1. (a) How large should $\lambda_0$ be so that the $c$–chart with 3–sigma control limits has a positive LCL? Elaborate on the importance of a positive LCL in such a context.

- Requested target value $\lambda_0$
  
  \[ LCL > 0 \Rightarrow \lambda_0 - 3 \times \sqrt{\lambda_0} > 0 \]
  
  \[ \lambda_0 > 9. \]

- Importance of a positive LCL
  
  It is essential that the $c$–chart has a positive LCL to be able detect decreases in the expected number of defects $\lambda$ (i.e., to detect a quality improvement) in a fairly quick fashion. If $LCL \neq 0$, we deal with an upper one-sided $c$–chart, whose out-of-control ARL in the presence of any decreases in $\lambda$ is surely (and unreasonably!) larger than the in-control ARL.

(b) Discuss the advantages of the ARL-unbiased $c$–chart, with in-control ARL equal to 370.4, when compared to the $c$–chart with 3–sigma control limits.

- Advantages of the ARL-unbiased $c$–chart...
  
  [Following the slides (2018-04-04-Slides-CandS2 charts.pdf, p. 16), we can add that...]

- As opposed to the $c$–chart with 3–sigma control limits:
  
  - the ARL-unbiased $c$–chart can take a pre-specified in-control ARL, in this case 370.4;
  
  - the associated ARL curve attains a maximum when $\lambda$ is on target;

- As opposed to the $c$–chart with null LCL and detects decreases in $\lambda$ in a timely fashion, by relying on the randomization probabilities.

2. The number of defective switches in samples of size 150 taken from a production line are shown below.

<table>
<thead>
<tr>
<th>Sample number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of defective switches</td>
<td>8</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>10</td>
<td>6</td>
</tr>
</tbody>
</table>

(a) Set up a $np$–chart with 3–sigma limits and in-control fraction defective equal to $p_0 = 0.02$. Could the production process be deemed in statistical control?

- Control statistic of the $np$–chart and its distribution
  
  \[ Y_n = \text{number of defective items in the } N \text{th sample, } N \in \mathbb{N} \]
  
  \[ Y_n \sim \text{Binomial}(n, p) \]

- 3–sigma control limits
  
  [The control statistic takes values in $[0,1,\ldots,150]$, thus the control limits are given by the following ceiling and floor functions of the target fraction of defective items, $p_0$]
  
  \[ LCL = \max(0, np_0 - 3 \times \sqrt{np_0(1 - p_0)}) \]
  
  \[ = \max(0, 150 \times 0.02 - 3 \times \sqrt{150 \times 0.02 \times (1 - 0.02)}) \]
  
  \[ = \max(0, -2.143929) \]
  
  \[ = 0. \]

(b) Let \( 1 - \zeta(\delta) = \frac{F_{\text{Binomial}(n,p_0)}(UCL) - \frac{F_{\text{Binomial}(n,p_0+\delta)}(LCL)}{1 - \frac{F_{\text{Binomial}(n,p_0)}(LCL-1)}}}{n} \), where $LCL$ and $UCL$ are the control limits of the $np$–chart.

Prove that, when $LCL > 0$, the root of \( \frac{d[1 - \zeta(\delta)]}{d\delta} = 0 \) is given by $\delta = \frac{\sqrt{\lambda_0}}{\sqrt{2}} - p_0$, where \( \lambda = \frac{np_0}{B(1-LCL)} \) and \( C = \frac{np_0}{1-LCL} \).

HINT: Recall that $F_{\text{Binomial}(n,p)}(a \cdot x)$, for $a, \beta \in \mathbb{N}$ and $x \in (0,1)$.

- Proof
  
  When $LCL > 0$,
  
  \[ 1 - \zeta(\delta) = \frac{F_{\text{Binomial}(n,p_0+\delta)}(UCL) - \frac{F_{\text{Binomial}(n,p_0+\delta)}(LCL-1)}}{n} \]
  
  \[ = \frac{[1 - F_{\text{Binomial}(n,p_0+\delta)}(LCL-1)] - [1 - F_{\text{Binomial}(n,p_0+\delta)}(LCL-1)]}{n} \]
  
  \[ = F_{\text{Beta}(n-p_0+\delta, LCL-1)}(p_0 + \delta) - F_{\text{Beta}(n-p_0+\delta, LCL-1)}(p_0) \]

By equating this derivative to zero and checking the formulae for the p.d.f. of the beta distribution, we get

\[ \frac{f_{\text{Beta}(n-p_0+\delta, LCL-1)}(x)}{f_{\text{Beta}(n-p_0+\delta, LCL-1)}(\theta)} = \frac{(p_0 + \delta)(1 - \theta)^{LCL-1}}{(p_0 + \delta)(1 - \theta)^{LCL-1} + (1 - p_0 + \delta)(p_0 + \delta)^{LCL-1}} \]

\[ = \frac{B(1-LCL)}{B(1-LCL) + 1} \]

\[ = \frac{p_0 + \delta}{1 - p_0 + \delta} \]

\[ = \frac{A^{c}}{1 - A^{c}} \]

\[ = \frac{\delta}{1 + A^{c} - p_0} \]

(c) The quality engineer in charge of the production anticipated both decreases and increases in $p$ and decided to adopt an ARL-unbiased $np$–chart with control limits $LCL^*$ and $UCL^*$ and randomization probabilities $Y_LCL^*$ and $Y_UCL^*$.

How should she use the ARL-unbiased $np$–chart? Sketch and comment its ARL profile.

- How to use the ARL-unbiased $np$–chart
  
  According to the slides (2018-08-28-Slides-np chart.pdf, p. 13), using a ARL-unbiased $np$–chart means to trigger a signal with:

  - probability one if the sample number of defective items is below $LCL^*$ or above $UCL^*$;

  - probability $Y_LCL^*$ (resp. $Y_UCL^*$) if the sample number of defective items is equal to $LCL^*$ (resp. $UCL^*$).

- (Sketch of the) ARL profile
  
  The plot below refers to $\delta = \{p_0, 3 \times p_0\}$ (a subset of the admissible values of $\delta$, $\{p_0, 1 - p_0\}$.}
3. The thickness of a printed circuit board is an important quality characteristic. Data on board thickness (in inches) refer to samples of size $n = 20$.

![Graph showing ARL curve](image)

**Comment**

The ARL curve achieves a maximum at $\delta = 0$ (in-control situation), thus this chart takes longer in average to trigger a false alarm than to detect any increase or decrease in the fraction defective.

(d) Obtain the corresponding out-of-control ARL of an ARL-unbiased $np$–chart, with $n^* = 20$. $LCL^* = 0$, $UCL^* = 4$, $Y_{LCL} = 0.003655$ and $Y_{UCL} = 0.394292$, when $p$ shifts from $p_0 = 0.02$ to $p = 0.01$.

- **Probability of a signal**

  When the fraction defective shifts from its target value $p_0 = 0.02$ to $p = 0 + \delta = 0.01$, the ARL-unbiased $np$–chart triggers a signal with probability

  \[
  \xi^*(\delta) = 1 \times P(Y_{UCL} \neq LCL^*, UCL^* | p = p_0 + \delta) + \gamma_{UCL} \times P(Y_{LCL} = LCL^* | p = p_0 + \delta) + \gamma_{LCL} \times P(Y_{UCL} = UCL^* | p = p_0)
  \]

  \[
  \xi^*(-0.01) = 1 - 0.003655 \times \left[ \text{F}_{\text{Binomial}}(20.02 - 0.0040; 0.02, 0.01) \right] - 0.003655 \times \left[ \text{F}_{\text{Binomial}}(20.02 - 0.0040; 0.02, 0.01) \right] + 0.394292 \times \left[ \text{F}_{\text{Binomial}}(20.02 - 0.0040; 0.02, 0.01) \right]
  \]

  \[
  = 0.002989
  \]

- **Out-of-control ARL**

  We are still dealing with a Shewhart chart, so the number of samples collected until this alternative chart triggers a signal given $\delta$, $RL^*(\delta)$, is such that $RL^*(\delta) = \text{Geometric}(\xi^*(\delta))$ and $ARL^*[\delta] = [RL^*(\delta)]^{-1}$.

  Thus, the requested out-of-control ARL is equal to

  \[
  ARL^*[-0.01] = 0.002989^{-1} = 334.56.
  \]

(a) Admit that the process mean suffered an upward shift, with magnitude $\delta = \sqrt{n} \frac{\mu_0 - \mu}{\sigma}$, and the standard deviation $\sigma$ is on-target. Compute the probability that at least one valid signal is triggered by the joint scheme within the first 10 samples.

- **Quality characteristic**

  $X =$ thickness of a printed circuit board

  $X \sim \text{Normal}(\mu, \sigma^2)$, where $\mu$ and $\sigma^2$ represent the process mean and variance, respectively.

- **Control statistics**

  \[
  \bar{X} = \text{mean of the Nth random sample of size n}
  \]

  $S^2 = \text{variance of the Nth random sample of size n}$

- **Distributions**

  \[
  X \sim \text{Normal}(\mu = \mu_0 + \delta, \sigma = \frac{\sigma}{\sqrt{n}})
  \]

  \[
  \delta = \frac{\mu_0 - \mu}{\sigma} \geq 0 \quad \text{(resp. } \mu_0 = \frac{\mu_0}{\sigma} \geq 1 \text{)}
  \]

- **Control limits of the individual charts**

  \[
  LCL = \mu_0 - \frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \mu_0 \text{ (in-control situation), thus this chart takes longer}
  \]

  \[
  UCL = \mu_0 + \frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \mu_0 \text{ (in-control situation), thus this chart takes longer}
  \]

- **Probabilities of triggering a signal**

  Taking into account the distribution of the control statistics, the STANDARD $\bar{X}$–chart and the UPPER ONE-SIDED $S^2$–chart trigger a signal with probabilities:

  \[
  \xi_{\text{up}}(\delta, \theta) = P\left( \frac{\bar{X} \neq \mu_0 \text{ UCL}_{\text{up}}}{\mu_0 \text{ LCL}_{\text{up}}} | \delta, \theta \right)
  \]

  \[
  \xi_{\text{up}}(\delta, \theta) = 1 - \phi \left( \frac{\mu_0 - \mu}{\sigma} \right) - \theta \cdot \phi \left( \frac{\mu_0 - \mu}{\sigma} \right), \quad \delta \in \mathbb{R}, \quad \theta \geq 1;
  \]

  \[
  \xi_{\text{up}}(\delta, \theta) = P\left( \frac{S^2 \neq \sigma^2 \mu_0 \text{ UCL}_{\text{up}}}{\mu_0 \text{ LCL}_{\text{up}}} | \theta \right)
  \]

  \[
  \xi_{\text{up}}(\delta, \theta) = 1 - \frac{1}{ARL_{\text{up}}(\theta)}, \quad \theta \geq 1.
  \]

  Furthermore, according to Exercise 10.38, the joint scheme triggers a signal with probability

  \[
  \xi_{\text{up}}(\delta, \theta) = P\left( \frac{\bar{X} \neq \mu_0 \text{ UCL}_{\text{up}}}{\mu_0 \text{ LCL}_{\text{up}}} | S^2 \neq \mu_0 \text{ UCL}_{\text{up}} \right)
  \]

  \[
  = \xi_{\text{up}}(\delta, \theta) + \xi_{\text{up}}(\theta, \delta) + \xi_{\text{up}}(\theta \cdot \delta, \theta), \quad \delta \in \mathbb{R}, \quad \theta \geq 1.
  \]

  Since the magnitude of the upward shift in the process mean is equal to $\delta = \sqrt{n} \frac{\mu_0 - \mu}{\sigma}$ and the process standard deviation $\sigma$ is on-target, i.e., $\theta = 1$, we get:

  \[
  \xi_{\text{up}}(0.25, 1) = 1 - \frac{1}{ARL_{\text{up}}(1)} = \frac{1}{1 - \left( \frac{32.5 - 3.25}{32.5} \right)} = 334.56
  \]

  \[
  \xi_{\text{up}}(0.25, 1) = 0.003577;
  \]

  \[
  \xi_{\text{up}}(1, 1) = 0.008559.
  \]
• Requested probability
Let \( W \) denote the number of valid signals triggered by the joint scheme in the first 10 samples. Since we are dealing with Shewhart charts (with no runs rules, etc.) then

\[
P(W \geq 1) = 1 - P(W = 0)
\]

\[
= 1 - \binom{10}{0} \cdot 0.0000005 \times (1 - 0.0000005)^{10 - 0}
\]

\[
= 1 - (1 - 0.0000005)^{10 - 0}
\]

\[
= 0.0823676.
\]

(b) Find the value of the probability of a misleading signal of Type IV when \( \delta = 0.25 \). Comment. (1.5)

**Hint:** You may find useful to know that \( \text{PMSIV}(\delta) = \frac{\zeta(\delta) + \zeta(1 - \delta)}{\psi(a^1) + \psi(\zeta(\delta))} \).

• Requested PMS of Type IV

Capitalizing on the hint, we get

\[
\text{PMSIV}(\delta) = \frac{\zeta(\delta) + \zeta(1 - \delta)}{\psi(a^1) + \psi(\zeta(\delta))}
\]

\[
\text{PMSIV}(0.25) \approx \frac{0.0005 \times (1 - 0.0003577)}{0.0008559} = 0.5820691.
\]

• Comment

The probability of misidentifying a shift in \( \mu \) with magnitude \( \delta \) is 1 by a shift in \( \sigma \), \( \text{PMSIV}(\delta) \), is larger than 0.5 because we have

\[
\text{ARL}_{\mu}(0.25, 1) - 1 = \frac{1}{0.0003577} - 1
\]

\[
= 278.564 > \text{ARL}_{\mu}(1) = 200.
\]

(See 2017-01-12-DetailedSolution-Test2.pdf.) [Alternatively, we could invoke that \( \text{ARL}_{\mu}(1) \) is much smaller than \( \text{ARL}_{\mu}(0, 1) \) and even though \( \text{ARL}_{\mu}(0.25, 1) \) is the signal quicker than the \( \xi \)-chart, in the presence of small shifts in \( \mu \).]

(c) Suppose the quality engineer decides to replace the upper one-sided \( S^2 \) chart by an upper one-sided EWMA chart for \( \sigma^2 \), with asymptotic control limits, no head-start, sample size \( n^* = 3 \), \( \lambda = 0.05 \), \( \theta = 0.7769 \), and in-control ARL approximately equal to 200 samples.\(^1\)

(i) Identify the control statistic and limit of the chart.

(ii) In the presence of an increase in \( \sigma \) with magnitude \( \theta = \sqrt{12.84/9.348} \), the out-of-control SDRL and CVRL of this EWMA chart are approximately equal to 10.549059 and 0.775839, respectively. How does this chart compare to the upper one-sided \( S^2 \) chart when it comes to the ability to detect such a shift?

• Control statistics/limits of the upper one-sided EWMA chart for \( \sigma^2 \)

Judging by Table 10.10 and given that this has no head-start, its control statistic is equal to:

\[
V_{\lambda} = \ln(\sigma^2) \max[N \ln(\sigma^2), (1 - \lambda_1) \times V_{\lambda_1} + \lambda_1 \times \ln(\sigma^2)_{\lambda_1}], \quad N \in \mathbb{N}.
\]

To obtain the control limits of this chart, recall that \( \lambda_1 = 0.05 \), \( \theta = 0.7769 \), \( n^* = 3 \), and

\[
\psi(\frac{\lambda_1}{1 - \lambda_1}) = \psi(1)^{10.31} = 1.6449340660.
\]

Thus, according to Table 10.10, we get:

\[
\text{LCL}_E = \ln(\sigma^2) + \ln(0.00065) = -14.8372;
\]

\[
\text{UCL}_E = \ln(\sigma^2) + \ln \left( \frac{\theta}{2 - \theta} \right) = -14.6963.
\]

• Out-of-control ARL

Recalling that \( \theta = \sqrt{12.84/9.348} \) and CVRL\( E \) are approximately equal to 10.549059 and 0.775839, respectively.

\[
\text{ARL}_E(\theta) = \frac{\text{SDRL}_E(\theta)}{\text{CVRL}_E(\theta)} = \frac{10.549059}{0.775839} = 13.59686.
\]

• Comment

Since

\[
\text{ARL}_E(\theta) = 10.40 > \text{ARL}_E(\theta) = 13.59686
\]

we can add that

\[
\text{ARL}_E(\theta) = 40.0 > \text{ARL}_E(\theta) = 13.59686.
\]

Consequently, the \( S^2 \)-chart tends to be, in average (and expectedly), slower than the EWMA chart in the detection of a small (or medium) size shift in \( \sigma \) such that \( \theta = \sqrt{12.84/9.348} = 1.17199.\)

4. A supplier ships components in lots of size \( N = 5000 \).

(a) A single sampling plan for attributes with \( n = 50 \) and \( c = 2 \) is being used for receiving inspection. Does this sampling plan comply with the producer’s and consumer’s risk points (1.5) \( (p_1 = AQL = 1\% , 1 - \alpha = 0.99) \) and \( (p_2 = LTPD = 5\% , \beta = 0.05) \)?

• Single sampling plan for attributes

\( n = 50 \) (sample size)

\( c = 2 \) (acceptance number)

• Auxiliary r.v., its approximate distribution and probabilities of acceptance

\[
D = \text{number of defective components in the sample}
\]

\[
p = \text{Binomial}(50, p)
\]

\[
P_s(p) = P(D \leq c)
\]

\[
= F_{\text{Binomial}(50, p)}(2)
\]

(Mathematica)

\[
\begin{align*}
0.986183, & \quad p = p_1 = AQL = 1\% \quad 0.540533, \quad p = p_2 = LTPD = 5\%.
\end{align*}
\]

• Comment

The sampling plan does NOT COMPLY with: the producer’s risk point, \( (p_1 = AQL = 1\% , 1 - \alpha = 0.99) \), because \( P_s(p_2) = 0.986183 < 0.99 \); or the consumer’s risk point, \( (p_2 = LTPD = 5\% , \beta = 0.05) \), in view of the fact that \( P_s(p_1) = 0.540533 > 0.05 \).
The quality engineer recommended: the derivation of a single sampling plan for attributes using the Wetherill and Brown procedure; that rejected lots are screened and all defective components are reworked and returned to the lot.

Obtain the associated sample size and acceptance number.

How should the engineer proceed if 10 defective components are to be found in a sample?

- **Producer's and consumer's risk points**
  
  \( p_1 = AQL, 1 - \alpha = (1\%, 0.99) \)
  
  \( p_2 = LT PD, \beta = (5\%, 0.05) \)
  
- **Obtaining the acceptance number and sample size**

  According to Wetherill and Brown (1991, p. 257) and page 129 of the lecture notes (in particular, formulae (13.11), (13.10) and (13.12), the acceptance number \( c \) and sample size \( n \) of a sampling plan for attributes, associated to risk points \( (p_1 = AQL, 1 - \alpha) \) and \( (p_2 = LT PD, \beta) \), can be approximately obtained:

  - \( c \) should be taken as the smallest integer satisfying
    \[
    r(c) = \frac{P^{-1}_1(1 - \beta)}{P^{-1}_2(p_1)} \leq n \leq \frac{P^{-1}_2(1 - \alpha)}{P^{-1}_2(p_1)},
    \]
    where
    \[
    r(c) = \frac{F^{-1}_2(p_2)}{P^{-1}_2(p_1)} (1 - \beta) \leq n \leq \frac{F^{-1}_2(p_2)}{P^{-1}_2(p_1)} (1 - \alpha)
    \]
  - \( n \) should be taken as the smallest integer verifying
    \[
    F^{-1}_2(p_2) (1 - \beta) \leq \frac{F^{-1}_2(p_2)}{P^{-1}_2(p_1)} (1 - \alpha)
    \]

  most likely the ceiling of the lower bound in the expression above.

  Using the tables to obtain \( F^{-1}_2(p_2) (1 - \beta) = 9.42 \) and \( F^{-1}_2(p_1) (1 - \alpha) = 0.01 \), we get

<table>
<thead>
<tr>
<th>( c )</th>
<th>( r(c) )</th>
<th>( Is r(c) \leq \frac{n}{p_2} \leq \frac{n}{p_1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.0</td>
<td>NO!</td>
</tr>
<tr>
<td>3</td>
<td>15.51</td>
<td>NO!</td>
</tr>
<tr>
<td>6</td>
<td>21.48</td>
<td>NO!</td>
</tr>
<tr>
<td>7</td>
<td>263</td>
<td>YES!</td>
</tr>
</tbody>
</table>

Consequently, \( c = 7 \). Moreover,

\[
\begin{align*}
\frac{F^{-1}_2(p_2)}{P^{-1}_2(p_1)} (1 - \beta) & = 9.42 \\
\frac{F^{-1}_2(p_2)}{P^{-1}_2(p_1)} (1 - \alpha) & = 0.01 \\
\end{align*}
\]

\[
\begin{align*}
n & = \frac{F^{-1}_2(p_2)}{P^{-1}_2(p_1)} (1 - \beta) \\
& = \frac{263 \times 0.05}{2} \\
& = 6.57
\end{align*}
\]

- **How to proceed...**

  The lot should be accepted iff the number of defective components, in a sample of \( n = 263 \), does not exceed \( c = 7 \), and that is NO! the case.

Since the lot has been reject and rectifying inspection has been adopted, the quality engineer has to:

(i) screen/inspect the remaining \( N - n = 5000 - 263 = 4737 \) components;

(ii) rework and return to the lot all the defective components found, namely the 10 defective components found in the sample.

(c) Suppose incoming lots of size \( N = 5000 \) contain 1% defective components. Find the corresponding \( AOQ \) (average outgoing quality) and \( ATI \) (average total inspection) of a single sampling plan with \( n = 263 \), \( c = 7 \) and **RECTIFYING INSPECTION**.

Comment on the values of \( AOQ \) and \( ATI \).

- **Single sampling plan for attributes with rectifying inspection**

  \( N = 5000 \) (batch size much larger than the sample size)

  \( n = 263 \) (sample size)

  \( c = 7 \) (acceptance number)

- **Auxiliary r.v. and its approximate distribution**

  \( D = \) number of defective defective components in the sample \( \approx \) Binomial(\( n, p \))

- **Requested probability of acceptance**

  \[
  P_D(p_2) = \mathbb{P}(D \leq c) = F_{Binomial(n,p)}(c)
  \]

  \[
  P_D(p_1) = \mathbb{P}(D_{Binomial(263,0.01)}(7)) = F_{Binomial(263,0.01)}(7)
  \]

  [Mathematica]

  \[
  = 0.994572
  \]

  [Alternatively, we could have used the Poisson approximation to get
  \[
  P_D(p_1) = \mathbb{P}(D_{Binomial(263,0.01)}(7)) = F_{Poisson(263 	imes 0.01)}(7) = 0.99947.
  \]

- **Requested average outgoing quality (AOQ) / Comment**

  \[
  AOQ(p_2) = \mathbb{P}(N - n)F_D(p_2)
  \]

  \[
  = \frac{1}{N} \times 0.01 \times (5000 - 263) \times 0.994572
  \]

  \[
  \approx 5000
  \]

  \[
  = 0.009423.
  \]

  Due to the rectifying inspection \( AOQ(p_1) \) is smaller than \( p = p_1 = 0.01 \). However, the relative reduction in the percentage defective is insignificant,

  \[
  \frac{1 - AOQ(p_1)}{p_1} = 100% \approx \frac{1 - 0.009423}{0.01} \times 100%
  \]

  \[
  = 5.77%.
  \]

  after all \( p = p_1 = AQL \).

- **Requested average total inspection (ATI) / Comment**

  Following, we get

  \[
  ATI(p_2) = \frac{n}{N}P_D(p_2) + N[1 - P_D(p_2)]
  \]

  \[
  ATI(p_1) = 263 \times 0.994572 + 5000 \times (1 - 0.994572)
  \]

  \[
  = 288.712.
  \]

  Since \( p = p_1 = AQL \), the probability of lot acceptance is larger than \( 1 - \alpha = 0.99 \). Consequently, we accept the lot more than 99% of the time and thus do not need to inspect the remaining \( N - n = 5000 - 263 \) items. Unsurprisingly, the average number of items we have to inspected is very close to the sample size, \( n = 263 \).