquad x(0) = x_0 \qquad u(t) \in R^m$$

The control that minimizes the quadratic cost over an infinite horizon

$$J = \frac{1}{2}\int_0^T \left[x'(t)Qx(t) + u'Ru\right]dt \qquad Q = Q' \geq 0 \qquad R = R' > 0$$

Is given by the state feedback with time varying gain:

$$u(t) = -K(t)x(t) \qquad K(t) = R^{-1}B'P(t)$$

Where $P(t)$ is a symmetric positive definite matrix that satisfies the Riccati differential equation

$$-\dot{P} = PA + A'P - PbR^{-1}b'P + Q \qquad P(T) = 0$$

Example (LQ Control of a 1st order system)

Consider the 1st order, open loop unstable system

$$\dot{x}(t) = x(t) + u(t) \qquad x(0) = 1$$

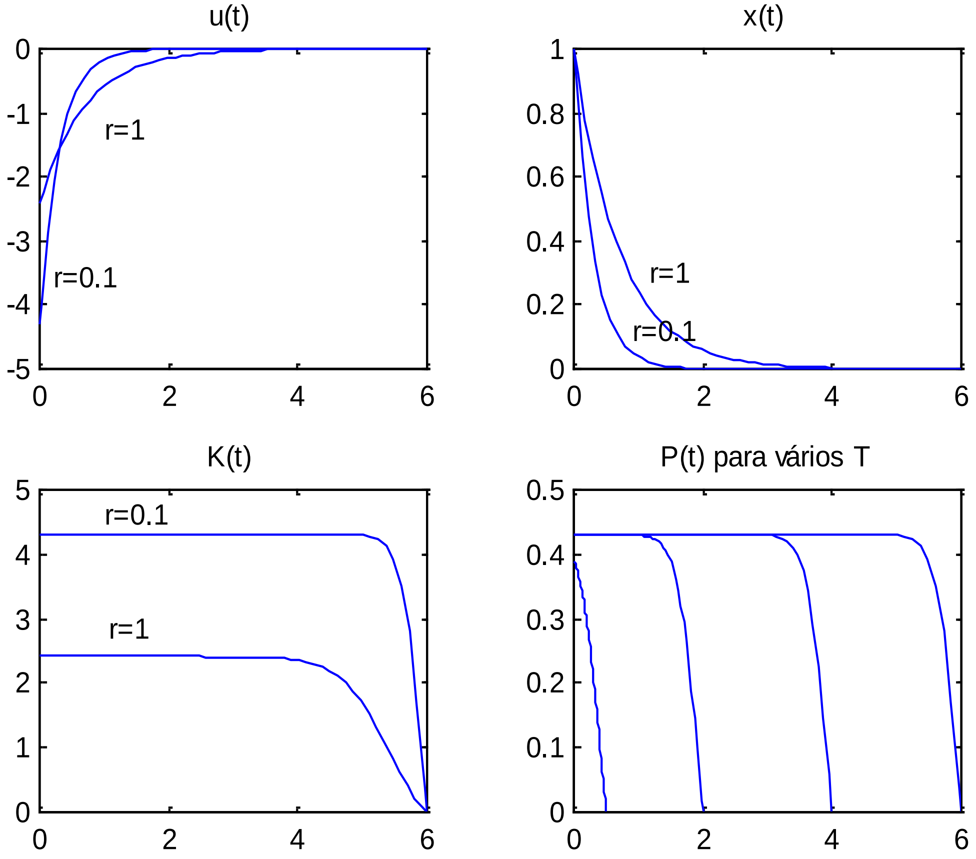
Find the control law that minimizes

$$J(u) = \frac{1}{2}\int_0^T \left[x^2(t) + ru^2(t)\right]dt \qquad T > 0, \quad r > 0$$

The solution is given by

$$\dot{p}(t) = -2p(t) + \frac{1}{r}p^2(t) - 1 \qquad p(T) = 0$$

$$u(t) = -K(t)x(t) \qquad K(t) = \frac{1}{r}p(t)$$


u(t)
r=1
r=0.1
x(t)
r=1
r=0.1
K(t)
r=0.1
r=1
P(t) para vários T

When the weight in the control action, r, decreases:

- The closed-loop becomes faster
- The controller gain increases

Increasing the horizon, T, the solution of the Riccati equation is initially a constant and there is a transient close to the end of T.

This suggests that, when $T \to \infty$ the solution of the Riccati equation becomes connsatnt for all times and the optimal control is a constant feedback of the state.

The previous example suggests the consideration of the problem that consists in minimizing a cost over an infinite horizon

$$J_{LQ\infty} = \int_0^{\infty} \left[x'(t)Qx(t) + u'(t)Ru(t) \right] dt$$

The solution is given by the constant state feedback ciontrol law

$$u(t) = -Kx(t) \qquad K = R^{-1}B'P$$

where P is the solution of the **algebraic Riccati equation**, given by

$$PA + A'P - PbR^{-1}b'P + Q = 0$$

If the system

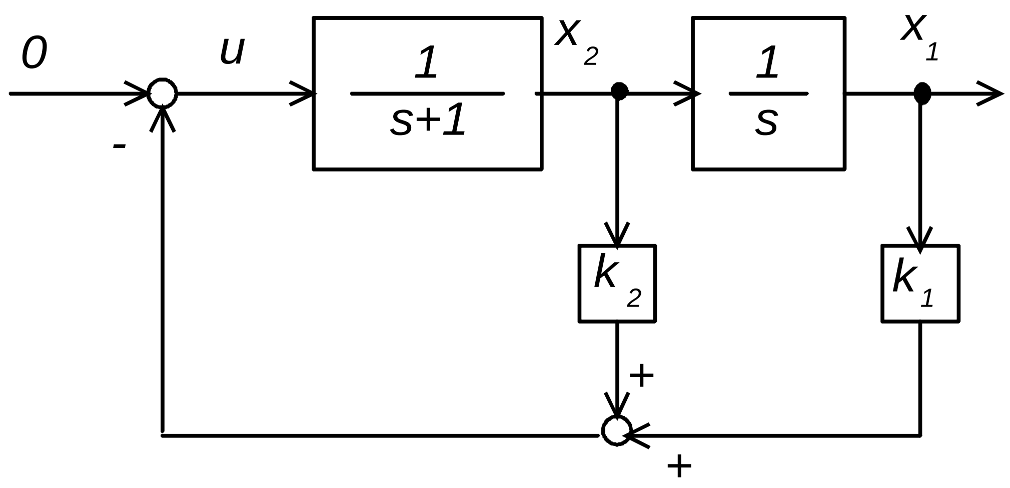
$$\dot{x}(t) = Ax(t) + bu(t)$$

Is stabilizable, *i. e.*, if there is a vector if gains F such that the closed-loop system

$$\dot{x}(t) = \left(A - bF\right)x(t)$$

Is stable, then the solution of the algebraic Riccati equation is positive semidefinite (at least) and corresponds to the limit of the solution of the Riccati differential equation when T increases.

Problem: Given the system defined by the block diagram



find the values of k_1 and k_2 that minimize

$$J = \int_0^{\infty} \left[x' Q x(t) + u' R u(t) \right] dt \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad R = 1$$

State model of the open-loop system

$$X_1(s) = \frac{1}{s} X_2(s) \quad \text{and hence} \quad \dot{x}_1(t) = x(2)$$

$$X_2(s) = \frac{1}{s+1} U(s) \quad \text{or} \quad sX_2(s) = -X_2(s) + U(s) \quad \text{and hence} \quad \dot{x}_2(t) = -x_2(t) + u(t)$$

The open-loop state model is thus

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

In this case, the algebraic Riccati equation

$$PA + A'P - PBR^{-1}C'P + Q = 0$$

becomes

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & p_{11} - p_{12} \\ 0 & p_{12} - p_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ p_{11} - p_{12} & p_{12} - p_{22} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & p_{11} - p_{12} \\ 0 & p_{12} - p_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ p_{11} - p_{12} & p_{12} - p_{22} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12} p_{22} \\ p_{12} p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the entries of the matrices in both members yields:

$$p_{12}^2 = 1$$

$$p_{11} - p_{12} - p_{12} p_{22} = 0$$

$$2(p_{12} - p_{22}) - p_{22}^2 + 0.1 = 0$$

The equation $p_{12}^2 = 1$ is verified by $p_{12} = \pm 1$. However, only the positive root leads to a positive definite matrix P. Therefore, $p_{12} = 1$.

$$p_{11} - p_{12} - p_{12} p_{22} = 0$$

$$2(p_{12} - p_{22}) - p_{22}^2 + 0.1 = 0$$

Being $p_{12} = 1$, these equations become

$$p_{11} - p_{22} = 1$$

$$p_{22}^2 + 2 p_{22} - 1.9 = 0$$

The 2nd equation has roots $-1 \pm \sqrt{2.9}$. Again, only the positive root leads to a positive definite P. Thus:

$$P = \begin{bmatrix} 1.7 & 1 \\ 1 & 0.7 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.7 & 1 \\ 1 & 0.7 \end{bmatrix}$$

The vector of optimal gains is given by

$$K = R^{-1} B' P$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1.7 & 1 \\ 1 & 0.7 \end{bmatrix} = \begin{bmatrix} 1 & 0.7 \end{bmatrix}$$

The optimal LQ control law is therefore

$$u(t) = -\left(x_1 + 0.76 x_2 \right)$$

This computation may also be performed with MATLAB (Control Systems Toolbox) using the function *lqr* (continuous time) or *dlqr* (discrete time).

Output quadratic regulation with infinite horizon

Model:

$$\dot{x}(t) = Ax(t) + bu(t) \qquad\qquad y(t) = Cx(t)$$

Cost functional

$$J_\infty = \int_0^\infty \left[y^2(t) + \rho u^2(t) \right] dt$$

Since

$$y^2(t) = x'(t) C' C x(t)$$

This problem reduces to the previous one by selecting Q as

$$Q = C' C$$

The solution of the problem that consists of minimizing

$$J_\infty = \int_0^\infty \left[y^2(t) + \rho u^2(t) \right] dt$$

where the system is modelled by

$$\dot{x}(t) = Ax(t) + bu(t) \qquad\qquad y(t) = Cx(t)$$

Is given by

$$u(t) = -Kx(t) \qquad K = R^{-1}B'P$$

where P is the unique positive definite solution of the algebraic Riccati equation

$$PA + A'P - \frac{1}{\rho} Pbb'P + C'C = 0$$

In relation to this controil law, we have the following theorem:

If the pair (A, B) is stabilizable, and the pair (A, C) is observable, the positive definite solution of the algebraic Rioccati equation exists and is unique, and the closed loop system is asymptotically stable.

The pair (*A*,*C*) is observable if

$$car\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \qquad n = \dim(x)$$

Definition

A matrix P is positive definite if

$$x'Px > 0 \qquad \forall x \neq 0$$

Is said to be positive semidefinite if

$$x'Px \geq 0 \qquad \forall x \neq 0$$

Problem: *What is the place of the closed-loop poles that corresponds to minimize J_∞ (for SISO systems)?*

Answer [Chang/Letov]: The poles of the optimal closed-loop system (with $\boldsymbol{T} = \infty$) are the $\boldsymbol{n}$ stable roots of the degree $\mathbf{2}\boldsymbol{n}$ polynomial $\boldsymbol{\Delta}(\boldsymbol{s})$

$$\Delta(s) = a(s)a(-s) + \frac{1}{\rho}b(s)b(-s)$$

where

$$b(s) = C\ adj(sI - A)B$$

Open-loop zeros

$$a(s) = det(\,sI - A)$$

Open-loop poles

$$\Delta(s) = a(s)a(-s) + \frac{1}{\rho} b(s)b(-s)$$

If $s = s_1$ is a root of $\Delta(s)$, then:

$$\Delta(s_1) = a(s_1)a(-s_1) + \frac{1}{\rho} b(s_1)b(-s_1) = 0$$

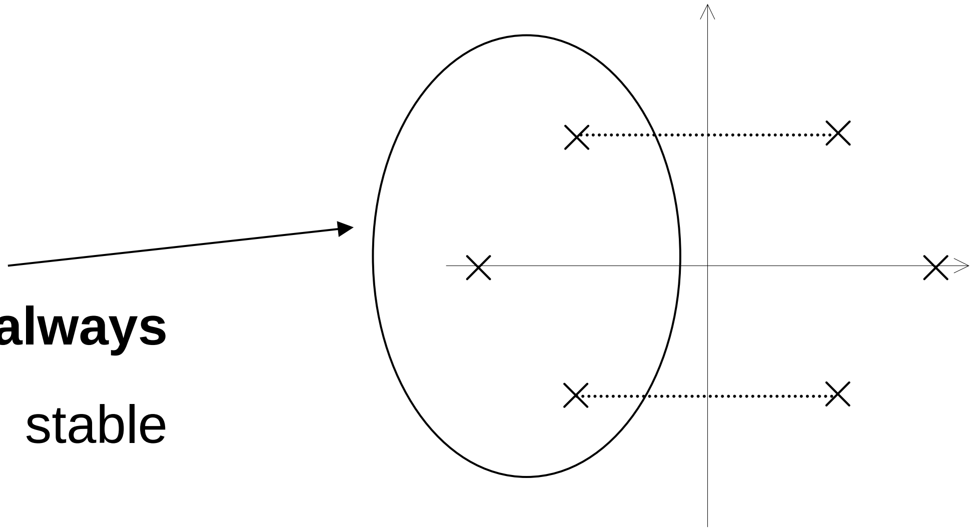
Hence, for $s = -s_1$:

$$\Delta(-s_1) = a(-s_1)a(s_1) + \frac{1}{\rho} b(-s_1)b(s_1) = 0$$

Meaning that if $s = s_1$ is a root of $\Delta(s)$, then $s = -s_1$ is also a root.

The roots of $\Delta(s)$ are symmetric with respect to the imaginary axis.

We can **always** select n stable poles



Since the poles of the controlled system are given by the roots of $\Delta(s)$ on the left-hand plane, then the system controlled with the LQ law with an infinite horizon is asymptotically stable.

Solution of the LQ ($T = \infty$) problem by pole placement

The solution of the infinite horizon LQ problem may be done as follows:

1. Compute the polynomial

$$\Delta(s) = a(s)a(-s) + \frac{1}{\rho} b(s)b(-s)$$

2. Compute the $n = \partial a(s)$ roots of $\Delta(s)$ on the left semiplane.

3. Compute the vector of controller gains such that the closed loop system has the poles coincident with the roots found in step 2.

Example

Given the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

find the state feedback control law that minimizes

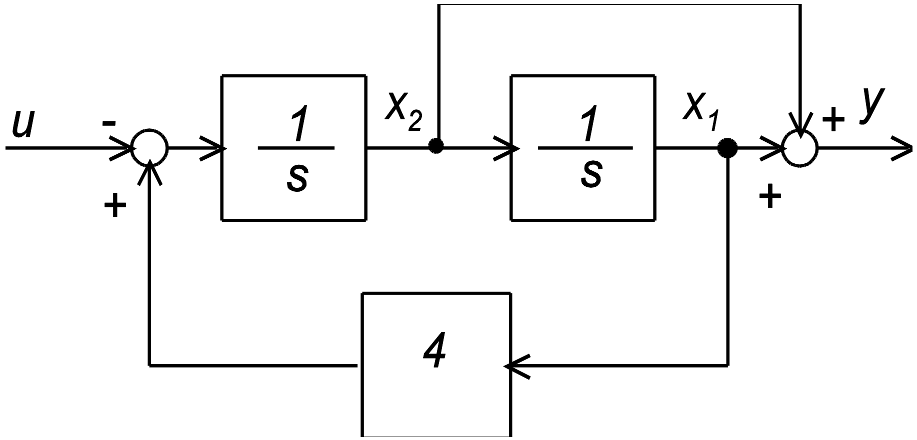
$$J_\infty = \int_0^\infty \left[y^2(t) + \rho u^2(t) \right] dt \qquad \rho = 10$$

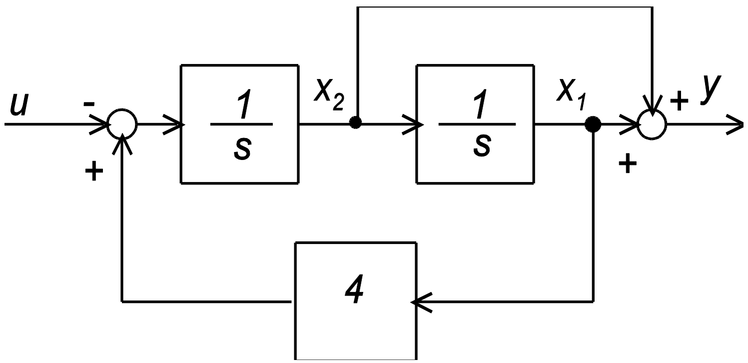
State equations

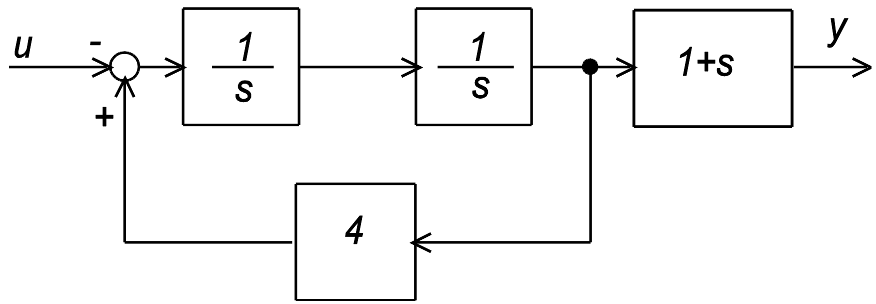
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 4x_1 - u$$

Equivalent block diagram







$$Y = \frac{-\frac{1}{s^2}}{1-\frac{4}{s^2}}(1+s)U \qquad\qquad Y = -\frac{1+s}{s^2-4}U$$

$$b(s) = -(1+s)$$

$$a(s) = s^2 - 4$$

The optimal poles are the stable roots of

$$\Delta(s) = a(s)a(-s) + \frac{1}{\rho} b(s)b(-s)$$

$$a(s) = s^2 - 4 \qquad b(s) = -(1 + s)$$

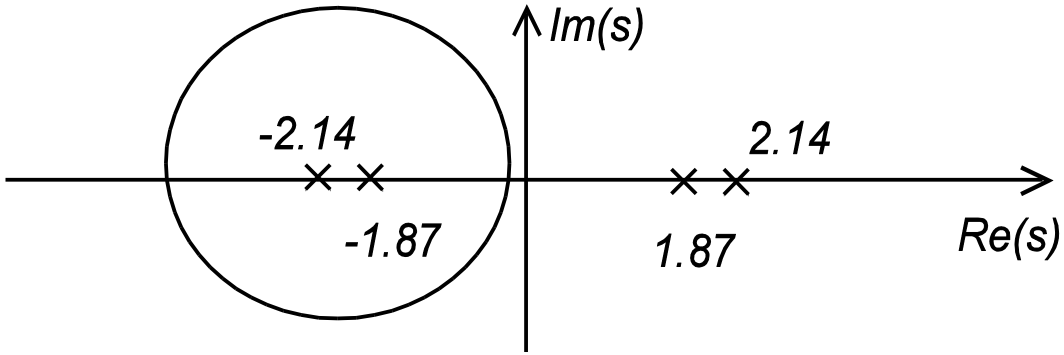
$$\Delta(s) = (s^2 - 4)^2 + \frac{1}{\rho}(1 + s)(1 - s) \quad = 1 - s^2$$

Change of variable

$$z = s^2 \qquad (z - 4)^2 + \frac{1}{\rho}(1 - z) = 0$$

$$z^2 - 8.1z + 16.1 = 0 \qquad z_1 = 4.6 \qquad z_2 = 3.5$$

$$s_1 = 2.14 \qquad s_2 = -2.14 \qquad s_3 = 1.87 \qquad s_4 = -1.87$$



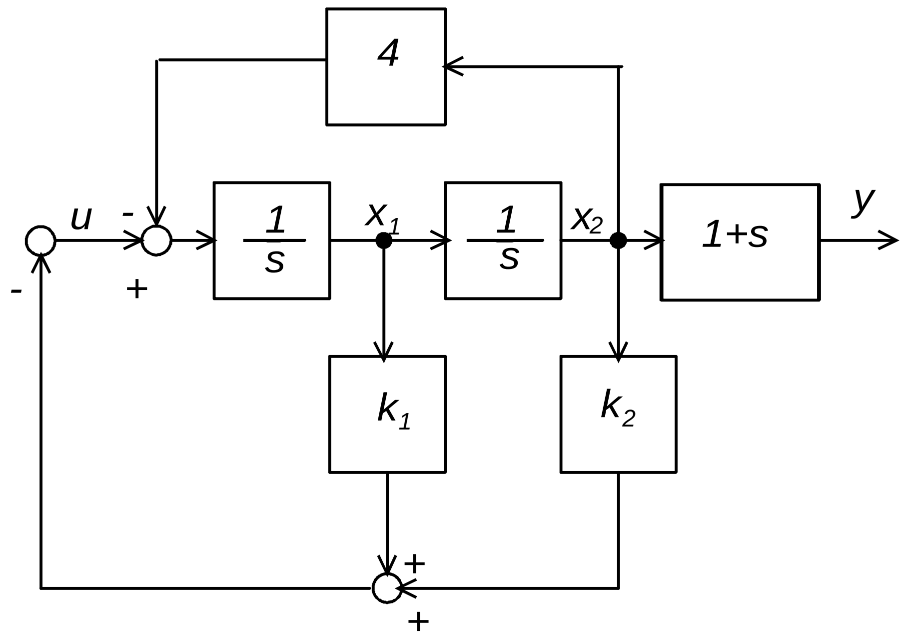
Estes são os pólos do sistema em cadeia fechada com o controlador óptimo

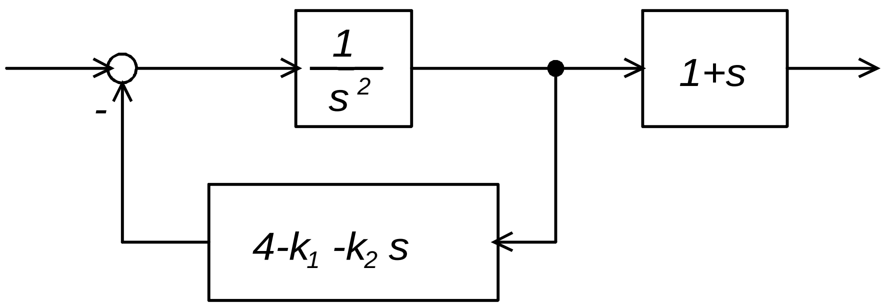
The optimal gain vector is computed such that the closed-loop poles are –2.14 and –1.87

The desired closed-loop polynomial is thus

$$\alpha(s) = (s + 2.14)(s + 1.87) = s^2 + 4.01s + 4$$

Block diagram of the closed-loop system with generic state feedback:





Closed-loop characteristic equation

$$1-\frac{1}{s^2}\left(4-k_1-k_2 s\right)=0$$

Closed-loop characteristic polynomial

$$\alpha_K(s)=s^2+k_2 s+k_1-4$$

Compare with the desired characteristic polynomial

$$\alpha(s)=s^2+4.01s+4$$

The optimal gain are obtained:

$$k_1^{opt}=8 \quad k_2^{opt}=4.01$$

Root square locus

The optimal closed-loop poles are the stable roots of

$$a(s)a(-s) + \frac{1}{\rho} b(s)b(-s) = 0$$

This equation may be written as

$$\frac{1}{\rho} \cdot \frac{b(s)b(-s)}{a(s)a(-s)} = -1$$

What happens to the roots of this equation when ρ varies?

$$a(s)a(-s)+\frac{1}{\rho}b(s)b(-s)=0$$

For ρ very big, the equation becomes approximatively

$$a(s)a(-s)=0$$

Thus, for ρ very big, the optimal poles are either the open loop poles if they are stable, or their symmetric if they are not.

$$a(s)a(-s) + \frac{1}{\rho} b(s)b(-s) = 0$$

What happens for ρ very little?

Root square locus **- example**

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The corresponding transfer function is

$$\frac{b(s)}{a(s)} = \frac{s+1}{s^2 - 0.25}$$

The root square locus is