# Information and Communication Theory: Exam

November 16, 2022

Name: ____________________________________________ Number: ________

**Duration:** 120 minutes. Part I scores: correct answer = 2/3 point; wrong answer = –1/3 points.

**Potentially useful facts**: $\log_2 3 \simeq 1.585$; $\log_{10}(2) \simeq 0.30$; $\log_a b = (\log_c b)/(\log_c a)$.

## Part I

1. Let $X, Y \in \{A\heartsuit, 2\heartsuit, ..., K\spadesuit\}$ be the two random variables representing the outcome of randomly drawing two cards from a deck of 52 cards (without replacement). Then,

   **a)** $H(X,Y) < 2\log(52)$ bits/symbol; ■
   **b)** $H(X,Y) = 2\log(52)$ bits/symbol; □
   **c)** $H(X,Y) > 2\log(52)$ bits/symbol. □

2. Let $Z, T \in \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$ be the suits of the two cards $X, Y$ drawn as described in the previous question. Then,

   **a)** $H(Z,T) < 4$ bits/symbol; ■
   **b)** $H(Z,T) = 4$ bits/symbol;; □
   **c)** $H(Z,T) > 4$ bits/symbol. □

3. Let $X \in \{A\heartsuit, 2\heartsuit, ..., K\spadesuit\}$ be the random variable defined in question 1 and $Z \in \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$ its suit. Then,

   **a)** $I(X;Z) < 2$ bits/symbol; □
   **b)** $I(X;Z) = 2$ bits/symbol; ■
   **c)** $I(X;Z) > 2$ bits/symbol. □

4. Let $X$ be the random variable defined in question 1, $Z \in \{\heartsuit, \diamondsuit, \clubsuit, \spadesuit\}$ its suit, and $U \in \{A, 2, ..., K\}$ its value. Then,

   **a)** $I(U;Z) < 2$ bits/symbol; ■
   **b)** $I(U;Z) = 2$ bits/symbol; □
   **c)** $I(U;Z) > 2$ bits/symbol. □

5. Let $C$ be an optimal binary code for the variable $X$ defined in question 1. Its expected length,

   **a)** $L(C) < 6$ bits/symbol; ■
   **b)** $L(C) = 6$ bits/symbol; □
   **c)** $L(C) > 6$ bits/symbol. □

6. Let $C$ be an optimal **ternary** code for the variable $Z$ defined in question 2. Its expected length,

   **a)** $L(C) < 7/4$ trits/symbol; ■
   **b)** $L(C) = 7/4$ trits/symbol; □
   **c)** $L(C) > 7/4$ trits/symbol. □

7. Let $X_1, ..., X_{52} \in \{A\heartsuit, 2\heartsuit, ..., K\spadesuit\}$ be the sequence of random variables corresponding to drawing the 52 cards of a deck, one by one, without replacement. The sequence $X_t$, for $t = 1, ..., 52$,

   **a)** is a stationary first-order Markov process; □
   **b)** is a non-stationary first-order Markov process; □
   **c)** is not a Markov process. ■

8. Let $X$ be the source defined in question 1. The efficiency of the Huffman **quarternary** code for the **second-order extension** of this source is

**a)** $\rho < 1$; ■

**b)** $\rho = 1$; □

**c)** $\rho > 1$. □

9. Let $X \in [1, 2]$ be a random variable with uniform probability density function and $Y = \log_2(X)$ another random variable. Then, the differential entropy $h(Y)$ satisfies

**a)** $h(Y) < 0$; ■

**b)** $h(Y) = 0$; □

**c)** $h(Y) > 0$. □

10. Let $X \in \mathbb{R}$ be a random variable with Gaussian probability density function, with zero mean and variance $\sigma^2$, and $Y$ another random variable given by $Y = 0$, if $X \leq 0$, and $Y = 1$, if $X > 0$. Then,

**a)** $I(X;Y) < 1$ bit/symbol; □

**b)** $I(X;Y) = 1$ bit/symbol; ■

**c)** $I(X;Y) > 1$ bit/symbol. □

11. Let $Y \in [-2, 2]$ be a random variable with uniform probability density function and $T, Z$ two other random variables defined as: $T = 0$, if $Y \leq 0$, and $T = 1$, if $Y > 0$; $Z = 0$, if $|Y| \leq 1$, and $Z = 1$, if $|Y| > 1$. Then,

**a)** $I(Y;T,Z) < 2$ bit/symbol; □

**b)** $I(Y;T,Z) = 2$ bit/symbol; ■

**c)** $I(Y;T,Z) > 2$ bit/symbol. □

12. Let $X_1$and $X_2$ be two independent Gaussian random variables, with variances $\sigma_1^2$ and $\sigma_2^2$. Let $U = X_1 + X_2$ be their sum. Then,

**a)** $h(U) < h(X_1)$, for any values of $\sigma_1^2$ and $\sigma_2^2$; □

**b)** $h(U) > h(X_1)$, for any values of $\sigma_1^2$ and $\sigma_2^2$; ■

**c)** none of the previous answers. □

13. Consider the so-called Z-channel, with the following matrix: $\begin{bmatrix} 1 & 0 \\ \beta & 1-\beta \end{bmatrix}$, with $\beta \in [0, 1]$. The capacity of this channel

**a)** is a monotonically decreasing function of $\beta$; ■

**b)** is a monotonically increasing function of $\beta$; □

**c)** none of the previous answers. □

14. Consider the channel with the following matrix: $\begin{bmatrix} 1-\varepsilon & 0 & \varepsilon \\ 0 & 1 & 0 \end{bmatrix}$. The capacity of this channel

**a)** is a monotonically decreasing function of $\varepsilon$; □

**b)** is a monotonically increasing function of $\varepsilon$; □

**c)** none of the previous answers. ■

15. Consider the channel given in question 13, with $\beta > 0$. Which of the two following words of a Hamming$(7, 4)$ code, $x_1 = 0000000$ and $x_2 = 1110000$, when transmitted through this channel, has the highest probability of being wrongly decoded?

**a)** $x_1$; □

**b)** $x_2$; ■

**c)** it depends on $\beta$. □

# Part II

### Problem 1

**a)** Consider a discrete memoryless source $X \in \{1, 2, 3, 4\}$ with probability mass function $\mathbf{f}_X = (1, 1, 2, 4)/8$. Consider $Y_1 = f(X)$, where $f$ is the function given by $f(1) = f(2) = 0$, $f(3) = f(4) = 1$, and $Y_2 = g(X)$, where $g(1) = g(3) = 0$, $g(2) = g(4) = 1$.

Determine $H(X)$, $H(Y_1)$, $H(Y_2)$, $H(Y_1, Y_2)$, $I(X; Y_1)$, $I(X; Y_2)$, and $I(X; Y_1, Y_2)$.

**Solution:**

$$H(X) = \frac{1}{4}\log_2 8 + \frac{1}{4}\log_2 4 + \frac{1}{2}\log_2 2 = \frac{7}{4} \text{ bits/symbol.}$$

$$\mathbb{P}[Y_1 = 0] = \frac{2}{8} = \frac{1}{4}, \text{ logo, } H(Y_1) = \frac{1}{4}\log_2 4 + \frac{3}{4}\log_2 \frac{4}{3} = 2 - \frac{3}{4}\log_2 3 \simeq 0.811 \text{ bits/symbol.}$$

$$\mathbb{P}[Y_2 = 0] = \frac{3}{8}, \text{ logo, } H(Y_2) = \frac{3}{8}\log_2 \frac{8}{3} + \frac{5}{8}\log_2 \frac{8}{5} = 3 - \frac{3}{8}\log_2 3 - \frac{5}{8}\log_2 5 \simeq 0.954 \text{ bits/symbol.}$$

$(Y_1, Y_2)$ is an injective function of $X$, thus $H(Y_1, Y_2) = H(X)$.

$I(X; Y_1) = H(Y_1) - H(Y_1|X) = H(Y_1)$, because $Y_1$ is a function of $X$.

$I(X; Y_2) = H(Y_2) - H(Y_2|X) = H(Y_2)$, because $Y_2$ is a function of $X$.

$I(X; Y_1, Y_2) = H(Y_1, Y_2) - H(Y_1, Y_2|X) = H(Y_1, Y_2)$, because $(Y_1, Y_2)$ is a function of $X$.

**b)** For the source $X$ defined in a), find an optimal **ternary** code and the corresponding expected length.

**Solution:** Possible optimal code: $C(1) = 20$, $C(2) = 21$, $C(3) = 1$, $C(4) = 0$. Expected length:

$$L(C) = \frac{2*1 + 2*1 + 1*2 + 1*4}{8} = \frac{10}{8} = \frac{5}{4} \text{ trits/symbol.}$$

**c)** Consider the so-called noisy binary erasure channel, with matrix

$$\begin{bmatrix} (1-\alpha)(1-\varepsilon) & \alpha(1-\varepsilon) & \varepsilon \\ \alpha(1-\varepsilon) & (1-\alpha)(1-\varepsilon) & \varepsilon \end{bmatrix}$$

Determine the capacity of this channel for $\varepsilon = 0$, for $\varepsilon = 1$, and for $\alpha = 1/2$.

**Solution:**

For $\varepsilon = 0$, the channel matrix is

$$\begin{bmatrix} (1-\alpha) & \alpha & 0 \\ \alpha & (1-\alpha) & 0 \end{bmatrix},$$

which corresponds to a binary symmetric channel, thus $C = 1 + \alpha\log_2\alpha + (1-\alpha)\log_2(1-\alpha)$ bits/transmission.

For $\varepsilon = 1$, the channel matrix is

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

which corresponds to a binary erasure channel with probability of erasure equal to 1, thus $C = 0$.

For $\alpha = 1/2$, the channel matrix is

$$\begin{bmatrix} (1-\varepsilon)/2 & (1-\varepsilon)/2 & \varepsilon \\ (1-\varepsilon)/2 & (1-\varepsilon)/2 & \varepsilon \end{bmatrix};$$

since the two rows of the matrix are equal, the output is independent of the input, thus the capacity is $C = 0$.

**Problem 2**

**a)** Consider a variable $X \in [0, 1]$ with uniform probability density function and another variable $Y \in \{1, ..., N\}$, which corresponds to the uniform discretization of $X$, with $N$ intervals, that is, $Y = i \Leftrightarrow X \in [(i-1)/N,\, i/N[$. Determine the differential entropy $h(X)$ and the entropy $H(Y)$.

**Solution:**

Differential entropy of a uniform density: $h(X) = \log_2(1) = 0$.

Variable $Y$ has probability $1/N$ to be in each of the $N$ intervals, thus $H(X) = \log_2 N$ bits/symbol.

**b)** Determine the mutual information $I(X; Y)$.

**Solution:** $I(X; Y) = H(Y) - H(Y|X) = H(Y) = \log_2 N$, since $Y$ is a function of $X$, thus $H(Y|X) = 0$.

**c)** Consider a variable $X \in [0, 1]$ with triangular probability density function, $f_X(x) = 2 - 2x$, and another variable $Y \in \{1, 2\}$ given by $Y = 1$, if $X \leq a$, and $Y = 2$, if $X > a$. For what value of $a$ is the mutual information $I(X; Y)$ maximal?

**Solution:** $I(X; Y) = H(Y) - H(Y|X) = H(Y)$, since since $Y$ is a function of $X$, thus $H(Y|X) = 0$.

This entropy (thus the mutual information) is maximized for $\mathbb{P}[Y = 1] = \mathbb{P}[Y = 2] = 1/2$.

$\mathbb{P}[Y = 2] = \int_a^1 (2-2x)\, dx$ is just the area of a triangle of base $1-a$ and height $2-2a$, thus the condition becomes

$$\frac{(1-a)(2-2a)}{2} = \frac{1}{2} \Leftrightarrow 2\,a^2 - 4\,a + 1 = 0 \Leftrightarrow a = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{1}{\sqrt{2}}.$$

Of the two solutions, only the smallest is in the interval [0, 1], thus the final solution is $1 - \frac{1}{\sqrt{2}} \simeq 0.293$.

**d)** Consider a channel with input $U \in \{0, 1\}$ and output $Z = U + X$, where $X$ is the random variable defined in question c). What is the capacity of this channel?

**Solution:** conditioned on $X = 0$, the density of $Y$ has support [0, 1]. Conditioned on $X = 1$, the density of $Y$ has support [1, 2]. Thus, knowing $Y$ implies knowing $X$, thus $H(X|Y) = 0$ and

$$C = \max_{f_X} I(X; Y) = \max_{f_X} H(X) - H(X|Y) = \max_{f_X} H(X) = 1 \text{ bits/transmission.}$$