DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

# Introduction to Stochastic Processes

**LEIC-A, LMAC, MECD, MEIC-A, MMA**

2nd. Semester – 2020/2021
2021/06/18 – 8AM
**Test 2**

Duration: 90 minutes

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

**Group 1 — Renewal Processes** 2.5 points

**1.** The number of inspections to an industrial site is governed by a delayed renewal process, $\{N_D(t): t \geq 0\}$, whose: i) renewal function is equal to $m_D(t) = \frac{\lambda t}{2} + \frac{1-e^{-\lambda t}}{2}$, $t \geq 0$; ii) inter-renewal times $X_i$, $i \in \mathbb{N}\backslash\{1\}$, are exponentially distributed with parameter $\frac{\lambda}{2}$.

(a) Derive the distribution of the inter-renewal time $X_1$ of the stochastic process $\{N_D(t): t \geq 0\}$. (2.0)

- **Delayed renewal process**
  $\{N_D(t): t \geq 0\}$
- **Inter-renewal times**
  $X_1 \sim G$
  $X_i \overset{i.i.d.}{\sim} X \sim F \sim \text{exponential}(\lambda/2), \quad i \in \mathbb{N}\backslash\{1\}$
- **LST of** $F$

$$\begin{aligned}\tilde{F}(s) &= \int_{0^-}^{+\infty} e^{-st}\, dF(t)\\ &= M_{exponential(\lambda/2)}(-s)\\ &\overset{\text{form.}}{=} \frac{\lambda/2}{\lambda/2+s}\\ &= \frac{\lambda}{\lambda+2s}\end{aligned}$$

- **LST of the renewal function**

$$\begin{aligned}\tilde{m}_D(s) &= \int_{0^-}^{+\infty} e^{-st}\, dm_D(t)\\ &= \int_{0^-}^{+\infty} e^{-st} \times \frac{\lambda}{2}\left(1+e^{-\lambda t}\right) dt\\ &= \frac{\lambda}{2} \times \left(LT[1,s] + LT[e^{-\lambda t}, s]\right)\\ &\overset{\text{form.}}{=} \frac{\lambda}{2} \times \left(\frac{1}{s} + \frac{1}{\lambda+s}\right)\\ &= \frac{\lambda(\lambda+2s)}{2s(\lambda+s)}\end{aligned}$$

- **Deriving the inter-renewal distribution** $G$
  Since $\tilde{m}_D(s) \overset{form.}{=} \frac{\tilde{G}(s)}{1-\tilde{F}(s)}$, the LST of the c.d.f. of the first inter-renewal time $(X_1)$ can be obtained as follows:

$$\begin{aligned}\tilde{G}(s) &= \int_{0^-}^{+\infty} e^{-sx}\, dG(x)\\ &= E\left(e^{-sX_1}\right)\\ &= \tilde{m}_D(s) \times [1-\tilde{F}(s)]\\ &= \frac{\lambda(\lambda+2s)}{2s(\lambda+s)} \times \left[1 - \frac{\lambda}{\lambda+2s}\right]\\ &= \frac{\lambda}{\lambda+s}\\ &\equiv M_{exponential(\lambda)}(-s),\end{aligned}$$

  i.e, $X_1 \sim \text{exponential}(\lambda)$.



(b) Show that the renewal function $m_D(t)$ verifies the elementary renewal theorem. (0.5)

- **Verification of the ERT**

$$\begin{aligned}\lim_{t\to+\infty} \frac{m_D(t)}{t} &= \lim_{t\to+\infty} \frac{\frac{\lambda t}{2} + \frac{1-e^{-\lambda t}}{2}}{t}\\ &= \lim_{t\to+\infty} \left(\frac{\lambda}{2} + \frac{1-e^{-\lambda t}}{2t}\right)\\ &= \frac{\lambda}{2}\\ &= \frac{1}{E(X_2)},\end{aligned}$$

  hence verifying the ERT for delayed renewal processes.

**Group 2 — Discrete time Markov chains** 9.5 points

**1.** A four state Bonus Malus automobile insurance system, for a policyholder whose yearly number of claims is a Poisson r.v. with mean $\frac{1}{2}$, is governed by a DTMC $\{X_n : n \in \mathbb{N}\}$ associated with the TPM

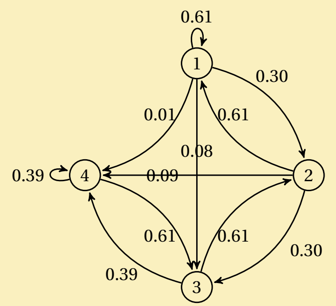
$$\mathbf{P} = \begin{bmatrix} 0.61 & 0.30 & 0.08 & 0.01\\ 0.61 & 0 & 0.30 & 0.09\\ 0 & 0.61 & 0 & 0.39\\ 0 & 0 & 0.61 & 0.39 \end{bmatrix}.$$

(a) Draw the associated transition diagram and classify the states of this DTMC. Are the states periodic? (1.5)

- **DTMC**
  $\{X_n : n \in \mathbb{N}\}$
  $X_n$ = state of the policyholder at period $n$
- **State space**
  $\mathcal{S} = \{1,2,3,4\}$
- **TPM**

$$\mathbf{P} = \begin{bmatrix} 0.61 & 0.30 & 0.08 & 0.01\\ 0.61 & 0 & 0.30 & 0.09\\ 0 & 0.61 & 0 & 0.39\\ 0 & 0 & 0.61 & 0.39 \end{bmatrix}$$

- **Transition diagram**




- **Classification of the states of the DTMC**
  – Judging by the transition diagram, all states communicate with one another, thus $\mathcal{S} = \{1,2,3,4\}$ is a single closed communicating class. Hence the DTMC has a finite state space and is irreducible. Consequently, all states are positive recurrent.

– The transition diagram leads to the conclusion that we can transition from state 1 to state 1 after 1,2,3,4,5,… transitions, thus $d(1) = gcd\{n \in \mathbb{N} : P_{11}^n > 0\} = 1$ and this state is aperiodic.
The same holds for the remaining states of this irreducible DTMC. [After all, periodicity is a class property.]

(b) Admit that the annual premiums are 200, 250, 400, and 600, for states 1, 2, 3, and 4 (respectively). Find the average annual premium paid by the policyholder (in the long-run).[1] (2.5)

- **Stationary distribution**
Since the DTMC is irreducible, positive recurrent and aperiodic we can add that the limit probabilities coincide with the unique stationary distribution, $\underline{\pi} = [\pi_j]_{j\in\mathcal{S}}$, given by

$$\underline{\pi} = \underline{1} \times (\mathbf{I} - \mathbf{P} + \mathbf{ONE})^{-1},$$

where: $\underline{1} = [1 \ \cdots \ 1]$ is a row vector with $\#\mathcal{S}$ ones; $\mathbf{I}$ = identity matrix with rank $\#\mathcal{S}$; **ONE** is the $\#\mathcal{S} \times \#\mathcal{S}$ matrix with all entries equal to one. Thus,

$$\begin{aligned}
\underline{\pi} &= \underline{1} \times \left( \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix} - \begin{bmatrix} 0.61&0.30&0.08&0.01\\0.61&0&0.30&0.09\\0&0.61&0&0.39\\0&0&0.61&0.39 \end{bmatrix} + \begin{bmatrix} 1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1 \end{bmatrix} \right)^{-1} \\
&= \underline{1} \times \begin{bmatrix} 1.39&0.7&0.92&0.99\\0.39&2&0.7&0.91\\1&0.39&2&0.61\\1&1&0.39&1.61 \end{bmatrix}^{-1} \\
&\simeq [1 \ 1 \ 1 \ 1] \times \begin{bmatrix} 1.767&-0.100&-0.623&-0.794\\0.415&0.675&-0.327&-0.513\\-0.595&0.029&0.751&0.065\\-1.211&-0.364&0.408&1.417 \end{bmatrix} \\
&= [0.376 \ \ 0.240 \ \ 0.209 \ \ 0.175].
\end{aligned}$$

- **Vector of costs**

$$\underline{c} = [c(j)]_{j\in\mathcal{S}} = \begin{bmatrix} 200\\250\\400\\600 \end{bmatrix}$$

- **Long-run average annual premium**

$$\begin{aligned}
\underline{\pi} \times \underline{c} &= \sum_{j\in\mathcal{S}} \pi_j \times c(j) \\
&= \underline{\pi} \times \underline{c} \\
&\simeq 0.376 \times 200 + 0.240 \times 250 + 0.209 \times 400 + 0.175 \times 600 \\
&= 323.8.
\end{aligned}$$

(c) What is the probability that the policyholder reaches state 4 before state 3, given that $X_1 = 1$? (2.0)
**Note**: You may have to consider states 3 and 4 absorbing, identify sub-stochastic matrices **Q** and **R** and calculate $(\mathbf{I} - \mathbf{Q})^{-1} \times \mathbf{R}$.

- **Important**
To calculate the requested probability, we have to consider another DTMC, whose states 3 and 4 are absorbing and whose associated TPM is

$$\mathbf{P}^\star = \begin{bmatrix} 0.61&0.30&0.08&0.01\\0.61&0&0.30&0.09\\0&0&1&0\\0&0&0&1 \end{bmatrix}.$$

The substochastic matrices governing the transitions between the transient states ($T = \{1,2\}$) of this DTMC and the transitions from the transient to the absorbing states ($\overline{T} = \{3,4\}$) are

$$\mathbf{Q} = \begin{bmatrix} 0.61&0.30\\0.61&0 \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} 0.08&0.01\\0.30&0.09 \end{bmatrix}$$

(respectively).

- **Requested probability**
Keeping in mind that

$$\begin{bmatrix} a&b\\c&d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d&-b\\-c&a \end{bmatrix},$$

we get

$$\begin{aligned}
\mathbf{U} &= [u_{ik}]_{i\in T, k\in\overline{T}} \\
&= [P(\text{reach absorbing state } k \mid X_1 = i)]_{i\in T, k\in\overline{T}} \\
&\overset{form.}{=} (\mathbf{I} - \mathbf{Q})^{-1} \times \mathbf{R} \\
&= \left( \begin{bmatrix} 1&0\\0&1 \end{bmatrix} - \begin{bmatrix} 0.61&0.30\\0.61&0 \end{bmatrix} \right)^{-1} \times \mathbf{R} \\
&= \begin{bmatrix} 0.39&-0.3\\-0.61&1 \end{bmatrix}^{-1} \times \mathbf{R} \\
&= \frac{1}{0.39 \times 1 - (-0.3) \times (-0.61)} \begin{bmatrix} 1&0.3\\0.61&0.39 \end{bmatrix} \times \mathbf{R} \\
&= \frac{1}{0.207} \begin{bmatrix} 1&0.3\\0.61&0.39 \end{bmatrix} \times \begin{bmatrix} 0.08&0.01\\0.30&0.09 \end{bmatrix} \\
&= \frac{1}{0.207} \begin{bmatrix} 0.17&0.037\\0.1658&0.0412 \end{bmatrix} \\
&\simeq \begin{bmatrix} 0.821256&0.178744\\0.800966&0.199034 \end{bmatrix}.
\end{aligned}$$

Hence the probability that the policyholder reaches state 4 before state 3, given $X_1 = 1$, is equal to

$$u_{14} = P(X_\tau = 4 \mid X_1 = 1) \simeq 0.178744.$$

[Alternatively, $u_{14} = (\text{1st row of } (\mathbf{I}-\mathbf{Q})^{-1}) \times (\text{2nd column of } \mathbf{R}) = \frac{1}{0.207}(1 \times 0.01 + 0.30 \times 0.09) = \frac{0.037}{0.207} \simeq 0.178744$.]

**2.** Let $\{X_n : n \in \mathbb{N}_0\}$ be a branching process, where $X_n$ represents the size of generation $n$. Admit that $X_0 = 1$ and the number of offspring per individual has p.g.f. given by

$$P(s) = \frac{1}{6} + \frac{1}{2}s + \frac{1}{3}s^3, \quad s \in [0,1].$$

(a) Determine the expected value of the total number of individuals that existed from generation 0 through generation 10. (1.0)

[1] The following result useful: $\begin{bmatrix} 1.39&0.7&0.92&0.99\\0.39&2&0.7&0.91\\1&0.39&2&0.61\\1&1&0.39&1.61 \end{bmatrix}^{-1} \simeq \begin{bmatrix} 1.767&-0.1&-0.623&-0.794\\0.415&0.675&-0.327&-0.513\\-0.595&0.029&0.751&0.065\\-1.211&-0.364&0.408&1.417 \end{bmatrix}.$

- **Branching process**

  $\{X_n : n \in \mathbb{N}_0\}$

  $X_n$ = size of generation $n$ a colony of cells

  $X_0 = 1$

  $X_n = \sum_{l=1}^{X_{n-1}} Z_l,\ n \in \mathbb{N}$

- **Number of offspring per individual and its p.g.f.**

  $Z_l \equiv Z_{l,n}$ = number of offspring of the $l^{th}$ individual of generation $n$

  $Z_l \overset{i.i.d.}{\sim} Z,\ l \in \mathbb{N}$

  $P(s) = E(s^Z) = \sum_j s^j \times P(Z=j) = 0.2 + 0.2\,s + 0.6\,s^3, \quad s \in [0,1]$

- **Expected number of offspring per individual**

$$\begin{aligned}
\mu &= E(Z) \\
&\overset{\text{form.}}{=} \left.\frac{dP(s)}{ds}\right|_{s=1} \\
&= \left.\frac{d\left(\frac{1}{6}+\frac{1}{2}s+\frac{1}{3}s^3\right)}{ds}\right|_{s=1} \\
&= \left.\left(\frac{1}{2}+s^2\right)\right|_{s=1} \\
&= 1.5.
\end{aligned}$$

- **Requested expected number of offspring per individual**

  Since $E(X_n \mid X_0 = 1) \overset{form.}{=} \mu^n$, the expected value of the total number of individuals that existed from generation 0 through generation 10 is given by

$$\begin{aligned}
E\left(\sum_{n=0}^{10} X_n \mid X_0 = 1\right) &= \sum_{n=0}^{10} E(X_n \mid X_0 = 1) \\
&= \sum_{n=0}^{10} \mu^n \\
&= \frac{1-\mu^{11}}{1-\mu} \\
&= \frac{1-1.5^{11}}{1-1.5} \\
&\simeq 170.995.
\end{aligned}$$

(b) Calculate the extinction probability. (1.0)

**Note**: $2s^3 - 3s + 1 = (s-1)(2s^2+2s-1)$.

- **Probability of extinction**

  Since $E(Z) > 1$, the probability of extinction, $\pi \overset{\text{form.}}{=} \lim_{n\to+\infty} P(X_n = 0 \mid X_0 = 1)$, is the smallest positive number satisfying

$$\begin{aligned}
s &\overset{form.}{=} \sum_{j=0}^{+\infty} s^j \times P_j \\
&= P(s) \\
s &= \frac{1}{6} + \frac{1}{2}s + \frac{1}{3}s^3 \\
6s &= 1 + 3s + 2s^3 \\
2s^3 - 3s + 1 &= 0 \\
(s-1)(2s^2+2s-1) &= 0 \\
s &= \frac{-2 \pm \sqrt{2^2 - 4 \times 2 \times (-1)}}{2 \times 2} \quad \text{or} \quad s = 1 \\
s &= \frac{-1 \pm \sqrt{3}}{2} \quad \text{or} \quad s = 1
\end{aligned}$$



thus $\pi = \frac{\sqrt{3}-1}{2} \simeq 0.366025$.

(c) Obtain the probability that the population is extinct by generation 3 but not by generation 2. (1.5)

- **Requested probability**

$$\begin{aligned}
P(X_3 = 0, X_2 > 0 \mid X_0 = 1) &= P(\{X_3 = 0\} \backslash \{X_2 = 0\} \mid X_0 = 1) \\
&= P(X_3 = 0 \mid X_0 = 1) - P(X_3 = 0, X_2 = 0 \mid X_0 = 1) \\
&= P(X_3 = 0 \mid X_0 = 1) - P(X_2 = 0 \mid X_0 = 1) \qquad [\pi_3 - \pi_2] \\
&= \left.\sum_{j=0}^{+\infty} s^j \times P(X_3 = j \mid X_0 = 1)\right|_{s=0} - \left.\sum_{j=0}^{+\infty} s^j \times P(X_2 = j \mid X_0 = 1)\right|_{s=0} \\
&= P_3(0) - P_2(0) \\
&\overset{form.}{=} P(P[P(0)]) - P[P(0)],
\end{aligned}$$

where

$$\begin{aligned}
P(0) &= \frac{1}{6} + \frac{1}{2} \times 0 + \frac{1}{3} \times 0^3 \\
&= \frac{1}{6} \\
P[P(0)] &= P(1/6) \\
&= \frac{1}{6} + \frac{1}{2} \times \frac{1}{6} + \frac{1}{3} \times \left(\frac{1}{6}\right)^3 \\
&= \frac{163}{648} \\
&\simeq 0.251543 \\
P(P[P(0)]) &= P(163/648) \\
&= \frac{1}{6} + \frac{1}{2} \times \frac{163}{648} + \frac{1}{3} \times \left(\frac{163}{648}\right)^3 \\
&= \frac{243046171}{816293376} \\
&\simeq 0.297744.
\end{aligned}$$

Consequently,

$$\begin{aligned}
P(X_3 = 0, X_2 > 0 \mid X_0 = 1) &= \frac{243046171}{816293376} - \frac{163}{648} \\
&= \frac{37713115}{816293376} \\
&\simeq 0.046200.
\end{aligned}$$

| **Group 3 — Continuous time Markov chains** | 8.0 points |
|---|---|

**1.** Let $\{X(t) : t \geq 0\}$ be a CTMC with state space $\mathscr{S} = \{0,1\}$, initial state $X(0) = 0$, and transition probability matrix given by

$$\mathbf{P}(t) = \begin{bmatrix} \frac{\mu + \lambda e^{-(\lambda+\mu)t}}{\lambda+\mu} & \frac{\lambda - \lambda e^{-(\lambda+\mu)t}}{\lambda+\mu} \\ \frac{\mu - \mu e^{-(\lambda+\mu)t}}{\lambda+\mu} & \frac{\lambda + \mu e^{-(\lambda+\mu)t}}{\lambda+\mu} \end{bmatrix},$$

where $\lambda, \mu > 0$.

(a) Determine $P[X(1) = 1]$. (1.0)

- **CTMC**

  $\{X(t) : t \geq 0\}$

- **State space and initial distribution of the CTMC**

  $\mathscr{S} = \{0,1\}$

$\underline{\alpha} = [P[X(0)=i]]_{i \in \mathscr{S}} = [1 \quad 0]$

- **TPM**
  See text above.

- **Requested probabilities**
  Since $[P[X(1)=i]]_{i\in\mathscr{S}} = \underline{\alpha} \times \mathbf{P}(1)$

$$\begin{aligned} P[X(1)=1] &= \underline{\alpha} \times \text{2nd. column of } \mathbf{P}(1) \\ &= [1 \quad 0] \times \begin{bmatrix} \frac{\lambda - \lambda e^{-(\lambda+\mu)}}{\lambda+\mu} \\ \frac{\lambda + \mu e^{-(\lambda+\mu)}}{\lambda+\mu} \end{bmatrix} \\ &= \frac{\lambda - \lambda\, e^{-(\lambda+\mu)}}{\lambda+\mu}. \end{aligned}$$

(b) Derive the infinitesimal generator $\mathbf{R}$ of this CTMC and draw the associated rate diagram. (1.5)

- **Instantaneous transition rates**
  By applying the L'Hôpital rule, we get $q_{ij} = \lim_{h\to 0^+} \frac{P_{ij}(h)}{h}$, $i \neq j$:

$$\begin{aligned} q_{01} &= \lim_{h\to 0^+} \frac{P_{01}(h)}{h} \\ &= \lim_{h\to 0^+} \frac{\frac{\lambda - \lambda e^{-(\lambda+\mu)h}}{\lambda+\mu}}{h} \\ &= \lim_{h\to 0^+} \frac{\frac{\lambda\times(\lambda+\mu)\times e^{-(\lambda+\mu)h}}{\lambda+\mu}}{1} \\ &= \lambda; \\ q_{10} &= \lim_{h\to 0^+} \frac{P_{10}(h)}{h} \\ &= \lim_{h\to 0^+} \frac{\frac{\mu - \mu e^{-(\lambda+\mu)h}}{\lambda+\mu}}{h} \\ &= \lim_{h\to 0^+} \frac{\frac{\mu\times(\lambda+\mu)\times e^{-(\lambda+\mu)h}}{\lambda+\mu}}{1} \\ &= \mu. \end{aligned}$$

- **Infinitesimal generator**
  This matrix has entries

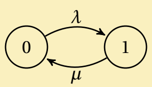
$$r_{ij} = \begin{cases} q_{ij}, & i \neq j \\ -\nu_i = -\sum_{m\in\mathscr{S}} q_{im}, & j = i \end{cases}$$

  and in this case $\mathbf{R} = [r_{ij}]_{i,j\in\mathscr{S}}$ is equal to

$$\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}.$$

  [Alternatively, we could find $\quad \mathbf{R} : \frac{d\,P(t)}{dt} = \mathbf{P}(t) \times \mathbf{R} = \mathbf{R} \times \mathbf{P}(t)$.]

- **Rate diagram**
  [Recall that the rate diagram of a CTMC is a directed graph — with no loops — in which each state is represented by a node and there is an arc going from node $i$ to node $j$ (if $q_{ij} > 0$) with $q_{ij}$ written on it. These rates coincide with the birth and death rates...]






(c) Obtain the equilibrium probabilities $P_j = \lim_{t\to+\infty} P[X(t)=j \mid X(0)=0]$, for $j \in \mathscr{S}$. (1.0)

- **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P[X(t)=j \mid X(0)=0]$

$$\begin{aligned} P_0 &= \lim_{t\to+\infty} P[X(t)=0 \mid X(0)=0] \\ &= \lim_{t\to+\infty} P_{00}(t) \\ &= \lim_{t\to+\infty} \left[\frac{\mu + \lambda\, e^{-(\lambda+\mu)\,t}}{\lambda+\mu}\right] \\ &= \frac{\mu}{\lambda+\mu} \\ P_1 &= 1 - P_0 \\ &= \frac{\lambda}{\lambda+\mu}. \end{aligned}$$

**2.** There are three ophthalmologists on duty at all times in the eye clinic of a large hospital. Admit that: patients arrive to this clinic according to a Poisson process with rate equal to 6 patients per hour; patients are taken by the ophthalmologists on a first-come, first-served basis; the consultation time is found to be exponentially distributed with an average of 20 minutes.

The hospital planners are interested in knowing the answers of the following questions.

(a) What is the fraction of time that there is at least one idle ophthalmologist, in the long-run? (1.5)
If this fraction exceeds 0.5 then the hospital planners recommend to transfer one of the ophthalmologists to another hospital. Is this the case?

- **Birth and death queueing system**
  $M/M/m$
  $\lambda = 6, \quad \mu = 3, \quad m = 3$

- **Birth and death rates**
  $\lambda_j = \lambda = 6, \quad j \in \mathbb{N}_0$

$$\mu_j = \begin{cases} j\,\mu = 3\,j, & j = 1,2 \\ m\,\mu = 9, & j = 3,4,\ldots \end{cases}$$

- **Traffic intensity; existence of limit probabilities**
  $\rho = \frac{\lambda}{m\mu} = \frac{6}{3\times 3} = \frac{2}{3} < 1 \qquad \checkmark$

- **Performance measure**
  $L_s$ = no. of patients an arriving patient sees in the eye clinic in the long-run

- **Requested probability**
  There is at least one idle ophthalmologist if $L_s = 0,\ldots,m-1$, thus the requested probability is:

$$\begin{aligned} P(L_s < m) &= 1 - P(L_s \geq m) \\ &\overset{form.}{=} 1 - C(m, m\rho) \\ &= 1 - \frac{\frac{(m\rho)^m}{m!(1-\rho)}}{\sum_{n=0}^{m-1} \frac{(m\rho)^n}{n!} + \frac{(m\rho)^m}{m!(1-\rho)}} \\ &= 1 - \frac{\frac{2^3}{3!(1-2/3)}}{1 + 2 + \frac{2^2}{2} + \frac{2^3}{3!(1-2/3)}} \\ &= \frac{5}{9} \qquad [= 0.55(5)]. \end{aligned}$$

  [Alternatively, we could have calculated the following sum: $P(L_s = 0) + P(L_s = 1) + P(L_s = 2)$, where the summands can be calculated by consulting the last page of the formulae (they are equal to $1/9$, $2/9$, and $2/9$).]

- **Comment**
  Since $P(L_s < m) = \frac{5}{9} > 0.5$ the hospital planners should recommend the transfer of one of the ophthalmologists to another hospital.

(b) What is the average number of people waiting for an ophthalmologist and the average amount of time a patient spends at the eye clinic, in the long-run? (1.5)

- **Performance measures**
  $L_q$ = no. of patients an arriving patient sees in the eye clinic waiting in line in the long-run

  $W_s$ = total time spent by an entering customer in the eye clinic in the long-run
- **Requested expected values**

$$
\begin{aligned}
E(L_q) &\overset{form.}{=} \frac{\rho}{1-\rho} \times C(m, m\rho) \\
&\overset{(a), form.}{\simeq} \frac{\frac{2}{3}}{1-\frac{2}{3}} \times \frac{4}{9} \\
&= \frac{8}{9} \\
E(W_s) &\overset{form.}{=} \frac{1}{\mu} + \frac{C(m, m\rho)}{m\mu(1-\rho)} \\
&\overset{(a)}{=} \frac{1}{3} + \frac{\frac{4}{9}}{3 \times 3 \times \left(1-\frac{2}{3}\right)} \\
&= \frac{13}{27} \\
&= 0.481(481).
\end{aligned}
$$

(c) What is the probability that a patient spends more than $E(W_q)$ hours waiting for an ophthalmologist? (1.5)

- **Performance measure**
  $W_q$ = time spent by an entering patient waiting for an ophthalmologist, in the long-run
- **Requested probability**
  Since

$$
\begin{aligned}
E(W_q) &= E(W_s) - E(\text{Service}) \\
&\overset{(b)}{=} \frac{13}{27} - \frac{1}{\mu} \\
&= \frac{4}{27},
\end{aligned}
$$

we get

$$
\begin{aligned}
P[W_q > t) &= 1 - F_{W_q}(t) \\
&\overset{form.}{=} C(m, m\rho) \times \left[1 - F_{exponential(m\mu(1-\rho))}(t)\right] \\
&= C(m, m\rho) \times e^{-m\mu(1-\rho)\,t} \\
&\overset{(a),\, t=E(W_q)}{=} \frac{4}{9} \times e^{-3\times3\times(1-2/3)\times4/27} \\
&= \frac{4}{9} \times e^{-4/9} \\
&\simeq 0.284969.
\end{aligned}
$$