Information and Communication Theory: Third Mini-Test

November 3, 2022

Name: ____ Number: _________

Duration: 45 minutes. Part I scores: correct answer = 3/2 point; wrong answer = –3/4 points.

Useful facts: $\log_a b = (\log_c b)/(\log_c a)$; $\log_2 3 \simeq 1.585$. Unless indicated otherwise, all logarithms are base-2.

Part I

1. Let $X \in [0, 1]$ be a source with probability density function $f_X(x) = 1 + \alpha(x - 1/2)$, for $\alpha \in [-2, 2]$ Then, the differential entropy $h(X)$

   a) is a monotonically decreasing function of α; ☐
   b) s a monotonically increasing function of α; ☐
   c) none of the previous answers. ■

2. Let $Y \in [0, 2]$ be a source with uniform probability density function, and $Z = Y^2/4$. Then,

   a) $h(Z) < h(Y)$; ■
   b) $h(Z) = h(Y)$; ☐
   c) $h(Z) > h(Y)$. ☐

3. Let $C(\beta)$ denote the capacity of a *binary symmetric channel* (BSC) with probability of error $\beta \in [0, 1]$. Then,

   a) $C(\beta/2) < C(\beta)$; ☐
   b) $C(\beta/2) > C(\beta)$ ☐
   c) none of the previous answers. ■

4. Let $C(\varepsilon)$ denote the capacity of an *binary erasure channel* (BSC) with probability of erasure $\varepsilon \in [0, 1]$. Then,

   a) $C(\varepsilon/2) < C(\varepsilon)$; ☐
   b) $C(\varepsilon/2) > C(\varepsilon)$ ■
   c) none of the previous answers. ■

   Explanation: For $\epsilon = 0 \in [0, 1]$, the correct answer is (c), since $C(0) = C(0)$. Since it is understandable that some people assumed this was not being considered, option (b) is also classified as correct, as it is for $\epsilon \in]0, 1]$.

5. Consider a channel with input alphabet $\mathcal{X} = \{1, 2, ..., 16\}$ and output alphabet $\mathcal{Y} = \{1, 2, ..., 8\}$. The capacity C of this channel necessarily satisfies

   a) $C \leq 3$ bits/transmission; ■
   b) $3 < C \leq 4$ bits/transmission; ☐
   c) $C > 4$ bits/transmission. ☐

6. Consider a Hamming$(7, 4)$ binary code. Which of the following is a valid codeword?

   a) 0001100; ☐
   b) 0111110; ☐
   c) 0001111. ■

7. Consider a binary symmetric channel with probability of error 0.1 used with a Hamming$(7, 4)$ code. The probability of a correct decoding satisfies is

   a) < 0.9; ■

   b) $= 0.9$; □

   c) > 0.9. □

8. Consider a Gaussian channel with power constraint P and noise power N. If the noise power is reduced from N to $N/2$, the capacity of the channel

   a) doubles; □

   b) increases by 50%; □

   c) none of the previous answers. ■

Part II

1. Consider the channel resulting from connecting the output of a binary symmetric channel with probability of error α with an erasure channel with probability of erasure ε.

   a) Draw the channel graph and write the channel matrix of the resulting channel.
   Solution:

$$\mathbf{P} = \begin{bmatrix} (1-\alpha)(1-\epsilon) & \epsilon & \alpha(1-\epsilon) \\ \alpha(1-\epsilon) & \epsilon & (1-\alpha)(1-\epsilon) \end{bmatrix}$$

   b) What is the capacity of the channel (as a function of ε) for $\alpha = 0$?
   Solution: For $\alpha = 0$, we simply have a Binary Erasure Channel (BEC), with capacity given by:

$$C(\epsilon) = 1 - \epsilon$$

   c) What is the capacity of the channel (as a function of α) for $\varepsilon = 0$?
   Solution: For $\epsilon = 0$, we simply have a Binary Symmetric Channel (BSC), with capacity given by:

$$C(\alpha) = 1 - H(\alpha, 1-\alpha)$$

   d) For $\varepsilon = 0.3$, is there any choice of α such that the capacity of the channel is 0.8 bits/transmission? Justify your answer.
   Solution: Let C_1 denote the capacity of the BSC, and C_2 the capacity of the BEC. For $\epsilon = 0.3$, we have $C_2 = 1 - 0.3 = 0.7$. Let C denote the capacity of the channel, which is the series connection of two channels, we know that:

$$C \leq min\{C_1, C_2\} \leq 0.7$$

   Therefore, there is no choice of α such that the capacity would be 0.8.