Chapter 6

Modelling fluidic systems

> Suppose a solid held above the surface of a liquid and partially immersed: a portion of the liquid is displaced, and the level of the liquid rises. But, by this rise of level, a little bit more of the solid is of course immersed, and so there is a new displacement of a second portion of the liquid, and a consequent rise of level. Again, this second rise of level causes a yet further immersion, and by consequence another displacement of liquid and another rise. It is self-evident that this process must continue till the entire solid is immersed, and that the liquid will then begin to immerse whatever holds the solid, which, being connected with it, must for the time be considered a part of it. If you hold a stick, six feet long, with its end in a tumbler of water, and wait long enough, you must eventually be immersed. The question as to the source from which the water is supplied—which belongs to a high branch of mathematics, and is therefore beyond our present scope—does not apply to the sea. Let us therefore take the familiar instance of a man standing at the edge of the sea, at ebb-tide, with a solid in his hand, which he partially immerses: he remains steadfast and unmoved, and we all know that he must be drowned.

Lewis CARROLL (1832 — †1898), *A tangled tale*, Knot IX

Pipe flow

In this chapter we are concerned with fluid flow in pipes (not with fluid flow with a free surface). Fluidic systems can be accurately modelled using the Navier-Stokes equations, which you learn in a different course. Fortunately, in many cases of fluid flow in pipes it is possible to use simplified equations as follows.

6.1 Energy, effort and flow

Energy E is given by the integral over distance x of the force F exerted by the fluid:

$$E = \int_0^x F \, \mathrm{d}x \tag{6.1}$$

Table 6.1: Effort, flow, accumulators and dissipators in fluidic systems

	Fluidic system	SI
effort e	pressure P	Pa
flow f	volume flow rate Q	m^3/s
effort accumulator	fluidic inductance with inertance L	$\mathrm{kg\,m^{-4}}$
accumulated effort $e_a = \int e\,\mathrm{d}t$	fluidic moment $\Gamma = \int p\,\mathrm{d}t$	$\mathrm{Pa\,s}$
relation between accumulated effort and flow $e_a = \varphi(f)$	fluidic moment $\Gamma = LQ$	
accumulated energy $E_e = \int e_a\,\mathrm{d}f$	kinetic energy of the flow $E_e = \frac{1}{2}LQ^2$	J
flow accumulator	reservoir with capacitance C	F
accumulated flow $f_a = \int f\,\mathrm{d}t$	volume $V = \int Q\,\mathrm{d}t$	m^3
relation between accumulated flow and effort $f_a = \varphi(e)$	volume $V = Cp$	
accumulated energy $E_f = \int f_a\,\mathrm{d}e$	potential energy of the flow $E_f = \frac{1}{2}Cp^2$	J
dissipator	fluidic resistance R	$\mathrm{kg\,s^{-1}\,m^{-4}}$
relation between effort and flow $e = \varphi(f)$	$p = RQ$	
dissipated energy $E_d = \int f\,\mathrm{d}e$	$E_d = \frac{1}{2}RQ^2$	J

The force is equal to the product of the pressure p and the cross-sectional area A, so

$$F = p\,A \Rightarrow E = \int_0^x p\,A\,\mathrm{d}x \tag{6.2}$$

Volume flow rate

The **volume flow rate** (or volumetric flow rate) Q is the derivative of the volume $V = A\,x$, given by

$$Q = A\frac{\mathrm{d}x}{\mathrm{d}t} \tag{6.3}$$

where we assume a constant A. With some abuse of notation, we can write $A = \frac{Q\,\mathrm{d}t}{\mathrm{d}x}$ and replace this in (6.2) to rewrite the integral in (6.1) as

$$E = \int_0^t pQ\,\mathrm{d}t \tag{6.4}$$

So **pressure** p and **volume flow rate** Q can be used as effort and flow. By an understandable universal convention, Q is always considered as the flow, and p as the effort.

Table 6.1 sums up the passing information and relations. The next section presents the basic components mentioned in that Table.

6.2 Basic components of a fluidic system

The basic components of a fluidic system are the following:

Reservoir
Tank

1. A **reservoir** or **tank**, which may either have a free surface (see Figure 6.1) or not. Tanks of the first case are often used with liquids; closed tanks are the only option when the fluid is a gas, since the gas might otherwise escape, even if its density is higher than that of the air.

   Open tank

   In the case of a tank with a free surface, there must be a pipe at the

Figure 6.1: A water reservoir at the Évora train station. (Source: Wikimedia.)

bottom (otherwise the tank could not be emptied). The pressure p at that point, as you know, is

$$p = \rho g h \tag{6.5}$$

where ρ is the fluid density, g is the acceleration of gravity, and h is the height of the fluid in the tank. But the volume of fluid in the tank is given by $V = A\,h$, and so, replacing $h = \frac{V}{A}$ in (6.5) and solving in order to V,

$$V = p \underbrace{\frac{A}{\rho g}}_{\text{capacitance } C} \tag{6.6}$$

Pressurised tank

In the case of reservoirs without a free surface, it can also be shown that $V = pC$, where the value of capacitance C will depend on whether the fluid is is a liquid, a gas undergoing an isothermal compression or expansion, or a gas undergoing an adiabatic compression or expansion. We need not worry with that, as long as the value of C is known.

Fluidic inductance

2. A **fluidic inductance**. This is in fact one of the two phenomena that take place in a pipe. Its model is an application of Newton's second law (4.1) to the fluid contained in a length ℓ of pipe (see Figure 6.2):

$$\underbrace{A\,p}_{\text{force}} = \underbrace{\rho A \ell}_{\text{mass}} \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \tag{6.7}$$

The force is the product of the cross-sectional area and the pressure (or rather the difference of pressures at the two extremities of the fluid separated by length ℓ). Integrating both sides, and introducing the **fluidic moment** $\Gamma = \int p\,\mathrm{d}t$,

$$\Gamma = \rho \ell \frac{\mathrm{d}x}{\mathrm{d}t} \tag{6.8}$$

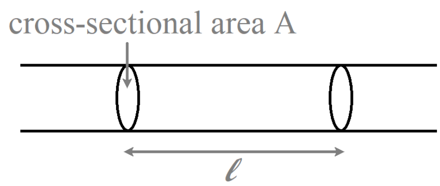

Figure 6.2: A pipe with cross-sectional area A.

From (6.3) we know that $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{Q}{A}$, so

$$\Gamma = \underbrace{\frac{\rho\ell}{A}}_{\text{inertance } L} Q \tag{6.9}$$

Pressure drop

Laminar flow

Turbulent flow

3. A **pressure drop**. This is the other phenomena taking place in any pipe, due to the resistance of viscous forces both between the fluid and the wall of the pipe and between fluid particles themselves. In another course you learn about the difference between **laminar flow** (i.e. a situation in which fluid particles move essentially in the direction of the flow only) and **turbulent flow** (i.e. a situation in which fluid particles move in a far more chaotic manner).

Hagen-Poiseuille equation for laminar flow

Here it suffices to notice that in laminar flow the **Hagen-Poiseuille equation** applies:

$$p = \underbrace{\frac{8\mu\ell}{\pi r^4}}_{\text{resistance } R} Q \tag{6.10}$$

Here p is the pressure drop over length ℓ of the pipe, μ is the dynamic viscosity, and r is the pipe radius. (If the cross-section of the pipe is not circular, then $r = \sqrt{\frac{A}{r}}$.) This expression was first determined experimentally, and then proved from the Navier-Stokes equations; all that we need to worry about is the value of the fluidic resistance.

The pressure drop is always higher for turbulent flow than for laminar flow, and the relation between p and Q is no longer linear. However, it may be linearised around a convenient point, so as to find an approximate value of resistance $R = \frac{p}{Q}$ valid in some range of values of these variables. (See Figure 4.3 again.)

Remark 6.1. Pipes have both inertance and resistance. Of course, it may be that one of the two is neglectable when compared to the other; but in reality both are present. □

Fluidic impedances

Remark 6.2. The impedances of these components are as follows:

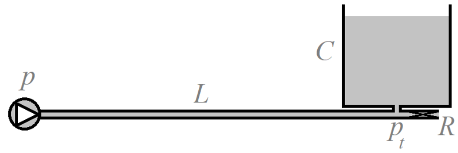

Figure 6.3: System of Example 6.1.

$$\frac{P(s)}{Q(s)} = \frac{1}{Cs} \tag{6.11}$$

$$\frac{P(s)}{Q(s)} = Ls \tag{6.12}$$

$$\frac{P(s)}{Q(s)} = R \quad \square \tag{6.13}$$

Pipe flow can be modelled putting together the equations describing these components with the conservation of mass.

Example 6.1. Consider the system in Figure 6.3, supplied with water by a pump providing a pressure p, that flows through a long pipe with inertance L and neglectable resistance, and either fills a tank with capacitance C or leaves the system through a valve with resistance R. We want to know the pressure below the tank p_t.

Let $q(t)$ be the volume flow rate through the long pipe, that is then divided into the flow feeding the tank $q_t(t)$ and the flow through the valve $q_v(t)$. Using the impedances, we get

$$\begin{cases} \frac{P(s)-P_t(s)}{Q(s)} = Ls \\ Q(s) = Q_t(s) + Q_v(s) \\ \frac{P_t(s)}{Q_t(s)} = \frac{1}{Cs} \\ \frac{P_t(s)}{Q_v(s)} = R \end{cases} \Rightarrow \begin{cases} P(s) - P_t(s) = LsP_t(s)\left(\frac{1}{R} + Cs\right) \\ Q(s) = P_t(s)\left(\frac{1}{R} + Cs\right) \\ Q_t(s) = P_t(s)Cs \\ Q_v(s) = \frac{1}{R}P_t(s) \end{cases} \tag{6.14}$$

The first equation then gives the desired answer:

$$P(s) = P_t(s)\left(1 + \frac{L}{R}s + LCs^2\right) \Leftrightarrow \frac{P_t(s)}{P(s)} = \frac{1}{1 + \frac{L}{R}s + LCs^2} \quad \square \tag{6.15}$$

Remark 6.3. Notice that (6.15) is similar to the model of a mass–spring–damper system (4.9) or the model of an RLC system (5.20). □

Remark 6.4. Liquids can be presumed to be incompressible, so ρ is constant and independent from p. Thus (6.5) shows that there is a one-to-one relation between p and h, where h is the **hydraulic head**. So p is often replaced by $\rho g h$. To do this, of course, the density of the liquid used in the system must be fixed in advance; the most usual cases are water, brine, and crude oil.

Hydraulic head

Model (6.9) of a fluidic inductance tells us that $\int p \, dt = \Gamma = LQ$; applying the Laplace transform, this becomes

Using the head instead of the pressure

Fluidic inductance

$$\frac{p}{s} = LQ \Leftrightarrow \frac{\rho g h}{s} = LQ \Leftrightarrow h = \frac{L}{\rho g} sQ \tag{6.16}$$

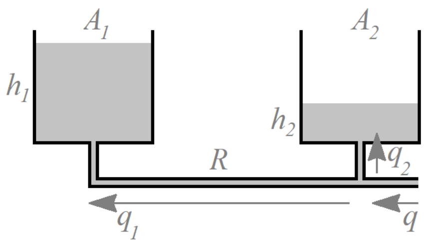

Figure 6.4: System of Example 6.2.

Here, $L^* = \frac{L}{\rho g}$ is the inertance relating hydraulic head and flow; L is the inertance relating pressure and flow. Notice that the SI units of L are $\mathrm{kg\,m^{-4}}$; those of L^* are $\mathrm{s^2\,m^{-2}}$.

Fluidic resistance

Likewise, model (6.10) of fluidic resistance tells us that $p = RQ$; so

$$\rho g h = RQ \Leftrightarrow h = \frac{R}{\rho g} Q \tag{6.17}$$

Here, $R^* = \frac{R}{\rho g}$ is the resistance relating hydraulic head and flow; R is the resistance relating pressure and flow. Notice that the SI units of R are $\mathrm{kg\,s^{-1}\,m^{-4}}$; those of R^* are $\mathrm{s\,m^{-2}}$. $\square$

Example 6.2. Consider the system in Figure 6.4, with two water reservoirs fed by a pump that delivers a flow q, and connected by a pipe with neglectable inertance and resistance R. We have

$$\begin{cases} q(t) = q_1(t) + q_2(t) \\ q_1(t) = A_1 \dot{h}_1(t) \\ q_2(t) = A_2 \dot{h}_2(t) \\ \frac{h_2(t) - h_1(t)}{R} = q_1(t) \end{cases} \Rightarrow \begin{cases} Q(s) = A_1 H_1(s) s + A_2 H_2(s) s \\ Q_1(s) = A_1 H_1(s) s \\ Q_2(s) = A_2 H_2(s) s \\ H_2(s) - H_1(s) = R A_1 H_1(s) s \end{cases} \tag{6.18}$$

Thus

$$\begin{cases} Q(s) = A_1 H_1(s) s + A_2 H_1(s)(R A_1 s + 1) s \\ H_2(s) = H_1(s)(R A_1 s + 1) \end{cases} \Leftrightarrow \begin{cases} Q(s) = H_1(s)(R A_1 A_2 s^2 + A_1 s + A_2 s) \\ H_2(s) = H_1(s)(R A_1 s + 1) \end{cases} \Leftrightarrow$$

$$\begin{cases} \dfrac{H_1(s)}{Q(s)} = \dfrac{1}{R A_1 A_2 s^2 + (A_1 + A_2) s} \\ \dfrac{H_2(s)}{Q(s)} = \dfrac{R A_1 s + 1}{R A_1 A_2 s^2 + (A_1 + A_2) s} \end{cases} \quad \square \tag{6.19}$$

6.3 Other components

Among the several other components that may be found in pipe flow systems, the hydraulic press deserves a passing mention. Its principle is shown in Figure 6.5. Because of (6.2), a similar pressure on both sides means that

$$\frac{F_1}{A_1} = \frac{F_2}{A_2} \Leftrightarrow F_2 = F_1 \frac{A_2}{A_1} \tag{6.20}$$

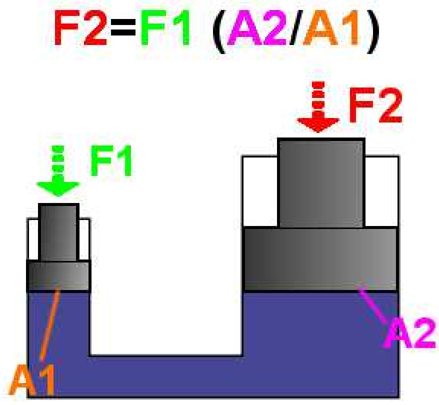

Figure 6.5: Principle of the hydraulic press.

where F_1 and F_2 are the forces exerted on the two pistons, with areas A_1 and A_2. This principle is used in presses such as that in Figure 6.6.

Glossary

> Había en el puerto gran multitud de buques de todas clases y tamaños, resplandeciendo entre ellos, llamando la atención y hasta excitando la admiración y la envidia de los españoles, un enorme y hermosísimo navío, construido con tal perfección, lujo y elegancia, que era una maravilla.
> Los españoles, naturalmente, tuvieron la curiosidad de saber quién era el dueño del navío y encargaron al secretario que, sirviendo de intérprete, se lo preguntase a algunos alemanes que habían venido a bordo.
> Lo preguntó el secretario y dijo luego a sus paisanos y camaradas:
> — El buque es propiedad de un poderoso comerciante y naviero de esta ciudad en que estamos, el cual se llama el señor Nichtverstehen.

Juan Valera (1824 — †1905), *Cuentos y chascarrillos andaluces*, El señor Nichtverstehen

fluidic inductance indutância fluídica
fluidic moment momento fluídico
fluidic resistance resistência fluídica
hydraulic head altura de coluna de fluido (de água, de água salgada, de crude)
hydraulic press prensa hidráulica
inertance indutância fluídica
pressure pressão
pressure drop perda de carga
reservoir reservatório, tanque
tank reservatório, tanque
volume flow rate caudal volumétrico
volumetric flow rate caudal volumétrico

Figure 6.6: Hydraulic press.

Exercises

1. Consider the system from Example 6.1, shown in Figure 6.3. Find its mechanical equivalent.

2. Consider the system in Figure 6.7, fed by a pump delivering volume flow rate $q_1(t)$. Tanks 1 and 2 are connected by a pipe with neglectable inertance and fluidic resistance R_1; tanks 2 and 3 are emptied through valves with resistances R_2 and R_3 respectively. Find transfer functions $\frac{H_1(s)}{Q_1(s)}$, $\frac{H_2(s)}{Q_1(s)}$ and $\frac{H_3(s)}{Q_1(s)}$.

3. Consider the system in Figure 6.8, fed by a pump delivering volume flow rate $q_1(t)$. Find transfer functions $\frac{H_1(s)}{Q_1(s)}$, $\frac{H_2(s)}{Q_1(s)}$, $\frac{H_3(s)}{Q_1(s)}$ and $\frac{Q_5(s)}{Q_1(s)}$.

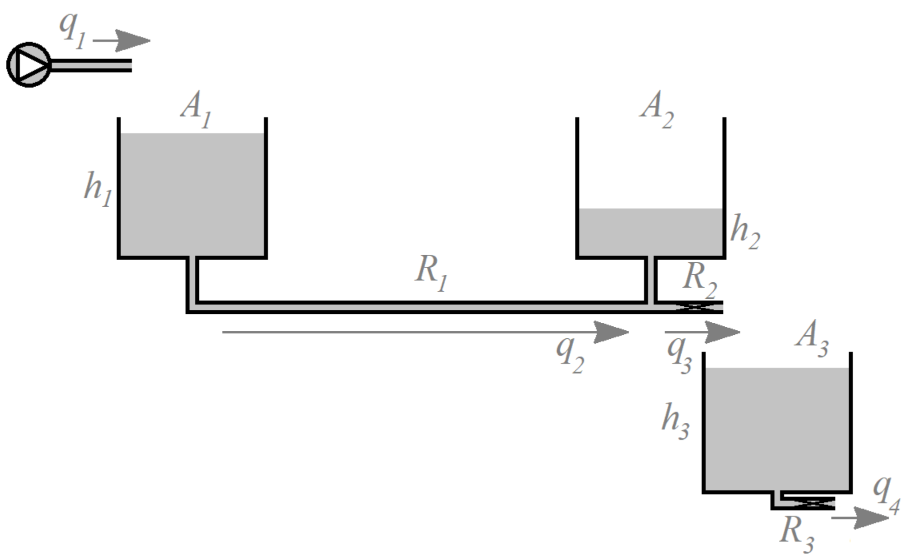

Figure 6.7: System of Exercise 2.

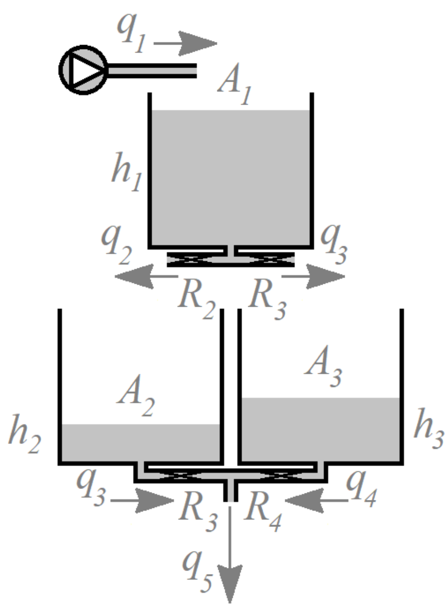

Figure 6.8: System of Exercise 3.