Optimal Linear State Feedback

Supplement to part 2

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Optimal Linear Quadratic design of the state feedback gains

Linear dynamics, quadratic cost → LQ control

$$\dot{x} = Ax + Bu$$

Select u so as to minimize (details to be studied later) the infinite horizon quadratic cost

$$J = \int_0^\infty (x^T Q_r x + R_r u^2) dt.$$

Consider the special case of u scalar.

Works also for multivariable plants with a minor modification.

$$\dot{x} = Ax + Bu \qquad J = \int_0^\infty (x^T Q_r x + R_r u^2)dt.$$

Solution

Find the unique positive definite matrix P that satisfies the Algebraic Riccati Equation (ARE)

$$A^T P + PA^T - PBR_r^{-1}B^T P + Q_r = 0,$$

This solution always exists if (A, B) is controllable.

Compute the state feedback gains K (a row vector) using

$$K = R_r^{-1}BP.$$

Can be computed with MATLAB function *lqr*

Apply to the plant the feedback control

$$u(t) = -Kx(t).$$

Understanding the cost functional

$$J = \int_0^\infty (x^T Q_r x + R_r u^2) dt.$$

Always yields an asymptotically stable closed loop for all $Q_r \succcurlyeq 0, R_r \succ 0$.

The cost entails a compromise between keeping the state low without much activity in control.

The diagonal entries of Q_r tell which entries of x are more important.

The higher the value of Q_{rii}, the lower the modulus of the entry x_{ii}

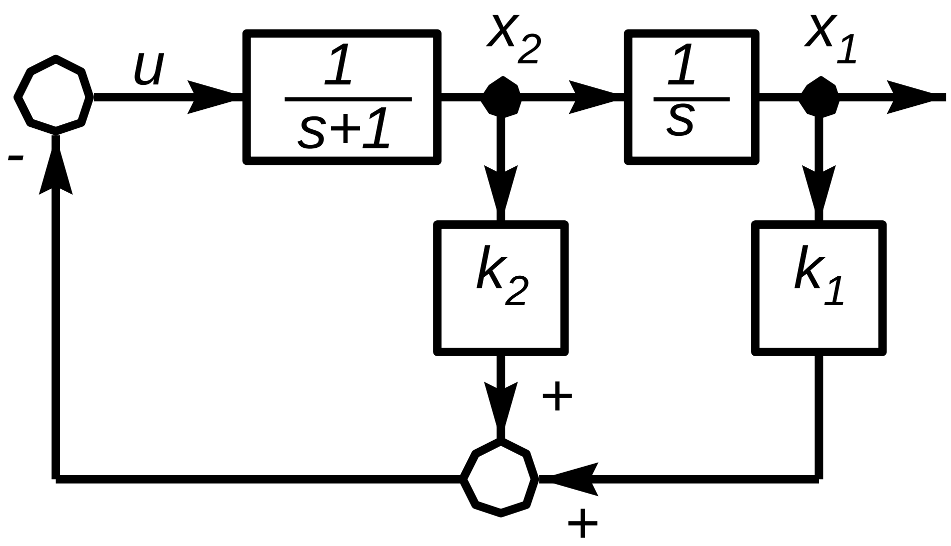
Increasing R_r decreases the modulus of u and allows x to have a higher value. The closed-loop system becomes slower (smaller bandwidth).

Example: Optimal control of a DC motor

$$\frac{dx_1}{dt} = x_2$$

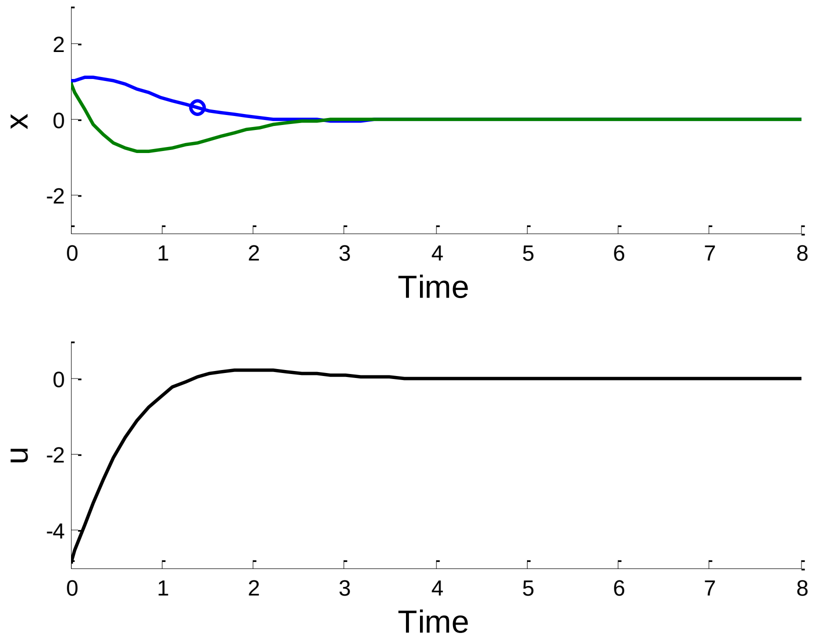
$$\frac{dx_2}{dt} = -x_2 + u$$

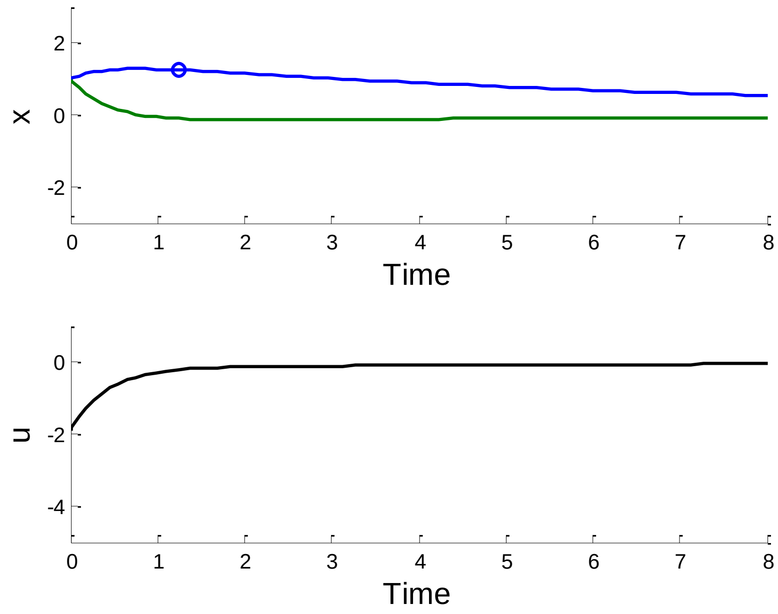
$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



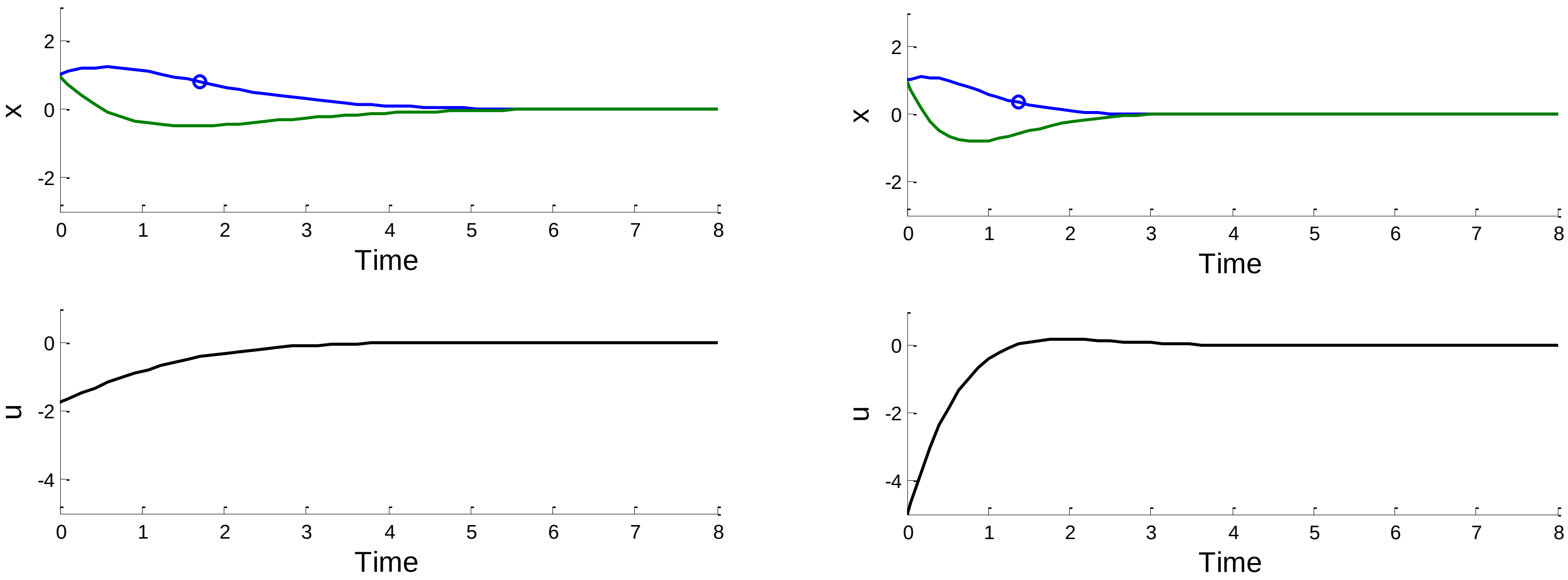
$$Q_r = \begin{bmatrix} 10 & 0 \\ 0 & 0{,}1 \end{bmatrix} \quad R_r = 1$$

$$Q_r = \begin{bmatrix} 0{,}1 & 0 \\ 0 & 5 \end{bmatrix} \quad R_r = 1$$





$$Q_r = \begin{bmatrix} 1 & 0 \\ 0 & 0{,}1 \end{bmatrix} \quad R_r = 1 \qquad\qquad Q_r = \begin{bmatrix} 1 & 0 \\ 0 & 0{,}1 \end{bmatrix} \quad R_r = 0.1$$



Increasing R_r makes the closed-loop faster. However, this trend can be a danger if there are un-modelled high-frequency dynamics.