Chapter 4

Modelling mechanical systems

> Lex I.
> Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare. (...)
> Lex II.
> Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur. (...)
> Lex III.
> Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

Isaac NEWTON (1643 — †1727), *Philosophiæ Naturalis Principia Mathematica*, Axiomata sive Leges Motus

> *Ut tensio sic vis*; That is, The Power of any Spring is in the same proportion with the Tension thereof: That is, if one power stretch or bend it one space, two will bend it two, and three will bend it three, and so forward.

Robert HOOKE (1635 — †1703), *Lectures* de Potentia Restitutiva *Or of Spring Explaining the Power of Springing Bodies*

In this and the following chapters, we will pass in review the basic concepts of system modelling, for different types of components. In this chapter, we concentrate upon mechanical components. Surely you will have already learned, if not all, at least most of these subjects in other courses. However, there are two reasons why a brief review is convenient at this point of your studies:

1. We will systematically resort to the Laplace transform to study dynamic systems, and after seeing too many equations with variable s it is easy to forget that we keep talking about real things — namely, in this course, mechatronic systems that are all around us in our daily life.

2. This is a good time to stress the similarities between apparently very different systems that can be described by the very same equations. We will see that thinking of any system as an energy converter helps to see those parallels.

4.1 Modelling the translation movement

Mechanical systems with movement along a straight line can usually be modelled using three components with the respective three equations:

Mass
Newton's second law

1. A **mass**. This component stores energy under the form of kinetic energy. To model them, apply Newton's second law (which you can read in Latin at the beginning of this chapter):

$$\sum F = \frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\,\dot{x}(t)\right) \tag{4.1}$$

Momentum

Here, $\sum F$ is the sum of all forces applied on the mass $m(t)$, which is at position $x(t)$. (Product $m(t)\,\dot{x}(t)$, as you know, is called **momentum**.) Because we are assuming a movement of translation, we need not bother to use vectors, but the forces must be applied along the direction considered; if not, their projection onto the said direction must be used. And, as we said in Section 3.1, we will only consider LTI systems; since the mass is, in (4.1), a parameter, this restriction means that it will not change with time, and so we are left with

$$\sum F = m\,\ddot{x}(t) \tag{4.2}$$

A mass is usually represented by m or M.

Spring
Hooke's law

2. A **spring**. This is a mechanical device that stores energy under the form of elastic potential energy (see Figure 4.1). A translation spring usually follows **Hooke's law** (which you can read in Latin and English at the beginning of this chapter):

$$F = k\,x \tag{4.3}$$

Here, F is the force exerted by the spring, x is the variation in length of the spring measured from the repose length, and k is the spring constant. This constant is usually represented by k or K, and its SI units are N/m.

Damper
Viscous damping

3. A **damper**. This is a mechanical device that dissipates energy (see Figure 4.2). The most usual model for dampers is **viscous damping**:

$$F = c\,\dot{x} \tag{4.4}$$

Here, F is the force exerted by the damper, $\dot{x}$ is the relative velocity of the extremities of the damper, and c is the damping constant. This constant is usually represented by c, C, b or B, and its SI units are N s/m.

Model (4.4) can also be used to model unintended energy dissipation, such as that due to friction. Notice that since energy dissipation is ubiquitous even a mechanical system consisting only of a mass and a spring will be more exactly modelled by a mass, a spring, and a damper, the latter to account for energy dissipation.

Remark 4.1. Unlike (4.2), Hooke's law (4.24) is often only an approximate model of the phenomenon it addresses. There are three ways in which reality usually deviates from (4.24).

Figure 4.1: Usual translation springs. Left: helical or coil spring; centre: volute spring; right: leaf spring. (Source: Wikimedia.)

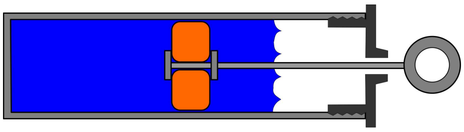

Figure 4.2: Dashpot damper (source: Wikimedia). There are other types of dampers. This one, because it contains a viscous fluid, follows (4.4) rather closely.

1. The relation between force and variation in length can be nonlinear. In any case, as long as the relation is continuous and has a continuous derivative, a linear approximation will be valid in a limited range of length variations (see Figure 4.3).

2. Springs that have a different behaviour for positive variations of length ($x > 0$, extension) and negative variations of length ($x < 0$, compression) are not uncommon.

3. In any case, Hooke's law is obviously valid only for a limited range of length variations. $\square$

Example 4.1. A stainless steel helicoidal spring is 10 cm long. When a traction force of 10 N is applied, its length increases to 12 cm. What force must be

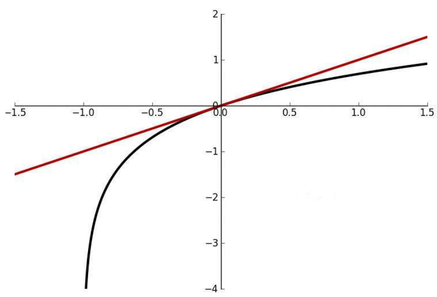

Figure 4.3: A linear approximation of a continuous function with a continuous derivative provides good results in some limited range (source: Wikimedia).

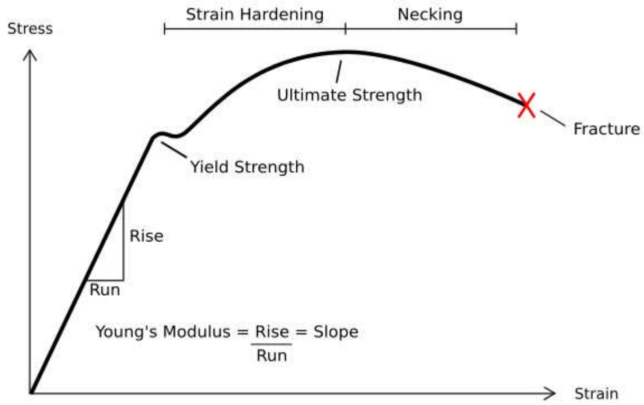

Figure 4.4: Schematic stress-strain curve of steel (source: Wikimedia).

applied so that its length increases to 15 cm? What force must be applied so that its length increases to 40×10^3 km?

A length increase of 2×10^{-2} m corresponds to a 10 N force, so $k = \frac{10}{2\times 10^{-2}} = 500$ N/m. The answer to the first question, when $x = 5$ cm $= 5 \times 10^{-2}$ m, is $F = 500 \times 5 \times 10^{-2} = 25$ N. In the second case, it should be obvious that a 10 cm helicoidal spring cannot be stretched to a length which is roughly the perimeter of the Earth. You should not have to calculate the ludicrous result $F = 500 \times 40 \times 10^6 = 2 \times 10^{10}$ N= 20 GN obtained applying the linear relation (4.24) to realise that the spring will surely break well before such a force is applied. You should have by now seen a sufficient number of diagrams such as the one in Figure 4.4 to realise this at once, without even having to look for the yield strength of stainless steel and coming up with an educated guess for the spring's cross-sectional area. □

Remark 4.2. Our models are approximations of reality. They are valid only for limited ranges of parameters. These important truths cannot be overstated. □

End of stroke

Remark 4.3. Viscous damping (4.4) is another model of reality that very often is only a rough approximation. Dashpot dampers such as the ones in Figure 4.2 follow this law more closely than other damping phenomena, where damping may be nonlinear or, if linear, proportional to another derivative of position x (some damping models even use fractional orders of differentiation). In any case, it is obvious that after a while x will reach its end of stroke, and (4.4) will no longer apply. □

Newton's third law

Combining (4.2)–(4.4) with Newton's third law — which states that when a body exerts a force on another, this latter body exerts an equal force, but opposite in direction, on the first body —, it is possible to find the differential equations that model translation mechanical systems.

Mass–spring–damper system

Example 4.2. One of the most simple, but also most useful, mechanical models is the so-called mass–spring–damper system, which can be used to model the behaviour of a mass, on which a force is applied, connected to an inertial referential by a spring and a damper (remember that any real spring also has

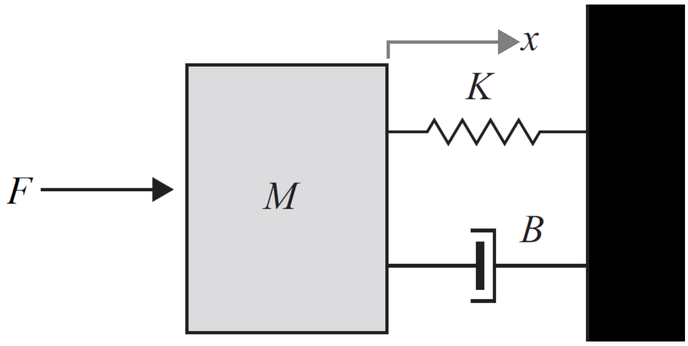

Figure 4.5: A mass–spring–damper system, with mass M, spring constant K, and damping coefficient B.

some damping, so, even in the absence of a dashpot damper or a similar device, energy dissipation must be accounted for). See Figure 4.5. This model can be applied to many systems, among which the vertical behaviour of a car's suspension (for which of course far more accurate, and complex, models can also be used — see Figure 4.6).

The force exerted on M by the spring is

$$F_K(t) = -K\,x(t) \tag{4.5}$$

or, omitting the dependence on time, $F_K = -K\,x$. There is a minus sign because, when x increases, the force on M opposes the increase of x. The force exerted on M by the damper is

$$F_B(t) = -B\,\dot{x}(t) \tag{4.6}$$

or $F_B = -B\,\dot{x}$ to simplify. There is a minus sign because, when x increases, $\dot{x}$ is positive, and the force on M opposes the increase of x. Thus

$$F(t) - K\,x(t) - B\,\dot{x}(t) = M\,\ddot{x}(t) \tag{4.7}$$

We will now assume that initial conditions are zero. Applying the Laplace transform,

$$F(s) - KX(s) - BsX(s) = Ms^2X(s) \tag{4.8}$$

which we can rearrange as

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + K} \quad \square \tag{4.9}$$

The form (4.9) in which the model of the mass–spring–damper system was put is called **transfer function**. It is very practical for the resolution of dynamic models.

Transfer function

Definition 4.1. Given a SISO system modelled by a differential equation, its transfer function is the ratio of the Laplace transform of the output (in the numerator) and the Laplace transform of the input (in the denominator), assuming that all initial conditions are zero. □

Figure 4.6: Independent suspension of a car's wheel (source: Wikimedia). The spring is clearly visible. The damper can be seen inside the coils of the spring.

Remark 4.4. When you see a transfer function, never forget that it is nothing but a differential equation under disguise. The transfer function is a rational function in s, which conceals a dynamic relation in time (or a relation in space, if the differential equation has derivatives in space rather than in time). □

Remark 4.5. Notice that it is necessary to assume zero initial conditions to obtain a transfer function. Otherwise, additional terms would appear, and it would be impossible to isolate on one side of the equation the ratio of the Laplace transforms of the output and the input. We will further study this subject in Chapter 8. □

Example 4.3. The transfer functions corresponding to (4.2)–(4.4) — i.e. to a mass, to a spring, and to a damper —, considering always a position $X(s)$ as the output and a force $F(s)$ as the input, are

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2} \tag{4.10}$$

$$\frac{X(s)}{F(s)} = \frac{1}{k} \tag{4.11}$$

$$\frac{X(s)}{F(s)} = \frac{1}{cs} \quad \square \tag{4.12}$$

Since transfer functions are functions of s, they are usually represented by one capital letter, such as F of G; when $F(s)$ is used to represent a transfer function, care must be taken not to use the same letter to represent the Laplace transform $F(s)$ of a force $f(t)$.

Example 4.4. Suppose that force $F(t) = \sin(t)$ is applied to the system in Figure 4.5, in which $M = 1$ kg, $B = 3.5$ N s/m, $K = 1.5$ N/m. What is the output $x(t)$?

The system's transfer function is

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + 3.5s + 1.5} = \frac{1}{(s+3)(s+0.5)} \tag{4.13}$$

We have

$$F(s) = \frac{1}{s^2+1} \tag{4.14}$$

and thus

$$\begin{aligned} X(s) &= \overbrace{\frac{1}{s^2+1}}^{\mathscr{L}\text{ of the input}} \overbrace{\frac{1}{(s+3)(s+0.5)}}^{\text{transfer function}} = \frac{as+b}{s^2+1} + \frac{c}{s+3} + \frac{d}{s+0.5} \qquad (4.15) \\ &= \frac{(as+b)(s^2+3.5s+1.5) + c(s^2+1)(s+0.5) + d(s^2+1)(s+3)}{(s^2+1)(s+3)(s+0.5)} \\ &= \frac{s^3(a+c+d) + s^2(3.5a+b+0.5c+3d) + s(1.5a+3.5b+c+d) + (1.5b+0.5c+d)}{(s^2+1)(s+3)(s+0.5)} \end{aligned}$$

whence

$$\begin{aligned} &\begin{cases} a+c+d=0 \\ 3.5a+b+0.5c+3d=0 \\ 1.5a+3.5b+c+d=0 \\ 1.5b+0.5c+3d=1 \end{cases} \Leftrightarrow \begin{cases} c+d=-a \\ 3.5a-0.5b=-1 \\ 0.5a+3.5b=0 \\ c+6d=2-3b \end{cases} \Leftrightarrow \begin{cases} c+d=-a \\ 50b=2 \\ a=-7b \\ c+6d=2-3b \end{cases} \\ &\Leftrightarrow \begin{cases} c+d=\frac{7}{25} \\ b=\frac{1}{25} \\ a=-\frac{7}{25} \\ c+6d=\frac{47}{25} \end{cases} \Leftrightarrow \begin{cases} c=\frac{7}{25}-d \\ — \\ — \\ 5d=\frac{40}{25} \end{cases} \Leftrightarrow \begin{cases} c=-\frac{1}{25} \\ — \\ — \\ d=\frac{8}{25} \end{cases} \end{aligned} \tag{4.16}$$

Finally,

$$\begin{aligned} x(t) &= \mathscr{L}^{-1}\left[\frac{-\frac{7}{25}s}{s^2+1} + \frac{\frac{1}{25}}{s^2+1} + \frac{-\frac{1}{25}}{s+3} + \frac{\frac{8}{25}}{s+0.5}\right] \\ &= -\frac{7}{25}\cos(t) + \frac{1}{25}\sin(t) - \frac{1}{25}e^{-3t} + \frac{8}{25}e^{-0.5t} \quad \square \end{aligned} \tag{4.17}$$

Example 4.5. Generalise the mass–spring–damper system of Example 4.2 to include three masses connected by springs and dampers as seen in Figure 4.7. The forces exerted by these components will be

$$\begin{cases} f_{K_1} = K_1(x_2 - x_1) \\ f_{K_2} = K_2(x_3 - x_2) \\ f_{B_1} = B_1(\dot{x}_2 - \dot{x}_1) \\ f_{B_2} = B_2(\dot{x}_3 - \dot{x}_2) \end{cases} \tag{4.18}$$

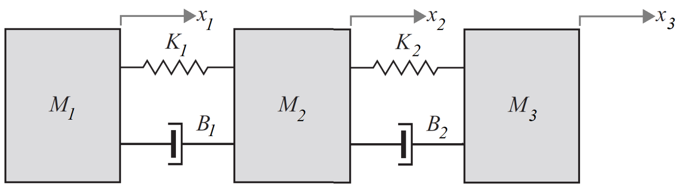

Figure 4.7: The system from Example 4.5, modelled by (4.20).

Applying Newton's law to the masses, we get

$$\begin{cases} K_1(x_2 - x_1) + B_1(\dot{x}_2 - \dot{x}_1) = M_1\dot{x}_1 \\ K_2(x_3 - x_2) + B_2(\dot{x}_3 - \dot{x}_2) - K_1(x_2 - x_1) - B_1(\dot{x}_2 - \dot{x}_1) = M_2\dot{x}_2 \\ -K_2(x_3 - x_2) - B_2(\dot{x}_3 - \dot{x}_2) = M_3\dot{x}_3 \end{cases} \tag{4.19}$$

Finally, the mathematical model of the system is

$$\begin{cases} M_1\ddot{x}_1 + K_1\left(x_1 - x_2\right) + B_1\left(\dot{x}_1 - \dot{x}_2\right) = 0 \\ M_2\ddot{x}_2 + K_1\left(x_2 - x_1\right) + B_1\left(\dot{x}_2 - \dot{x}_1\right) + K_2\left(x_2 - x_3\right) + B_2\left(\dot{x}_2 - \dot{x}_3\right) = 0 \\ M_3\ddot{x}_3 + K_2\left(x_3 - x_2\right) + B_2\left(\dot{x}_3 - \dot{x}_2\right) = 0 \end{cases} \tag{4.20}$$

To find a transfer function from the equations above, we would have to know which of the three positions x_1, x_2 and x_3 is the input and which is the output. □

Remark 4.6. Remember that it is somewhat irrelevant if positive displacements are assumed to be in one direction or the other. In the example above, positive displacements were arbitrarily assigned to the direction from the left to the right; the opposite could have been assumed, and signs would then be changed in such a way that the resulting model would still be correct. □

4.2 Simulating transfer functions in Matlab

There are two ways of creating a transfer function with Matlab:

- `tf` creates a transfer function, represented by two vectors with the coefficients of the polynomials in the numerator and in the denominator (in decreasing order of the exponent);
- `s = tf('s')` creates the Laplace transform variable s, which can then be manipulated using algebraic operators.

Matlab*'s command* `tf`

Example 4.6. Transfer function (4.13) from Example 4.4 can be created as

```
>> G = tf(1,[1 3.5 1.5])

G =

                1
```

```
  -----------------
  s^2 + 3.5 s + 1.5

Continuous-time transfer function.
```

or else as

```
>> s = tf('s')

s =

  s

Continuous-time transfer function.

>> G = 1/(s^2+3.5*s+1.5)

G =

          1
  -----------------
  s^2 + 3.5 s + 1.5

Continuous-time transfer function.
```

□

Simulation

Command `lsim` (linear simulation) uses numerical methods to solve the differential equation represented by a transfer function for a given input. In other words, it **simulates** the LTI represented by the transfer function.

MATLAB's command `lsim`

Example 4.7. The output found in Example 4.4 can be obtained and displayed as follows, if transfer function $G(s)$ has been created as above:

```
>> t = 0 : 0.01 : 50;
>> f = sin(t);
>> x = lsim(G, f, t);
>> figure,plot(t,x, t,-7/25*cos(t)+1/25*sin(t)-1/25*exp(-3*t)+8/25*exp(-0.5*t))
>> xlabel('t [s]'), ylabel('x [m]')
```

See Figure 4.8. Notice that we plotted two curves: the first was created with `lsim`, the second is (4.17). As expected, they coincide (there is a small numerical difference, too small to show up in the plot), and only one curve can be seen. □

Remark 4.7. The result (4.17) from Example 4.4 is exact. So the second curve in Figure 4.8 only has those numerical errors resulting from the implementation of the functions. The first curve has the errors resulting from the numerical method with which the differential equation was solved. Of course, both curves are based upon the same transfer function, and thus will suffer from any errors that there may be in that model (e.g. imprecise values of parameters M, B, and K, or neglected non-linearities in the spring or the damper). Do you still remember Remark 4.2? □

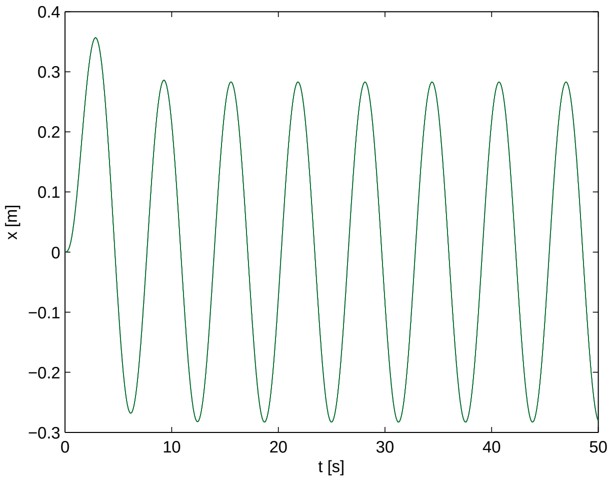

Figure 4.8: Results of Example 4.4.

4.3 Modelling the rotational movement

Mechanical systems with movement of rotation can usually be modelled using three components with the respective three equations:

Moment of inertia

Newton's second law for rotation

1. A **moment of inertia**. For these components, apply Newton's second law for rotation:

   Theorem 4.1.

   $$\sum \tau = \frac{\mathrm{d}}{\mathrm{d}t}\left(J(t)\,\dot{\omega}(t)\right) \qquad (4.21)$$

   Here, $\sum \tau$ is the sum of all torques applied on the moment of inertia $J(t)$, which is at angular position $\omega(t)$.

   Proof. Let r be the radius of rotation for which are applied the tangential forces $\sum F$ that cause the torque (see Figure 4.9). Because $x = r\omega$, then $\dot{x} = r\dot{\omega}$, and Newton's second law (4.1) becomes

   $$\sum F = \frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\,r\dot{\omega}(t)\right) \Leftrightarrow r\sum F = \frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)\,r^2\dot{\omega}(t)\right) \qquad (4.22)$$

   Because the torque τ of a force F is rF, and the moment of inertia J of mass m is mr^2, (4.21) follows for a point-like mass. In the case of a distributed mass, integrating (4.22) over the volume occupied will then yield the desired result. □

   Corollary 4.1. Considering an LTI system, J will not change with time, and so we are left with

   $$\sum \tau = J\,\ddot{\omega}(t) \qquad (4.23)$$

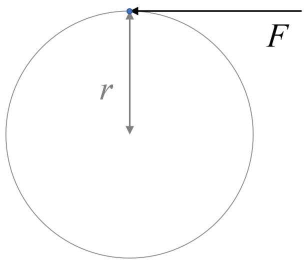

Figure 4.9: Tangential force F for a rotation of radius r applied on a point-like mass.

Figure 4.10: A torsion spring mounted on a mousetrap (source: Wikimedia). Notice that Hooke's law for torsion springs will only apply in the range of angles comprised within the ends of stroke (say, from 0 rad to π rad).

A moment of inertia is usually represented by J or I (the latter letter is avoided when it can be confounded with an electrical current); its SI units are $\mathrm{kg\,m^2}$. A torque is usually represented by τ or T; its SI units are $\mathrm{N\,m}$.

2. A **torsion spring**. This is a mechanical device that stores energy (see Figure 4.10) and usually follows the **angular form of Hooke's law**:

Torsion spring

Angular form of Hooke's law

$$\tau = \kappa\,\omega \tag{4.24}$$

Here, τ is the torque exerted by the spring, ω is the angular variation of the extremities of the spring measured from the repose position, and k is the spring constant. This constant is usually represented by the Greek character κ or K (to avoid confusion with a translation spring, for which a Latin character is used), and its SI units are N/rad.

3. A **rotary damper**, or **torsional damper**. The most usual model for this mechanical device that dissipates energy is **viscous damping**:

Rotary (or torsional) damper

$$\tau = c\,\dot{\omega} \tag{4.25}$$

Here, τ is the torque exerted by the damper, $\dot{\omega}$ is the relative angular velocity of the extremities of the damper, and c is the damping constant. This constant is usually represented by c, C, b or B, just like for the translation case, but its SI units are $\mathrm{N\,s/rad}$.

Also like (4.4), model (4.25) can be used to model unintended energy dissipation, such as that due to friction.

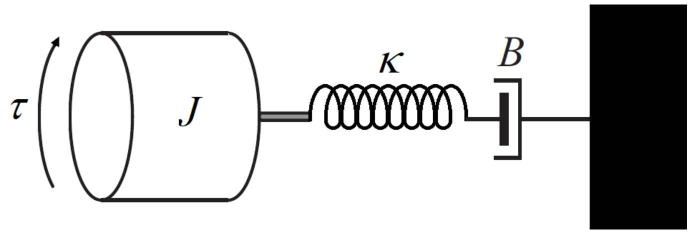

Figure 4.11: A mechanical system comprising a moment of inertia J, a torsion spring with constant κ, and a rotational damper with constant B.

Remark 4.8. When dealing with rotation, take care with angular units. Confusion about values in degrees, radians, and rotations is a common source of error. This is true also for angular speed, angular velocity, angular spring constants, etc.. □

Example 4.8. Consider the system in Figure 4.11. The torque exerted on J by the spring is

$$\tau_\kappa(t) = -\kappa\,\omega(t) \tag{4.26}$$

where ω is the rotation of J in the sense of rotation in which the applied torque τ is positive. The torque exerted by the damper is

$$\tau_B(t) = -B\,\dot{\omega}(t) \tag{4.27}$$

Thus

$$\tau(t) - \kappa\,\omega(t) - B\,\dot{\omega}(t) = J\,\ddot{\omega}(t) \tag{4.28}$$

Applying the Laplace transform (and assuming, once more, that all initial conditions are zero),

$$T(s) - \kappa\Omega(s) - Bs\Omega(s) = Js^2\Omega(s) \Leftrightarrow \frac{\Omega(s)}{T(s)} = \frac{1}{Js^2 + Bs + \kappa} \quad \square \tag{4.29}$$

4.4 Energy, effort and flow

A comparison of transfer functions (4.9) and (4.29) shows us that different systems can have similar models. Actually, if the numerical values of M and J, and of B (in translation) and B (in rotation), and of K and κ, are the same, then the model will be the same.

As you surely know by now, this happens not only with mechanical systems, but also with systems of other, different types, as we shall see in the following chapters. One of the best ways of studying this parallelism is to see systems as energy converters, and energy E as the integral of the product of two variables, called **effort variable** e and **flow variable** f:

Effort
Flow

$$E(t) = \int_0^t e(t)\,f(t)\,\mathrm{d}t \tag{4.30}$$

er

In other words, the product $e(t) \times f(t)$ is the instantaneous **power** $\dot{E}(t)$.

In the case of a translation movement,

$$\dot{E}(t) = F(t)\,\dot{x}(t) \Leftrightarrow E(t) = \int_0^t F(t)\,\dot{x}(t)\,\mathrm{d}t \tag{4.31}$$

In the case of a rotation movement,

$$\dot{E}(t) = \tau(t)\,\dot{\omega}(t) \Leftrightarrow E(t) = \int_0^t \tau(t)\,\dot{\omega}(t)\,\mathrm{d}t \tag{4.32}$$

We will consider force F and torque τ as the flow variable, and velocity $\dot{x}$ and angular velocity $\dot{\omega}$ as the effort variable. But notice that it would make no difference if it were the other way round. In any case, their product will be the power. Both choices can be found in published literature.

Effort accumulator
Flow accumulator
Dissipator

The components of a system are the **effort accumulator**, the **flow accumulator**, and the **Energy dissipator**, as seen in Table 4.1. For both accumulators, energy is the integral of accumulated flow or accumulated effort: elastic potential energy in the case of effort, and kinetic energy in the case of flow. The dissipator dissipates energy and it makes no difference whether it is kinetic or potential energy that it dissipates. Table 4.1 also includes the relations between these quantities.

Impedance
Admittance

Definition 4.2. A transfer function of a system that has the flux as input and the effort as output is called **impedance** of that system. A transfer function of a system that has the effort as input and the flux as output is called **admittance** of that system. Consequently, the admittance is the inverse of the impedance. □

Mechanical impedance

Transfer functions (4.10)–(4.12) can be rewritten so as to give the mechanical impedance of a mass, a spring, and a damper:

$$\frac{sX(s)}{F(s)} = \frac{1}{ms} \tag{4.33}$$

$$\frac{sX(s)}{F(s)} = \frac{s}{k} \tag{4.34}$$

$$\frac{sX(s)}{F(s)} = \frac{1}{c} \tag{4.35}$$

4.5 Other components

Among the several other components that may be found in mechanical systems, the following ones, because of their general use and of their linearity, deserve a passing mention:

Belt

- A **transmission belt**. It converts rotation movement into another rotation movement:

$$\frac{\omega_1}{\omega_2} = \frac{\dot{\omega}_1}{\dot{\omega}_2} = \frac{\ddot{\omega}_1}{\ddot{\omega}_2} = \frac{r_1}{r_2} \tag{4.36}$$

  Here ω_1 and ω_2 are the angular positions of the two wheels connected by the belt, and r_1 and r_2 are the respective radius. See Figure 4.12.

Table 4.1: Effort, flow, accumulators and dissipators in mechanical systems

	Translation mechanical system	SI	Rotation mechanical system	SI
effort e	velocity $\dot{x}$	$\mathrm{m\,s^{-1}}$	angular velocity $\dot{\omega}$	$\mathrm{rad\,s^{-1}}$
flow f	force F	N	torque τ	$\mathrm{N\,m}$
effort accumulator	spring, with spring constant K	N/m	angular spring, with spring constant κ	N/rad
accumulated effort $e_a = \int e\,\mathrm{d}t$	position $x = \int \dot{x}\,\mathrm{d}t$	m	angular position $\omega = \int \dot{\omega}\,\mathrm{d}t$	rad
relation between accumulated effort and flow $e_a = \varphi(f)$	position $x = \frac{1}{K}F$		angular position $\omega = \frac{1}{\kappa}\tau$	
accumulated energy $E_e = \int e_a\,\mathrm{d}f$	elastic potential energy $E_e = \frac{1}{2K}F^2$	J	elastic potential energy $E_e = \frac{1}{2\kappa}\tau^2$	J
flow accumulator	mass M	kg	moment of inertia J	$\mathrm{kg\,m^2}$
accumulated flow $f_a = \int f\,\mathrm{d}t$	momentum $p = \int F\,\mathrm{d}t$	$\mathrm{kg\,m\,s^{-1}}$	angular momentum $h = \int \tau\,\mathrm{d}t$	$\mathrm{kg\,m^2\,s^{-1}}$
relation between accumulated flow and effort $f_a = \varphi(e)$	momentum $p = M\dot{x}$		angular momentum $h = J\dot{\omega}$	
accumulated energy $E_f = \int f_a\,\mathrm{d}e$	kinetic energy $E_f = \frac{1}{2}M\dot{x}^2$	J	kinetic energy $E_f = \frac{1}{2}J\dot{\omega}^2$	J
dissipator	damper, with damping constant b	N s/m	rotary damper, with damping constant b	N s/rad
relation between effort and flow $e = \varphi(f)$	$\dot{x} = \frac{1}{b}F$		$\dot{\omega} = \frac{1}{b}\tau$	
dissipated energy $E_d = \int f\,\mathrm{d}e$	$E_d = \frac{1}{2}b\dot{x}^2$	J	$E_d = \frac{1}{2}b\dot{\omega}^2$	J

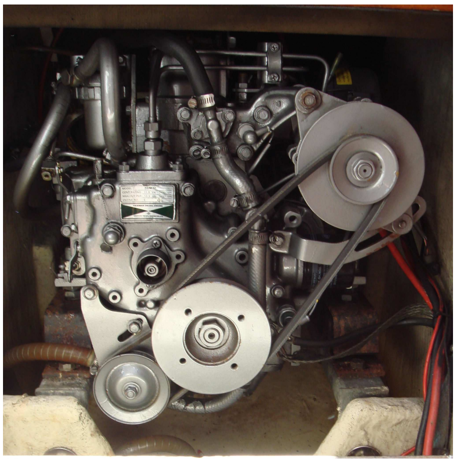

Figure 4.12: Transmission belts in a Diesel engine (source: Wikimedia).

- **Cogwheels**, or gears. These wheels also convert rotation movement into another rotation movement, but the wheels need not be in the same plane. For two external cogwheels, the sense of rotation is inverted, whereas for one external and one internal cogwheel it is not (see Figure 4.13): *Cogwheels*

$$\frac{\omega_1}{\omega_2} = \frac{\dot{\omega}_1}{\dot{\omega}_2} = \frac{\ddot{\omega}_1}{\ddot{\omega}_2} = \pm\frac{r_1}{r_2} = \pm\frac{n_1}{n_2} \tag{4.37}$$

  Here n_1 and n_2 are the numbers of cogs, or teeth, in the cogwheels.

- A **rack and pinion**. It converts rotation movement into translation movement and vice-versa: *Rack and pinion*

$$x = \omega r \Leftrightarrow \dot{x} = \dot{\omega} r \Leftrightarrow \ddot{x} = \ddot{\omega} r \tag{4.38}$$

  Here x is the distance of the translation movement, r the radius of the wheel, and ω the angle of the rotation movement. See Figure 4.14.

- A **harmonic drive**. It converts rotation movement into another rotation movement, using an outside internal circular gear, inside which there is an external elliptical gear, to which an elliptical shaft is connected through a rolling bearing: *Harmonic drive*

$$\frac{\omega_1}{\omega_2} = \frac{\dot{\omega}_1}{\dot{\omega}_2} = \frac{\ddot{\omega}_1}{\ddot{\omega}_2} = -\frac{n_1 - n_0}{n_0} \tag{4.39}$$

  Here ω_2 is the angular position of the elliptical shaft inside the elliptical gear, ω_1 is the angular position of the shaft connected to the elliptical gear, n_1 is the number of teeth of the said gear, and n_0 is the number of teeth of the outside internal gear (which is fixed). See Figure 4.14.

Remark 4.9. Linear models (4.36)–(4.39) omit nonlinear effects that may appear, such as backlash due to gaps betweens cogs (see Figure 4.15). These effects may sometimes be important but are not our concern here. □

 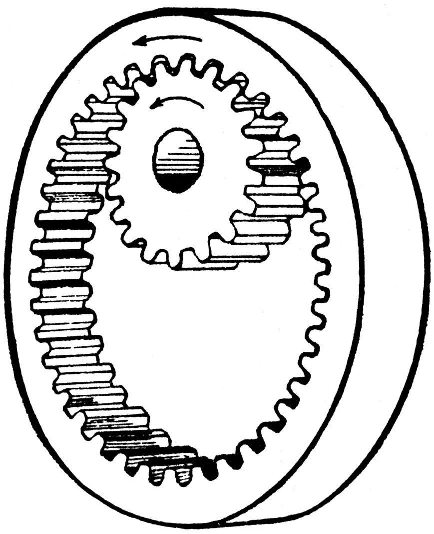

Figure 4.13: Left: two external cogwheels (source: Wikimedia). Right: two cogwheels, the outside one being an internal cogwheel (because the cogs are on the inside), and the inside one being an external cogwheel (because the cogs are on the outside; source: `https://etc.usf.edu/clipart`).

 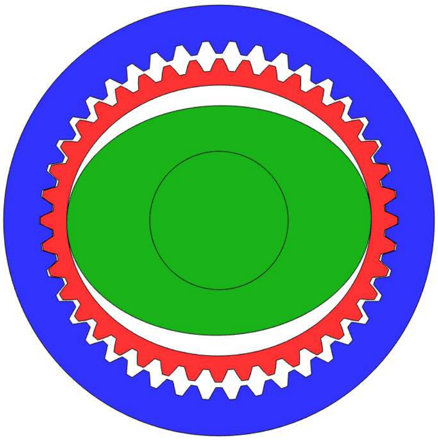

Figure 4.14: Left: rack and pinion in a canal gate; the rack is the linear element (in this case, vertical). Right: schematic of a harmonic drive. (Source: Wikimedia.)

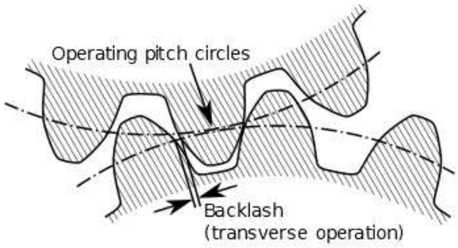

Figure 4.15: Backlash (source: Wikimedia).

Glossary

> Als Storm weer bijkomt is het eerste wat hij ziet het vriendelijke gezicht van de jager.
> „Wat wilt u dat ik voor u doe, god?"
> „Ik... kan je verstaan, maar... maar spreek ik nu jouw taal of jij de mijne? En wat bedoel je met god? Ik ben geen god!"
> „Maar natuurlijk bent u dat! Alles wijst erop. U sprak wartaal en begreep mij niet. Nou, het is duidelijk dat de goden de mensen niet begrijpen, anders hadden ze ze allang uitgeroeid! En nu u van de parel der kennis hebt gegeten, begrijpt u er nog niets van. Dat bewijst dat u gek bent! En de goden moeten gek zijn anders hadden ze de wereld nooit zo gemaakt als hij is."

Don Lawrence (1928 — †2003), Martin Lodewijk (1939 — ...), *Storm*, De kronieken van Pandarve 10, De piraten van Pandarve

accumulated effort potencial acumulado
accumulated flow fluxo acumulado
admittance admitância
cogwheel roda dentada
coil spring mola helicoidal
compression compressão
damper amortecedor
damping amortecimento
dashpot amortecedor viscoso
dissipator dissipador
effort accumulator acumulador de potencial
effort variable variável de potencial
end of stroke fim de curso
energy energia
extension extensão
flow accumulator acumulador de fluxo
flow variable variável de fluxo
gear roda dentada
harmonic drive redutor harmónico
helical spring mola helicoidal
impedance impedância
leaf spring mola de folhas, mola de lâminas

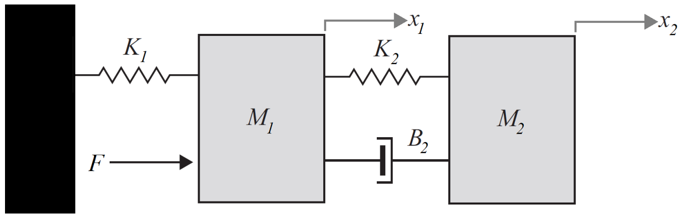

Figure 4.16: System of Exercise 1.

mass massa
mechanical impedance impedância mecânica
moment of inertia momento de inércia
momentum quantidade de movimento, momento linear
point-like mass massa pontual
power potência
rack and pinion pinhão e cremalheira
rotary damper amortecedor rotativo, amortecedor de torção
simulation simulação
spring mola
spring constant constante de mola
torque binário, torque (bras.)
torsion spring mola de torção
torsional damper amortecedor rotativo, amortecedor de torção
transfer function função de transferência
transmission belt correia de transmissão
volute spring mola de volutas, mola voluta

Exercises

1. Consider the system in Figure 4.16.

   (a) Find the differential equations that model the system.

   (b) From the result above, knowing that $M_1 = 1$ kg, $M_2 = 0.5$ kg, $K_1 = 10$ N/m, $K_2 = 2$ N/m, and $B_2 = 4$ N s/m, find transfer function $\frac{X_2(s)}{F(s)}$.

2. Consider the system in Figure 4.17. The wheels have neglectable mass and inertia; they are included to show that the masses move without friction.

   (a) Find the differential equations that model the system.

   (b) From the result above, knowing that $M_1 = 100$ kg, $M_2 = 10$ kg, $K = 50$ N/m, and $B = 25$ N s/m, find transfer function $\frac{X_2(s)}{X_1(s)}$.

3. Consider the system in Figure 4.18. The wheels have neglectable mass and inertia; they are included to show that the masses move without friction.

   (a) Find the differential equations that model the system.

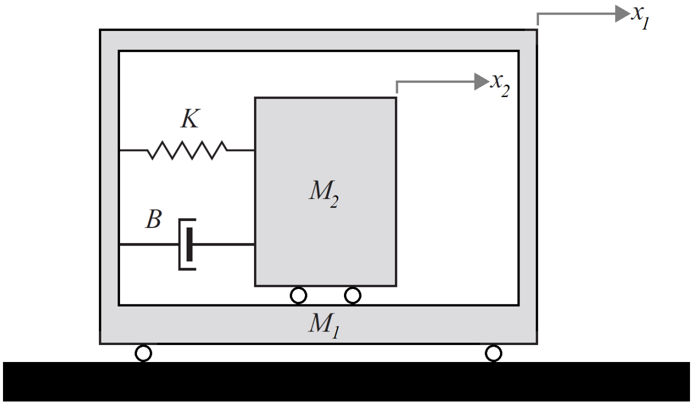

Figure 4.17: System of Exercise 2.

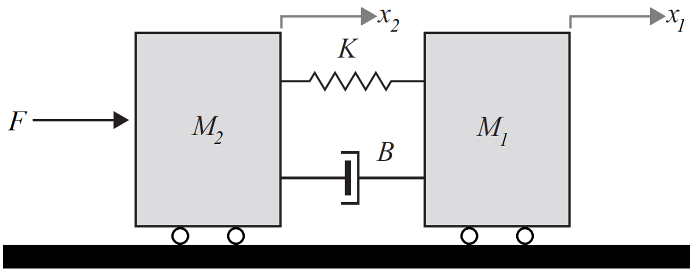

Figure 4.18: System of Exercise 3.

(b) From the result above, knowing that $M_1 = M_2 1$ kg, $K = 5$ N/m, and $B = 10$ N s/m, find transfer function $\frac{X_1(s)}{F(s)}$.

(c) For the same constants, find transfer function $\frac{X_2(s)}{F(s)}$.

4. Consider the system in Figure 4.19.

   (a) Find the differential equations that model the system.

   (b) From the result above, knowing that $M_1 = 1$ kg, $M_2 = 2$ kg, $M_3 = 3$ kg, $K_1 = 10$ N/m, $K_2 = 20$ N/m, $K_3 = 30$ N/m, $B_1 = 5$ N s/m, $B_2 = 10$ N s/m, and $B_3 = 15$ N s/m, find transfer function $\frac{X_1(s)}{F(s)}$.

   (c) For the same constants, find transfer function $\frac{X_2(s)}{F(s)}$.

   (d) For the same constants, find transfer function $\frac{X_3(s)}{F(s)}$.

5. Consider the system in Figure 4.20. The cogwheels have neglectable moments of inertia, when compared to J. Let N_u be the number of cogs in the upper cogwheel, and N_l the number of cogs in the lower cogwheel.

   (a) Find the differential equations that model the system.

   (b) From the result above, knowing that $J = 50$ kg m^2, $N_u = 20$, $N_l = 30$, and $B = 40$ N s/m, find transfer function $\frac{\Omega(s)}{T(s)}$.

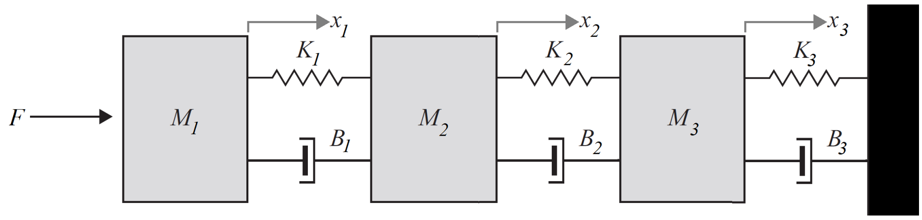

Figure 4.19: System of Exercise 4.

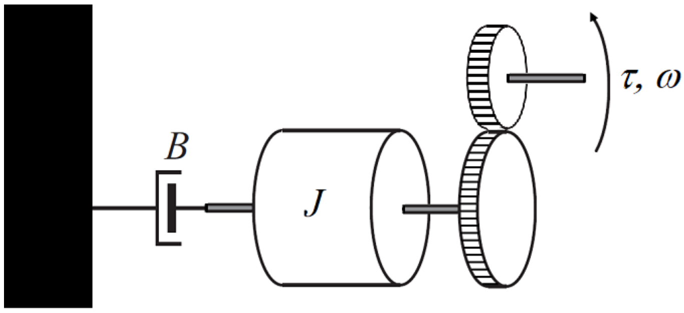

Figure 4.20: System of Exercise 5.

6. Consider the system in Figure 4.21. The pinion's centre is fixed, and its moment of inertia I includes the lever actuated by force F.

   (a) Find the differential equations that model the system.

   (b) From the result above, knowing that $r = 0.2$ m, $I = 0.8$ kg m^2, $m = 20$ kg, $k = 1000$ N/m, and $b = 480$ N s/m, find transfer function $\frac{X(s)}{F(s)}$.

7. Consider the system in Figure 4.22. Force F is applied through a bar of neglectable mass, connected by the spring and the damper to mass m, affected by friction force F_a that follows the law of viscous damping with constant b_a. The bar has velocity v_F; mass m has velocity F.

   (a) Find the differential equations that model the system.

   (b) From the result above, find transfer function $\frac{V_F(s)}{F(s)}$.

8. Consider the system in Figure 4.23. The position of mass M is $x(t)$.

   (a) Find the differential equations that model the system.

   (b) From the result above, find transfer function $\frac{X(s)}{T(s)}$.

9. For all the systems in the exercises above, find:

   (a) the effort variables;

   (b) the effort accumulators;

   (c) the flow variables;

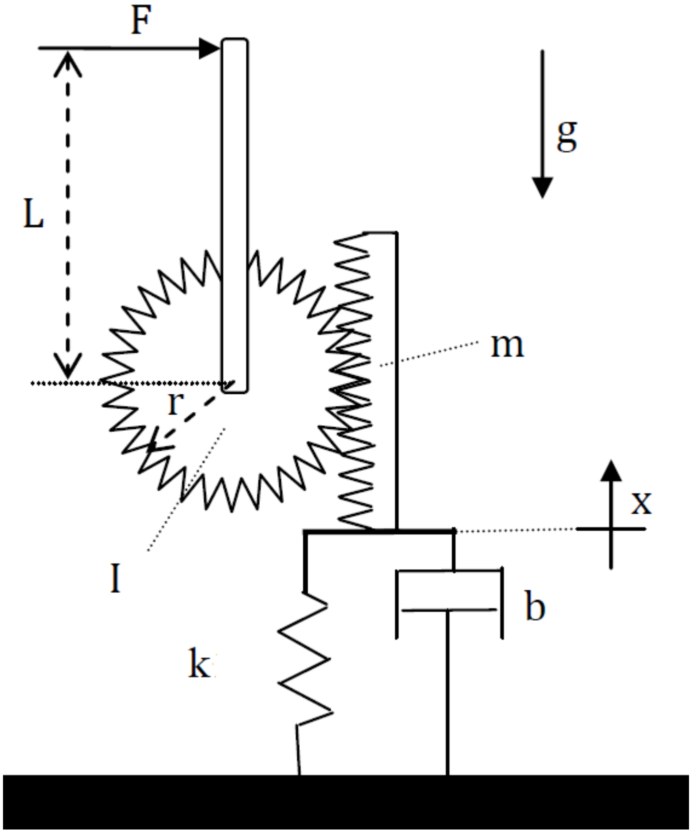

Figure 4.21: System of Exercise 6.

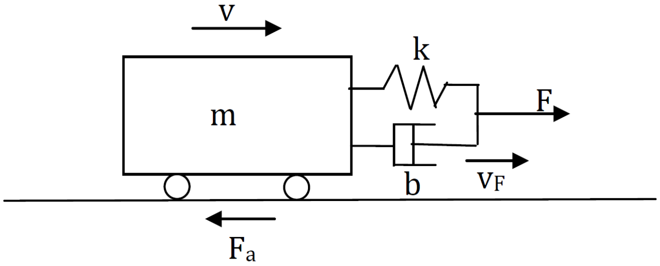

Figure 4.22: System of Exercise 7.

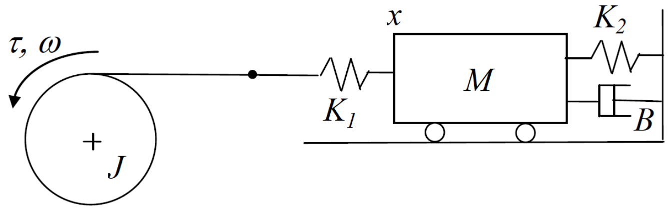

Figure 4.23: System of Exercise 8.

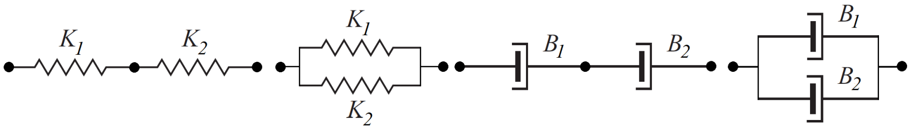

Figure 4.24: From left two right: two springs in series; two springs in parallel; two dampers in series; two dampers in parallel.

(d) the flow accumulators;

(e) the dissipators.

10. Find the $\frac{X(s)}{F(s)}$ transfer functions, as in (4.10)–(4.12), of the following systems (see Figure 4.24):

(a) two springs, with constants K_1 and K_2, in series;

(b) two springs, with constants K_1 and K_2, in parallel;

(c) two dampers, with constants B_1 and B_2, in series;

(d) two dampers, with constants B_1 and B_2, in parallel.