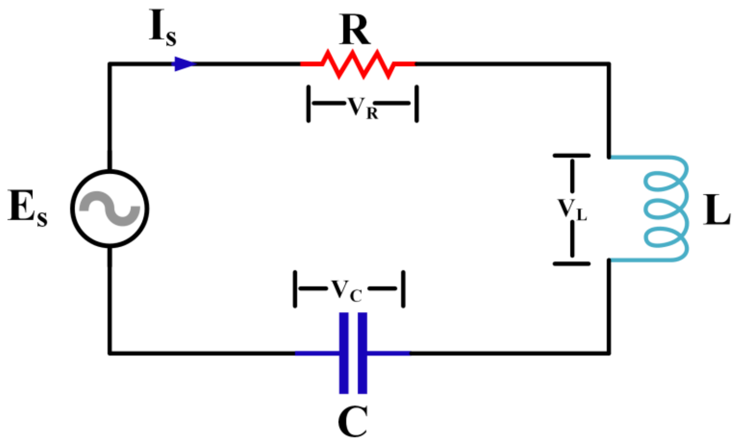
AC Electrical Circuits for non-Electrical Engineers



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FOREWORD

This text is intended to provide basic AC electrical circuits knowledge to students that do not hold a background on Electrical Engineering. The main target are the foreigner students that are enrolled in Técnico's non-Electrical Engineering courses.

The inspiring source is the text "Noções de Eletrotecnia" published in 1992 by IST's late full professor Domingos Moura. In some aspects, the reference book "Redes de Energia Elétrica" from IST's emeritus professor José Sucena Paiva has been consulted too.

Certainly, the topics are not approached with the same theoretical groundings as when they are presented in the Electrical Engineering course. Actually, we are concerned in just introducing the basic operating principles, so that a non-electrical engineer is able to understand them, and providing the main tools to deal with the typical electrical related problems that an engineer may have to face in his/her job. All in all, the text slightly approaches the main electrical concerned topics and by no means introduces them with the accurate required scientific deepness.

This document is organized in 6 chapters. In the first one, an overview of the power system is offered, approaching the main organizing blocks: generation, transmission, distribution, retailing and consumption. Then, the AC circuits topic is introduced in the second chapter, pointing out the need for complex numbers to ease the analysis, and presenting the consequent complex amplitude (phasor) concept. The power in AC circuits (active, reactive, complex) is approached in Chapter 3. Chapter 4 is concerned with balanced three-phase circuits, where, for instance, phase and line voltages are introduced. The per unit system, which facilitates the power system analysis, is the topic of the fifth chapter. Finally, the last chapter presents the induction motor as an application case of AC electrical circuits. To illustrate the methods, there are several numeric examples, spread all over the text.

This is a draft version, as so, the figures have been picked up from the internet. The proper reference source is supplied in the caption of the figures. Usually, more information or a different approach of the same concept, can be found in the provided reference.

LIST OF ACRONYMS

CCGT – Combined Cycle Gas Turbine.

CHG – GreenHouse Gas.

CHP – Combined Heat and Power.

DSO – Distribution System Operator.

EHV – Extremely High Voltage.

emf – electromotive force.

HV – High Voltage.

HVAC – Heating, Ventilation and Air Conditioning.

KCL – Kirchoff Current Law.

KVL – Kirchoff Current Law.

LV – Low Voltage.

MIBEL –Iberian Electricity Market.

MV – Medium Voltage.

NZEB – Near Zero Energy Buildings.

PHS – Pumped Hydroelectricity Storage.

PV – Photovoltaic.

RES – Renewable Energy Source.

TSO – Transmission System Operator.

WTG – Wind Turbine Generator.

INDEX

1 THE POWER SYSTEM

1.1 GENERAL CONCEPT

The power system is a very complex system, which is designed with the main objective of delivering electricity to the consumers. The electricity, or electrical energy, is produced[1] in power plants, which are usually located far from the places where the consumers are concentrated. As so, it is necessary to transport the energy from the places where it is produced to the locations where it is consumed. Electricity transportation requires a physical infrastructure[2], composed namely by overhead power lines and underground cables. This infrastructure includes further the electrical devices that allow the power transmission to be performed with reduced losses; these devices are the transformers. Finally, the electricity is delivered to the customers with the appropriate power quality.

1.1.1 Fundamental quantities

In order to observe, study and understand the operation of the power system, two fundamental quantities are used: the current and the voltage. The electric current is an organized flow of electrons in a material, usually, a metal. In the metals, there are electrons that can get out of the electron cloud structure and move across the material, therefore, giving rise to an electric current. To keep the electrons' flow, it is necessary to continuously supply energy, because the electrons loose energy in the collisions with the material structure. This is why the wires get hot when there is a current flowing across them. The voltage is somehow (*lato sensu*) a measure of the energy that should be supplied to

[1] Actually, the electricity is not produced, nor generated, but instead it is converted (transformed) from other forms of energy. For simplicity, we will use the words "production" (or "generation") and "consumption", but we should keep in mind that we are referring to the word "conversion" or "transformation".

[2] It is technically possible to have wireless electricity transmission, using advanced technologies, as direct induction or resonant magnetic induction. However, at the present time, there are still no commercial high power applications, mainly due to the low efficiency of the process: only a small parcel of the transmitted power is received by the electrical recipient.

keep the current, i.e. to keep the electrons moving on. There is a third quantity, named power, that is related to the product of voltage and current. Power is the availability to produce electricity. When we refer to the intended full-load sustained output of a facility, such as a power plant, we use the term capacity.

To represent the voltage, the letters *V* or *U* are used. Here we will use letter *V*. Volt (V) is the International System of Units (SI) for voltage. Current is represented by *I* and is measured in Ampère (A). As for power, the representative letter is *S* and the SI unit is Volt-Ampère (VA)[3].

1.1.2 Requirements

As mentioned before, the power system operation is a very complex task, for several reasons, one of the main ones being the necessity of assuring that a fundamental condition is met: power production must be equal, at each time instant, to power consumption plus transmission losses, at that very same instant. The accomplishment of this condition is essential, because electricity characteristics do not make it appropriate to be stored.

Actually, electricity can be stored, using for this purpose the well-known batteries. However, with the currently economical technology, the energy that be stored into batteries is minimal, as compared with the electricity movements associated to the consumption at a country-level. As it is not possible to store electricity in significant amounts, the mentioned condition must be assured at every time instant.

In spite of advanced load[4] forecasting techniques being currently used, it is virtually impossible to avoid a mismatch between production and consumption. Therefore, sophisticated control systems are required to regulate the power production in the plants holding that feature, as well as to manage the power flow in the interconnections.

Besides this fundamental requirement, which is related to the overall operation of the power system, there are other second order requirements, but nevertheless important:

- Electricity should be supplied at every place where it is required.

[3] The multiples are: kilo (k) = 10^3; Mega (M) = 10^6; Giga (G) = 10^9; Tera (T) = 10^{12} and the submultiples are mili (m) = 10^{-3}; micro (μ) = 10^{-6}; nano (n) = 10^{-9}; pico (p) = 10^{-12}.

[4] Load, or demand, are words used to refer to electricity consumption.

- Electricity should meet quality criteria: constant frequency; voltage under control, sinusoidal waveform[5], high reliability.
- Production costs must be minimal.
- Environmental impact must be low.

1.1.3 Structure and components

Figure 1-1 shows a schematic of the power system structure.

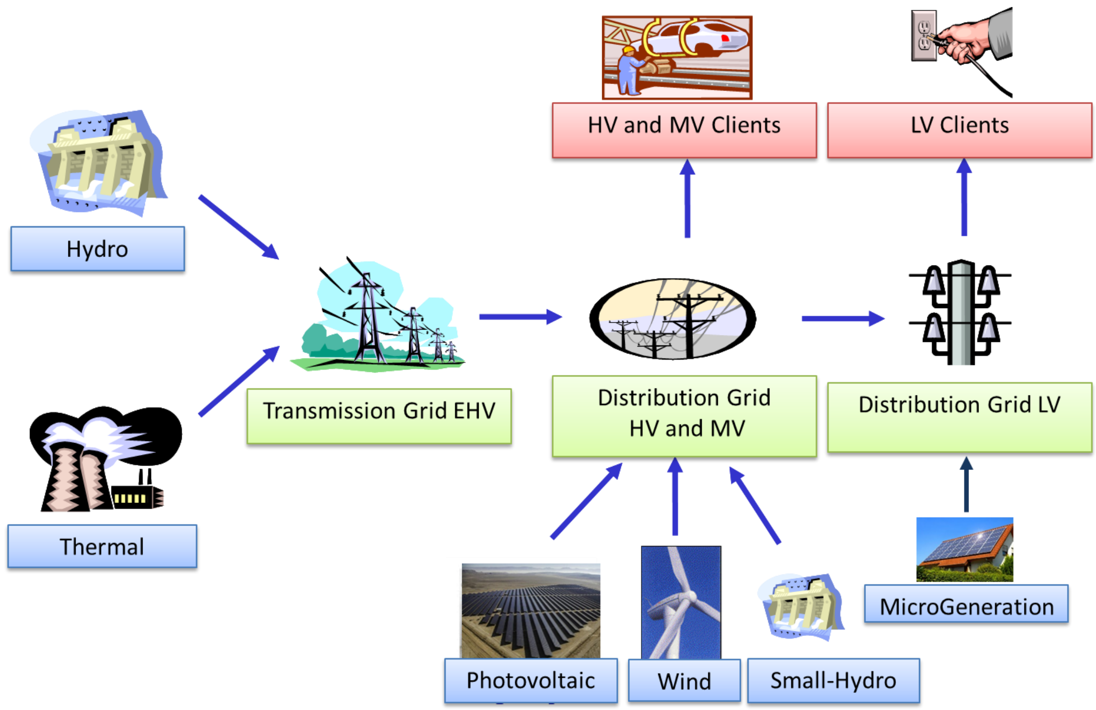

Figure 1-1: Structure of a power system. Source: Adapted from: J.P. Sucena Paiva, Redes de Energia Elétrica: Uma Análise Sistémica, IST Press, 2007.

Power is produced in the three following types of infrastructures, ranked by capacity level (from the highest to the lowest).

- Large power stations with rated capacity of hundreds or one thousand MVA. These power stations can be: (i) thermal power plants, when the thermal energy associated to a fossil fuel (coal or natural gas) combustion is converted into electricity; (ii) hydro plants, when the kinetic and potential energy associated to a river waterfall is transformed into electrical energy. The location of thermal and hydro

[5] We will see later on what these concepts mean.

power plants is concentrated in specific sites, which must hold certain characteristics: for instance, easy fuel supply and availability of refrigeration water, in the case of thermal, and a waterfall and a water flow, in the hydro case.

- Medium or Low power stations, with rated capacity of tens of MVA. These power stations can be: small-hydro power plants, wind power, which use the wind energy, photovoltaic (PV), which transform the sun irradiation. These generating units are called dispersed power production, because they are not concentrated in specific sites, instead they are distributed by several places, selected due to the prime mover abundance.
- Micro power stations, with rated capacity around several kVA. These power plants (also known as micro-generation) are located at the consumption point, that is usually a household or a small industry. Normally, these power plants are of PV type, installed on the rooftops of the buildings.

The power produced in the large thermal and hydro power plants is delivered to the transmission network, composed by Extremely High Voltage (EHV) overhead power lines. EHV is the voltage level appropriate to the transportation of large electricity amounts over long distances, because the transmission losses decrease when the voltage increases[6]. The transformers are the devices that allow to change the voltage level, so that the optimal losses are attained from both the technical and economic points of view.

Extremely High Voltages are not economically adequate to transport electricity at a regional or local level, where both the distances and the amounts of electricity involved are smaller. Therefore, the electricity is transferred to the distribution grid that operates in High Voltage (HV), Medium Voltage (MV) and Low Voltage (LV). Wind and PV parks are usually connected in HV or MV. As for the micro-generation, it is connected to the LV level.

1.2 BASIC OPTIONS

The power system was developed based on three basic options that have been decided a long time ago: the power system operates in Alternating Current (AC), at the frequency

[6] We will prove this later on in this course.

of 50 Hz[7] and is three-phase. Let us look at the meaning of these concepts and the basic options that were taken to converge in the current situation.

1.2.1 Alternating Current Versus Direct Current

In an AC system, the fundamental quantities change in time, following a sinusoidal waveform; in a Direct Current (DC) system, these quantities are constant over time.

The power system operates in Alternating Current (AC). In fact, the electrical quantities, as voltages and currents, change periodically in time as sinusoidal functions (sine or cosine waves). An example of an alternating current is depicted in Figure 1-2. It can be seen that the period is 0.02 s and the maximum value is 1 A (we recall that current is measured in A – Ampère).

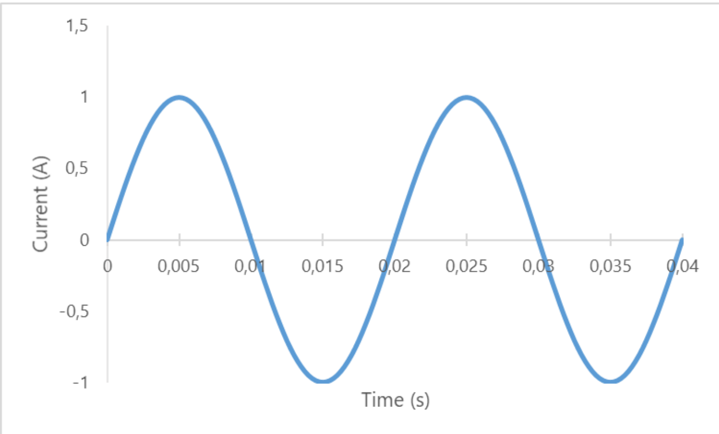

Figure 1-2: Example of an alternating current.

The current instantaneous values repeat periodically with period *T*:

$$i(t) = i(t+T) \qquad \text{equation 1.1}$$

It is well-known the relationship between period (*T*) and frequency (*f*):

$$f = \frac{1}{T} \qquad \text{equation 1.2}$$

The frequency is measured in Hertz (Hz – Hertz). As the period is 0.02 s, the frequency is 50 Hz. 1 Hz is 1 cycle per second, meaning that in one second there are 50 full-cycles or

[7] In some parts of the world, as the USA and part of Asia, 60 Hz are used, as we will see later on.

50 periods. 50 Hz is the frequency used in Europe, but, for instance, in the USA, the frequency is 60 Hz.

The angle (rad – radians), i.e. the argument of the sinusoidal function, is given by:

$$\theta = \omega t \qquad \text{equation 1.3}$$

in which, ω is the angular frequency (rad/s) and t is the time (s – second).

During one full-cycle ($t=T$), an angle of 2π is described. Therefore:

$$\omega = 2\pi f \qquad \text{equation 1.4}$$

In the example of Figure 1-2, the current as a function of time is given by (note that f = 50 Hz):

$$i(t) = I_{\text{MAX}} \sin\theta = I_{\text{MAX}} \sin(\omega t) = 1\sin(100\pi t)\ \text{A} \qquad \text{equation 1.5}$$

I_{MAX} = 1 A is the maximum value or peak-value or amplitude of current $i(t)$. The angular frequency is ω = 100π rad/s, because T = 0.02 s and, consequently, f = 50 Hz.

We use small letters to represent the time evolution of the quantities. For instance, $i(t)$ is the current time evolution.

We denote by I (capital letter) the Root Mean Square (RMS) value of current $i(t)$, i.e. the square root of the mean square (the arithmetic mean of the squares of the current instantaneous values), in a period T:

$$I = \left[\frac{1}{T}\int_0^T [i(t)]^2\,dt\right]^{\frac{1}{2}} \qquad \text{equation 1.6}$$

For an alternating electric current, RMS is equal to the value of the direct current (DC) that would produce the same average heat power dissipation in a resistive load, in a period T.

RMS values are of utmost importance. Usually, common measuring instruments, voltmeters and ammeters (from Ampère meter), measure the RMS values of the voltage and current, respectively[8]. Moreover, the RMS values are the ones that effectively contribute

[8] Note that it makes no sense to measure the average value, because it would be zero.

to the transfer of useful power between two systems. The useful power is normally represented by letter *P*.

From equation 1.6, one can write:

$$I^2 = \frac{1}{T}\int_0^T [I_{MAX}\sin(\omega t)]^2\,dt = \frac{I_{MAX}^2}{T}\int_0^T [\sin^2(\omega t)]dt \qquad \text{equation 1.7}$$

As:

$$\sin^2(\omega t) = \frac{1-\cos(2\omega t)}{2} \qquad \text{equation 1.8}$$

equation 1.7 becomes:

$$I^2 = \frac{1}{T}\frac{I_{MAX}^2}{2}T + \frac{1}{T}\frac{I_{MAX}^2}{2}\left[\frac{\sin(2\omega t)}{2\omega}\right]_0^T = \frac{I_{MAX}^2}{2} \qquad \text{equation 1.9}$$

and we conclude that:

$$I = \frac{I_{MAX}}{\sqrt{2}} \qquad \text{equation 1.10}$$

The amplitude of a sinusoidal current (or a sinusoidal voltage) is equal to square root of 2 times the RMS value. In this way, equation 1.5 can be written as:

$$i(t) = \sqrt{2}I\sin(\omega t) = \sqrt{2}\frac{1}{\sqrt{2}}\sin(100\pi t)\text{ A} \qquad \text{equation 1.11}$$

the current RMS value being:

$$I = \frac{1}{\sqrt{2}} = 0.707\text{ A} \qquad \text{equation 1.12}$$

We recall that a capital letter denotes the RMS value of a quantity. For instance, *I* is the current RMS. A graphic showing the time evolution of the alternating current that we have been using as an example with the RMS value marked is depicted in Figure 1-3.

We will see later other applications where the RMS values play a very important role in AC electrical circuits.

The main advantage to justify the AC option is that it facilitates electricity transmission. Actually, transmission losses are inversely dependent on the square of the voltage as we will now demonstrate in a simplified way.

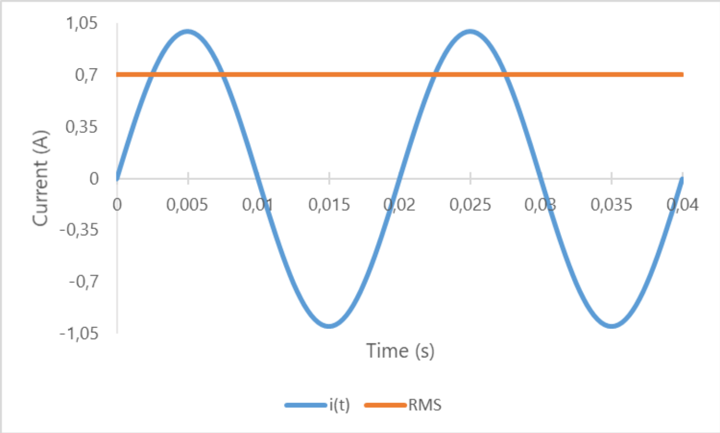

Figure 1-3: Example of an alternating current with RMS value marked.

It is known that the transmitted power P depends on the product of RMS voltage and current:

$$P = k_1 VI \qquad \text{equation 1.13}$$

Moreover, we know that power losses P_L depend on the square of the RMS current:

$$P_L = k_2 I^2 \qquad \text{equation 1.14}$$

As so, we can write:

$$P_L = \frac{k_2 P^2}{k_1^2} \frac{1}{V^2} = k \frac{1}{V^2} \qquad \text{equation 1.15}$$

This means that the voltage should be as high as necessary to keep the losses under an acceptable value. Transformers are the electrical equipment that allows for changing the voltage level, whether it is to increase (to transport in the transmission grid) or to decrease (to distribute to the customers in the distribution grid). The issue is that conventional transformers only operate in AC; they do not perform the required function in DC[9].

Another reason to justify the use of AC is that it facilitates the current interruption. Sometimes, namely when a fault in the grid occurs, it is necessary to interrupt the current to clear the fault. High currents are hard to interrupt due to the high magnetic energy that is stored in the circuits. As an alternating current is periodically equal to zero, circuit breakers take advantage of this AC characteristic to perform current interruption.

[9] Actually, there are currently DC transformers. They perform the DC voltage level change using power electronics devices. However, they are still not an economic solution for EHV and HV.

1.2.2 50 Hz Versus 60 Hz

In the countries that followed the European trends, the power system frequency of 50 Hz is used. In the USA and in the countries it technologically influences, 60 Hz is the frequency. This is an unfortunate circumstance, which is nowadays irreversible. One can ask why frequencies lower than 50 Hz or higher than 60 Hz are not used.

Frequencies lower than 50 Hz produce an inconvenient flicker in traditional lights, used in the past. Frequencies higher than 60 Hz would give rise to high losses in the magnetic circuits (losses increase with the frequency as we will see later on in this course).

1.2.3 Single-phase Versus Three-phase

The power system is three-phase, i.e. to transmit power between a source and a recipient, three wires are used; in certain conditions, a return wire – called neutral wire – is required. The advantages of using a three-phase system as compared to a single-phase one, in which one active wire and one return wire are used, can be found at the generation, transmission and use of energy levels. However, the most important advantage is at the transmission level, as we will explain bellow.

Let us suppose we want to transmit power P at the distance d. We need a wire with length $2d$, because a return path is required to close the loop. If, instead we use three active wires, the return can be made by a single common return wire, and we are able to transmit $3P$ with a $4d$ length conductor. This means that we multiply the transmitted power by 3, but the wire length is only multiplied by 2. This is a significant advantage of three-phase systems. Moreover, in some conditions, if the system is said to be balanced, the return wire can be suppressed, because it would carry a current equal to zero (we will return to this issue later on). In these circumstances, the economy is even more apparent: the conductors' length is increased by 50% ($3d$) and the transmitted power is multiplied by 3. This is the main reason why the power system is three-phase.

The power system being three-phase, two voltages can be defined: a phase-to-neutral voltage (V_{p-n}) and a phase-to-phase voltage (V_{p-p}). We will prove later that the relationship between these two voltage is:

$$V_{p-p} = \sqrt{3} V_{p-n} \qquad \text{equation 1.16}$$

1.3 Organization

As mentioned before, the power system is a highly complex system, governed by a fundamental constraint: at each instant, production must match the consumption added of the losses. In the past, the generating system was composed of centralized and controlled power plants. However, over the last years, the structure of the generating system has undergone significant changes: nowadays, distributed generation is playing an increasing important role in the modern power system. Distributed generation is composed by many low-power units, dispersed along the network, generally dependent on non-controllable renewable resources, like the sun and the wind. This structural change is motivating a paradigm shift from a consumption-driven system to a generation-driven system. Finding the best strategies to deal with the new paradigm, which seems irreversible, is a challenging task for the electrical engineers.

The current power system is usually divided into five blocks: 1) Generation; 2) Transmission; 3) Distribution; 4) Retailing; 5) Consumption.

1.3.1 Generation

Power production is a market-driven activity, open to the private initiative, but the participants must be certified. The role of the State is to create the adequate conditions for market development and to monitor the system to ensure security of supply.

Power is generated in power plants. There are several technologies used for power production.

1.3.1.1 Thermal power plants

Thermal power plants are usually divided into three main groups: coal-fired (conventional), combined cycle and nuclear.

In conventional coal-fired power plants, a fossil fuel, the coal (which has been ground into a fine powder by a pulveriser), is burned in order to obtain thermal energy (heat). The heat, which is released from the coal combustion, heats water that runs through a series of pipes, located inside the boiler, therefore making the water to change from liquid phase to steam. The high-pressure steam is carried to a steam turbine, turning the turbine's blades, so that to obtain mechanical power (power associated to the rotation of the turbine's blades). An electrical generator[10] is mounted in the same shaft where the turbine rotates, therefore converting mechanical power into electrical power[11]. Meanwhile, the low-pressure vapour is cooled down into a condenser[12], returning to the liquid phase. This water is pumped back into the boiler and the process is reinitiated. Figure 1-4 shows a schematic of a typical coal-fired power plant.

The efficiency of coal-fired power plants, i.e. the ratio between the output electricity and the input heat, is low, around 35%. Furthermore, coal is an extremely pollutant fossil fuel, significantly contributing to the greenhouse gas (GHG) emissions. From the fossil fuels used to produce electricity, coal is the most aggressive from an environmental point-of-view. For this reason, no coal-fired power plants were built in Portugal, for many years.

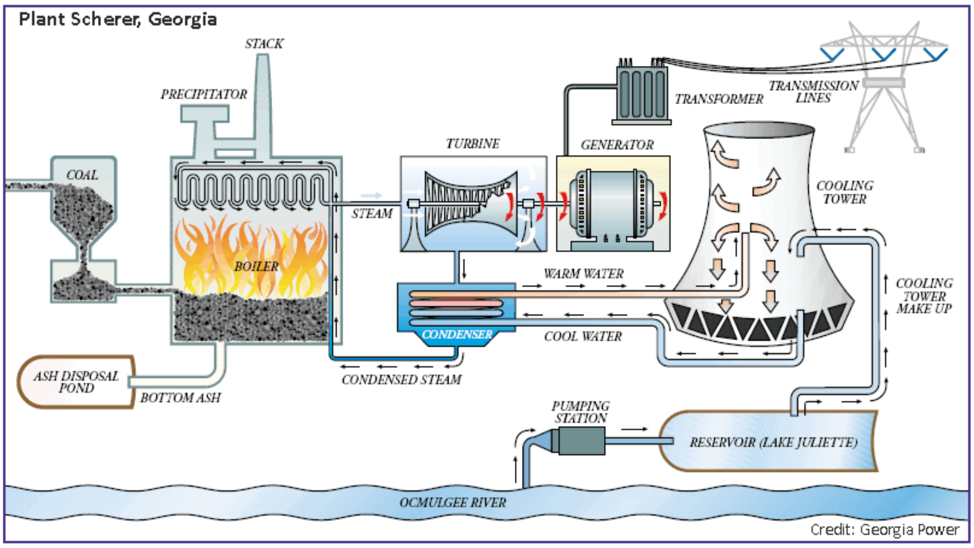

Figure 1-4: Diagram of a coal-fired power plant. Example of the 3500 MW Georgia Power's Plant Scherer that provides electricity for Georgia. Source: https://water.usgs.gov/edu/wupt-coalplant-diagram.html.

CCGT (Combined Cycle Gas Turbine) is the technology that replaced coal technology. In a CCGT power plant, atmospheric air is compressed and passed into a combustion chamber where it is burned with natural gas. The mixture of air and combustion gases is carried to a gas turbine, where it turns the turbine's blades and produces mechanical power. An

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[10] Electrical generators in thermal and hydro power plants are called alternators. This corresponds to a type of electrical generator that rotates at constant speed, related to the constant 50 Hz frequency, the so-called synchronous speed.

[11] We will see later on how this conversion is actually performed.

[12] Do not confuse condenser with capacitor. The former is a heat exchanger, in which the vapour is condensed; the latter is an electrical device able of storing electrical energy.

alternator then converts this movement energy into electricity. The efficiency of this process is low, about 30%. This means that a significant part of the input heat is not converted into electrical energy and is available to be used elsewhere. This heat available in the hot exhaust gases is transferred to a heat recovery boiler, where steam is obtained by heating water that runs in a pipeline inside the boiler. The blades of a second turbine, this time a steam turbine, are turned by the steam, therefore producing rotational movement, which drives a second alternator to produce more electricity. We highlight that this technology uses two thermodynamic cycles: the gas cycle and the steam cycle, hence the designation of combined cycle. In Figure 1-5, we can see a diagram of a CCGT power plant.

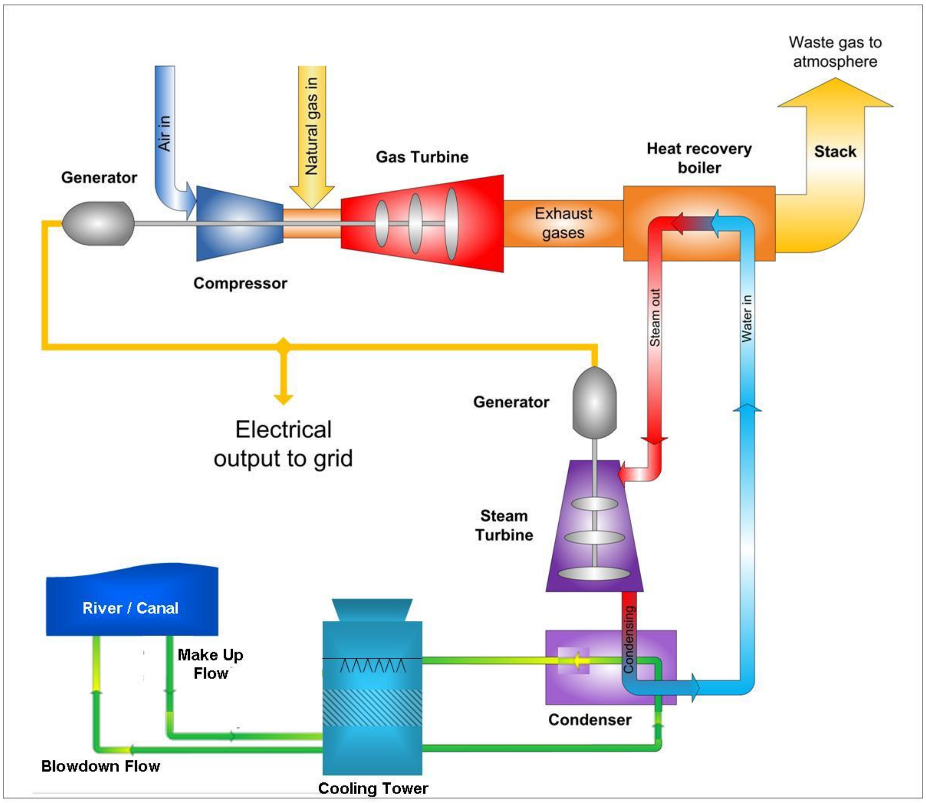

Figure 1-5: Diagram of a CCGT power plant. Source: http://www.knottingleypower.co.uk/faqs.

The efficiency, i.e. the ratio between the total output electricity (natural gas plus steam based) and the input natural gas heat (the only fossil fuel used) is now far higher, around 55 to 60%. Moreover, natural gas is less pollutant than coal, typically natural gas GHG emissions (namely CO2 emissions) are one half of the coal's[13]. This is the reason why the latest thermal power plants installed in Portugal use CCGT technology.

An extension of the combined cycle concept is the Combined Heat and Power (CHP). In these low power plants (typically some MW sized) the heat that is not converted into electricity is used in a useful application. Useful applications are, for instance, hot water for Heating, Ventilation and Air Conditioning (HVAC) in buildings, or process heat in the industry. Instead of a turbine, an internal combustion engine, usually fed by natural gas, is used as the thermal machine. An alternator then converts the mechanical power into electricity. Heat exchangers are used to recover the heat available, for instance, in the exhaust gases rejected to the stack of the engine.

A nuclear power plant is just like a coal-fired power plant with one fundamental difference. Instead of burning a fossil fuel to produce heat, the heat source in the nuclear power plant is a nuclear reactor. A nuclear reactor produces and controls the release of energy from splitting the atoms of certain elements (nuclear fission), with uranium being the dominant choice of fuel in the world today. As for the rest of the process, it is exactly the same: the heat is used to generate steam, which drives a steam turbine connected to a generator, which produces electricity.

Figure 1-6 shows a diagram of a typical nuclear power plant. Portugal has no nuclear power plants in its territory, but the Spanish Almaraz 2000 MW nuclear power plant is located near the border.

Nuclear power plants do not release carbon or pollutants like nitrogen and sulphur oxides into the air. This evident benefit of using nuclear power is tempered by a number of issues that need to be considered, including the safety of nuclear reactors, and the disposal of radioactive waste.

[13] Typically, a coal fired power plant emits 0.9 ton of CO2 for each MWh of electricity produced. This figure reduces to 0.4 ton/MWh in a CCGT power plant. MWh (Megawatt-hour) is the unit in which electrical energy is usually measured: it corresponds to an average power of 1 MW used during 1 hour, as we will show later on in this course.

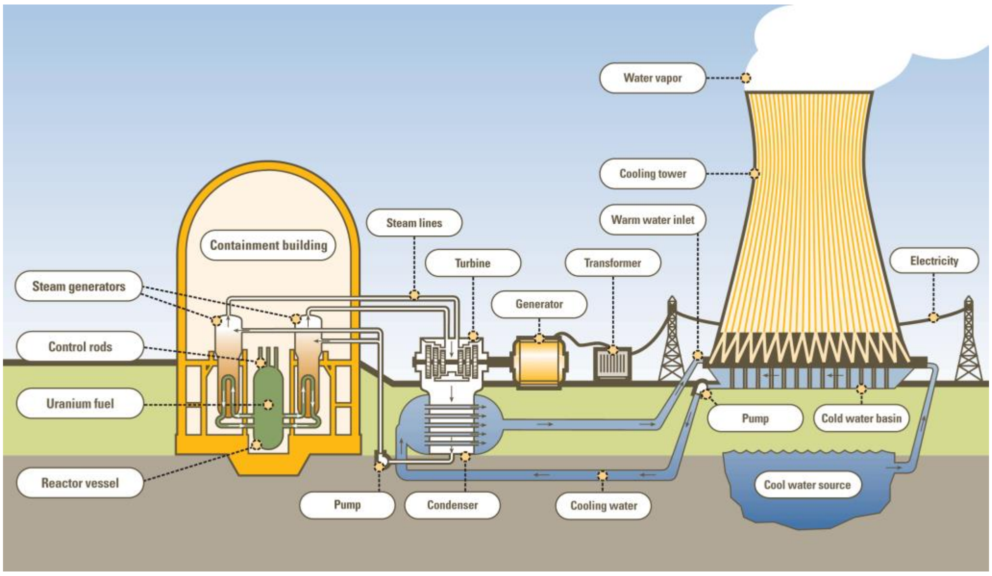

Figure 1-6: Diagram of a nuclear power plant. Source: https://www.nuclear-power.net/nuclear-power-plant/.

1.3.1.2 Hydro power plants

In hydro power plants, one takes advantage of a river waterfall and a water flow to turn the blades of a hydro turbine and obtain electricity through an alternator mounted in the same shaft of the hydro turbine (see Figure 1-7). The water flow is the volume of water that crosses a section of the river in the time unit; it is measured in m^3/s.

The efficiency of the power generation process is far higher than in the case of the thermal power plants, reaching, in this case, values around 80%. Furthermore, a Renewable Energy Source (RES), the water, is used, which turns these power plants into valuable assets.

The main issue is that they cannot be located everywhere, instead they require proper geographic conditions to be met. Also, large hydro power plants require the construction of big reservoirs of water, which do interfere with aquatic life, namely by blocking the migration of animals. These are environmental impacts that should not be neglected. In this sense, small hydro plants are less aggressive to the environment.

There are two types of hydro power plants: run-of-river and reservoir. In run-of-river plants, the resource is used as it comes, little or no water storage is provided. When the resource is not enough, the power plant stops; when it exceeds the power plant rated capacity, the surplus energy is wasted. Reservoir power plants are much more valuable. The reservoir is a natural lake, made possible due to adequate geographic conditions, in which incoming water can be stored. In this way, an optimal management of the power plant is achieved, as it can perform power regulation or even operate in dry periods, using the stored water.

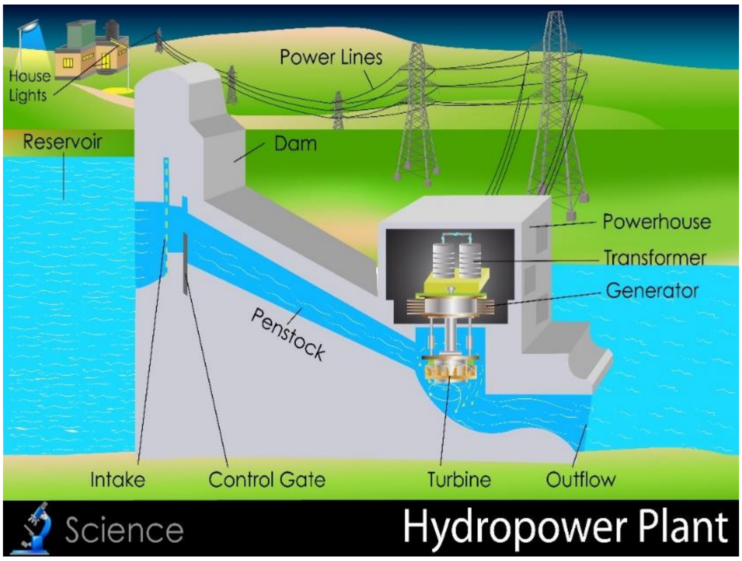

Figure 1-7: Diagram of a hydro power plant. Source: https://www.renfrewpg.ca/generation-power/.

Reservoir power plants allow for Pumped Hydroelectricity Storage (PHS), in which water is pumped from a lower reservoir to the higher one, consuming excess power during off-peak periods. Then, at peak hours, when electricity is more valuable, the stored water is released through the turbines to produce power, as it is explained in Figure 1-8.

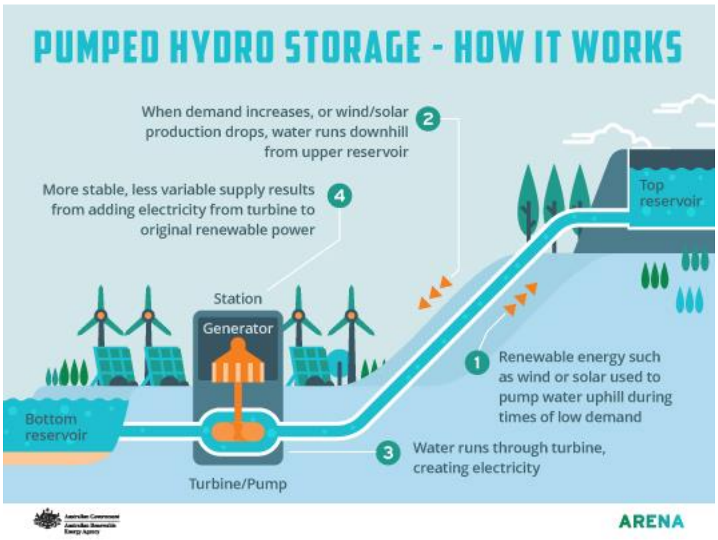

Figure 1-8: Diagram of a pumped hydroelectricity storage power plant.
Source: https://arena.gov.au/blog/pumped-hydro/

Even though the overall process has an efficiency less than 100%, this is a way of storing energy, which is a key asset for power systems operators, and facilitates the integration of other RES, like wind and solar, for instance.

1.3.1.3 Wind power plants

Wind power plants convert the kinetic energy associated to the wind speed into mechanical energy, through the spinning of the wind turbine blades. Once again, an electrical generator is mounted on the common shaft, which allows to obtain electrical energy. Figure 1-9 shows the components inside the nacelle of a Wind Turbine Generator (WTG).

The type of electrical generator that equips the wind converters is different from the alternators that produce electricity in thermal and hydro power plants. In order to increase the efficiency of the wind energy conversion, variable speed operation is required. This imposes the electrical generators to be connected to the grid through power electronics converters. The electrical generator itself can be of synchronous type or, more frequently, of asynchronous type[14]. Both of them are operated at variable speed.

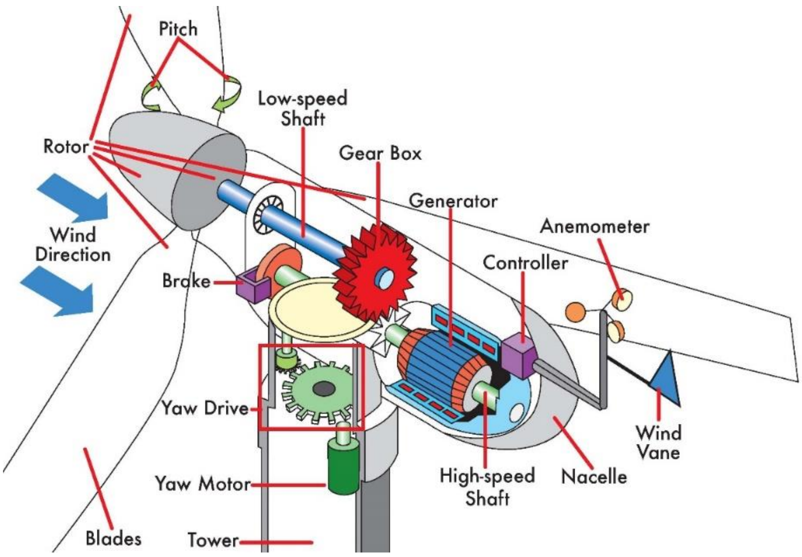

Figure 1-9: Wind Turbine Generator components. Source: https://sites.lafayette.edu/egrs352-sp14-wind/technology/.

A modern WTG may have a rated capacity of 3 MW, the rotor blades being 45 m long. The rotor blades speed may change typically between 9 and 18 rpm[15], depending on the

[14] Doubly-Fed Induction Generator.

[15] Revolutions per minute.

wind speed. In most WTG, there is a gearbox that converts the low speed shaft to the high speed shaft where the generator is mounted.

When wind speeds reach about 4 m/s, the WTG begins generating electricity. Rated power is obtained at about 15 m/s. The WTG control system regulates then the output power to the rated power till the cut-off wind speed (typically around 25 m/s) is attained. For higher wind speeds the WTG shuts down and turns out of the wind to protect from over speed failures.

The efficiency depends on the wind speed. For the most frequent wind speeds, the efficiency lays in the interval 40% to 50%. Of course, the main advantage of wind power is that it is carbon emissions free. On the drawbacks side, they are unable to regulate the output power in accordance to the system load needs. As a matter of fact, the WTG output power is linked to the wind speed resource, which is uncontrollable. Thermal and reservoir hydro are able to perform this important task that allows to balance generation and consumption. As we have seen before, this is an essential condition to the power system operation.

With the progressive shortness of ideal windy sites onshore, the next step of wind energy development is offshore wind, i.e. the deployment of WTG in open sea. There are several countries, like for instance, the UK, that are betting strong in offshore to take advantage of higher and more uniform (unperturbed) winds. Of course, installation and operation and maintenance costs are also higher, due to harsh conditions at open sea. For the time being, the cost-benefit analysis is still doubtful, but the situation is expected to reverse in the coming years.

In Portugal, a demonstration offshore WTG for deep waters was installed, under the so-called *WindFloat* project, shown in Figure 1-10.

Figure 1-10: WindFloat project. Source: https://www.edp.com/pt-pt/noticias/windfloat-regressa-ao-porto-de-pois-de-missao-bem-sucedida-no-mar.

1.3.1.4 PV power plants

In all technologies that we have approached so far – thermal, hydro, wind – the conversion principle is always the same: a turbine-generator group is used to perform the conversion. Photovoltaic (PV) power plants configure a totally different operating principle, because there is no turbine, nor generator.

In PV power plants, the sun light is directly converted into electricity. A material – the silicon –, after proper treatment, shows special characteristics when exposed to the sun. Actually, the photons, which are the particles that compose the sun irradiation, are able to displace electrons that acquire the capacity to move, therefore giving rise to an electrical current (Figure 1-11). This current is DC, therefore, to connect the PV power station to the AC grid, an interface device is required. This power electronics device is called an inverter and performs DC to AC conversion.

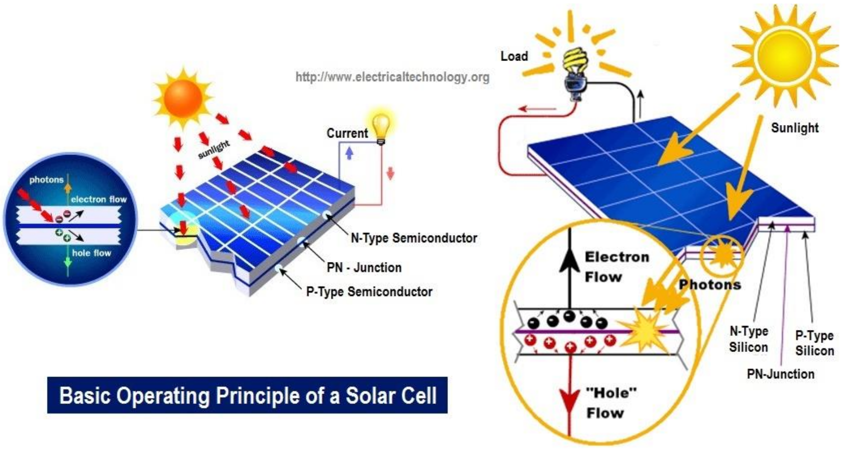

Figure 1-11: PV cell operating principle. Source: https://www.engineerwing.com/2017/06/operating-principle-of-solar-panel-and.html.

PV cells are assembled in modules. The capacity of a PV module is a few hundreds of Watts (200 – 300 W). To obtain more power, PV modules are assembled in PV arrays. Lots of PV arrays compose a PV park with installed capacity that can reach tens of MW. In Figure 1-12, a photo of the 46 MW largest PV power plant in Portugal is depicted.

Figure 1-12: Amareleja (Portugal) PV power plant. Source: http://geoelvas.blogspot.pt/2013/08/central-solar-fotovoltaica-de-amareleja.html

Historically, PV power presented two significant weaknesses: high cost and low efficiency. Nowadays, the first drawback is completely overcome: PV cost is in-line with the other forms of electricity production, namely with wind power. As for the efficiency, the advances were not so impressive, the efficiency still laying on around 15%. This low efficiency, together with the fact that there is no sun at night, make the production cost of each MWh of PV electricity still a little bit higher than in wind power stations, in a utility-scale level.

PV power is the preferred technology for micro generation applications in PV buildings, where it is extremely cost effective. PV facades or PV rooftops are more and more disseminated elements of the landscape. Also, Near Zero Energy Buildings (NZEB), which make intensive use of PV power, are becoming common (Figure 1-13).

Figure 1-13: Near Zero Energy Buildings at National Renewable Energy Laboratory in Golden, Colorado. Source: https://www.rdmag.com/article/2017/10/how-achieve-zero-energy-building.

1.3.2 Transmission

The electricity transmission is an exclusive public service concession. In Portugal, the concession was awarded to REN – Redes Energéticas Nacionais[16]. REN is the Transmission System Operator (TSO), who is also in charge of the overall technical management of the power system. For the use of the transmission network, the TSO is paid by a tariff set by ERSE – Entidade Reguladora dos Serviços Energéticos[17], the National Energy Regulator.

The transmission grid is the EHV network (Figure 1-14). In Portugal, there are three EHV voltage levels – 400 kV, 220 kV and 150 kV[18]. The transmission grid is mostly composed by overhead lines; some underground cables are used near the main cities.

In the transmission grid there are transformers with the purpose of performing voltage transformation between EHV voltage levels, as well as transformers to interface the transmission grid with the distribution grid (EHV/HV). The transformers are located in the so-called substations. Figure 1-15 shows a schematic of a EHV/HV substation with the identification of its main components.

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[16] www.ren.pt.

[17] www.erse.pt.

[18] These numbers are the RMS value of the phase-to-phase voltage. In power systems, when we refer a voltage by a number, it is always the RMS value of the phase-to-phase voltage.

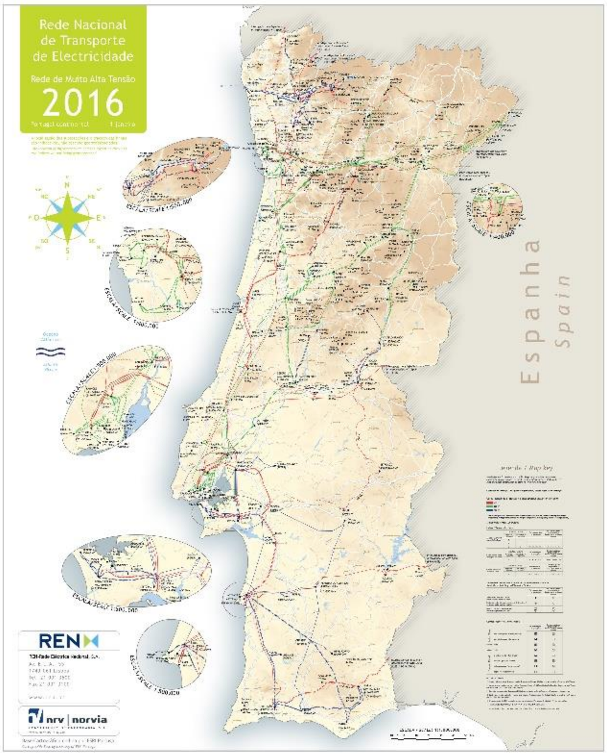

Figure 1-14: Portuguese transmission grid. Source: http://www.centrodeinformacao.ren.pt/PT/InformacaoTecnica/Paginas/MapaRNT.aspx.

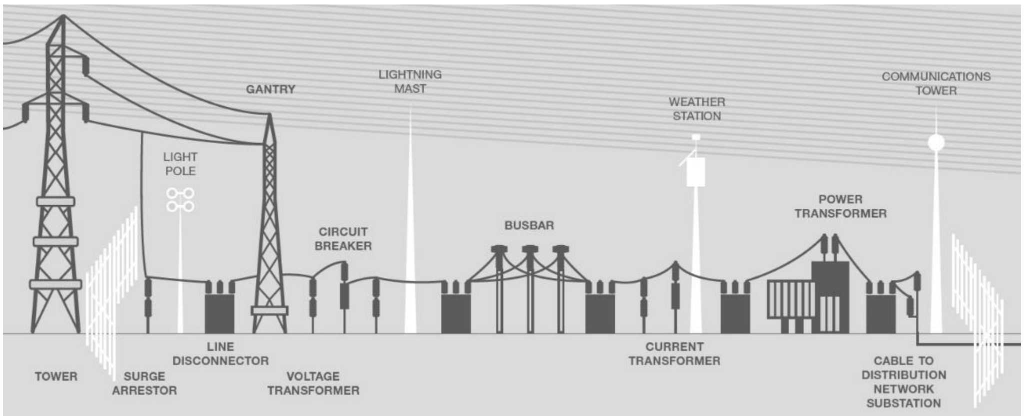

Figure 1-15: Schematic of a transmission substation. Source: https://www.electranet.com.au/our-approach/safety/transmission-substations/.

1.3.3 Distribution

As for the case of transmission, the electricity distribution is also an exclusive public service concession. In Portugal, the concession was awarded to EDP–Distribuição – Energias de Portugal – Distribuição[19]. EDP–Distribuição is the Distribution System Operator (DSO), who is in charge of the operation and maintenance of the distribution grid and of the

[19] https://www.edpdistribuicao.pt/.

respective power flow management. EDP–Distribuição is remunerated for the use of the distribution network by a tariff set by the energy regulator.

The HV and MV distribution network is composed by 80% of overhead lines and 20% of underground cables. In Portugal, 60 kV are used in HV, and 30 kV, 15 kV and 10 kV are the MV voltage levels. Beyond the lines and cables, the distribution grid includes, the substations (where the HV/MV transformers are installed) and the transformation stations (where the MV/LV transformers are located). A piece of the HV distribution grid in the Lisbon area is depicted in Figure 1-16.

The LV distribution grid is a separated entity form the HV and MV distribution grid, because it is a municipal concession. Yet, EDP–Distribuição is the holder of the great majority of the concessions, which were awarded by the municipalities. The operating voltage in LV is 400 V[20].

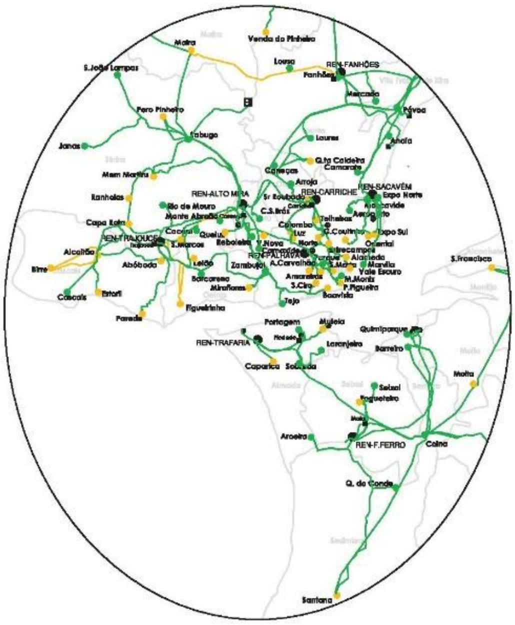

Figure 1-16: A piece of the HV distribution grid in the Lisbon area. Source: http://edp-distribuicao-rede.wntech.com/.

1.3.4 Retailing

The retailing activity is a separated activity from distribution, being the last electricity related activity in the supply chain to the consumers. As the generation, retailing is a

[20] 400 V is the RMS phase-to-phase voltage, which correspond to 230 V, RMS phase-to-neutral voltage.

market-driven activity, open to the private initiative, but the participants must be certified. In Portugal, there are several retailing companies, that operate in the framework of the electricity market.

The electricity market is the Iberian Electricity Market (MIBEL – Mercado Ibérico da Eletricidade). In MIBEL, at every hour, the power producers offer to sell electricity at a certain price; the retailers offer to purchase electricity at another price. The selling offers are placed in an increasing price order; the purchase offers are ordered price-decreasingly. The intersection point between the two curves (supply and demand) defines the market clearing price. All the producers that have been selected to operate (because their offers match the demand) will receive at the market price, and the retailers will pay also the market price. Hence, the market price changes on an hourly basis, despite the consumer tariffs still do not reflect this hourly price changing.

The retailer is the clients' representative in the market. It buys electricity at the market price, pays the tariffs for the use of the transmission and distribution grids, pays other power system costs and reflects all these parcels in the tariffs it charges to the clients.

1.3.5 Consumption

The domestic consumers are connected in the LV distribution network. The big consumers are usually connected in HV distribution network; the medium industrial consumers are normally connected in the MV network.

The load diagram is the representation of the load power[21] as a function of time. Load diagram supply very important information for the power system operation; they allow to find the peak-power (maximum power) and the consumed electricity (area below the load diagram)[22].

The load diagram on the day of annual peak demand for the Portuguese power system (2016 and 2017) is presented in Figure 1-17. We can see that Portugal was exporting energy to Spain during the whole day. Also, note the typical behaviour of the daily load

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[21] In power systems engineering the power demanded by the consumers is usually known as load power.

[22] We will see later than the electrical energy is the integral of the electrical power.

diagram of the Portuguese power system: during the night the load power decreases to about a half of the peak power; then, a first peak occurs at about noon, and finally the peak power is attained by dinner time.

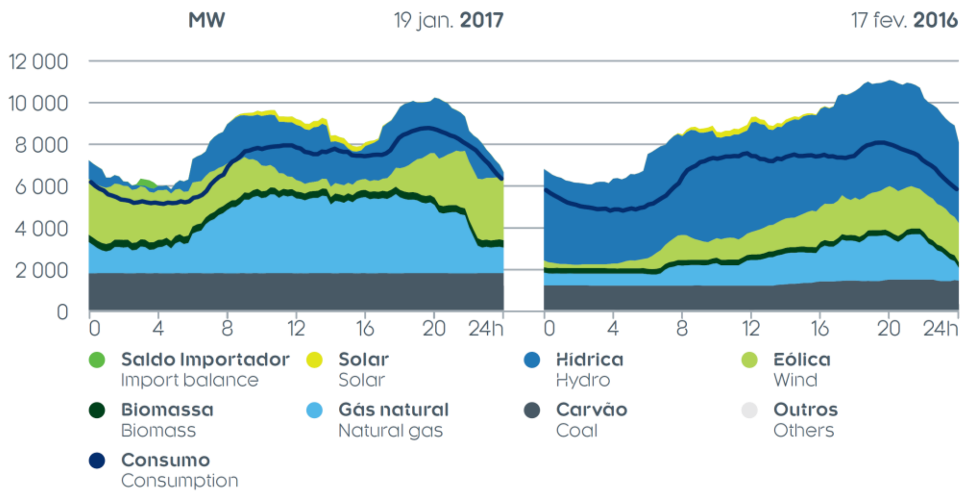

Figure 1-17: Load diagram on the day of annual peak demand (Portuguese power system; 2016 and 2017). Source: http://www.centrodeinformacao.ren.pt/PT/InformacaoTecnica/Paginas/DadosTecnicos.aspx.

1.4 Balancing Generation and Consumption

The energy in its electrical form is not appropriate for large scale storage. As so, as already mentioned, the electricity that is obtained in the power stations has to match, in each time instant, the consumption plus the losses.

The power demanded by the consumers is always changing, as a result of a random switch on and switch off of the electrical devices. Moreover, some electricity production is time variable in an uncontrolled way. This is, namely, the case of wind power, which depends on the available wind resource, and in a somehow less extent, PV power, which is, nevertheless, more predictable than wind.

To compensate for all these changes, the conventional power stations regulators are constantly controlling the valves that supply water to the hydro turbines, or fossil fuels to the thermal power plants, so that a perfect match between generation and consumption is attained at each instant.

The prime movers (water in the hydro power plants; water steam in the coal-fired power plants; gases mix in the gas power plants) turn the turbine's blades and originate the driving torque that moves the alternator. The driving torque is opposed by the resistance torques. When the driving torque is equal to the summation of the resistance torques, the turbine and the generator rotate at constant speed[23]: the electricity produced from the resources is equal to the electricity consumed by the receivers and the grid frequency is constant.

However, the regulators are not fast enough to keep a constant speed. During the time of opening or closing the valves, the driving and the resistance torques are not equal. As so, if the consumption is reduced, the resistance torque is reduced and the machines experiment a transient speed increase (acceleration). The opposite happens when the consumption increases.

When the angular speed changes, so changes the grid frequency. This is most undesirable, because it disturbs the electricity supply to the consumers. As so, frequency must be kept constant. This is accomplished by designing the rotating masses with a large moment of inertia. It is the kinetic energy of the rotating masses that compensates for the unbalance between production and consumption, while the regulators are performing the required valves opening or closing actions. Let us look at the so called swing equation:

$$T_d - T_r = J\frac{d\omega}{dt} \qquad \text{equation 1.17}$$

where T_d and T_r are the driving torque and the resistance torque, respectively, J is the moment of inertia and ω is the angular speed ($d\omega/dt$ is the angular acceleration).

It is possible to conclude that even for a high unbalance in the torques, a moderate angular acceleration can be achieved providing that the moment of inertia is high enough. The design of conventional power plants includes choosing the appropriate values for the moment of inertia of the rotating masses, so that the frequency changes are kept within a very narrow range.

[23] The so-called synchronous speed.

2 AC ELECTRICAL CIRCUITS

Time variation of the electrical quantities makes it more difficult to analyse the steady-state behaviour of AC electrical circuits. However, Electrical Engineering developed some methods to smooth this difficulty. We will present in the sequence the most used concepts and methods to analyse the steady-state behaviour of AC electrical circuits.

2.1 THE SINUSOIDAL VOLTAGE

Figure 2-1 shows a uniform (constant) magnetic field created by permanent magnets (represented by magnetic poles N and S). Inside this magnetic field, a rectangular winding is rotating at constant angular speed ω (rad/s) imposed by an external agent.

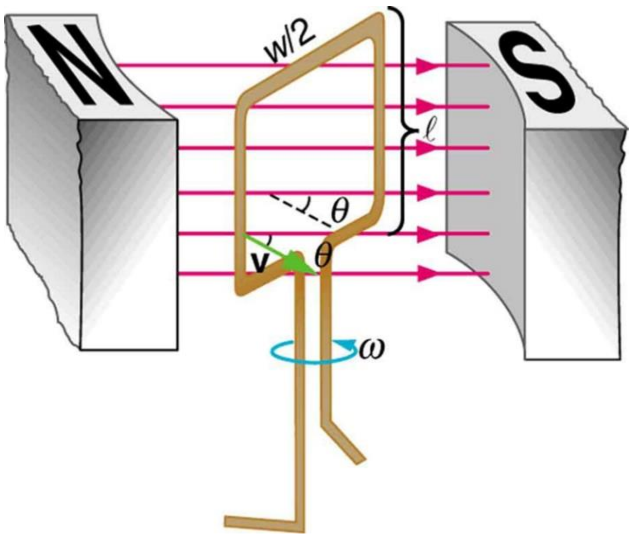

Figure 2-1: Rectangular winding rotating inside a uniform magnetic field. Source: http://philschatz.com/physics-book/contents/m42408.html.

Now look at Figure 2-2: when the angle θ is 0 (dimmed winding), the flux crossing the winding is maximum; when the angle θ is $\pi/2$ (lighten winding), the flux is null.

As so, the magnetic flux, ϕ (Wb – Webber), can be expressed as:

$$\phi(t) = \phi_{MAX} \cos\theta = \phi_{MAX} \cos(\omega t) \qquad \text{equation 2.1}$$

where, ϕ_{MAX} is the maximum flux and θ is the angle between the magnetic field and a perpendicular to the winding V (nothing to do with voltage) in Figure 2-1.

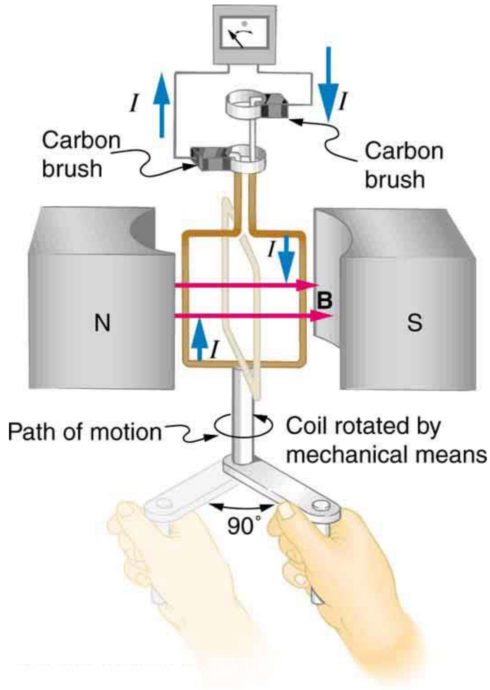

Figure 2-2: Winding rotation makes the magnetic flux change from maximum to zero. Source: http://philschatz.com/physics-book/contents/m42408.html

In accordance with a known electrical law (Faraday's Law), an electromotive force (emf), with the dimensions of a voltage (V – Volt) is induced in the winding, which is given by:

$$e(t) = -\frac{d\phi}{dt} = \omega\phi_{MAX}\sin(\omega t) = E_{MAX}\sin(\omega t) = \sqrt{2}E\sin(\omega t) \qquad \text{equation 2.2}$$

where E_{MAX} is the maximum emf, linearly dependent on the maximum flux and on the angular speed. The emf RMS value is E. It should be noted that no emf is produced if the winding is static (ω=0).

We conclude that the rotation of a winding inside a uniform magnetic field produces a sinusoidal emf. This is the operating principle of a AC generator. To transmit power from the rotating winding to a stationary circuit, slip rings and carbon brushes are used: slip rings are rotating metallic devices connected to the winding and carbon brushes are fixed contacts connected to an external static circuit (see Figure 2-3).

At the terminals of the external circuit, a sinusoidal voltage, that slightly differs from the emf due to circuit losses, is obtained:

$$v(t) = \sqrt{2}V\sin(\omega t) \qquad \text{equation 2.3}$$

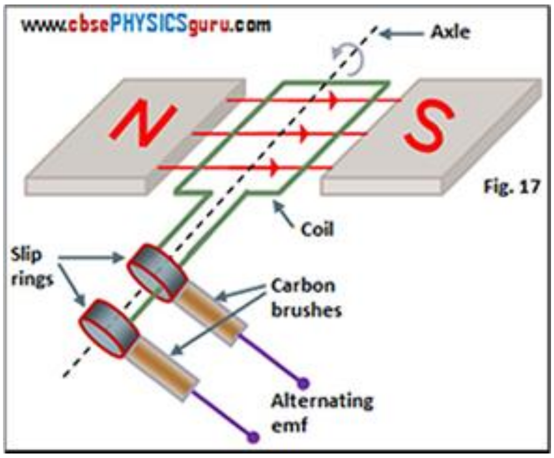

Figure 2-3: Slip rings and carbon brushes. Source: http://www.cbseguess.com/ebooks/xii/physics/ac-generator_dynamo.php.

2.2 PURE RESISTIVE CIRCUIT

At this stage, no current exists. To obtain a current, a load must be connected to the external circuit. If the simplest load, a constant resistance, R (Ω – Ohm), is connected, a current will flow through it. In accordance with another electrical law (Ohm's Law), this current, i_R, will be given by:

$$i_R(t) = \frac{v_R(t)}{R} = \sqrt{2}\frac{V_R}{R}\sin(\omega t) = \sqrt{2}I_R\sin(\omega t) \qquad \text{equation 2.4}$$

Let us look at a numerical example. Let us suppose that a V_R = 230 V RMS voltage is produced in a larger but similar system to the one in Figure 2-2. Then, a resistance R = 23 Ω, is connected to the system through an external circuit, as seen in Figure 2-4. The current RMS value is $I_R = V_R/R$ = 10 A. The time evolution of the voltage and current are presented in Figure 2-5.

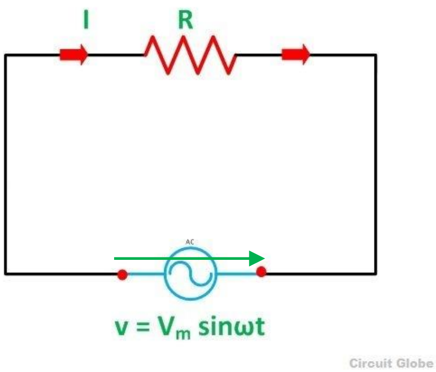

Figure 2-4: Resistive circuit. $I = I_R$; $V_m = V_R$; Source: https://www.quora.com/What-is-a-purely-resistive-AC-circuit.

From this figure, we can see that voltage and current are zero at the same time instants; and that they are maximum also at the same time instants. We say that voltage and current are “in phase”.

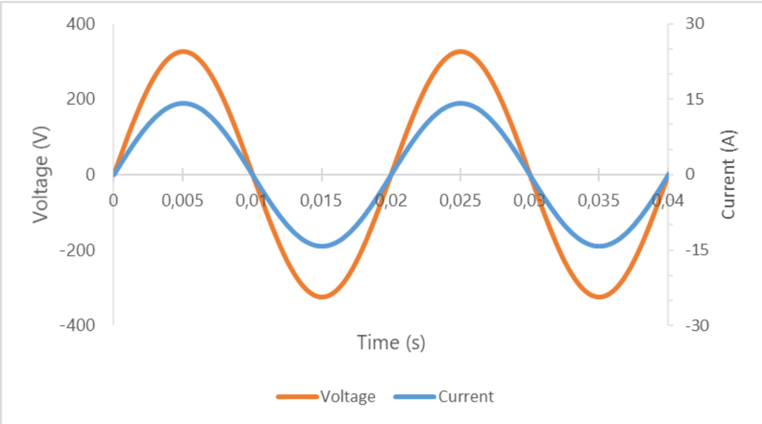

Figure 2-5: Voltage and current changing as a function of time in a resistive circuit.

We can now use the knowledge we have just learned to investigate what is going on as regards to emf and flux. The situation is quite different: when the flux is maximum, the emf is zero and when the emf is maximum, the flux is zero. This means that these two quantities are not in phase, actually they are “out of phase”, which is due to the derivative relationship between the cause (flux) and the effect (emf) (see equation 2.2).

In Figure 2-6, it is depicted the time evolution of the flux (equation 2.1) and emf (equation 2.2), where the out of phase of these two quantities is apparent. One can say that the flux leads the emf (and the emf lags the flux). For instance, the maximum flux occurs at t = 0, and the maximum emf occurs later, at t = 0.005 s, this is why the flux leads the emf.

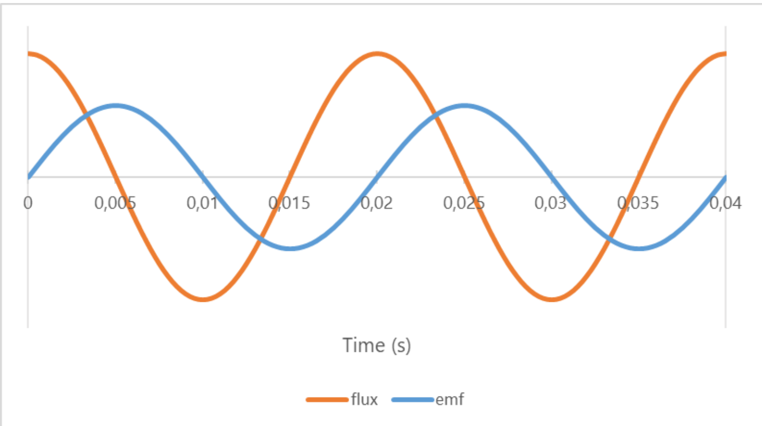

Figure 2-6: Flux and emf changing as a function of time.

2.3 Pure Inductive Circuit

It is known that when a current $i_L(t)$ flows through a rolled up conductor, called an inductor, (AB in Figure 2-7), a magnetic flux, $\phi(t)$, is created.

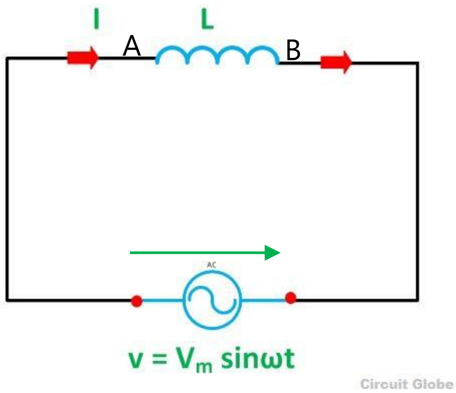

Figure 2-7: Pure inductive circuit. $I = I_L$; $V_m = V_L$; Source: https://circuitglobe.com/what-is-pure-inductive-circuit.html.

If, in the space where the flux is established, no ferromagnetic materials exist, the flux is proportional to the current, the proportionally constant being L, as follows:

$$\phi(t) = Li_L(t) \qquad \text{equation 2.5}$$

L is the inductance; it is a measure of the induced magnetic flux due to the current that flows across the circuit itself. The unit of the coefficient of inductance is H – Henry[24].

From Faraday's Law, we know that, providing the flux changes in time, a emf is produced between AB terminals, which is given by:

$$e(t) = -\frac{d\phi}{dt} = -L\frac{di_L}{dt} \qquad \text{equation 2.6}$$

Looking back to Figure 2-7, it shows a circuit in which a voltage $v(t)$ is applied to a coil with an inductance L; we suppose the resistance of the coil is null. Under these circumstances, a time changing current $i_L(t)$ flows across the coil. This current creates a time changing magnetic flux, $\phi(t)$, which, in turn, induces an emf in the coil itself.

Applying Kirchoff Voltage Law (KVL), we obtain:

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[24] For the majority of the applications, an inductance equal to 1 H is very high. This is why normally we use the milliHenry (1 mH = 10^{-3} H).

$$v_L(t) + e(t) = 0 \qquad \text{equation 2.7}$$

and then, taking equation 2.6 into account:

$$v_L(t) = L\frac{di_L}{dt} \qquad \text{equation 2.8}$$

It is possible to demonstrate that a sinusoidal voltage causes a sinusoidal current. It is a usual procedure to assume the voltage angle at t = 0 to be equal to zero. As so, the voltage is given by:

$$v_L(t) = \sqrt{2}V_L \sin(\omega t + 0) = \sqrt{2}V_L \sin(\omega t) \qquad \text{equation 2.9}$$

We will prove that, for the circuit of Figure 2-7, the current is:

$$i_L(t) = \sqrt{2}I_L \sin(\omega t - \frac{\pi}{2}) \qquad \text{equation 2.10}$$

From equation 2.8, we write:

$$v_L(t) = L\frac{d}{dt}\left(\sqrt{2}I_L \sin(\omega t - \frac{\pi}{2})\right) = \sqrt{2}\omega L I_L \cos(\omega t - \frac{\pi}{2}) \qquad \text{equation 2.11}$$

and so, knowing that:

$$\cos(\omega t - \frac{\pi}{2}) = \sin(\omega t) \qquad \text{equation 2.12}$$

we obtain

$$v_L(t) = \sqrt{2}\omega L I_L \sin(\omega t) \qquad \text{equation 2.13}$$

which proves our hypothesis.

Comparing equation 2.9 and equation 2.13, we conclude that:

$$V_L = \omega L I_L = X_L I \qquad \text{equation 2.14}$$

where X_L is known as reactance, in this case, an inductive reactance.

The units of the reactance are the same as for the resistance., i.e. Ω – Ohm, because, as for the case of the resistance, the reactance is a ratio between the RMS values of the voltage and current.

In a resistive circuit, we have:

$$R = \frac{V_R}{I_R} \qquad \text{equation 2.15}$$

and in a pure inductive circuit, it is:

$$\omega L = X_L = \frac{V_L}{I_L} \qquad \text{equation 2.16}$$

Furthermore, looking at equation 2.10 and equation 2.13, we conclude that, in a pure inductive circuit, the current is lagging the voltage, or the voltage is leading the current, by and angle of $\pi/2$.

Let us look again at a numerical example. Consider that the RMS voltage is V_L = 230 V, f = 50 Hz, and a coil with a reactance X_L = 23 Ω (L = 73.21 mH), is connected to the voltage source. The current RMS value is $I_L = V_L/X_L$ = 10 A. The time evolution of the voltage and current are presented in Figure 2-8.

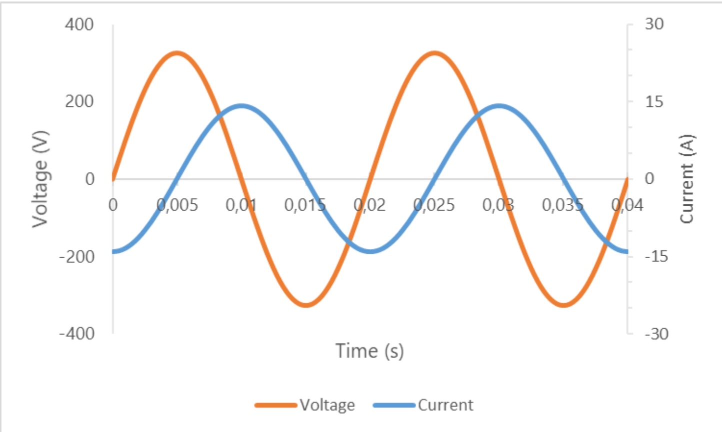


Figure 2-8: Voltage and current changing as a function of time in a pure inductive circuit.

It can be seen that the current is lagging the voltage, because the peak-current occurs after the peak-voltage.

2.4 Pure Capacitive Circuit

A capacitor consists of two conductive plates which are separated by a dielectric (insulator) medium. The electrical charge of a capacitor, $q(t)$, is directly proportional to the voltage, $v_C(t)$, at the capacitor's terminals. Hence, we can write:

$$q(t) = Cv_C(t) \qquad \text{equation 2.17}$$

where C is the proportionality coefficient, known as capacitance. The capacitance is measured in F – Farad[25] and the electrical charge in C – Coulomb.

The electrical charge is the charge in one of the two conductors that compose a capacitor (Figure 2-9). In the other conductor, there is a symmetrical (with a minus sign) but equal charge. However, the relevant charge is q, and not $2q$, which is the charge that is moving, during the charge and discharge processes.

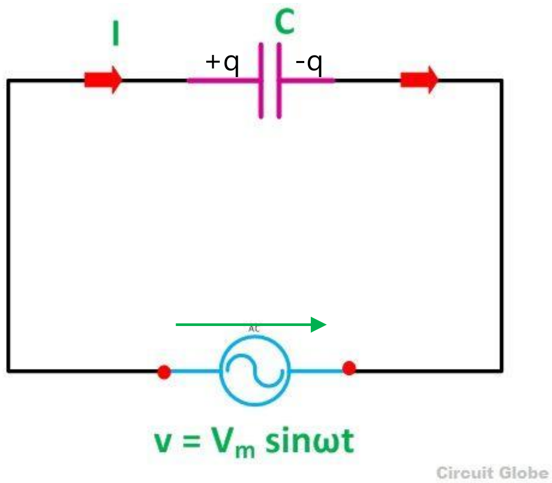


Figure 2-9: Pure capacitive circuit. $I = I_C$; $V_m = V_C$; Source: https://circuitglobe.com/what-is-pure-capacitor-circuit.html.

The current $i_C(t)$ in each cross section of a conductor is the quantity of electricity (electrical charge) flow that crosses that section in each time instant. This definition allows us to write:

$$i_C(t) = \frac{dq}{dt} \qquad \text{equation 2.18}$$

As a physical quantity, current is the rate at which charge flows past a point on a circuit. For the circuit in Figure 2-9, the current $i_C(t)$ measures the rate at which que capacitor's electrical charge is changing.

If one admits that the capacitance, C, does not change in time, which is true most of the times, from equation 2.17, we can conclude that:

$$i_C(t) = C\frac{dv_C}{dt} \qquad \text{equation 2.19}$$

[25] Usually, the capacitance is given in microFarad (1μF = 10^{-6} F).

As for the case of the pure inductive circuit, let us now prove for the pure capacitive circuit, that if the source voltage is $v(t) = \sqrt{2}V\sin(\omega t)$, the current will be:

$$i_C(t) = \sqrt{2}I_C \sin(\omega t + \frac{\pi}{2}) \qquad \text{equation 2.20}$$

From equation 2.19, we obtain:

$$i_C(t) = C\frac{d}{dt}\left(\sqrt{2}V_C \sin(\omega t)\right) = \sqrt{2}\omega C V_C \cos(\omega t) \qquad \text{equation 2.21}$$

and so, as:

$$\cos(\omega t) = \sin(\omega t + \frac{\pi}{2}) \qquad \text{equation 2.22}$$

It follows that:

$$i_C(t) = \sqrt{2}\omega C V_C \sin(\omega t + \frac{\pi}{2}) \qquad \text{equation 2.23}$$

which proves our hypothesis.

From equation 2.20 and equation 2.23, we conclude that:

$$\begin{aligned} I_C &= \omega C V_C = B_C V_C \\ V_C &= \frac{1}{\omega C} I_C = X_C I_C \end{aligned} \qquad \text{equation 2.24}$$

Where B_C is known as susceptance and X_C is our already known reactance, in this case a capacitive reactance.

The units of the susceptance are S – Siemens. In a pure capacitive circuit, a susceptance is the ratio between the RMS values of the current and voltage and the reactance is the ratio between the RMS values of the voltage and current.

If we look at equation 2.9 and equation 2.23, we notice that, the current is leading the voltage, or the voltage is lagging the current, by and angle of $\pi/2$. This holds true for a pure capacitive circuit.

Let us focus again at a numerical example. The RMS voltage is V_C = 230 V, f = 50 Hz, and a capacitor with a reactance X_C = 23 Ω (B_C = 43.48 mS; C = 138.40 μF), is connected to the voltage source. The current RMS value is $I_C = V_C/X_C$ = 10 A. The time evolution of the voltage and current are presented in Figure 2-10.

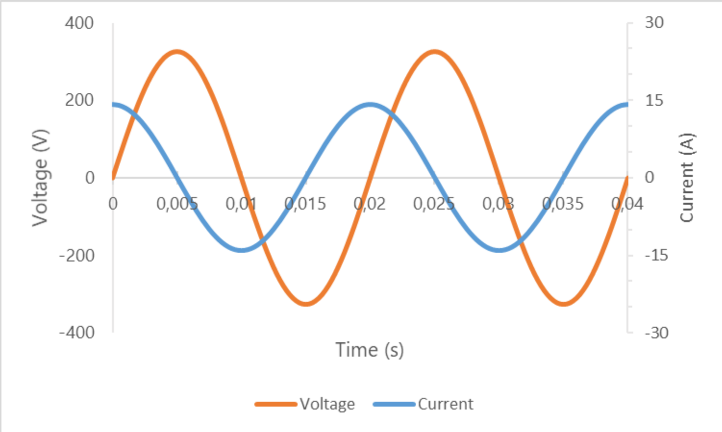


Figure 2-10: Voltage and current changing as a function of time in a pure capacitive circuit.

It is apparent that the current is leading the voltage, because the peak-current occurs before the peak-voltage.

2.5 The Need for Complex Numbers

We have seen that knowing the resistance (in a pure resistive circuit) or the reactance (in a pure inductive or capacitive circuit) allows to know the RMS value of the current from the knowledge of the RMS value of the voltage. However, this is not enough to completely describe the sinusoidal current.

In general, a sinusoidal current can be written as (let us recall that for a pure inductive circuit, $\varphi = \pi/2$; for a pure capacitive circuit, $\varphi = -\pi/2$; for a pure resistive circuit, $\varphi = 0$):

$$i(t) = \sqrt{2}I\sin(\omega t - \varphi) \qquad \text{equation 2.25}$$

To describe $i(t)$, we need:

- the RMS value of the current, I;
- the frequency, f, that allows us to compute the angular ferquency, $\omega = 2\pi f$;
- the initial phase, φ, i.e., the angle at the origin of times, $t = 0$.

We conclude that we need three informations to completely describe a sinusoidal current (for a voltage, the same applies). However, in many situations, we are interested in assessing the power system in steady-state. In steady-state, and for linear circuits (constant parameters), the frequency is always equal to $f = 50$ Hz. As a consequence, $\omega = 100\pi$, and we need only to know two informations to completely describe a sinusoidal current (or voltage): RMS and initial phase.

As so, we need numbers able to carry two informations. These numbers are the complex numbers.

2.6 REVIEW ON COMPLEX NUMBERS

We start from that there is an (imaginary) number, *j*, which, multiplied by itself, equals –1:

$$j = \sqrt{-1} \qquad \text{equation 2.26}$$

We use *j* for the imaginary unit. Many books use the letter *i*, but this letter is normally assigned in electrical engineering to denote the current.

A so-called complex number:

$$z = a + jb \qquad \text{equation 2.27}$$

has both a real part (Re(*z*) = *a*) and an imaginary part (Im(*z*) = *b*). The format in equation 2.27 is called the rectangular format of complex numbers. The representation of this complex number in the complex plan is shown in Figure 2-11.

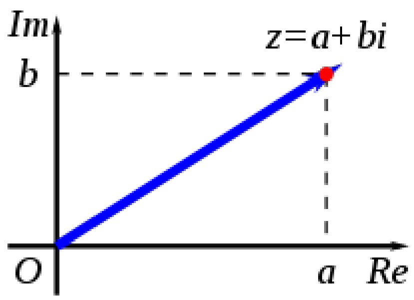


Figure 2-11: Representation of a complex number in the complex plane. Source: Wikipedia.

The complex number is represented by the red dot in Figure 2-11. We can assign to that complex number a vector with a certain length (amplitude) and angle with respect to the x-axis (phase).

Leonhard Euler (1707 – 1783) discovered the relation, which relates complex numbers to the (periodic) trigonometric functions, known as Euler's formula:

$$e^{j\varphi} = \cos\varphi + j\sin\varphi \qquad \text{equation 2.28}$$

With the aid of Euler's formula, it is possible to transform any complex number in the rectangular format into the polar format:

$$z = re^{j\varphi} \quad \text{equation 2.29}$$

where r is the amplitude and φ is the phase, as in Figure 2-12.

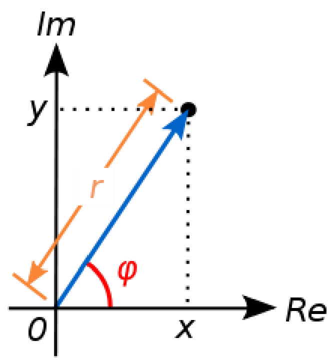


Figure 2-12: Representation of a complex number in the polar format in the complex plane. Source: Wikipedia.

A complex number with unitary modulus (r = 1) and angle φ is represented in Figure 2-13.

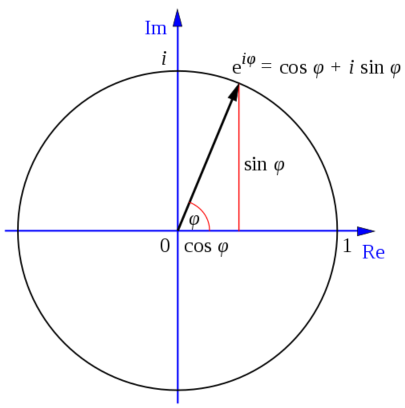


Figure 2-13: Representation of a complex number with unitary modulus and angle φ in the complex plane. Source: Wikipedia.

Plugging equation 2.28 into equation 2.29, we obtain:

$$z = r\cos\varphi + jr\sin\varphi = a + jb \quad \text{equation 2.30}$$

where:

$$\begin{aligned} \mathrm{Re}\{z\} &= a = r\cos\varphi \\ \mathrm{Im}\{z\} &= b = r\sin\varphi \end{aligned} \quad \text{equation 2.31}$$

Solving the system of two equations in equation 2.31 (dividing the second equation by the first and then squaring and summing the two equations), we are led to:

$$r = \sqrt{a^2 + b^2}$$
$$\varphi = \tan^{-1}\left(\frac{b}{a}\right) \qquad \text{equation 2.32}$$

The same result is reached by visual inspection of Figure 2-13, taking trigonometric relationships and Pythagoras theorem into account (Figure 2-14).

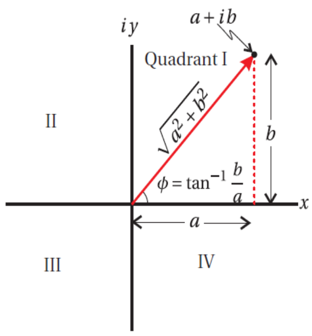


Figure 2-14: A complex number can be represented either in the rectangular or polar formats. Source: https://www.physics.byu.edu/faculty/peatross/homework/complex145.pdf.

Let us now recall the basic operations with complex numbers. Consider two complex numbers:

$$z_1 = a_1 + jb_1 = r_1 e^{j\varphi_1}$$
$$z_2 = a_2 + jb_2 = r_2 e^{j\varphi_2} \qquad \text{equation 2.33}$$

Summations and subtractions are easier performed in rectangular format:

$$z_1 + z_2 = a_1 + jb_1 + a_2 + jb_2 = (a_1 + a_2) + j(b_1 + b_2)$$
$$z_1 - z_2 = a_1 + jb_1 - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2) \qquad \text{equation 2.34}$$

Products and divisions are easier performed in polar format:

$$z_1 z_2 = r_1 e^{j\varphi_1} r_2 e^{j\varphi_2} = r_1 r_2 e^{j(\varphi_1 + \varphi_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1 e^{j\varphi_1}}{r_2 e^{j\varphi_2}} = \frac{r_1}{r_2} e^{j(\varphi_1 - \varphi_2)} \qquad \text{equation 2.35}$$

We define complex conjugate of a complex number z = $a + jb = re^{j\varphi}$ as:

$$z^* = (a - jb) = re^{-j\varphi} \qquad \text{equation 2.36}$$

and the reciprocal as:

$$\frac{1}{z} = \frac{1}{a + jb} = \frac{1e^{j0}}{re^{j\varphi}} = \frac{1}{r}e^{-j\varphi} \qquad \text{equation 2.37}$$

Now some singular cases that will be important in the sequence (a>0, b>0) :

$$\begin{aligned} a &= ae^{j0} \\ jb &= be^{j\frac{\pi}{2}} \\ -a &= ae^{j\pi} \\ -jb &= be^{-j\frac{\pi}{2}} \end{aligned} \qquad \text{equation 2.38}$$

Multiplying a vector by j, moves the vector forward by 90º; multiplying by $-j$, holds the vector back by –90º.

2.7 COMPLEX AMPLITUDES TO SOLVE AC CIRCUITS

We have seen that complex numbers can be associated to vectors. When complex numbers are used to solve AC circuits, the quantities that are to be represented by complex numbers are, among others, voltages and currents. It is quite apparent that a sinusoidal voltage cannot be represented by a vector, because it has no associated direction in space as, for instance, a force, a velocity or an accelaration. However, as for the rest, the vector representation is applicabble. Because of this, the nomenclature "phasors" will be used instead of "vectors".

One way of representing an AC quantity is in terms of a rotating phasor, (we use bold capital letters with an upper bar to denote rotating phasors, as shown in Figure 2-15). A rotating phasor is rather like the hand on a clock, though the phasors we will consider will all rotate in the anticlockwise direction. A rotating phasor $\overline{\mathbf{A}}$ has a magnitude A and rotates at a fixed angular speed ω, so that the angle $\theta = \omega t$ from the x–axis to the phasor increases with time. The projection of the rotating phasor on to the y–axis (or in the x-axis) is a sinusoidal function of the angle θ. The projection in the y-axis results in a sine function, whereas we obtain a cosine from the projection in the x-axis. Therefore, a rotating phasor is a complex number that is able to represent the amplitude and phase of a sinusoid.

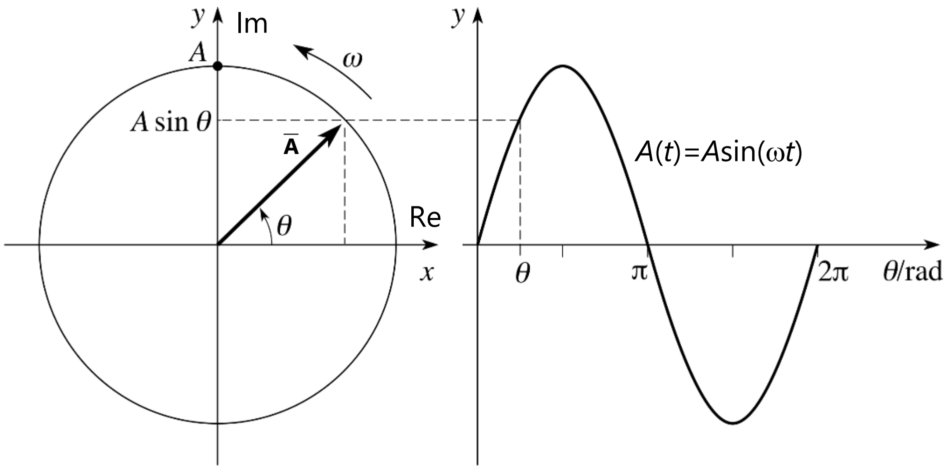


Figure 2-15: A phasor as a representation of a sinusoidal wave with φ = θ(0) = 0. Source: http://www.met.reading.ac.uk/pplato2/h-flap/phys5_4.html.

In this example, in t = 0, the angle $\theta = \omega t$ is zero. As so, the initial phase is $\varphi = 0$. However, we could easily generalize for $\varphi = \theta(0) \neq 0$, as seen in Figure 2-16.

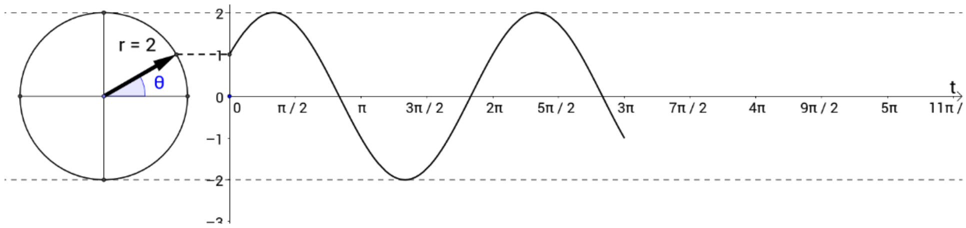


Figure 2-16: A phasor as a representation of a sinusoidal wave with φ = θ(0) ≠ 0. Source: https://www.geogebra.org/m/MRJcswr5.

Based on what you have presented whatsoever, let us consider a sinusoidal time dependent current.

$$i(t) = \sqrt{2}I\sin(\omega t - \varphi) \qquad \text{equation 2.39}$$

We have showed that this is equivalent to having the projection on the y-axis of the rotating phasor:

$$\bar{\mathbf{I}} = \sqrt{2}I\cos(\omega t - \varphi) + j\sqrt{2}I\sin(\omega t - \varphi) \qquad \text{equation 2.40}$$

According to Euler's formula, this is equal to:

$$\bar{\mathbf{I}} = \sqrt{2}Ie^{-j\varphi}e^{j\omega t} \qquad \text{equation 2.41}$$

When multiplied by $e^{j\omega t}$, the phasor starts rotating anticlockwise in the complex plane. As the sinusoidal function has a constant frequency (for steady-state power systems f = 50 Hz always), the phasor is rotating at constant rate. If observed in a reference frame rotating at the same rate, the rotating phasor become standing still, i.e., we can simply drop the time-varying component $e^{j\omega t}$, and represent the function in terms of a static phasor with an amplitude and initial phase only. Moreover, as most of the times we are interested in RMS values (namely because these are the values a measuring device measures), we can drop the constant square root of two factor too.

This static phasor is therefore given by (we use bold capital letters to denote the static phasor, or simply, the phasor):

$$\mathbf{I} = Ie^{-j\varphi} = I\cos\varphi - jI\sin\varphi \qquad \text{equation 2.42}$$

It is important to highlight that the phasor **I** carries two informations as intended to solve AC electrical circuits: the RMS *I* and the initial phase φ. In Electrical Engineering the phasor is named “**Complex Amplitude**”.

To recover the original sinusoidal current (equation 2.39), we simply compute:

$$i(t) = \mathrm{Im}\{\bar{\mathrm{I}}\} = \mathrm{Im}\{\sqrt{2}\mathbf{I}e^{j\omega t}\} = \sqrt{2}I\sin(\omega t - \varphi) \qquad \text{equation 2.43}$$

2.8 COMPLEX IMPEDANCE

Let us consider an AC electrical circuit, whose voltage and current complex amplitudes are:

$$\begin{aligned} \mathbf{V} &= Ve^{j\alpha} \\ \mathbf{I} &= Ie^{j\beta} \end{aligned} \qquad \text{equation 2.44}$$

It is common practice to consider the angle’s reference located in the voltage complex amplitude, therefore the voltage initial phase is usually assumed to be 0. Here, for generality, we have considered it to be equal to α.

As seen before, equation 2.44 contains all the necessary information to define the phasors **I** and **V**, as well as the time evolution *i*(*t*) and *v*(*t*), providing the angular frequency is known, which is common.

The ratio between the complex amplitude of the voltage and the complex amplitude of the current is a complex number known as complex impedance and denoted as **Z**:

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{Ve^{j\alpha}}{Ie^{j\beta}} = \frac{V}{I}e^{j(\alpha-\beta)} = Ze^{j\varphi} = Z\cos\varphi + jZ\sin\varphi = R + jX \qquad \text{equation 2.45}$$

The impedance modulus is $Z=V/I$ (ratio between the respective RMS values) and the impedance angle, φ, is the difference between the voltage angle (α) and the current angle (β). On the other hand, the real part of an impedance is the resistance and the imaginary part is the reactance.

The complex impedance may be regarded as an operator that allows to convert one quantity into another one.

The phi angle (the impedance angle) is a very important angle. Its cosine is called power factor:

$$pf = \cos\varphi \qquad \text{equation 2.46}$$

Phi ranges from $-\pi/2$ (pure capacitive circuit) to $\pi/2$ (pure inductive circuit); if $\varphi = 0$, the circuit is resistive. We will retake this subject later on.

The reciprocal (inverse) of the complex impedance is the complex admittance, denoted by **Y**:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{\mathbf{I}}{\mathbf{V}} = \frac{Ie^{j\beta}}{Ve^{j\alpha}} = \frac{I}{V}e^{j(\beta-\alpha)} == \frac{1}{Z}e^{-j\varphi} = Ye^{-j\varphi} = G + JB \qquad \text{equation 2.47}$$

The real part of the admittance is the conductance and the imaginary part is the susceptance. The relationship between impedance and admittance can be further developed as:

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = \frac{1}{R+jX} = \frac{R-jX}{(R+jX)(R-jX)} = \frac{R}{R^2+X^2} - j\frac{X}{R^2+X^2} = G + jB \qquad \text{equation 2.48}$$

We highlight that neither the impedance, neither the admittance, represent sinusoidal waves. They are just complex numbers.

2.9 APPLICATION TO BASIC AC CIRCUITS

We will now apply the complex amplitude method to the pure resistive, pure inductive and pure capacitive circuits that we have studied before. As usually, we consider that the voltage holds the angles' reference, therefore the initial phase of the voltage is zero.

We recall that in a pure resistive circuit the voltage and current are in phase. As so, we can write the respective complex amplitudes as:

$$\mathbf{V_R} = V_R e^{j0} = V_R$$
$$\mathbf{I_R} = I_R e^{j0} = I_R \qquad \text{equation 2.49}$$

The complex impedance is:

$$\mathbf{Z_R} = \frac{\mathbf{V_R}}{\mathbf{I_R}} = \frac{V_R}{I_R} = R \qquad \text{equation 2.50}$$

In a pure resistive circuit, the complex impedance is the so-called resistance, which is a real number (see Figure 2-17). In this case, the phi (φ) angle is zero.

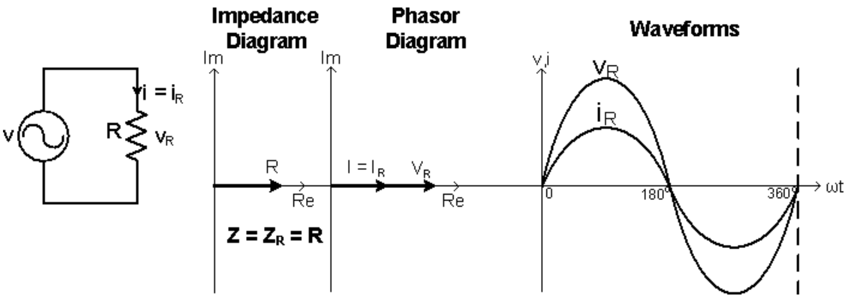


Figure 2-17: Voltage, current and impedance in a pure resistive circuit. Source: http://macao.communications.museum/eng/exhibition/secondfloor/MoreInfo/2_4_4_PhaseShift.html.

As far as a pure inductive circuit is concerned, the current lags the voltage by 90º. Therefore, we can write:

$$\mathbf{V_L} = V_L e^{j0} = V_L$$
$$\mathbf{I_L} = I_L e^{-j\frac{\pi}{2}} \qquad \text{equation 2.51}$$

The complex impedance is (take note that $j\omega L = \omega L e^{j\pi/2}$):

$$\mathbf{Z_L} = \frac{\mathbf{V_L}}{\mathbf{I_L}} = \frac{V_L}{I_L} e^{j\frac{\pi}{2}} = \omega L e^{j\frac{\pi}{2}} = j\omega L \qquad \text{equation 2.52}$$

Please take note that in a pure inductive circuit, the phi angle (the impedance's angle) is 90º.

Let us recall here for convenience the fundamental equation of the pure inductive circuit (equation 2.8):

$$v_L(t) = L\frac{di_L}{dt} \qquad \text{equation 2.53}$$

In terms of complex amplitudes, from equation 2.52, one can write:

$$\mathbf{V_L} = \mathbf{Z_L I_L} = j\omega L \mathbf{I_L} \qquad \text{equation 2.54}$$

Comparing equation 2.53 and equation 2.54, we conclude that in the complex amplitude framework the derivative *d/dt* is translated by the operator $j\omega$. As a matter of fact, when we multiply the phasor $\mathbf{I_L}$ by $j\omega L$, this is equivalent to rotate phasor $\mathbf{I_L}$ by 90º in the anti-clockwise direction, therefore obtaining phasor $\mathbf{V_L}$ (see Figure 2-18, but please disregard the power curve, we will get back to this concept later on).

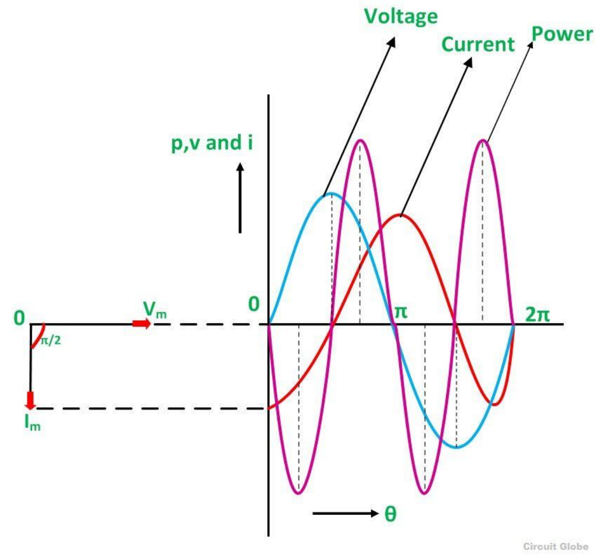


Figure 2-18: Voltage, current and impedance in a pure inductive circuit. Source : https://circuitglobe.com/what-is-pure-inductive-circuit.html.

In what concerns the pure capacitive circuit, it was previously demonstrated that the current leads the voltage by 90º. Therefore, the complex amplitudes of the voltage and current are given by:

$$\mathbf{V_C} = V_C e^{j0} = V_C$$
$$\mathbf{I_C} = I_C e^{j\frac{\pi}{2}} \qquad \text{equation 2.55}$$

The complex impedance is take note that $-j(1/\omega C) = (1/j\omega C) = (1/\omega C)e^{-j\pi/2}$):

$$\mathbf{Z_C} = \frac{\mathbf{V_C}}{\mathbf{I_C}} = \frac{V_C}{I_C} e^{-j\frac{\pi}{2}} = \frac{1}{\omega C} e^{-j\frac{\pi}{2}} = -j\frac{1}{\omega C} \qquad \text{equation 2.56}$$

It is apparent that the phi angle is –90º in a pure capacitive circuit.

In a pure capacitive circuit, it is common to express the relationship between the complex amplitudes of the voltage and current in the opposite way, leading to the complex admittance:

$$\mathbf{Y_C} = \frac{\mathbf{I_C}}{\mathbf{V_C}} = \frac{I_C}{V_C} e^{j\frac{\pi}{2}} = \omega C e^{j\frac{\pi}{2}} = j\omega C \qquad \text{equation 2.57}$$

At this point, we recall the fundamental equation of the pure capacitive circuit, for convenience (equation 2.19):

$$i_C(t) = C\frac{dv_C}{dt} \qquad \text{equation 2.58}$$

We can write equation 2.57 in the form:

$$\mathbf{I_C} = j\omega C\mathbf{V_C} \qquad \text{equation 2.59}$$

Comparing equation 2.58 and equation 2.59, we conclude again that the multiplication by the operator $j\omega$ performs a 90° rotation in the anticlockwise direction.

Figure 2-19 shows the representation of the voltage and current (static) phasors, as well as the time evolution of these two quantities in a pure capacitive circuit (again disregard the power curve, at this stage).

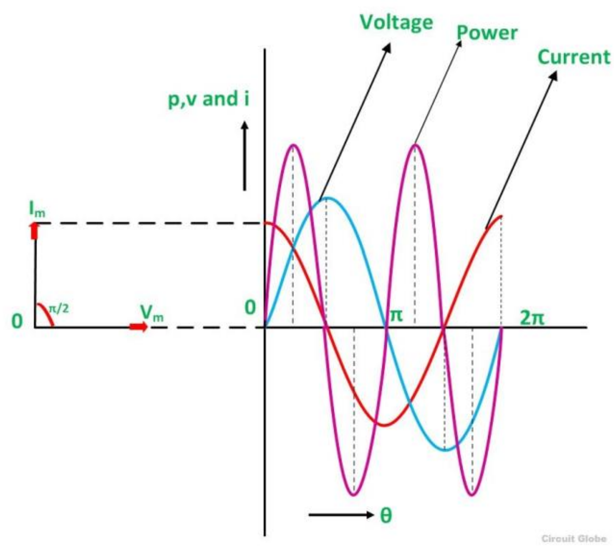


Figure 2-19: Voltage, current and impedance in a pure capacitive circuit. Source : https://circuitglobe.com/what-is-pure-capacitor-circuit.html.

2.10 RLC Circuit

With what we have learnt so far, we are now able to solve the RLC series circuit, presented in Figure 2-20.

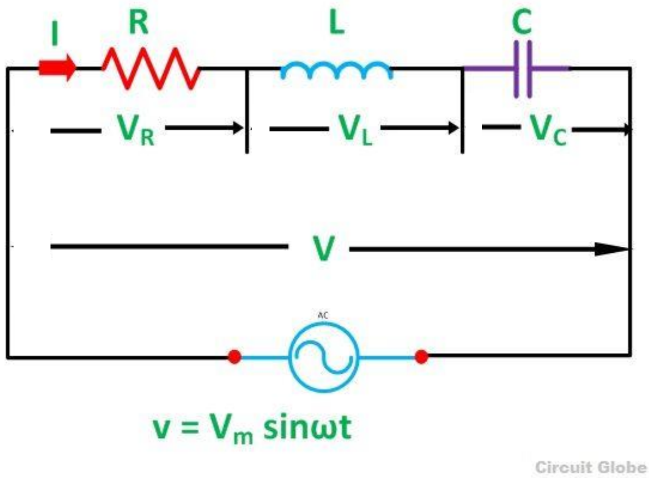


Figure 2-20: RLC series circuit. Source: https://circuitglobe.com/what-is-rlc-series-circuit.html.

At this stage, it should be apparent that it is equivalent to write the time evolution equations of the voltage and current or the respective complex amplitudes. Of course, we will use the complex amplitudes because it is much easier.

KVL[26] allows us to write:

$$\mathbf{V} = \mathbf{V_R} + \mathbf{V_L} + \mathbf{V_C} \qquad \text{equation 2.60}$$

And now we can write the complex amplitudes of the three voltages as (we recall that $-j = 1/j$):

$$\mathbf{V} = R\mathbf{I} + j\omega L\mathbf{I} + \frac{1}{j\omega C}\mathbf{I} = \left(R + j\omega L - j\frac{1}{\omega C}\right)\mathbf{I} = \mathbf{ZI} \qquad \text{equation 2.61}$$

The equivalent impedance of the RLC series circuit is:

$$\mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right) = R + jX \qquad \text{equation 2.62}$$

The same conclusion could be immediately reached if we recall a basic electrical knowledge stating that the equivalent impedance of a set of series-connected impedances is equal to the sum of the individual impedances:

$$\mathbf{Z_{eq}} = \mathbf{Z_1} + \mathbf{Z_2} + \mathbf{Z_3} = \mathbf{Z_R} + \mathbf{Z_L} + \mathbf{Z_C} \qquad \text{equation 2.63}$$

[26] Kirchoff Laws – KVL and KCL – also apply to complex amplitudes: in a closed loop $\sum \mathbf{V_i} = 0$ (KVL) and in a node $\sum \mathbf{I_i} = 0$ (KCL).

We will use equation 2.62 to state that an impedance is a complex number whose real part is called resistance and the imaginary part is the reactance. In this case, the equivalent circuit's resistance and reactance are:

$$\begin{aligned} R &= R \\ X &= \omega L - \frac{1}{\omega C} \end{aligned} \qquad \text{equation 2.64}$$

The impedance modulus and phase are, respectively (recall equation 2.32 and equation 2.45):

$$\begin{aligned} Z &= \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \\ \varphi &= \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \end{aligned} \qquad \text{equation 2.65}$$

As seen, reactance *X* depends on *L* and *C*; therefore, *X* can be negative, zero or positive, meaning that the behaviour of the RLC circuit can be capacitive, resistive or inductive, respectively.

The impedance, resistance and reactance are all measured in Ω – ohm. We highlight that the complex impedance is not a time varying quantity. It is given by the ratio of two complex amplitudes, and does not represent a quantity that is changing with time.

The complex amplitude of the current is given by:

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{Ve^{j0}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\, e^{j\varphi}} = \frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} e^{-j\varphi} \qquad \text{equation 2.66}$$

The correspondent current sine wave is:

$$i(t) = \sqrt{2}\frac{V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \sin(\omega t - \varphi) \qquad \text{equation 2.67}$$

Let us put some numbers on this. Consider that the RMS voltage is V = 230 V, f = 50 Hz, R = 10 Ω; L = 63.67 mH, C = 636.5 μF. Under these circumstances, $\omega L = 2\pi f L = X_L = 20\ \Omega$, $1/\omega C = X_C = 5\ \Omega$. The impedance will be:

$$\mathbf{Z} = 10 + j15 = 18.03e^{j56.31^\circ}\ \Omega$$

The current (in fact it is the complex amplitude of the current, but we will drop the "complex amplitude" mention when there no misunderstanding risk) is:

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = 12.76e^{-j56.31^\circ}\ \text{A}$$

considering, as usually, that the angles' reference is the zero voltage angle:

$$\mathbf{V} = 230e^{j0}\ \text{V}$$

As the current's phase is negative (–56.31º), it follows that the current lags the voltage. Therefore, the circuit is inductive.

The phi angle (remember that this is the impedance's angle) is positive (56.31º). The power factor is:

$$pf = \cos(56.31^\circ) = 0.55\ \text{ind.}$$

You may have noticed that we added "ind." (inductive) to the numerical value of the power factor. Why? Let us open a parenthesis to approach this.

Let us now assume that $X_L = 5\ \Omega$ and $X_C = 20\ \Omega$, while keeping $R = 10\ \Omega$. In this case:

$$\mathbf{Z_2} = 10 - j15 = 18.03e^{-j56.31^\circ}\ \Omega$$

and the current would be:

$$\mathbf{I_2} = \frac{\mathbf{V}}{\mathbf{Z_2}} = 12.76e^{j56.31^\circ}\ \text{A}$$

The current's initial angle is now 56.31º meaning that the current is leading the voltage (remember that the voltage's angle is 0º). The circuit is now said to be capacitive. However, notice that the numerical value of the power factor is the same. We therefore write ("cap." meaning capacitive):

$$pf_2 = \cos(-56.31^\circ) = 0.55\ \text{cap.}$$

Closing parenthesis and retaking the inductive circuit case, we can compute the three voltages:

$$\mathbf{V_R} = R\mathbf{I} = 10 \times 12.76e^{-j56.31^\circ} = 127.6e^{-j56.31^\circ} \text{ V}$$
$$\mathbf{V_L} = jX_L\mathbf{I} = 20e^{j90^\circ}12.76e^{-j56.31^\circ} = 255.16e^{j33.69^\circ} \text{ V}$$
$$\mathbf{V_C} = -jX_C\mathbf{I} = 5e^{-j90^\circ}12.76e^{-j56.31^\circ} = 63.79e^{-j146.31^\circ} \text{ V}$$

Of course, $\mathbf{V_R}$ is in phase with **I**, and $\mathbf{V_L}$ is advanced by 90º and $\mathbf{V_C}$ is delayed by 90º, both with respect to the current's complex amplitude **I**.

If we want to recover the time evolution of the current, we would write (56.31°=0.98rad):

$$i(t) = \sqrt{2}12.76\sin(100\pi t - 0.98) \text{ A}$$

We will now show that the alternative to solve the very same problem is much harder. Let us assume that we do not want to use the concepts of complex amplitude, and want to solve the RLC circuit the hard way. Looking at Figure 2-20 and recalling what we have learned so far, we can write, without complex amplitudes (the unknown is the current $i(t)$):

$$v(t) = Ri(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt \qquad \text{equation 2.68}$$

We expect the current to oscillate with the same frequency, but at a different phase, so we admit the solution (trial solution) would be of the form:

$$i(t) = A\sin\omega t - B\cos\omega t \qquad \text{equation 2.69}$$

Plugging in equation 2.68, we get:

$$v(t) = R(A\sin\omega t - B\cos\omega t) + L\frac{d(A\sin\omega t - B\cos\omega t)}{dt} + \frac{1}{C}\int(A\sin\omega t - B\cos\omega t)dt \qquad \text{equation 2.70}$$

or:

$$\begin{aligned}\sqrt{2}V\sin\omega t &= R(A\sin\omega t - B\cos\omega t) + \omega L(A\cos\omega t + B\sin\omega t) + \\ &+ \frac{1}{\omega C}(-A\cos\omega t - B\sin\omega t)\end{aligned} \qquad \text{equation 2.71}$$

The sine and cosine coefficients must be equal in both terms. Therefore:

$$\begin{aligned}\sqrt{2}V &= AR + B\omega L - B\frac{1}{\omega C} \\ 0 &= -BR + A\omega L - A\frac{1}{\omega C}\end{aligned} \qquad \text{equation 2.72}$$

Now, from the second equation, we write:

$$B = \frac{A}{R}\left(\omega L - \frac{1}{\omega C}\right) \qquad \text{equation 2.73}$$

and replacing on the first, and back substituting, yields to:

$$A = \frac{\sqrt{2}VR}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$
$$B = \frac{\sqrt{2}V\left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \qquad \text{equation 2.74}$$

We will now factoring out our trial solution in equation 2.69 as:

$$i(t) = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}}\sin\omega t - \frac{B}{\sqrt{A^2 + B^2}}\cos\omega t\right) \qquad \text{equation 2.75}$$

$\frac{A}{\sqrt{A^2 + B^2}}$ and $\frac{B}{\sqrt{A^2 + B^2}}$ are the cosine and the sine of a φ angle defined on a right triangle with legs *A* and *B*. This angle is given as:

$$\varphi = \tan^{-1}\left(\frac{B}{A}\right) = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right) \qquad \text{equation 2.76}$$

and equation 2.75 can be written as:

$$i(t) = \sqrt{A^2 + B^2}\left(\cos\varphi\sin\omega t - \sin\varphi\cos\omega t\right) = \sqrt{A^2 + B^2}\sin(\omega t - \varphi) \qquad \text{equation 2.77}$$

On the other way, note that:

$$\sqrt{A^2 + B^2} = \frac{\sqrt{2}V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \qquad \text{equation 2.78}$$

which allow us to finally conclude that:

$$i(t) = \frac{\sqrt{2}V}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\sin(\omega t - \varphi) \qquad \text{equation 2.79}$$

The reader is invited to compare this equation with equation 2.67. They are equal, as expected, but the calculations burden involved in the hard way is much higher than when we have used the complex amplitude approach. We hope that, at this time, the reader is convinced that the complex amplitude method is the smart way of solving AC circuits.

Now let us take a look at the RLC parallel circuit depicted in Figure 2-21. Of course, we will use, from now on, only the complex amplitude method.

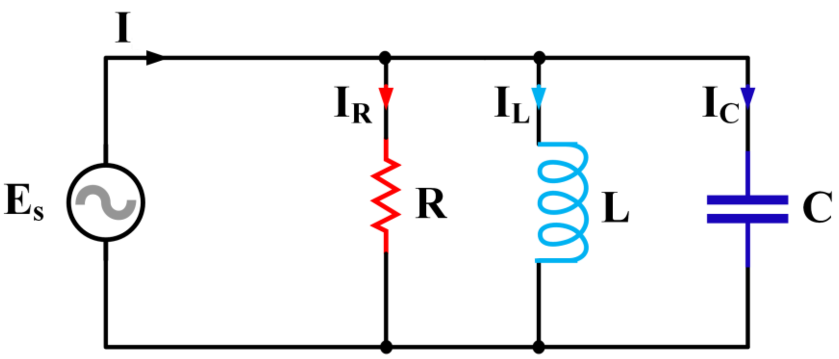


Figure 2-21: RLC parallel circuit. Source: http://electricalacademia.com/basic-electrical/passive-components-ac-circuits-equations/attachment/figure-11-parallel-rlc-circuit/.

We will keep working with complex amplitudes, because the computations are easier using this AC circuits solving method. Using Kirchoff Current Law (KCL), we can write about the total current that:

$$\mathbf{I} = \mathbf{I_R} + \mathbf{I_L} + \mathbf{I_C} \qquad \text{equation 2.80}$$

We have learned to relate the complex amplitudes of current and voltage in resistive, inductive and capacitive circuits. As so, equation 2.80 becomes (again we recall that $-j = 1/j$):

$$\mathbf{I} = \frac{1}{R}\mathbf{V} + \frac{1}{j\omega L}\mathbf{V} + j\omega C\mathbf{V} = \left(\frac{1}{R} - j\frac{1}{\omega L} + j\omega C\right)\mathbf{V} = \mathbf{YV} \qquad \text{equation 2.81}$$

The equivalent admittance of the circuit is:

$$\mathbf{Y} = \left[\frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)\right] = G + jB \qquad \text{equation 2.82}$$

When the elements of the circuit are parallel-connected it is more comfortable to work with admittances instead of impedances. The equivalent admittance of a set of admittances connected in parallel is the sum of all the individual admittances.

$$\mathbf{Y_{eq}} = \mathbf{Y_1} + \mathbf{Y_2} \quad \text{equation 2.83}$$

This is equivalent to:

$$\begin{aligned} \frac{1}{\mathbf{Z}} &= \frac{1}{\mathbf{Z_1}} + \frac{1}{\mathbf{Z_2}} \\ \frac{\mathbf{1}}{\mathbf{Z_{eq}}} &= \frac{\mathbf{Z_2} + \mathbf{Z_1}}{\mathbf{Z_1 Z_2}} \\ \mathbf{Z_{eq}} &= \frac{\mathbf{Z_1 Z_2}}{\mathbf{Z_1} + \mathbf{Z_2}} \end{aligned} \quad \text{equation 2.84}$$

An admittance is a complex number whose real part is called conductance (*G*) and the imaginary part is the susceptance (*B*). In this case, the conductance and susceptance are respectively:

$$\begin{aligned} G &= \frac{1}{R} \\ B &= \omega C - \frac{1}{\omega L} \end{aligned} \quad \text{equation 2.85}$$

The admittance modulus and phase are, respectively:

$$\begin{aligned} Y &= \sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2} \\ \theta &= \tan^{-1}\left[R\left(\omega L - \frac{1}{\omega C}\right)\right] \end{aligned} \quad \text{equation 2.86}$$

The admittance, conductance and susceptance are all measured in S – Siemens.

Let us retake the same numbers we have considered before: the RMS voltage is V = 230 V, f = 50 Hz, R = 10 Ω; L = 63.7 mH, C = 634 μF. Under these circumstances, $1/\omega L = B_L$= 0.05 S, $\omega C = B_C$ = 0.2 S. The admittance will be:

$$\mathbf{Y} = 0.1 + j0.15 = 0.18e^{j56.31^\circ}\ S$$

The current is (we recall that $\mathbf{V}=230e^{j0}$ V):

$$\mathbf{I} = \mathbf{YV} = 41.46e^{j56.31^\circ}\ \mathrm{A}$$

In this case, the current's phase is positive (56.31º), meaning that the current leads the voltage. Therefore, the circuit is capacitive.

The power factor is (recall that the phi angle is negative):

$$pf = \cos(-56.31^{o}) = 0.55 \text{ cap.}$$

We can now compute the three currents:

$$\mathbf{I_R} = G\mathbf{V} = 23e^{j0} \text{ A}$$
$$\mathbf{I_L} = -jB_L\mathbf{V} = 11.50e^{-j90^{\circ}} \text{ A}$$
$$\mathbf{I_C} = jX_C\mathbf{V} = 45.40e^{j90^{\circ}} \text{ A}$$

Of course, $\mathbf{I_R}$ is in phase with **V**, and $\mathbf{I_L}$ is delayed by 90º and $\mathbf{I_C}$ is advanced by 90º, both with respect to the voltage's complex amplitude **V**.

If we want to recover the time evolution of the current, we would write (56.31º=0.98rad):

$$i(t) = \sqrt{2}\,41.46 \sin(100\pi t + 0.98) \text{ A}$$

Example 2—1:

A sinusoidal voltage V = 48 V, f = 50 Hz, feeds the circuit depicted in Figure 2-22. The given parameters of the circuit are: R = 2 Ω; L = 0.05 H, C = 900 μF. Compute:

1. *The equivalent complex impedance of the circuit.*
2. *Current **I** and voltages $\mathbf{V_R}$ and $\mathbf{V_X}$.*
3. *Currents $\mathbf{I_L}$ and $\mathbf{I_C}$.*

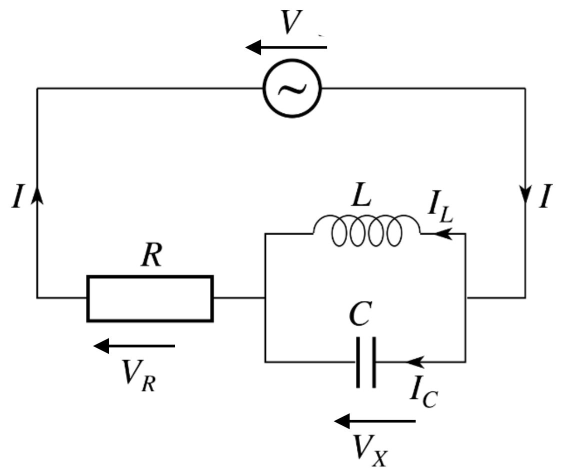


Figure 2-22: Circuit of Example 2—1: LC parallel with losses circuit. Source: http://www.met.reading.ac.uk/pplato2/h-flap/phys5_4.html.

1. Let us begin by computing the impedances of the elements, which are:

$$R = 2\,\Omega$$
$$X_L = \omega L = 15.71\,\Omega$$
$$X_C = \frac{1}{\omega C} = 3.54\,\Omega$$

Therefore, the equivalent impedance is (we recall that $j^2 = -1$ and equation 2.84):

$$\mathbf{Z} = R + jX_{LC} = R + \frac{-jX_L jX_C}{jX_L - jX_C} = R - j\left(\frac{X_L X_C}{X_L - X_C}\right) \qquad \textit{equation 2.87}$$

Taking the imaginary part of and replacing the expressions of the reactances, we obtain:

$$\frac{X_L X_C}{X_L - X_C} = \frac{\dfrac{\omega L}{\omega C}}{\omega L - \dfrac{1}{\omega C}} = \frac{\dfrac{L}{C}}{\dfrac{\omega^2 LC - 1}{\omega C}} = \frac{\omega L}{\omega^2 LC - 1} \qquad \textit{equation 2.88}$$

C and L could be such that the denominator is zero ($\omega^2 LC = 1$). In this case the reactance would be infinity, meaning an open circuit. This is not the case of the proposed problem.

Replacing values, we obtain:

$$\mathbf{Z} = 2 - j4.56 = 4.98e^{-j66.34°} \ \Omega$$

*2. Current **I** can be easily computed by:*

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{48e^{j0}}{4.98e^{-j66.34°}} = 9.63e^{j66.34°} \ \text{A}$$

We assumed that the reference for the angles is, as usually, located in the voltage. The circuit is capacitive, because the phi angle is negative (the current leads the voltage).

The voltage at the resistance terminals, $\mathbf{V_R}$, is:

$$\mathbf{V_R} = \mathrm{R}\mathbf{I} = 19.26e^{j66.34°} = 7.73 + j17.64 \ \text{V}$$

This voltage is in phase with the current.

As for voltage $\mathbf{V_X}$, we may write, using KVL:

$$\mathbf{V} - \mathbf{V_R} - \mathbf{V_X} = 0$$
$$\mathbf{V_X} = \mathbf{V} - \mathbf{V_R} = 40.27 - j17.64 = 43.96e^{-j23.66°} \ \text{V}$$

Another way of obtaining the same result is:

$$\mathbf{V_X} = jX_{LC}\mathbf{I} = 4.56e^{-j90°}9.63e^{j66.34°} = 43.96e^{-j23.66°} \ \text{V}$$

3. Currents $\mathbf{I_L}$ and $\mathbf{I_C}$ are respectively (we recall that $jX_L = X_L e^{j90°}$):

$$\mathbf{I_L} = \frac{\mathbf{V_X}}{jX_L} = 2.80e^{-j113.66°}\ \text{A}$$
$$\mathbf{I_C} = \frac{\mathbf{V_X}}{jX_C} = 12.43e^{j66.34°}\ \text{A}$$

If required, we are now able to compute the time dependent equations of voltages and currents as:

$$v(t) = \sqrt{2}48\sin(100\pi t + 0)\ \text{V}$$
$$v_R(t) = \sqrt{2}19.26\sin(100\pi t + 1.16)\ \text{V}$$
$$v_X(t) = \sqrt{2}43.96\sin(100\pi t - 0.41)\ \text{V}$$
$$i(t) = \sqrt{2}9.63\sin(100\pi t + 1.16)\ \text{A}$$
$$i_L(t) = \sqrt{2}2.80\sin(100\pi t - 1.98)\ \text{A}$$
$$i_C(t) = \sqrt{2}12.43\sin(100\pi t + 1.16)\ \text{A}$$

Figure 2-23 depicts the correspondent graphics.

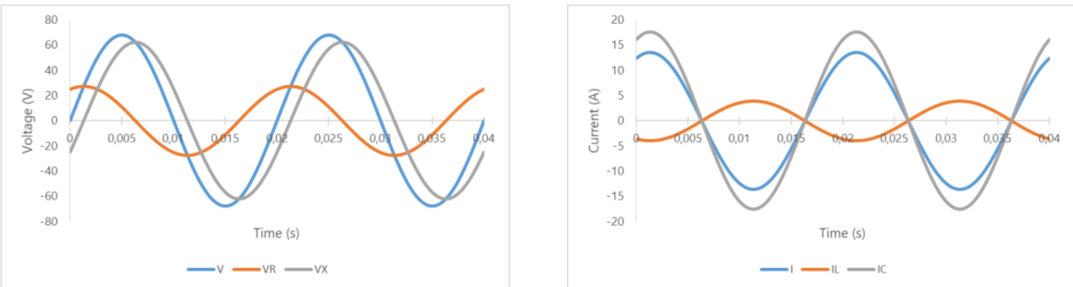


Figure 2-23: Time evolution of voltages (on the left) and currents (on the left) on the circuit of Example 2—1.

3 POWER IN AC CIRCUITS

When to a capacitor of capacitance C is applied a voltage v, an energy W_e is supplied to the capacitor:

$$W_e = \frac{1}{2}Cv^2 \qquad \text{equation 3.1}$$

This energy is stored in the electrical field created by the charges in the conductive plates of the capacitor.

Moreover, when a current i flows into an inductor, a magnetic field is created in the space outside the conductor. In this magnetic field, the magnetic energy W_m is stored:

$$W_m = \frac{1}{2}Li^2 \qquad \text{equation 3.2}$$

where L is the inductance of the inductor.

In AC circuits, the voltages and currents are periodically changing with time. This implies that the energies stored in the magnetic and electric fields are periodically changing, giving rise to energy exchanges that do not correspond to irreversible energy delivery to the loads. Loads are the usual designation for the electricity consumption at the end-user level. The source is the electricity feeding system, normally the public electricity grid.

Therefore, in the AC circuits we can distinguish three types of power, in each time instant:

- The so-called active power, which is related to the energy consumption in the load, or, better saying, to the conversion of electricity in other forms of energy that takes place in the load.
- A power given by:

  $$P_c = \frac{dW_e}{dt} = Cv\frac{dv}{dt} \qquad \text{equation 3.3}$$

  which corresponds to the variation of the energy stored in the electrical fields between the load and the source.
- A power given by:

$$P_m = \frac{dW_m}{dt} = Li\frac{di}{dt} \qquad \text{equation 3.4}$$

which corresponds to the variation of the energy stored in the magnetic fields between the load and the source.

3.1 Active Power and Reactive Power

Let us suppose that we apply a sinusoidal voltage to a whatever circuit:

$$v(t) = \sqrt{2}V\sin(\omega t) \qquad \text{equation 3.5}$$

and that the corresponding current is:

$$i(t) = \sqrt{2}I\sin(\omega t - \varphi) \qquad \text{equation 3.6}$$

The power delivered to the circuit is:

$$p(t) = v(t)i(t) = 2VI\sin(\omega t)\sin(\omega t - \varphi) \qquad \text{equation 3.7}$$

Applying some trigonometric rules, after some manipulation we obtain:

$$p(t) = VI\cos\varphi(1-\cos(2\omega t)) - VI\sin\varphi\sin(2\omega t) = p_a(t) + p_r(t) \qquad \text{equation 3.8}$$

A graphic representation is given in Figure 3-1 of $v(t)$, $i(t)$ and $p(t)$ (V = 230 V, I = 100 A, f = 50 Hz, $\varphi = \pi/3$).

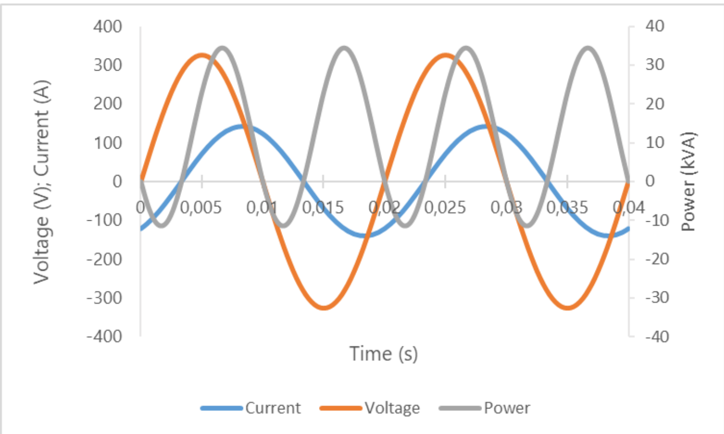


Figure 3-1: Voltage, current and power in a AC circuit.

Figure 3-2 depicts the graphical representation of $p(t)$, $p_a(t)$ and $p_r(t)$.

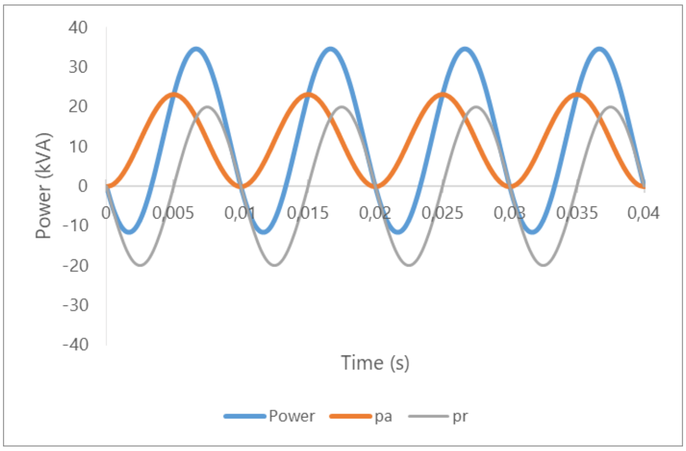


Figure 3-2: Power in a AC circuit.

Power $p(t)$ changes periodically in time and can be divided into two parcels as shown in equation 3.8. The angular speed of the first parcel, $p_a(t)$, is 2ω and the mean value is:

$$P = VI\cos\varphi = \{p_a(t)\}_{avg} \qquad \text{equation 3.9}$$

We denote the mean value of $p(t)$ by P. It is called active power and is measured in W – Watt, its multiples used in power systems being kW (1 kW = 10^3 W) and MW (1 MW = 10^6 W). The active power may be also called real power. The integration in time of the active power (the active power is constant in a particular time instant, but changes over time) is the energy that is irreversibly delivered to the load circuit:

$$E = \int_{t_1}^{t_2} P(t)dt \qquad \text{equation 3.10}$$

This energy will produce work or heat; this is why it is called active energy.

One note on the practical use of equation 3.10, when the active power can be considered constant over a time interval. In this case:

$$E = P(t_2 - t_1) = P\Delta t \qquad \text{equation 3.11}$$

If P is constant over two time intervals (P_1 in the first interval and P_2 in the second), it is:

$$E = P_1(t_2 - t_1) + P_2(t_3 - t_2) = P_{avg}(t_3 - t_1) \qquad \text{equation 3.12}$$

in which P_{avg} is the average power between P_1 and P_2.

This clearly shows why electricity is usually measured in MWh. 1 MWh corresponds to an average power of 1 MW used during 1 hour.

The angular speed of the second parcel, $p_r(t)$, is also 2ω, but its mean value is zero. The maximum value of the second parcel is:

$$Q = VI\sin\varphi = \{p_r(t)\}_{MAX} \qquad \text{equation 3.13}$$

Q is called the reactive power and is measured in var –Volt-Ampère reactive. The multiples used in power systems are kvar and Mvar. It corresponds to the energy that oscillates between the voltage source and the electric and magnetic fields of the load circuit. This is related to the two power parcels, P_c and P_m, that we have identified previously. The time integration of $p_r(t)$ over an integer number of cycles is zero, the reactive power being defined as the peak-value of $p_r(t)$. We call this power as reactive power in the sense that it is a non-active power.

From equation 3.9 and equation 3.13, an important relationship between active and reactive power may be derived:

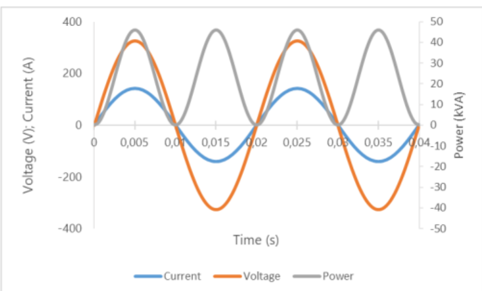
$$\frac{Q}{P} = \frac{VI\sin\varphi}{VI\cos\varphi} = \tan\varphi \qquad \text{equation 3.14}$$

3.2 Active Power and Reactive Power in the Basic AC Circuits

In the pure resistive circuit, it is $\varphi = 0$, therefore (take note that $V_R = RI_R$):

$$\begin{aligned} &p(t) = p_a(t) \\ &p_r(t) = 0 \\ &P = VI = RI^2 \\ &Q = 0 \end{aligned} \qquad \text{equation 3.15}$$

The relevant graphics are shown in Figure 3-3 (V = 230 V, I = 100 A, f = 50 Hz, φ = 0).



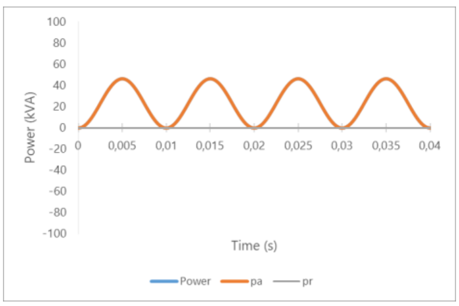


Figure 3-3: Time evolution of the relevant quantities (voltage, current and power) in a pure resistive circuit.

We highlight that the energy flows always from the voltage source to the resistive load; there are no energy oscillations.

As for the pure inductive circuit, it is $\varphi = \pi/2$, which originates (recall that $V_L = \omega L I_L$):

$$\begin{aligned} p(t) &= p_r(t) \\ p_a(t) &= 0 \\ P &= 0 \\ Q &= VI = \omega L I^2 \end{aligned} \qquad \text{equation 3.16}$$

Figure 3-4 shows the quantities of interest regarding the pure inductive circuit (V = 230 V, I = 100 A, f = 50 Hz, $\varphi = \pi/2$).

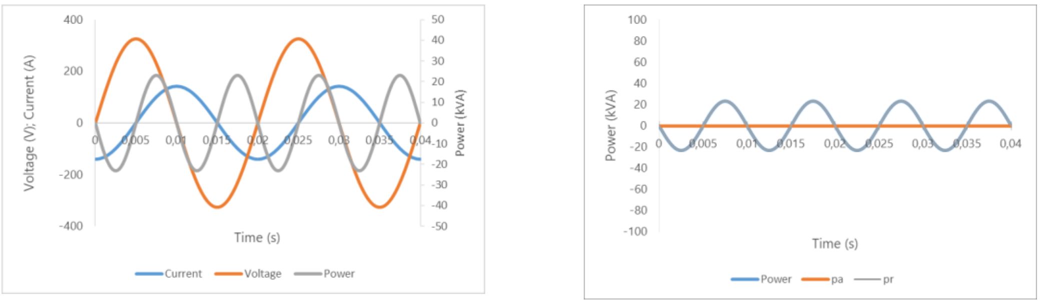


Figure 3-4: Time evolution of the relevant quantities (voltage, current and power) in a pure inductive circuit.

The coil, that composes the pure inductive circuit, sometimes supplies energy to the source, other times receives energy from it, with a null mean value. It follows that no energy is supplied or received irreversibly. Now you should be able to understand Figure 2-18.

We note that the reactive power is positive. This is the usual conventional: the reactive power is positive when the load exchanges magnetic energy with the source. By convention, we say that reactive power is absorbed in an inductive circuit.

As far as the pure capacitive circuit is concerned, we recall that $\varphi = -\pi/2$. Under these circumstances, we can write (do not forget that $I_C = \omega C V_C$):

$$\begin{aligned} p(t) &= p_r(t) \\ p_a(t) &= 0 \\ P &= 0 \\ Q &= -VI = -\omega C V^2 \end{aligned} \qquad \text{equation 3.17}$$

The monitored quantities are displayed in Figure 3-5 (V = 230 V, I = 100 A, f = 50 Hz, $\varphi = -\pi/2$).

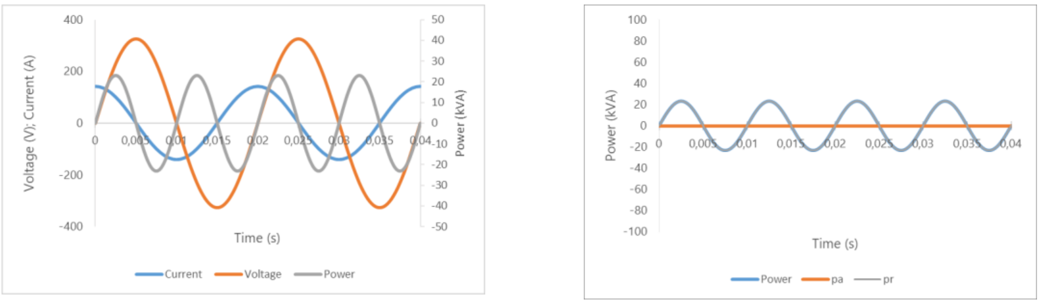


Figure 3-5: Time evolution of the relevant quantities (voltage, current and power) in a pure capacitive circuit.

Sometimes the capacitor, that composes the pure capacitive circuit, behaves like a load (stores electrical energy), other times, it behaves as a generator (supplies electrical energy), with null mean value. The conclusion is that no electrical energy is irreversibly stored in the capacitor. You should take a new look at Figure 2-19.

Now the reactive power is negative. This is the usual convention when electrical energy is exchanged. In this case, by convention, we say that reactive power is supplied in a pure capacitive circuit.

In Table 3-1, we summarize the main aspects of we have learned so far about active and reactive power in basic AC circuits.

Table 3-1: Phi angle, active power and reactive power for several load types.

Load type	Phi angle	Active Power	Reactive Power
R	$\varphi = 0$	$P = VI > 0$	$Q = 0$
L	$\varphi = \pi/2$	$P = 0$	$Q = VI > 0$
C	$\varphi = -\pi/2$	$P = 0$	$Q = -VI < 0$
RL series	$0 < \varphi < \pi/2$	$P = VI\cos\varphi > 0$	$Q = VI\sin\varphi > 0$
RL parallel	$0 < \varphi < \pi/2$	$P = VI\cos\varphi > 0$	$Q = VI\sin\varphi > 0$
RC series	$-\pi/2 < \varphi < 0$	$P = VI\cos\varphi > 0$	$Q = VI\sin\varphi < 0$
RC parallel	$-\pi/2 < \varphi < 0$	$P = VI\cos\varphi > 0$	$Q = VI\sin\varphi < 0$

Example 3—1:

Retake the circuit of Example 2—1 depicted in Figure 2-22 and compute:

1. *The active power, the reactive power and the power factor of the circuit.*
2. *The reactive power "absorbed" by the coil and the reactive power "supplied" by the capacitor.*

Recall that V = 48 V, f = 50 Hz, R = 2 Ω; L = 0.05 H, C = 900 μF.

We recall that in Example 2—1 we computed:

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{48e^{j0}}{4.98e^{-j66.34°}} = 9.63e^{j66.34°} \text{ A}$$

$$\mathbf{V_X} = jX_{LC}\mathbf{I} = 4.56e^{-j90°}9.63e^{j66.34°} = 43.96e^{-j23.66°} \text{ V}$$

$$\mathbf{I_L} = \frac{\mathbf{V_X}}{jX_L} = 2.80e^{-j113.66°} \text{ A}$$

1.

$$\begin{aligned} P &= VI\cos\varphi = 48\times 9.63\cos(-66.34°) = 185.55 \text{ W} \\ Q &= VI\sin\varphi = 48\times 9.63\sin(-66.34°) = -423.46 \text{ var} \\ pf &= \cos(-66.34°) = 0.40 \text{ cap.} \end{aligned}$$

2.

$$\begin{aligned} Q_L &= \omega L I_L^2 = 123.05 \text{ var} \\ Q_C &= -\omega C V_X^2 = -546.52 \text{ var} \\ Q &= Q_L + Q_C = 423.46 \text{ var} \end{aligned}$$

3.3 Complex Power

Let us consider a voltage and a current, whose complex amplitudes are given by:

$$\begin{aligned} \mathbf{V} &= Ve^{j\alpha} \\ \mathbf{I} &= Ie^{j\beta} \end{aligned} \qquad \text{equation 3.18}$$

We define the quantity "complex power" as:

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* \qquad \text{equation 3.19}$$

where **I*** is the complex conjugate of **I**:

$$\mathbf{I}^* = Ie^{-j\beta} \qquad \text{equation 3.20}$$

Developing equation 3.19, we obtain (we recall that $\alpha-\beta=\varphi$):

$$\mathbf{S} = \mathbf{VI}^* = Ve^{j\alpha}Ie^{-j\beta} = VIe^{j(\alpha-\beta)} = VIe^{j\varphi} \qquad \text{equation 3.21}$$

Recurring to Euler's formula, leads to:

$$\mathbf{S} = VIe^{j\varphi} = VI\cos\varphi + jVI\sin\varphi = P + jQ \qquad \text{equation 3.22}$$

The complex power is a complex number whose real part is the active power and the imaginary part is the reactive power.

The modulus (absolute value) of the complex power is the apparent power:

$$S = |\mathbf{S}| = VI = \sqrt{P^2 + Q^2} \qquad \text{equation 3.23}$$

The apparent power is measured in VA – Volt-Ampère, or kVA or MVA, for power systems applications.

3.4 Power Factor Correction

In a transmission line, the transmission losses (Joule losses) are given by (R_L is the line resistance):

$$P_L = R_L I^2 \qquad \text{equation 3.24}$$

Taking equation 3.23 into account, we can write:

$$P_L = R_L I^2 = R\frac{P^2 + Q^2}{V^2} \qquad \text{equation 3.25}$$

For instance, if R_L is the resistance of a power line that transmits power P, at voltage V, the transmissions losses are strongly dependent on reactive power flow Q. Minimum losses are achieved if Q = 0, meaning no reactive power flow in the line. As so, the reactive power needs should be locally generated, therefore avoiding reactive power flowing in the lines, at all, or at least, minimizing reactive power flow in order to achieve reduced losses.

Consider the circuit represented in Figure 3-6, but disregard the capacitor C, for the time being. The line resistance is R_L. The circuit is inductive (RL load), therefore the current is lagging the voltage, which we admit, as usually, to hold the zero reference angle. The phase shift is φ_1 and the RMS current is I_1, the circuit is consuming reactive power. The active power, reactive power and transmission losses are given, respectively, by:

$$\begin{aligned} P_1 &= VI_1 \cos\varphi_1 \\ Q_1 &= VI_1 \sin\varphi_1 \\ P_{L1} &= R_L I_1^2 \end{aligned} \qquad \text{equation 3.26}$$

Now consider that we install a capacitor *C* and we connect it in parallel with the RL load. As the capacitor is submitted to voltage **V**, a current $\mathbf{I_C}$ flows in the capacitor branch. Applying KCL, we obtain (note that these are the complex amplitudes of the currents):

$$\mathbf{I_2} = \mathbf{I_C} + \mathbf{I_1} \qquad \text{equation 3.27}$$

Current $\mathbf{I_C}$ has an angle equal to 90º, because a capacitive current leads the voltage by 90º. The new line current is $\mathbf{I_2}$ which has an angle φ_2. The phasor diagram is shown in Figure 3-6. Note that $I_2 < I_1$ and $\varphi_2 < \varphi_1$.

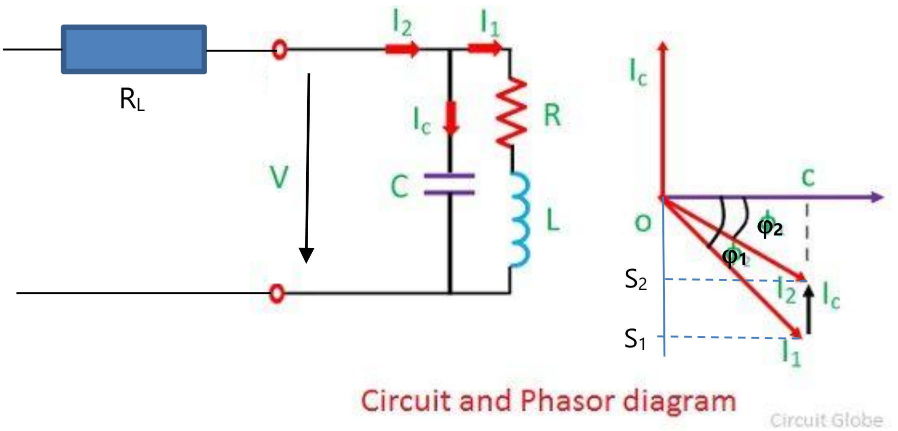


Figure 3-6: Power factor correction. Source: https://circuitglobe.com/power-factor-improvement.html.

From the phasor diagram of Figure 3-6, it is apparent that:

$$\begin{aligned} &I_2 \cos\varphi_2 = I_1 \cos\varphi_1 = \overline{OC} \\ &I_2 \sin\varphi_2 = \overline{OS_2} < I_1 \sin\varphi_1 = \overline{OS_1} \\ &I_2^2 < I_1^2 \end{aligned} \qquad \text{equation 3.28}$$

The new active power, reactive power and transmission losses are now given, respectively, by:

$$
\begin{aligned}
&P_2 = VI_2 \cos\varphi_2 = P_1 = VI_1 \cos\varphi_1 \\
&Q_2 = VI_2 \sin\varphi_2 < Q_1 = VI_1 \sin\varphi_1 \\
&P_{L2} = R_L I_2^2 < P_{L1} = R_L I_1^2
\end{aligned}
\qquad \text{equation 3.29}
$$

The reduction of the line current (from I_1 to I_2), while keeping the transmitted active power unchanged ($P_1 = P_2$) is called the power factor correction. A full corrected power factor is a power factor equal to 1. In this case, we have not performed full power factor correction, actually we have improved it (make it closer to 1):

$$
\begin{aligned}
&\varphi_2 < \varphi_1 \\
&pf_2 = \cos\varphi_2 > pf =_1 \cos\varphi_1
\end{aligned}
\qquad \text{equation 3.30}
$$

We note that the transmitted active power remains unchanged, while the transmission losses decreased due to the decrease in the line current. The reactive power flow across the line also decreased because the capacitor partly supplied the reactive power consumed by the coil. In the event the capacitor supplies all the reactive power absorbed by the coil, the reactive power demanded to the source would be zero, the power factor would be 1 and the losses would be minimum.

In order to keep the losses at the minimum possible, the grid operators want to avoid reactive power flow in their networks. Therefore, they charge heavily the consumers with a poor power factor. That is why the consumers install capacitors to locally supply the reactive power needs. This holds true mainly for industries, factories, medium/big consumers connected in MV or HV.

Example 3—2:

A device fed by a RMS voltage V = 230 V, f = 50 Hz, absorbs I_1 = 16 A with a power factor equal to pf_1 = 0.6 ind. Compute the capacitance of a capacitor to be connected in parallel with the device so that the power factor is corrected to pf_2 = 0.9 ind.

Please refer to the phasor diagram of Figure 3-6 and recall some trigonometric relationships.

$$I_2 \cos\varphi_2 = I_1 \cos\varphi_1$$

$$I_2 = I_1 \frac{\cos\varphi_2}{\cos\varphi_1} = I_1 \frac{pf_2}{pf_1} = 10.67 \text{ A}$$

$$\varphi_1 = \cos^{-1}(0.6) = 53.13^{\circ}$$

$$\varphi_2 = \cos^{-1}(0.9) = 25.84^{\circ}$$

$$I_C = I_1 \sin\varphi_1 - I_2 \sin\varphi_2 = 8.15 \text{ A}$$

$$C = \frac{I_C}{\omega V} = 112.8\ \mu\text{F}$$

The same result could be obtained, if we deal with the reactive power concept.

$$Q_1 = VI_1 \sin\varphi_1 = 230 \times 16 \sin(53.13^{\circ}) = 2944.0 \text{ var}$$

$$Q_2 = VI_2 \sin\varphi_2 = 230 \times 10.67 \sin(25.84^{\circ}) = 1069.38 \text{ var}$$

$$Q_C = Q_2 - Q_1 = -1874.62 \text{ var}$$

$$C = \frac{|Q_C|}{\omega V^2} = 112.8\ \mu\text{F}$$

4 BALANCED THREE-PHASE AC CIRCUITS

4.1 THREE-PHASE SYSTEMS

Consider three identical coils, a_1a_2, b_1b_2 and c_1c_2, as shown in Figure 4-1. a_1, b_1 and c_1 are the starting terminals, whereas a_2, b_2 and c_2 are the end terminals of the three coils. The spatial phase difference of 120° ($2\pi/3$ rad) has to be maintained between the starting terminals a_1, b_1 and c_1.

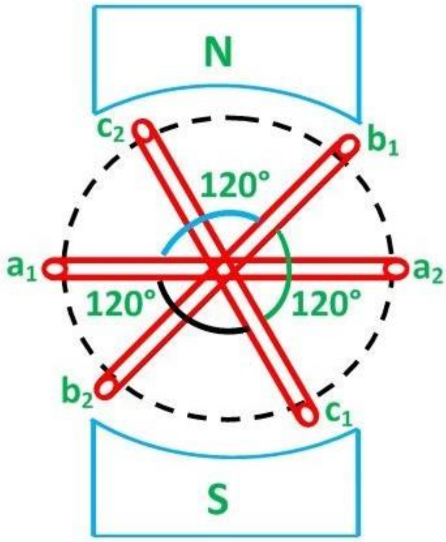


Figure 4-1: Three coils in a three-phase system. Source: https://circuitglobe.com/generation-of-3-phase-power-in-3-phase-circuits.html.

Now, let the three coils, mounted on the same axis, rotate at ω rad/s, inside the uniform magnetic field created by the two poles NS. We have seen in Chapter 2 that, for one rotating coil, a sinusoidal emf is induced in the coil. Similarly, three sinusoidal 120° phase-shifted emfs are now induced in the coils, whose equations are given by:

$$\begin{aligned} e_1(t) &= \sqrt{2}E\sin(\omega t) \\ e_2(t) &= \sqrt{2}E\sin(\omega t - \frac{2\pi}{3}) \\ e_3(t) &= \sqrt{2}E\sin(\omega t - \frac{4\pi}{3}) \end{aligned} \qquad \text{equation 4.1}$$

if the time origin for $e_1(t)$ is conveniently chosen so that at t = 0, $e_1(t)$ = 0. The time evolution of the emfs is shown in Figure 4-2.

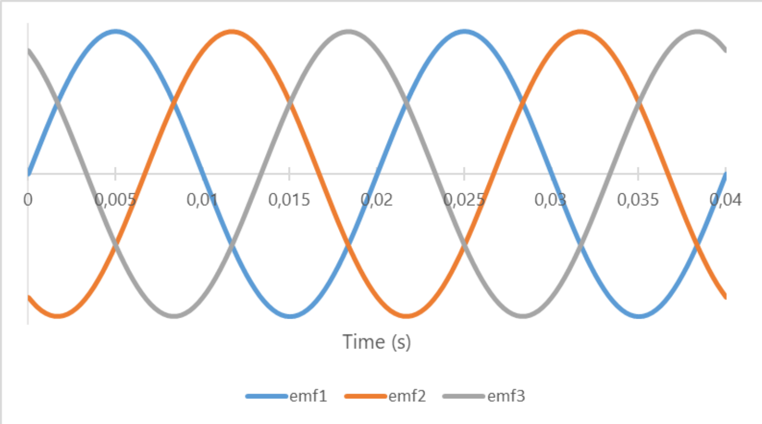


Figure 4-2: Emf in a three-phase system.

The emf is the no-load voltage. If loads are connected, the load voltages will slightly differ from the emfs, due to circuit losses, and will be given by:

$$\begin{aligned} \mathbf{V_1} &= Ve^{j0} \\ \mathbf{V_2} &= Ve^{-j120°} \\ \mathbf{V_3} &= Ve^{-j240°} \end{aligned} \qquad \text{equation 4.2}$$

If each coil is connected to an impedance $\mathbf{Z}=Ze^{j\varphi}$, the current (complex amplitude) in each coil will be:

$$\begin{aligned} \mathbf{I_1} &= \frac{\mathbf{V_1}}{\mathbf{Z}} = \frac{V}{Z}e^{-j\varphi} = Ie^{-j\varphi} \\ \mathbf{I_2} &= \frac{\mathbf{V_2}}{\mathbf{Z}} = \frac{V}{Z}e^{j(-120-\varphi)} = Ie^{-j(120+\varphi)°} \\ \mathbf{I_3} &= \frac{\mathbf{V_3}}{\mathbf{Z}} = \frac{V}{Z}e^{j(-240-\varphi)} = Ie^{-j(240+\varphi)°} \end{aligned} \qquad \text{equation 4.3}$$

We say that $\mathbf{I_1}$ is the complex amplitude of the current in phase 1.

Let us now suppose a three-phase system composed by a three-phase generator (the coils, or windings, are the three phases of the generator) connected to a three-phase load. The three loads are equal, therefore, we say that the system is balanced as represented in Figure 4-3. For the time being, 4 wires are used, the 4th acting as a return path, called the the neutral wire Nn. This return path can be common for the three phases.

If the load impedance is equal in the three-phases ($\mathbf{Z}=Ze^{j\varphi}$), the currents are given by equation 4.3. Applying KCL, the current flowing in the neutral wire is the sum of the three phasors representing the complex amplitudes of the currents. By performing this calculation, we will obtain zero.

$$\mathbf{I_n} = \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3} = 0 \qquad \text{equation 4.4}$$

Reporting to Figure 4-3, the positive directions for the voltages and currents are indicated. We use indexes 1,2,3 to refer to the phases instead of a,b,c as in Figure 4-3.

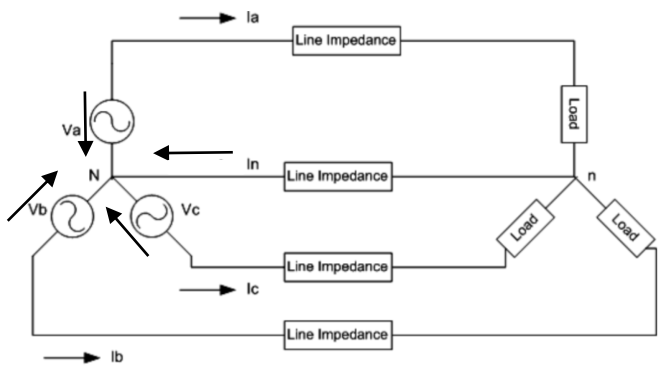


Figure 4-3: Balanced three-phase system with 4 wires. Source: http://farhek.com/jd/0l15f040/three-figure/104f0a5/.

Note that equation 4.4 holds true if the three-phase system is symmetrical and balanced. Symmetrical means that the three currents (and voltages) are 120º phase-shifted; balanced means that the three loads are equal and so the currents hold the same RMS value and are 120º shifted. As a consequence, the neutral wire can be suppressed and nothing changes in the electric circuit. We say that we have a 3 wire three-phase system as in Figure 4-4.

In the transmission network, the system is symmetrical and balanced, so 3 conductors are used.

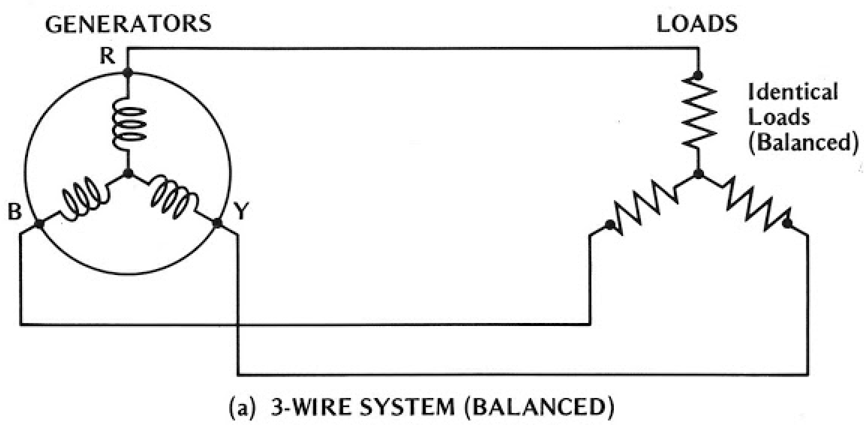


Figure 4-4: Balanced three-phase system with 3 wires. Source: http://emadrlc.blogspot.pt/2012/12/chapter-2-star-and-delta-connections.html.

Let us now suppose that we have three different loads (the system is unbalanced).

$$\mathbf{Z_1} = Z_1 e^{j\varphi_1}$$
$$\mathbf{Z_2} = Z_2 e^{j\varphi_2} \qquad \text{equation 4.5}$$
$$\mathbf{Z_3} = Z_3 e^{j\varphi_3}$$

The three currents would be now (the system is unsymmetrical):

$$\mathbf{I_1} = \frac{\mathbf{V_1}}{\mathbf{Z_1}} = I_1 e^{-j\varphi_1}$$
$$\mathbf{I_2} = \frac{\mathbf{V_2}}{\mathbf{Z_2}} = I_2 e^{-j(120°+\varphi_2)}$$
$$\mathbf{I_3} = \frac{\mathbf{V_3}}{\mathbf{Z_3}} = I_3 e^{-j(240°+\varphi_3)}$$

equation 4.6

The neutral current is now different from zero and the 4th wire cannot be suppressed[27], as shown in Figure 4-5.

$$\mathbf{I_n} = \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3} \neq 0 \qquad \text{equation 4.7}$$

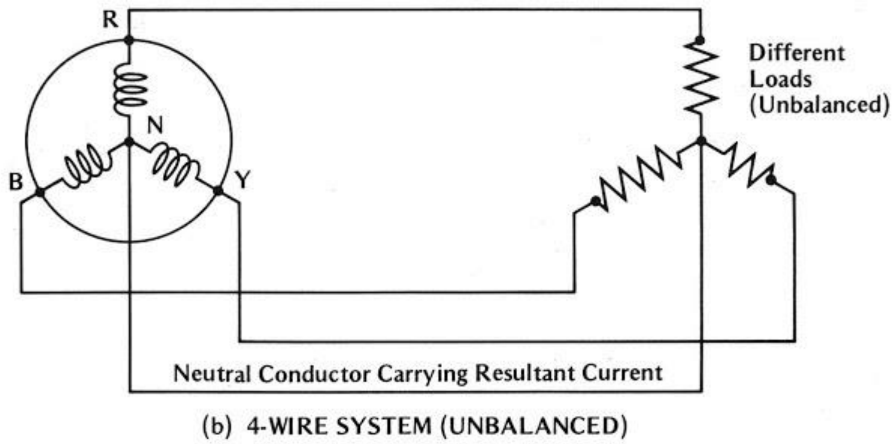


Figure 4-5: Unbalanced three-phase system with 4 wires. Source: http://emadrlc.blogspot.pt/2012/12/chapter-2-star-and-delta-connections.html.

In the distribution network the system is unbalanced, and so 4 wires are used.

Example 4—1:

A three-phase symmetrical, balanced, 4 wire system, with V = 230 V, f = 50 Hz, feeds a three-phase balanced load $\mathbf{Z_1} = \mathbf{Z_2} = \mathbf{Z_3} = 14.14 + j14.14\ \Omega$*. Compute the neutral current.*

The three-voltages are:

[27] If we want equation 4.6 to hold true.

$$\mathbf{V_1} = 230e^{j0} \text{ V}$$
$$\mathbf{V_2} = 230e^{-j120°} \text{ V}$$
$$\mathbf{V_3} = 230e^{-j240°} \text{ V}$$

Each impedance is:

$$\mathbf{Z} = 14.14 + j14.14 = \sqrt{14.14^2 + 14.14^2}\, e^{j\tan^{-1}\left(\frac{14.14}{14.14}\right)} = 20e^{j45°}\ \Omega$$

The three currents are:

$$\mathbf{I_1} = \frac{\mathbf{V_1}}{\mathbf{Z_1}} = 11.5e^{-j45°} \text{ A}$$
$$\mathbf{I_2} = \frac{\mathbf{V_2}}{\mathbf{Z_2}} = 11.5e^{-j165°} \text{ A}$$
$$\mathbf{I_3} = \frac{\mathbf{V_3}}{\mathbf{Z_3}} = 11.5e^{-j285°} \text{ A}$$

or in the rectangular form:

$$\mathbf{I_1} = 11.5\cos(-45°) + j11.5\sin(-45°) = 8.13 - j8.13 \text{ A}$$
$$\mathbf{I_2} = 11.5\cos(-165°) + j11.5\sin(-165°) = -11.11 - j2.98 \text{ A}$$
$$\mathbf{I_3} = 11.5\cos(-285°) + j11.5\sin(-285°) = 2.98 + j11.11 \text{ A}$$

The neutral current is:

$$\mathbf{I_n} = \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3} = 0$$

Example 4—2:

Retake Example 4—1 but assume now that the impedance is unbalanced. The modulus of the impedance remains the same, while the angles changes: $\mathbf{Z_1} = 20\ \Omega$; $\mathbf{Z_2} = j20\ \Omega$; $\mathbf{Z_3} = -j20\ \Omega$. *Compute the neutral current.*

The three currents are:

$$\mathbf{I_1} = \frac{\mathbf{V_1}}{\mathbf{Z_1}} = 11.5e^{-j0°} \text{ A}$$
$$\mathbf{I_2} = \frac{\mathbf{V_2}}{\mathbf{Z_2}} = 11.5e^{-j210°} \text{ A}$$
$$\mathbf{I_3} = \frac{\mathbf{V_3}}{\mathbf{Z_3}} = 11.5e^{-j150°} \text{ A}$$

or in the rectangular form:

$$\mathbf{I_1} = 11.5\cos(0°) + j11.5\sin(0°) = 11.5 + j0 \text{ A}$$
$$\mathbf{I_2} = 11.5\cos(-210°) + j11.5\sin(-210°) = -9.96 + j5.75 \text{ A}$$
$$\mathbf{I_3} = 11.5\cos(-150°) + j11.5\sin(-150°) = -9.96 - j5.75 \text{ A}$$

The neutral current is:

$$\mathbf{I_n} = \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3} = -8.42 + j0 = 8.42e^{j180°} \text{ A}$$

4.2 PHASE-TO-NEUTRAL VOLTAGE AND PHASE-TO-PHASE VOLTAGE

Let us assume that the loads, instead of being connected as in Figure 4-4, we now connect them as in Figure 4-6. We say that the loads are connected in delta or in triangle. In the previous case, we say that the loads are connected in star.

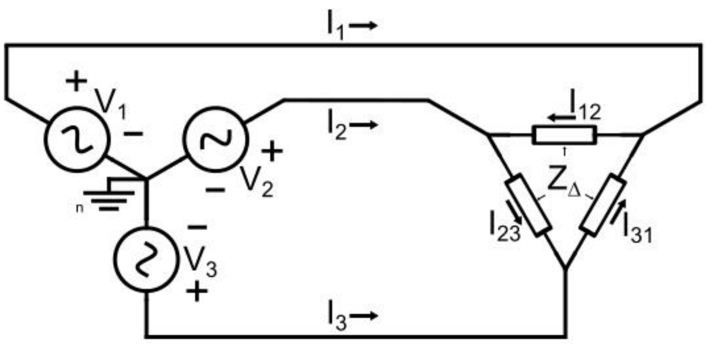


Figure 4-6: Three-phase system with loads delta connection. Source: https://commons.wikimedia.org/wiki/File:3_Phase_Power_Connected_to_Delta_Load.svg.

In this case (delta-connection), the loads are connected between two phases.

Let us consider that the three symmetrical, balanced voltages are:

$$\begin{aligned} \mathbf{V_1} &= Ve^{j0} \\ \mathbf{V_2} &= Ve^{-j120°} \\ \mathbf{V_3} &= Ve^{-j240°} \end{aligned} \qquad \text{equation 4.8}$$

Appling KVL, we obtain:

$$\begin{aligned} \mathbf{V_1} - \mathbf{V_2} &= \mathbf{V_{12}} = \mathbf{Z_\Delta I_{12}} \\ \mathbf{V_2} - \mathbf{V_3} &= \mathbf{V_{23}} = \mathbf{Z_\Delta I_{23}} \\ \mathbf{V_3} - \mathbf{V_1} &= \mathbf{V_{31}} = \mathbf{Z_\Delta I_{31}} \end{aligned} \qquad \text{equation 4.9}$$

For instance, let us take as an example, voltages $\mathbf{V_1}$ and $\mathbf{V_2}$. Voltage $\mathbf{V_1}$ is taken between a phase and the neutral point; as so, it is called phase-to-neutral voltage, or phase voltage. Voltage $\mathbf{V_{12}}$ is the difference between voltage $\mathbf{V_1}$ and $\mathbf{V_2}$ and is taken between two phases; as so, it is called phase-to-phase voltage or line voltage.

Let us keep going on with phase voltages $\mathbf{V_1}$ and $\mathbf{V_2}$. For these two voltages, we can write:

$$\mathbf{V_1} = V\cos(0^\circ) + jV\sin(0^\circ) = V$$

$$\mathbf{V_2} = V\cos(-120^\circ) + jV\sin(-120^\circ) = -\frac{1}{2}V - j\frac{\sqrt{3}}{2}V$$

$$\mathbf{V_{12}} = \mathbf{V_1} - \mathbf{V_2} = \frac{3}{2}V + j\frac{\sqrt{3}}{2}V \qquad \text{equation 4.10}$$

$$\mathbf{V_{12}} = \sqrt{\left(\frac{3}{2}V\right)^2 + \left(\frac{\sqrt{3}}{2}V\right)^2}\, e^{j\tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}V}{\frac{3}{2}V}\right)} = \sqrt{\frac{12}{4}V^2}\, e^{j\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)} = \sqrt{3}Ve^{j30^\circ}$$

If we make the same computations for the other line voltages, we would find that it is possible to always write:

$$V_{p-p} = \sqrt{3}V_{p-n} \qquad \text{equation 4.11}$$

meaning that the RMS phase-to-phase voltage ($V_{p\text{-}p}$) is square root of three times higher than the RMS phase-to-neutral voltage ($V_{p\text{-}n}$).

Here are all the three line voltages:

$$\begin{aligned} \mathbf{V_1} - \mathbf{V_2} &= \mathbf{V_{12}} = \sqrt{3}Ve^{j30^\circ} \\ \mathbf{V_2} - \mathbf{V_3} &= \mathbf{V_{23}} = \sqrt{3}Ve^{-j90^\circ} \\ \mathbf{V_3} - \mathbf{V_1} &= \mathbf{V_{31}} = \sqrt{3}Ve^{j150^\circ} \end{aligned} \qquad \text{equation 4.12}$$

We highlight that the line voltages are also 120º phase-shifted.

In LV, the phase-to-neutral RMS voltage is 230 V and the RMS phase-to-phase voltage is 400 V.

4.3 Star Connection and Delta Connection

From what we have learned, we conclude that if the loads are star connected, as in Figure 4-5, for instance, the loads are subjected to the phase-to-neutral voltages. Also, the current that flows in the line (connecting one phase of the generator to the respective load) is equal to the current across the load. We say that, in this case, the line current is equal to the load (or phase) current (see current $\mathbf{I_1}$ in Figure 4-7).

As far as the delta connected loads are concerned, the situation is different. At each load, the phase-to-phase voltage is applied, as we have seen. Regarding Figure 4-7 and applying KCL, we obtain:

$$\begin{gathered}\mathbf{I_1}+\mathbf{I_{31}}=\mathbf{I_{12}}\\ \mathbf{I_1}=\mathbf{I_{12}}-\mathbf{I_{31}}\end{gathered} \qquad \text{equation 4.13}$$

In a similar way to what we have seen in the case of the relationship between phase-to phase and phase-to-neutral voltages (equation 4.10), here the same relationship applies: the RMS line current (for instance $\mathbf{I_1}$) is square root of 3 times higher than the RMS delta load (or phase) current:

$$I_{line\Delta}=\sqrt{3}I_{phase\Delta} \qquad \text{equation 4.14}$$

This holds true for a balanced, symmetrical, three-phase system with delta connected loads.

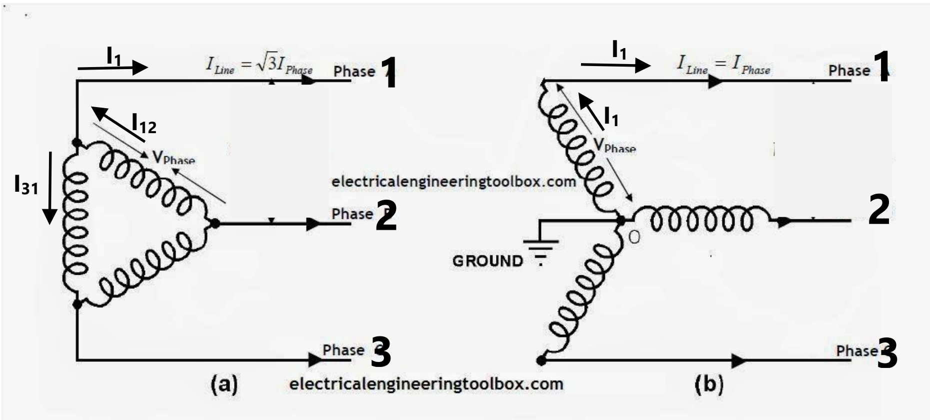


Figure 4-7: Loads with star connection (right) and delta connection (left). Source: https://www.quora.com/What-is-the-difference-between-a-star-and-a-delta-connection-What-happens-when-it-is-star-connected-or-delta-connected-Are-the-circuits-designed-in-our-homes-star-or-delta-connected.

4.4 Three-Phase Power in Balanced and Symmetrical Systems

Let us recall the unbalanced system, star-connected, depicted in Figure 4-5. The three-phase active power delivered to the loads is:

$$P_{3ph}=V_1I_1\cos\varphi_1+V_2I_2\cos\varphi_2+V_3I_3\cos\varphi_3 \qquad \text{equation 4.15}$$

where V_1, V_2, V_3 are the RMS phase-to-neutral voltages, I_1, I_2, I_3 are the RMS line currents and φ_1, φ_2, φ_3 are the phase-shifts between the phase-to-neutral voltage and the corresponding current.

If the system is symmetrical and balanced, the three RMS voltages are equal (V_{p-n}), the three RMS line currents are equal (I_{line})and the three phase-shifts are equal (φ_Y). Therefore, for a star connected system, we will write, recalling the relationship between phase-to-phase and phase-to-neutral voltages:

$$P_{3phY} = 3V_{p-n}I_{line}\cos\varphi_Y = \sqrt{3}V_{p-p}I_{line}\cos\varphi_Y \qquad \text{equation 4.16}$$

Now, let us recall the balanced, delta connected, system displayed in Figure 4-6. The three-phase active power delivered to the loads is given by:

$$P_{3ph\Delta} = 3V_{p-p}I_{phase}\cos\varphi_\Delta \qquad \text{equation 4.17}$$

where V_{p-p} is the RMS phase-to-phase voltage, I_{phase} is the RMS phase (load) current and φ_Δ is the phase-shift between the phase-to-phase voltage and the phase current.

We recall that:

$$\varphi_\Delta = \varphi_Y = \varphi \qquad \text{equation 4.18}$$

that is, the angle between the phase-to-neutral voltage and the line current is equal to the angle between the phase-to-phase voltage and the phase current.

Recalling equation 4.14 ($I_{line\Delta} = \sqrt{3}I_{phase\Delta}$) and that $3/\sqrt{3} = \sqrt{3}$, we conclude that whether the loads are star or delta connected, the three-phase active power delivered to the loads is always:

$$P_{3ph} = \sqrt{3}V_{p-p}I_{line}\cos\varphi \qquad \text{equation 4.19}$$

The same logic can be applied to derive the three-phase apparent power and the three-phase reactive power:

$$\begin{aligned} S_{3ph} &= \sqrt{3}V_{p-p}I_{line} \\ Q_{3ph} &= \sqrt{3}V_{p-p}I_{line}\sin\varphi \end{aligned} \qquad \text{equation 4.20}$$

Example 4—3:

Suppose that the three balanced loads of Example 4—1, instead of being star connected, are now delta connected. Compute the phase currents, the line currents and the three phase active power delivered to the loads. Compare with the star connected loads' case.

Let us begin by recalling the voltages and currents in the star connected case (Example 4—1):

$$\mathbf{V_1} = 230e^{j0}\ \text{V}$$
$$\mathbf{V_2} = 230e^{-j120°}\ \text{V}$$
$$\mathbf{V_3} = 230e^{-j240°}\ \text{V}$$

$$\mathbf{I_{1Y}} = 11.5e^{-j45°}\ \text{A}$$
$$\mathbf{I_{2Y}} = 11.5e^{-j165°}\ \text{A}$$
$$\mathbf{I_{3Y}} = 11.5e^{-j285°}\ \text{A}$$

And now the line voltages:

$$\mathbf{V_{12}} = \sqrt{3}230e^{j30°}\ \text{V}$$
$$\mathbf{V_{23}} = \sqrt{3}230e^{-j90°}\ \text{V}$$
$$\mathbf{V_{31}} = \sqrt{3}230e^{j150°}\ \text{V}$$

The phase currents for the delta connected case are:

$$\mathbf{I_{12}} = \frac{\mathbf{V_{12}}}{\mathbf{Z}} = 19.92e^{-j15°}\ \text{A}$$
$$\mathbf{I_{23}} = \frac{\mathbf{V_{23}}}{\mathbf{Z}} = 19.92e^{-j135°}\ \text{A}$$
$$\mathbf{I_{31}} = \frac{\mathbf{V_{31}}}{\mathbf{Z}} = 19.92e^{j105°}\ \text{A}$$

The RMS value of the current in the loads is square root of three times higher if they are delta connected than if they are star connected (compare, for instance, I_{12} with I_{1Y}).

The line currents would be:

$$\mathbf{I_{1\Delta}} = \mathbf{I_{12}} - \mathbf{I_{31}} = 34.50e^{-j45°}\ \text{A}$$
$$\mathbf{I_{2\Delta}} = \mathbf{I_{23}} - \mathbf{I_{12}} = 34.50e^{-j165°}\ \text{A}$$
$$\mathbf{I_{3\Delta}} = \mathbf{I_{31}} - \mathbf{I_{23}} = 34.50e^{j75°}\ \text{A}$$

The RMS value of the line current is three times higher if they are delta connected than if they are star connected (compare, for instance, $I_{1\Delta}$ with I_{1Y}).

We take the opportunity to prove equation 4.18. In fact:

$$\varphi_Y = \text{angle}(\mathbf{V_1}) - \text{angle}(\mathbf{I_1}) = 0 + 45 = 45^\circ = \varphi$$
$$\varphi_\Delta = \text{angle}(\mathbf{V_{12}}) - \text{angle}(\mathbf{I_{12}}) = 30 + 15 = 45^\circ = \varphi$$

In what regards the active power delivered to the loads, it is:

$$P_Y = 3V_1 I_{1Y} \cos\varphi = \sqrt{3} V_{12} I_{1Y} \cos\varphi = 5610.9 \text{ W}$$
$$P_\Delta = 3V_{12} I_{12} \cos\varphi = \sqrt{3} V_{12} I_{1\Delta} \cos\varphi = 16{,}833 \text{ W}$$

We conclude that the active power delivered to the loads is three times higher if they are delta connected than if they are star connected. Of course, this holds true if the three loads are balanced and equal in both connections.

5 PER UNIT (PU) SYSTEM

In electrical systems, the four fundamental quantities are: voltage, current, impedance and power. In per unit (pu) notation, the physical quantity is expressed as a fraction of a reference (base) value. Let us take a general quantity *X* expressed in its own SI units. The pu value is given by:

$$X_{pu} = \frac{X}{X_{base}} \qquad \text{equation 5.1}$$

The base value is a reference value for the magnitude of the quantity. As a consequence, it is a real value. The quantity *X* is a quantity expressed in SI units: it can be a magnitude, a phasor, or a complex number, or even an instantaneous value.

Given the basic relationships between the four fundamental quantities, it is only necessary to specify the base values for two of the four quantities; the other two will follow directly. It is common practice to specify the base power and the base voltage. In general, these two base values can be arbitrarily specified, but it is also common practice to specify the base values as the nominal (nameplate) values of the quantities.

For a single-phase system, the nominal values are the single-phase power and the phase-to-neutral (phase) voltage; for a three-phase system, the nominal values are the three-phase power and the phase-to-phase (line) voltage.

The known relationships between the fundamental quantities also apply in the pu system:

$$\begin{aligned}
&\text{single phase}\begin{cases} S_b = V_b I_b \\ V_b = Z_b I_b \end{cases} \\
&\text{three phase}\begin{cases} S_b = \sqrt{3} V_b I_b \\ \dfrac{V_b}{\sqrt{3}} = Z_b I_b \end{cases}
\end{aligned} \qquad \text{equation 5.2}$$

We highlight that for the single-phase system, the base power is the single-phase power and the base voltage is the phase voltage, whereas for the three-phase system, the base power is the three-phase power and the base voltage is the line voltage.

As so, for a single phase system, we specify the base power (S_b) and the base voltage (V_b). The corresponding base current (I_b) and base impedance (Z_b) are:

$$I_b = \frac{S_b}{V_b}$$
$$Z_b = \frac{V_b}{I_b} = \frac{V_b^2}{S_b} \qquad \text{equation 5.3}$$

For a three-phase system, the base three-phase power (S_b) and the base line voltage (V_b) should be specified. The corresponding base current (I_b) and base impedance (Z_b) are:

$$I_b = \frac{S_b}{\sqrt{3}V_b}$$
$$Z_b = \frac{\frac{V_b}{\sqrt{3}}}{I_b} = \frac{V_b^2}{S_b} \qquad \text{equation 5.4}$$

A word is worth to clarify the base impedance. If the system is three-phase and the loads are star-connected (this is the usual situation), the phase voltage is applied to each load and the single-phase power is delivered to each one. Defining the base voltage as the line voltage and the base power as the three-phase power, as usually in three-phase system, yields:

$$Z_b = \frac{V_b^2}{S_b} = \frac{\left(\frac{V_b}{\sqrt{3}}\right)^2}{\left(\frac{S_b}{3}\right)} = \frac{V_b^2}{S_b} \qquad \text{equation 5.5}$$

If the system is three-phase and the loads are delta-connected, the line voltage is applied to each load and the single-phase power is delivered. Again, defining the base voltage as the line voltage and the base power as the three-phase power leads to:

$$Z_b = \frac{V_b^2}{S_b} = \frac{V_b^2}{\left(\frac{S_b}{3}\right)} = 3\frac{V_b^2}{S_b} \qquad \text{equation 5.6}$$

One important conclusion is that the usual $\sqrt{3}$ factor that appears in three-phase system in SI units, disappear in the pu system. Let us see why:

$$\text{single phase}\left\{ S_{pu} = \frac{(VI)_{SI}}{V_b I_b} = V_{pu} I_{pu} \right.$$
$$\text{three phase}\left\{ S_{pu} = \frac{\left(\sqrt{3}VI\right)_{SI}}{\sqrt{3}V_b I_b} = V_{pu} I_{pu} \right. \qquad \text{equation 5.7}$$

This means that the power is computed using the same equation, whether it is a single-phase or a three-phase system.

For a connected system, it is obvious that the same bases should be used for the whole network, such that the normal circuit theorems and equations also apply to per unit values. As so, a single base power and a single base voltage (usually the nominal values) should be specified for a connected circuit.

Example 5—1

Solve again Example 4—3 using the pu system. As the system is balanced, solve for phase 1 only.

Let us begin by defining the base (three-phase) power and base (line) voltage:

$$S_b = 10 \text{ kVA}$$
$$V_b = \sqrt{3}230 \text{ V}$$

Then, we compute, the base current and the base impedances:

$$I_b = \frac{S_b}{\sqrt{3}V_b} = 14.49 \text{ A}$$
$$Z_{bY} = \frac{V_b^2}{S_b} = 15.87\ \Omega$$
$$Z_{b\Delta} = 3\frac{V_b^2}{S_b} = 47.61\ \Omega$$

The voltage and the impedances in pu are:

$$\mathbf{V} = \frac{\sqrt{3}230}{\sqrt{3}230} e^{j0} = 1 \text{ pu}$$
$$\mathbf{Z_Y} = \frac{20}{15.87} e^{j45^\circ} = 1.26 e^{j45^\circ} \text{ pu}$$
$$\mathbf{Z_\Delta} = \frac{20}{47.61} e^{j45^\circ} = 0.42 e^{j45^\circ} \text{ pu}$$

Note that pu conversion affects the magnitudes, not the angles.

As a consequence, the line currents are:

$$\mathbf{I_Y} = \frac{\mathbf{V}}{\mathbf{Z_Y}} = 0.79e^{-j45^\circ} \text{ pu} \Rightarrow \mathbf{I_Y} = 0.79 \times 14.43e^{-j45^\circ} = 11.5e^{-j45^\circ} \text{ A}$$
$$\mathbf{I_\Delta} = \frac{\mathbf{V}}{\mathbf{Z_\Delta}} = 2.38e^{-j45^\circ} \text{ pu} \Rightarrow \mathbf{I_\Delta} = 2.38 \times 14.43e^{-j45^\circ} = 34.5e^{-j45^\circ} \text{ A}$$

The active power delivered to the loads is (compare with Example 4—3 results):

$$P_Y = VI_Y \cos\varphi = 0.57 \text{ pu} \Rightarrow P_Y = 0.56 \times 10 \text{ kW} = 5.6 \text{ kW}$$
$$P_\Delta = VI_\Delta \cos\varphi = 1.7 \text{ pu} \Rightarrow P_\Delta = 1.68 \times 10 \text{ kW} = 16.8 \text{ kW}$$

In a transformer, two circuits are not directly connected but magnetically coupled. The voltages of the windings are in the ratio of turns and currents in inverse ratio[28]. For the coupled circuit, we should then choose the same base power and base voltages in the ratio of turns. Therefore, in a circuit with transformers, all the nominal voltage levels are base voltages and a single base power should be chosen. For instance, if we have a 400V/15kV transformer, both 400 V and 15 kV are base voltages.

The main advantages of the pu system may be summarized as follows:

- Normally we are dealing with numerics near unity rather than over a wide range.
- Provides a more meaningful comparison of parameters of machines with different ratings.
- As the pu values of parameters of a rotating machine or a transformer normally falls within a certain range, a typical value can be used if such parameters are not provided.
- Calculations involving transformers are much easier, because there is no need to refer circuit quantities to one or other side of the transformer.
- The pu values clearly represent the relative values of the circuit quantities. Many of the ubiquitous constants are eliminated.

[28] In fact, the voltages of the windings are almost in the ratio of turns and currents in inverse ratio, because the transformers are not ideal transformers.

6 AC CIRCUIT APPLICATION: THE INDUCTION MOTOR

The induction motor is perhaps the most common type of electric motor in the world, because it is simple, robust, reliable and cheap. An electric motor converts electrical energy, that it receives from the grid, into mechanical energy (energy associated to the rotation of a shaft, for instance), that it delivers to a load.

An induction motor has 2 main parts: the stator and rotor. The stator is the stationary part (Figure 6-1) and the rotor, which sits inside the stator, is the rotating part (Figure 6-2). There is also a small gap between rotor and stator, known as air-gap. The value of the radial air-gap may vary from 0.5 to 2 mm.

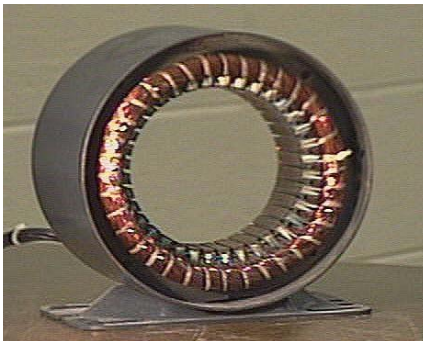


Figure 6-1: Stator of an induction motor. Source: http://www.egr.unlv.edu/~eebag/InductionMotors.pdf.

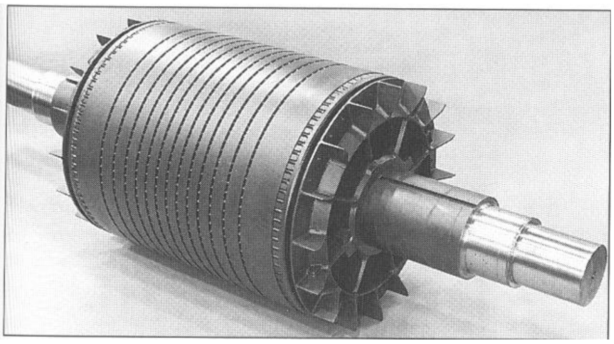


Figure 6-2: Short-circuited (squirrel cage) rotor of an induction motor. Source: http://www.egr.unlv.edu/~eebag/InductionMotors.pdf.

In Figure 6-3, we show an exploded view of a three-phase induction motor.

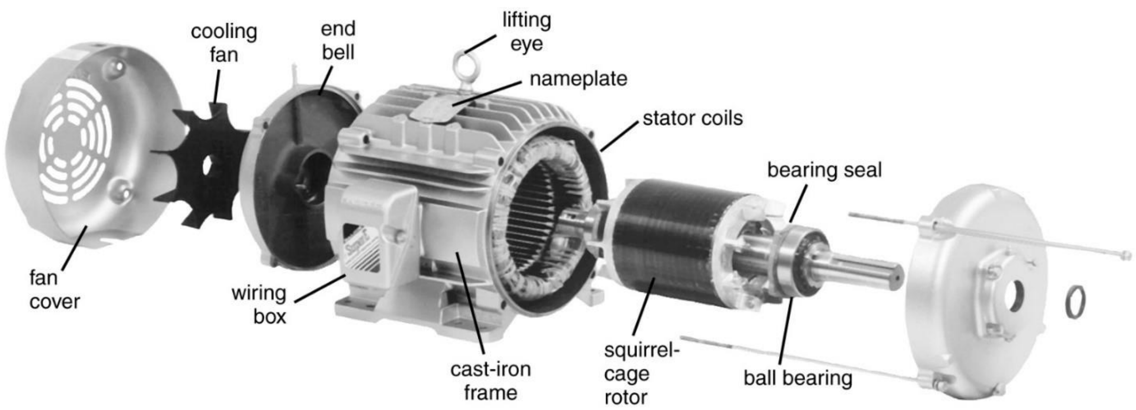


Figure 6-3: Short-circuited rotor of an induction motor. Source: http://electricalacademia.com/induction-motor/three-phase-induction-motor-construction/

6.1 Basic Operating Principle

Suppose that a magnetic pole is moving at angular speed ω_s in relation to a closed winding. In these conditions, a current *i* will flow in the winding. The interaction between the current and the magnetic field created by the rotating pole causes a force. This force tries to keep the relative position of the winding in relation to the changing magnetic field. As a consequence, the winding will rotate at an angular speed $\omega_r < \omega_s$, because there are forces that oppose to the movement, friction forces, for instance. You can think of the rotor frantically trying to "catch up" with the rotating magnetic field in an effort to eliminate the difference in motion between them.

If $\omega_r = \omega_s$, the winding would not "see" any magnetic field variation, as so, no induced emf, nor induced current would be produced, and no force would drag the winding. As, in practice, $\omega_r < \omega_s$, it follows that there is a slip, between the rotating winding (which rotates at ω_r) and the rotating pole (that rotates at synchronous speed, ω_s). This is why the induction motor is also called an asynchronous motor.

This is the basic operating principle of an induction motor. In a real induction motor, the poles do not actually rotate. What rotates is a magnetic field distribution, which is identical to the one that would be obtained if the poles were rotating. This is the so called Rotating Magnetic Field and is obtained from the grid AC balanced and symmetrical power supply to the three stator coils 120º spatially phase-shifted. Actually, there is a ring of grid-connected coils arranged around the outside (making up the stator), which are specially designed to produce a rotating magnetic field. Inside the stator, there is a loop of wires assembled in a squirrel cage made of metal bars and interconnections or some other freely rotating metal part that can conduct electricity. This is the most used design of induction motors, the so-called squirrel cage induction motor, or short-circuited rotor induction motor.

The electrical currents that flow in the rotor are obtained by induction; no galvanic connection exists between stator and rotor. The coupling is purely magnetic; the rotating magnetic field drags the rotor.

6.2 DERIVING THE EQUIVALENT CIRCUIT

For quantitative predictions about the behaviour of the induction machine, under various operating conditions, it is convenient to represent it as an equivalent circuit under sinusoidal steady state. Since the operation is balanced, a single-phase equivalent circuit is sufficient for most purposes.

We will begin by defining the induction motor slip as:

$$s = \frac{\omega_s - \omega_r}{\omega_s} \qquad \text{equation 6.1}$$

where ω_s is the synchronous speed and ω_r is the rotor speed. When s = 1, the rotor is blocked; when s = 0, the motor is in no load condition (no current is induced).

Let us now look at the stator circuit of an induction motor. It should contain inductances to account for the leakage (flux that do not link stator and rotor) and mutual (magnetization) fluxes and a resistance to account for the resistance of the stator winding.

Figure 6-4 shows the equivalent circuit of one phase of the stator of the induction motor. The elements R_s and $X_{ls} = \omega_s L_{ls}$ are the stator winding resistance and leakage reactance, and X_m is the magnetizing reactance.

Using complex amplitudes, we can write:

$$\mathbf{V_s} = (R_s + j\omega_s L_{ls})\mathbf{I_s} + \mathbf{E_s} \qquad \text{equation 6.2}$$

where $\mathbf{V_s}$ is the stator voltage, $\mathbf{I_s}$ is the stator current and $\mathbf{E_s}$ is the emf induced in the stator coil by the mutual flux.

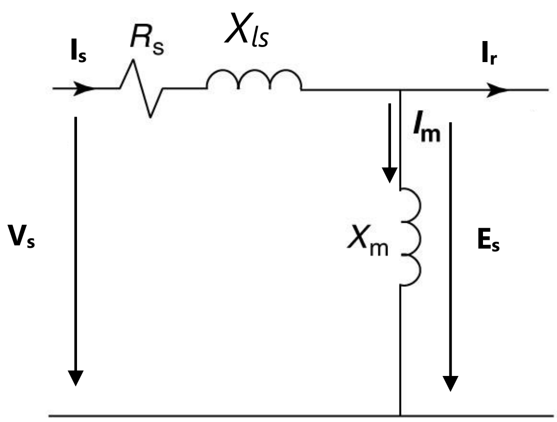


Figure 6-4: Stator equivalent circuit of an induction motor. Source: http://electricalacademia.com/induction-motor/three-phase-induction-motor-equivalent-circuit/

We now need to add the rotor to the equivalent circuit. The issue is that the rotor is rotating. A first consequence of this is that the relative angular speed of the stator field and rotor coil is $s\omega_s = \omega_s - \omega_r$ instead of ω_s. Let us recall that (remember equation 2.2):

$$E_s = k\omega_s\phi \qquad \text{equation 6.3}$$

Therefore:

$$E_r = ks\omega_s\phi = sE_s \qquad \text{equation 6.4}$$

i.e. the emf induced in the rotor will be $E_r = sE_s$.

A second consequence is that since the frequency of the rotor currents is $s\omega_s$, the leakage rotor reactance will have a value of $s\omega_s L_{lr}$.

These two consequences of rotor turning, yields the rotor equivalent circuit of Figure 6-5.

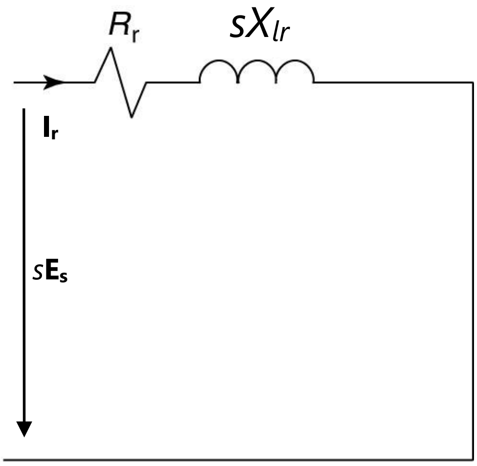


Figure 6-5: Rotor equivalent circuit of an induction motor. Source: http://electricalacademia.com/induction-motor/three-phase-induction-motor-equivalent-circuit/

Again using phasor notation, we can write (note that the rotor is short-circuited in our induction motor):

$$0 = (R_r + js\omega_s L_{lr})\mathbf{I_r} - s\mathbf{E_s} \qquad \text{equation 6.5}$$

The elements R_r and $X_{lr} = \omega_s L_{lr}$ are the rotor winding resistance and leakage reactance and $\mathbf{I_r}$ is the rotor current.

Dividing by the slip, one gets:

$$\mathbf{E_s} = \left(\frac{R_r}{s} + j\omega_s L_{lr}\right)\mathbf{I_r} \qquad \text{equation 6.6}$$

which leads to the rotor equivalent circuit of Figure 6-6.

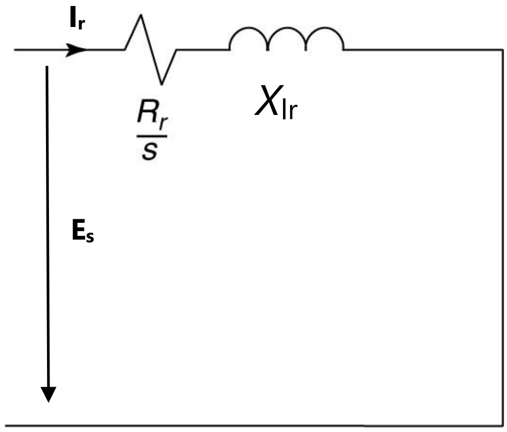


Figure 6-6: Rotor equivalent circuit of an induction motor reduced to the stator by frequency. Source: http://electricalacademia.com/induction-motor/three-phase-induction-motor-equivalent-circuit/

Now, we can connect the stator circuit (Figure 6-4) with the rotor circuit (Figure 6-6). Before that, it is instructive to split the resistance into two separate components. For convenience, we can write:

$$\frac{R_r}{s} = R_r + \left(\frac{1-s}{s}\right)R_r \qquad \text{equation 6.7}$$

and the induction motor equivalent circuit is depicted in Figure 6-7 (for simplification, the emf is represented by **E**).

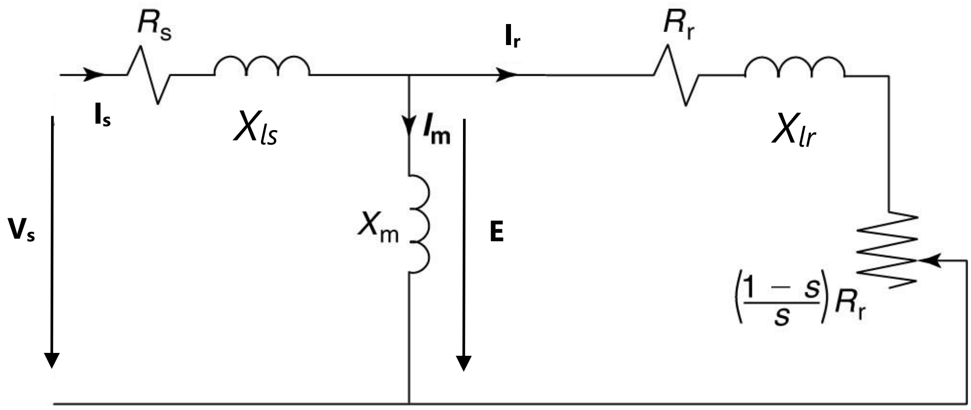


Figure 6-7: Equivalent circuit of an induction motor. Source: http://electricalacademia.com/induction-motor/three-phase-induction-motor-equivalent-circuit/

A couple of remarks should be made regarding the equivalent circuit of Figure 6-7:

- The rotor quantities and parameters are referred to the stator, meaning that they are affected by the turns ratio of stator and rotor. This inconvenience disappears if the circuit is solved in pu: in pu the turns ratio is 1. Nevertheless, the stator and rotor turns are usually the same, therefore, the turns ratio is frequently 1.
- The core (iron) losses can be represented by a resistance in parallel with the magnetizing reactance. In the deducted equivalent circuit, we have neglected these losses.
- An induction motor has also mechanical losses, due to friction and windage[29]. If considered, the useful power output (P_{out}) is computed by subtracting these losses from the mechanical power delivered in the shaft (P_{mec}).

6.3 Mechanical Power and Efficiency

From the equivalent circuit, one can see that the dissipation in R_s represents the stator losses. Therefore, the power absorption indicated by the rotor part of the circuit in Figure 6-7 must represent all other means of power consumption, namely the actual mechanical output, friction and windage losses and the rotor copper losses. Since the dissipation in

[29] Windage losses refer to the losses sustained by a machine due to the resistance offered by air to the rotation of the shaft.

R_r is rotor copper losses, the power dissipation in $\left(\frac{1-s}{s}\right)R_r$ is the sum total of the remaining (mechanical power plus friction and windage losses). In standard terminology (SI units; please note that I_r is the RMS value of the rotor current):

$$\begin{aligned} P_{ag} &= 3\frac{R_r}{s}I_r^2 \\ P_{Lr} &= 3R_r I_r^2 \\ P_{mec} &= 3R_r\left(\frac{1-s}{s}\right)I_r^2 = P_{ag} - P_{LR} \\ P_{out} &= P_{mec} - P_{Lfw} \end{aligned} \quad \text{equation 6.8}$$

where, P_{ag} is the air gap power, P_{Lr} is the rotor copper loss, P_{mec} is the mechanical power output, P_{out} is the useful power output and P_{Lfw} and the friction and windage losses. Out of the power P_{ag} transferred at the air gap, a fraction s is dissipated in the rotor and (1 − s) is delivered as output at the shaft. If there are no mechanical losses, this represents the power delivered in the shaft and available to the load.

The input electrical power is given by:

$$P_{in} = 3\left(R_s I_s^2 + \frac{R_r}{s}I_r^2\right) = 3V_s I_s \cos\varphi \quad \text{equation 6.9}$$

and the efficiency is:

$$\eta = \frac{P_{out}}{P_{in}} = \frac{3\frac{1-s}{s}R_r I_r^2 - P_{Lfw}}{3\left(R_s I_s^2 + \frac{R_r}{s}I_r^2\right)} \quad \text{equation 6.10}$$

The absorbed reactive power is:

$$Q_{in} = 3\left(X_m I_m^2 + X_{ls}I_s^2 + X_{lr}I_r^2\right) = 3V_s I_s \sin\varphi \quad \text{equation 6.11}$$

Example 6—1:

A 50 Hz, 460 V, 5 HP, 0.86 pf, 90% efficiency, induction motor has the following equivalent circuit parameters: R_s=1.21Ω; R_r=0.742Ω; X_{lr}=2.41Ω; X_{ls}=3.10Ω; X_m=65.6Ω. Find the starting and no-load currents for this machine.

1 HP (horse-power) is a power unit equivalent to 746 W. This motor has a rated power output of 3.73 kW. The rated current is:

$$I_N = \frac{P_N}{\sqrt{3}V_N \cos(\varphi_N)\eta} = 6.05 \text{ A}$$

At starting, the rotor speed is zero, meaning that the slip is 1. From an equivalent impedance as seen from the stator point-of-view, the stator impedance is in series with the parallel of the magnetizing impedance and rotor impedance. As so, we can write:

$$\mathbf{Z_{eq}} = (R_s + jX_{ls}) + \frac{jX_m(R_r + jX_{lr})}{jX_m + (R_r + jX_{lr})} = 1.90 + j5.43\ \Omega = 5.75e^{j70.72°}\ \Omega$$

Taking the equivalent circuit of Figure 6-7 into consideration, the absorbed current is:

$$\mathbf{I_{st}} = \frac{\mathbf{V_s}}{\mathbf{Z_{eq}}} = \frac{\frac{460}{\sqrt{3}}e^{j0}}{5.75e^{j70.72°}} = 46.15e^{-j70.72°} \text{ A}$$

The starting current is 7.6 times higher than the nominal current.

In no-load, the slip is zero, because the rotor speed is equal to the synchronous speed (neglecting mechanical losses). Therefore, the rotor part equivalent circuit is an open-circuit and the equivalent impedance is

$$\mathbf{Z_{eq2}} = R_s + jX_{ls} + jX_m = 1.21 + j68.70\ \Omega = 68.71e^{j88.89°}\ \Omega$$

The no-load current is given by:

$$\mathbf{I_{nl}} = \frac{\mathbf{V_s}}{\mathbf{Z_{eq2}}} = \frac{\frac{460}{\sqrt{3}}e^{j0}}{68.71e^{j88.89°}} = 3.87e^{-j88.89°} \text{ A}$$

Example 6—2:

A 460 V, 25 hp, 50 Hz, Y-connected induction motor has the following impedances in ohms per phase referred to the stator circuit: R_s=0.641Ω; R_r=0.332Ω; X_{ls}=1.106Ω; X_{lr}=0.464Ω; X_m=26.3Ω. The total rotational losses are 1100 W and are assumed to be constant. The core loss is lumped in with the rotational losses. For a rotor slip of 2.2 percent at the rated voltage and rated frequency, find the motor's efficiency.

First, let us remark that the rated power output of this motor is 25 hp = 18.65 kW.

To compute the efficiency, we have to compute the output power and the input power. Let us start with the latter.

The equivalent impedance as seen from the stator terminals is given by:

$$\mathbf{Z_{eq}} = (R_s + jX_{ls}) + \frac{jX_m\left(\frac{R_r}{s} + jX_{lr}\right)}{jX_m + \left(\frac{R_r}{s} + jX_{lr}\right)} = 11.70 + j7.80\ \Omega = 14.06e^{j33.67^\circ}\ \Omega$$

This enables the computation of the stator current as:

$$\mathbf{I_s} = \frac{\mathbf{V_s}}{\mathbf{Z_{eq}}} = 18.89e^{-j33.67^\circ}\ \text{A}$$

The input electrical power is:

$$P_{in} = 3V_s I_s \cos\varphi = 3 \times \frac{460}{\sqrt{3}} \times 18.89 \times \cos(33.67) = 12{,}526\ \text{W}$$

As for the mechanical power computation, there are several options. We will present two paths:

A simpler path immediately arises after equation 6.9, where can write:

$$I_r^2 = \frac{\frac{P_{in}}{3} - R_s I_s^2}{\frac{R_r}{s}} = 261.51\ \text{A}^2$$

and the mechanical power and output power are, respectively:

$$P_{mec} = 3R_r\left(\frac{1-s}{s}\right)I_r^2 = 11{,}579\ \text{W}$$
$$P_{out} = P_{mec} - P_{Lfw} = 10{,}479\ \text{W}$$

At this operating point the output useful power is 10.479 kW, meaning that it is not operating at rated power (18.65 kW). It is roughly half-load.

The efficiency is given by:

$$\eta = \frac{P_{out}}{P_{in}} = 83.66\%$$

A more complicated path, but that can be useful in solving other exercises, is related to the rotor current computation.

We may begin by computing the emf as (see equation 6.2):

$$\mathbf{E} = \mathbf{V_s} - (R_s + jX_{ls})\mathbf{I_s} = 243.92 - j10.67\ V = 244.15e^{-j2.51^\circ}\ \text{V}$$

The magnetizing current is:

$$\mathbf{I_m} = \frac{\mathbf{E}}{jX_m} = 9.28e^{-j92.51^\circ} \text{ A}$$

Applying KCL, we obtain the rotor current, which is:

$$\mathbf{I_r} = \mathbf{I_s} - \mathbf{I_m} = 16.13 - j1.20 \text{ A} = 16.17e^{-j4.25^\circ} \text{ A}$$

Still another way to obtain the rotor current is to apply equation 6.6:

$$\mathbf{I_r} = \frac{\mathbf{E}}{\frac{R_r}{s} + jX_{lr}} = 16.17e^{-j4.25^\circ} \text{ A}$$

6.4 ROTOR ANGULAR SPEED

We have seen that a symmetrical and balanced system of currents feeding three coils 120º phase-shifted is able to produce a rotating magnetic field with angular speed ω_s = $2\pi f$ = 100π, for a 50 Hz frequency. This holds true if the induction motor has 2 poles (1 pair of poles). Usually, common induction motors have 4 poles (2 pairs).

In general, for an induction motor with *p* pairs of poles, the angular speed of the rotating magnetic field (also called synchronous speed) is:

$$\omega_p = \frac{\omega_s}{p} = \frac{2\pi f}{p} \text{ rad/s} \qquad \text{equation 6.12}$$

In rpm (revolutions per minute), the synchronous speed is:

$$N_p = \frac{60\omega_s}{p2\pi} \text{ rpm} \qquad \text{equation 6.13}$$

For instance, for an induction machine with 2 pairs of poles, the angular speed of the rotating magnetic field is:

$$\begin{aligned} \omega_p &= \frac{\omega_s}{p} = \frac{2\pi 50}{2} = 50\pi \text{ rad/s} \\ N_p &= \frac{60\omega_s}{p2\pi} = \frac{60 \times 50}{2} = 1500 \text{ rpm} \end{aligned} \qquad \text{equation 6.14}$$

This means that the generalized equation for the slip is:

$$s = \frac{\omega_p - \omega_r}{\omega_p} = \frac{N_p - N_r}{N_p} \qquad \text{equation 6.15}$$

Example 6—3:

The nominal slip of a four pole induction motor is 6%. Find the nominal revolutions.

From equation 6.15, one gets:

$$N_r = N_p(1-s) = 1500(1-0.06) = 1410 \text{ rpm}$$

6.5 OPERATION AS A GENERATOR

We have seen that if a symmetrical and balanced AC supply is connected to the stator three 120° phase-shifted coils, a rotating magnetic field is produced in the stator which pulls the rotor to run behind it and the machine is acting as a motor.

Now, suppose that the rotor is accelerated to the synchronous speed by means of a prime mover. Under these circumstances, the slip will be zero and hence no emf is induced, the rotor current will become zero and no output mechanical power is produced.

If the rotor is made to rotate at a speed higher than the synchronous speed, the slip becomes negative. A rotor current is generated in the opposite direction, due to the rotor conductors cutting the stator rotating magnetic field, therefore producing a rotating magnetic field in the rotor. This causes a stator voltage which pushes current flowing out of the stator winding. Thus, the machine is now working as an induction generator.

The induction machine, whether it is operating as a motor or as a generator, always takes reactive power from the AC power supply[30]. When in generator operating mode, it supplies active power back into the grid. Reactive power is needed for producing the rotating magnetic field.

[30] Just take a look at the equivalent circuit. Remember that the coils absorb reactive power, which must be supplied by the AC power system.