Reliability and Quality Control

MMA / LMAC

TEST 2 ("Recurso")
2nd. Semester — **2017/18**
2018/07/05 — **9:30 AM**

- Duration: **1h30m**
- **Please justify your answers.**
- This test has **two pages** and **four questions**. The total of points is **20.0**.

1. (a) How large should λ_0 be so that the $c-$chart with 3–sigma control limits has a positive LCL? Elaborate on the importance of a positive LCL in such a context. **(0.5)**

- **Requested target value λ_0**

$$LCL > 0 \quad \Leftrightarrow \quad \lambda_0 - 3 \times \sqrt{\lambda_0} > 0$$
$$\lambda_0 > 9.$$

- **Importance of a positive LCL**
  It is essential that the $c-$chart has a positive LCL to be able detect decreases in the expected number of defects λ (i.e., to detect a quality improvement) in a fairly quick fashion.
  If $LCL \not> 0$, we deal with an upper one-sided $c-$chart, whose out-of-control ARL in the presence of any decreases in λ is surely (and unreasonably!) larger than the in-control ARL.

(b) Discuss the advantages of the ARL-unbiased $c-$chart, with in-control ARL equal to 370.4, when compared to the $c-$chart with 3–sigma control limits. **(1.0)**

- **Advantages of the ARL-unbiased $c-$ chart...**
  [Following the slides (2018-04-04-Slides-CandS2 charts.pdf, p. 16), we can add that...]
  As opposed to the c-chart with 3–sigma control limits:
  – the ARL-unbiased c-chart can take a pre-specified in-control ARL, in this case 370.4;
  – the associated ARL curve attains a maximum when λ is on target;
  – the ARL-unbiased c-chart tackles the curse of the null LCL and detects decreases in λ in a timely fashion, by relying on the randomization probabilities.

2. The number of defective switches in samples of size 150 taken from a production line are shown below.

Sample number	1	2	3	4	5	6	7	8	9	10
Number of defective switches	8	1	3	0	2	4	0	1	10	6

(a) Set up a $np-$chart with 3–sigma limits and in-control fraction defective equal to $p_0 = 0.02$. Could the production process be deemed in statistical control? **(1.0)**

- **Control statistic of the $np-$chart and its distribution**
  Y_N = number of defective items in the N^{th} sample, $N \in \mathbb{N}$
  $Y_N \sim \text{Binomial}(n, p)$
- **3-sigma control limits**
  [The control statistic takes values in $\{0, 1, \ldots, 150\}$, thus the control limits are given by the following ceiling and floor functions of the target fraction of defective items, p_0:]

$$\begin{aligned}
LCL &= \left\lceil \max\{0, np_0 - 3 \times \sqrt{np_0(1-p_0)}\} \right\rceil \\
&= \left\lceil \max\left\{0, 150 \times 0.02 - 3 \times \sqrt{150 \times 0.02 \times (1-0.02)}\right\} \right\rceil \\
&= \lceil \max\{0, -2.143929\} \rceil \\
&= 0
\end{aligned}$$

$$\begin{aligned}
UCL &= \left\lfloor np_0 + 3 \times \sqrt{np_0(1-p_0)} \right\rfloor \\
&= \left\lfloor 150 \times 0.02 - 3 \times \sqrt{150 \times 0.02 \times (1-0.02)} \right\rfloor \\
&= \lfloor 8.143929 \rfloor \\
&= 8.
\end{aligned}$$

- **Examining the process for statistical control**
  Since $y_9 \not\in [LCL, UCL] = [0, 8]$, we deem the production process out-of-control.

(b) Let $1 - \xi(\delta) = F_{Binomial(n, p_0+\delta)}(UCL) - F_{Binomial(n, p_0+\delta)}(LCL - 1)$, $\delta \in (-p_0, 1-p_0)$, where LCL and UCL are the control limits of the $np-$chart. **(2.0)**
Prove that, when $LCL > 0$, the root of $\frac{d[1-\xi(\delta)]}{d\delta} = 0$ is given by $\delta = \frac{A^C}{1+A^C} - p_0$, where $A = \frac{B(UCL+1, n-UCL)}{B(LCL, n-LCL+1)}$ and $C = \frac{1}{UCL-LCL+1}$.

Hint: Recall that $F_{Binomial(\alpha+\beta-1, x)}(\alpha - 1) = 1 - F_{Beta(\alpha,\beta)}(x)$, for $\alpha, \beta \in \mathbb{N}$ and $x \in (0,1)$. Moreover, $f_{Beta(\alpha,\beta)}(x) = \frac{1}{B(\alpha,\beta)}\, x^{\alpha-1}\,(1-x)^{\beta-1}$, for $x \in (0,1)$ and $\alpha, \beta > 0$.

- **Proof**
  When $LCL > 0$,

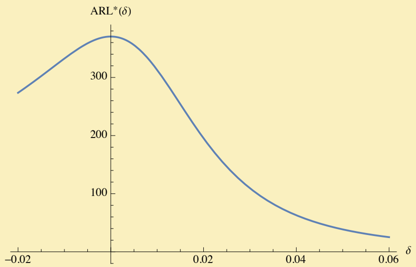
$$\begin{aligned}
1 - \xi(\delta) &= F_{Binomial(n,p_0+\delta)}(UCL) - F_{Binomial(p_0+\delta)}(LCL-1) \\
&= [1 - F_{Beta(UCL+1, n-UCL)}(p_0+\delta)] - [1 - F_{Beta(LCL, n-LCL+1)}(p_0+\delta)] \\
&= F_{Beta(LCL, n-LCL+1)}(p_0+\delta) - F_{Beta(UCL+1, n-UCL)}(p_0+\delta) \\
\frac{d\,[1-\xi(\delta)]}{d\delta} &= f_{Beta(LCL, n-LCL+1)}(p_0+\delta) - f_{Beta(UCL+1, n-UCL)}(p_0+\delta).
\end{aligned}$$

By equating this derivative to zero and checking the formulae for the p.d.f. of the beta distribution, we get

$$\begin{aligned}
\frac{(p_0+\delta)^{LCL-1}\,[1-(p_0+\delta)]^{n-LCL+1-1}}{B(LCL, n-LCL+1)} &= \frac{(p_0+\delta)^{UCL+1-1}\,[1-(p_0+\delta)]^{n-UCL-1}}{B(UCL+1, n-UCL)} \\
\left[\frac{p_0+\delta}{1-(p_0+\delta)}\right]^{LCL-UCL-1} &= \frac{B(LCL, n-LCL+1)}{B(UCL+1, n-UCL)} \\
\frac{p_0+\delta}{1-(p_0+\delta)} &= A^C \\
\delta\,(1+A^C) &= A^C - p_0\,(1+A^C) \\
\delta &= \frac{A^C}{1+A^C} - p_0.
\end{aligned}$$

(c) The quality engineer in charge of the production anticipated both decreases and increases in p and decided to adopt an ARL-unbiased $np-$chart with control limits $LCL^\star$ and $UCL^\star$ and randomization probabilities $\gamma_{LCL^\star}$ and $\gamma_{UCL^\star}$. **(1.0)**
How should she use the ARL-unbiased $np-$chart? Sketch and comment its ARL profile.

- **How to use the ARL-unbiased $np-$chart**
  According to the slides (2016-08-28-Slides-np chart.pdf, p. 13), using a ARL-unbiased $np-$chart means to trigger a signal with:
  – probability one if the sample number of defective items is below $LCL^\star$ or above $UCL^\star$;
  – probability $\gamma_{LCL^\star}$ (resp. $\gamma_{UCL^\star}$) if the sample number of defective items is equal to $LCL^\star$ (resp. $UCL^\star$).
- **(Sketch of the) ARL profile**
  The plot below refers to $\delta \in (-p_0.3 \times p_0)$ (a subset of the admissible values of δ, $(-p_0, 1-p_0)$.



- **Comment**
  The ARL curve achieves a maximum at $\delta = 0$ (in-control situation), thus this chart takes longer in average to trigger a false alarm than to detect any increase or decrease in the fraction defective.

(d) Obtain the corresponding out-of-control ARL of an ARL-unbiased $np-$chart, with $n^\star = 20$, $LCL^\star = 0$, $UCL^\star = 4$, $\gamma_{LCL^\star} = 0.003655$ and $\gamma_{UCL^\star} = 0.394292$, when p shifts from $p_0 = 0.02$ to $p = 0.01$. **(1.5)**

- **Probability of a signal**
  When the fraction defective shifts from its target value $p_0 = 0.02$ to $p = p_0 + \delta = 0.01$, the ARL-unbiased $np-$chart triggers a signal with probability
$$\begin{aligned}\xi^\star(\delta) &= 1 \times P(Y_N^\star \not\in [LCL^\star, UCL^\star] \mid p = p_0+\delta)\\ &\quad + \gamma_{LCL^\star} \times P(Y_N^\star = LCL^\star \mid p = p_0+\delta)\\ &\quad + \gamma_{UCL^\star} \times P(Y_N^\star = UCL^\star \mid p = p_0+\theta)\\ \xi^\star(-0.01) &= 1 - \left[F_{Binomial(20,0.02-0.01)}(4) - F_{Binomial(20,0.02-0.01)}(0-1)\right]\\ &\quad +0.003655 \times \left[F_{Binomial(20,0.02-0.010)}(0) - F_{Binomial(20,0.02-0.01)}(0-1)\right]\\ &\quad +0.394292 \times \left[F_{Binomial(20,0.02-0.01)}(4) - F_{Binomial(20,0.02-0.01)}(4-1)\right]\\ &\stackrel{tables}{=} 1-(1.0000-0)+0.003655\times(0.8179-0)+0.394292\times(1.0000-1.0000)\\ &\simeq 0.002989\end{aligned}$$

- **Out-of-control ARL**
  We are still dealing with a Shewhart chart, thus the number of samples collected until this alternative chart triggers a signal given δ, $RL^\star(\delta)$, is such that $RL^\star(\delta) \sim \text{Geometric}(\xi^\star(\delta))$ and
$$ARL^\star(\delta) = [\xi^\star(\delta)]^{-1}.$$
  Thus, the requested out-of-control ARL is equal to
$$\begin{aligned}ARL^\star(-0.01) &\simeq 0.002989^{-1}\\ &\simeq 334.56.\end{aligned}$$

3. The thickness of a printed circuit board is an important quality characteristic. Data on board thickness (in inches) refer to samples of $n = 4$ boards each. Admit that this quality characteristic has a normal distribution with nominal mean and variance equal to $\mu_0 = 0.06$ and $\sigma_0^2 = 0.0006^2$ (respectively). Furthermore, the control limits of the individual:

- $\bar{X}-$chart are equal to $LCL_\mu = \mu_0 - \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$ and $UCL_\mu = \mu_0 + \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$, where $\gamma_\mu = \Phi^{-1}\left(1 - \frac{1}{2\times ARL_\mu(0,1)}\right)$ and $ARL_\mu(0,1) = 370.4$;
- S^2-chart are given by $LCL_\sigma = 0$ and $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$, where $\gamma_\sigma = F^{-1}_{\chi^2_{(n-1)}}\left(1 - \frac{1}{ARL_\sigma(1)}\right)$ and $ARL_\sigma(1) = 200$.

(a) Admit that the process mean suffered an upward shift, with magnitude $\delta = \sqrt{n}\,\frac{\mu-\mu_0}{\sigma_0} = 0.25$, and the standard deviation σ is on-target. **(2.5)**
Compute the probability that at least one valid signal is triggered by the joint scheme within the first 10 samples.

- **Quality characteristic**
  X = thickness of a printed circuit board
  $X \sim \text{Normal}(\mu, \sigma^2)$, where μ and σ^2 represent the process mean and variance, respectively.
- **Control statistics**
  $\bar{X}_N$ = mean of the N^{th} random sample of size n
  S_N^2 = variance of the N^{th} random sample of size n
- **Distributions**
  $\bar{X}_N \sim \text{Normal}\left(\mu = \mu_0 + \delta \times \frac{\sigma_0}{\sqrt{n}}, \frac{\sigma^2}{n} = \frac{(\theta\sigma_0)^2}{n}\right)$,
  $\delta = \frac{\mu-\mu_0}{\sigma_0/\sqrt{n}} \le 0$ (resp. $\theta = \frac{\sigma}{\sigma_0} \ge 1$) is the magnitude of the shift in μ (resp. upward shift in σ).
  $\frac{(n-1)S_N^2}{(\theta\sigma_0)^2} \sim \chi^2_{(n-1)}$
- **Control limits of the individual charts**
  $LCL_\mu = \mu_0 - \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$
  $UCL_\mu = \mu_0 + \gamma_\mu \times \frac{\sigma_0}{\sqrt{n}}$, where $\gamma_\mu = \Phi^{-1}\left(1 - \frac{1}{2\times ARL_\mu(0,1)}\right) = \cdots = 3$
  $LCL_\sigma = 0$
  $UCL_\sigma = \frac{\sigma_0^2}{n-1} \times \gamma_\sigma$, where $\gamma_\sigma = F^{-1}_{\chi^2_{(n-1)}}\left(1 - \frac{1}{ARL_\sigma(1)}\right) = \cdots \stackrel{tables}{=} 12.84.$
- **Probabilities of triggering a signal**
  Taking into account the distribution of the control statistics, the STANDARD $\bar{X}-$chart and the UPPER ONE-SIDED S^2-chart trigger a signal with probabilities:
$$\begin{aligned}\xi_\mu(\delta,\theta) &= P\left(\bar{X}_N \not\in [LCL_\mu, UCL_\mu] \mid \delta, \theta\right)\\ &= \ldots\\ &= 1 - \left[\Phi\left(\frac{\gamma_\mu - \delta}{\theta}\right) - \Phi\left(\frac{-\gamma_\mu - \delta}{\theta}\right)\right], \quad \delta \in \mathbb{R}, \quad \theta \ge 1;\\ \xi_\sigma(\theta) &= P\left(S_N^2 \not\in [LCL_\sigma, UCL_\sigma] \mid \theta\right)\\ &= \ldots\\ &= 1 - F_{\chi^2_{(n-1)}}\left(\frac{\gamma_\sigma}{\theta^2}\right)\\ &= \frac{1}{ARL_\sigma(\theta)}, \quad \theta \ge 1.\end{aligned}$$
  Furthermore, according to Exercise 10.38, the joint scheme triggers a signal with probability
$$\begin{aligned}\xi_{\mu,\sigma}(\delta,\theta) &= P\left(\bar{X}_N \not\in [LCL_\mu, UCL_\mu] \text{ or } S_N^2 \not\in [LCL_\sigma, UCL_\sigma] \mid \delta, \theta\right)\\ &= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta) \times \xi_\sigma(\theta), \quad \delta \in \mathbb{R}, \quad \theta \ge 1.\end{aligned}$$
  Since the magnitude of the upward shift in the process mean is equal to $\delta = \sqrt{n}\,\frac{\mu-\mu_0}{\sigma_0} = 0.25$ and the process standard deviation σ on-target, i.e., $\theta = 1$, we get:
$$\begin{aligned}\xi_\mu(0.25,1) &= 1 - \left[\Phi\left(\frac{3-0.25}{1}\right) - \Phi\left(\frac{-3-0.25}{1}\right)\right]\\ &\simeq 1 - [\Phi(2.75) - \Phi(-3.25)]\\ &\stackrel{tables}{=} 1 - [0.9970 - (1 - 0.999423)]\\ &= 0.003577;\\ \xi_\sigma(1) &= \frac{1}{ARL_\sigma(1)}\\ &= 0.005;\\ \xi_{\mu,\sigma}(1,1) &\simeq 0.003577 + 0.005 - 0.003577 \times 0.005 \simeq 0.008559.\end{aligned}$$

- **Requested probability**
  Let W denote the number of valid signals triggered by the joint scheme in the first 10 samples. Since we are dealing with Shewhart charts [with no runs rules, etc.] then

$$\begin{aligned} W &\sim \text{Binomial}\left(10, \xi_{\mu,\sigma}(0.25,1) \simeq 0.008559\right) \\ P(W \geq 1) &= 1 - P(W=0) \\ &= 1 - \binom{10}{0} \times 0.008559^0 \times (1-0.008559)^{10-0} \\ &\simeq 1-(1-0.008559)^{10} \\ &\simeq 0.0823676. \end{aligned}$$

(b) Find the value of the probability of a misleading signal of Type IV when $\delta = 0.25$. Comment. **(1.5)**
Hint: You may find useful to know that $PMS_{IV}(\delta) = \frac{\xi_\sigma(1)\times[1-\xi_\mu(\delta,1)]}{\xi_\mu(\delta,1)+\xi_\sigma(1)-\xi_\mu(\delta,1)\times\xi_\sigma(1)}$.

- **Requested PMS of Type IV**
  Capitalizing on the hint, we get

$$\begin{aligned} PMS_{IV}(\delta) &= \frac{\xi_\sigma(1)\times[1-\xi_\mu(\delta,1)]}{\xi_\mu(\delta,1)+\xi_\sigma(1)-\xi_\mu(\delta,1)\times\xi_\sigma(1)} \\ PMS_{IV}(0.25) &\overset{(a)}{\simeq} \frac{0.005\times(1-0.003577)}{0.008559} \\ &\simeq 0.582091. \end{aligned}$$

- **Comment**
  The probability of misidentifying a shift in μ with magnitude $\delta = 1$ by a shift in σ, $PMS_{IV}(\delta)$, is larger than 0.5 because we have

$$\begin{aligned} ARL_\mu(0.25,1) - 1 &\simeq \frac{1}{0.003577} - 1 \\ &\simeq 278.564 \\ &> ARL_\sigma(1) = 200. \end{aligned}$$

  (See 2017-01-12-DetailedSolution-Test2.pdf.)
  [Alternatively, we could invoke that $ARL_\sigma(1)$ is much smaller that $ARL_\mu(0,1)$ and that even though $ARL_\mu(\delta,1) \downarrow_{\delta\geq 0}$ the S^2-chart triggers a signal quicker than the $\bar{X}-$chart, in the presence of small shifts in μ.]

(c) Suppose the quality engineer decides to replace the upper one-sided S^2 chart by an upper one-sided EWMA chart for σ^2, with asymptotic control limits, no head-start, sample size $n^\star = 3$, $\lambda_E = 0.05$, $\gamma_E = 0.7769$, and in-control ARL approximately equal to 200 samples.[1] **(3.0)**
(i) Identify the control statistic and limit of this chart.
(ii) In the presence of an increase in σ with magnitude $\theta = \sqrt{12.84/9.348}$, the out-of-control SDRL and CVRL of this EWMA chart are approximately equal to 10.549059 and 0.775839, respectively. How does this chart compare to the upper one-sided S^2 chart when it comes to the ability to detect such a shift?

- **Control statistics/ limits of the upper one-sided EWMA chart for σ^2**
  Judging by Table 10.10 and given that this has no head-start, its control statistic is equal to:

$$V_N = \begin{cases} \ln(\sigma_0^2), & N = 0 \\ \max\left\{\ln(\sigma_0^2), (1-\lambda_E)\times V_{N-1} + \lambda_E \times \ln(S_N^2)\right\}, & N \in I\!N. \end{cases}$$

  To obtain the control limits of this chart, recall that $\lambda_E = 0.05$, $\gamma_E = 0.7769$, $n^\star = 3$, and $\psi'\left(\frac{n^\star-1}{2}\right) = \psi'(1) \overset{(10.31)}{=} 1.6449340668$. Thus, according to Table 10.10, we get:

[1] $(50+1)$ transient states were used to obtain this ARL approximation.

$$\begin{aligned} LCL_E &= \ln(\sigma_0^2) \\ &= \ln(0.0006^2) \\ &\simeq -14.8372; \\ UCL_E &= \ln(\sigma_0^2) + \gamma_E \times \sqrt{\psi'\left(\frac{n^\star-1}{2}\right)\times\frac{\lambda_E}{2-\lambda_E}} \\ &= \ln\left(0.0006^2\right) + 0.7769 \times \sqrt{1.6449340668 \times \frac{0.05}{2-0.05}} \\ &= -14.6963. \end{aligned}$$

- **Out-of-control ARL**
  Recalling that $\theta = \sqrt{12.84/9.348}$ and $CVRL_E(\theta) = \frac{SDRL_E(\theta)}{ARL_E(\theta)}$, we get:

$$\begin{aligned} ARL_E(\theta) &= \frac{SDRL_E(\theta)}{CVRL_E(\theta)} \\ &\simeq \frac{10.549059}{0.775839} \\ &\simeq 13.59696. \end{aligned}$$

- **Comment**
  Since

$$ARL_\sigma(\theta) = \frac{1}{\xi_\sigma(\theta)} = \frac{1}{1-F_{\chi^2_{(4-1)}}\left(\frac{F^{-1}_{\chi^2_{(4-1)}}(1-\frac{1}{200})}{\sqrt{12.84/9.348}^2}\right)} = \frac{1}{1-F_{\chi^2_{(3)}}(9.348)} \overset{tables}{=} \frac{1}{1-0.975} = 40.0,$$

  we can add that

$$ARL_\sigma(\theta) = 40.0 > ARL_E(\theta) \simeq 13.59696.$$

  Consequently, the S^2-chart tends to be, in average [and expectedly!], slower than the EWMA chart in the detection of a small (or medium) size shift in σ such as $\theta = \sqrt{12.84/9.348} \simeq 1.17199$.

4. A supplier ships components in lots of size $N = 5000$.

(a) A single sampling plan for ATTRIBUTES with $n = 50$ and $c = 2$ is being used for receiving inspection. Does this sampling plan comply with the producer's and consumer's risk points $(p_1 = AQL = 1\%, 1-\alpha = 0.99)$ and $(p_2 = LTPD = 5\%, \beta = 0.05)$? **(1.5)**

- **Single sampling plan for attributes**
  $n = 50$ (sample size)
  $c = 2$ (acceptance number)
- **Auxiliary r.v., its approximate distribution and probabilities of acceptance**

$$\begin{aligned} D &= \text{number of defective components in the sample} \\ &\overset{a}{\sim} \text{Binomial}(50, p) \\ P_a(p) &= P(D \leq c) \\ &\simeq F_{Binomial(50,p)}(2) \\ &\overset{Mathematica}{=} \begin{cases} 0.986183, & p = p_1 = AQL = 1\% \\ 0.540533, & p = p_2 = LTPD = 5\%. \end{cases} \end{aligned}$$

- **Comment**
  The sampling plan DOES NOT COMPLY with: the producer's risk point, $(p_1 = AQL = 1\%, 1-\alpha = 0.99)$, because $P_a(AQL) = 0.986183 < 0.99$; or the consumer's risk point, $(p_2 = LTPD = 5\%, \beta = 0.05)$, in view of the fact that $P_a(LTPD) = 0.540533 > 0.05$.

(b) The quality engineer recommended: the derivation of a single sampling plan for attributes using the Wetherill and Brown procedure; that rejected lots are screened and all defective components are reworked and returned to the lot. **(2.5)**

Obtain the associated sample size and acceptance number.

How should the engineer proceed if 10 defective components are to be found in a sample?

- **Producer's and consumer's risk points**
  $(p_1 = AQL, 1-\alpha) = (1\%, 0.99)$
  $(p_2 = LTPD, \beta) = (5\%, 0.05)$

- **Obtaining the acceptance number and sample size**
  According to Wetherill and Brown (1991, p. 257) and page 129 of the lecture notes (in particular, formulae (13.11), (13.10) and (13.12)), the acceptance number c and sample size n of a sampling plan for attributes, associated to risk points $(p_1 = AQL, 1-\alpha)$ and $(p_2 = LTPD, \beta)$, can be approximately obtained:
  - c should be taken as the smallest integer satisfying
  $$r(c) \leq \frac{p_2}{p_1},$$
  where $r(c) = \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}$;
  - n should be taken as the smallest integer verifying
  $$\frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{2p_2} \leq n \leq \frac{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}{2p_1},$$
  most likely the ceiling of the lower bound in the expression above.

  Using the tables to obtain $F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta = 0.95)$ and $F^{-1}_{\chi^2_{(2(c+1))}}(\alpha = 0.01)$, we get

c	$r(c) = \frac{F^{-1}_{\chi^2_{(2(c+1))}}(1-\beta)}{F^{-1}_{\chi^2_{(2(c+1))}}(\alpha)}$	Is $r(c) \leq \frac{p_2}{p_1} = \frac{0.05}{0.01} = 5$?
0	$\frac{F^{-1}_{\chi^2_{(2(0+1))}}(1-0.05)}{F^{-1}_{\chi^2_{(2(0+1))}}(0.01)} = \frac{5.991}{0.0201} \simeq 298.06$	NO!
3	$\frac{F^{-1}_{\chi^2_{(2(3+1))}}(1-0.05)}{F^{-1}_{\chi^2_{(2(3+1))}}(0.01)} = \frac{15.51}{1.647} \simeq 9.42$	NO!
6	$\frac{F^{-1}_{\chi^2_{(2(6+1))}}(1-0.05)}{F^{-1}_{\chi^2_{(2(6+1))}}(0.01)} = \frac{23.68}{4.660} \simeq 5.08$	NO!
7	$\frac{F^{-1}_{\chi^2_{(2(7+1))}}(1-0.05)}{F^{-1}_{\chi^2_{(2(7+1))}}(0.01)} = \frac{26.30}{5.812} \simeq 4.52512$	**YES!**

  Consequently, $c = 7$. Moreover,
  $$\begin{aligned} n &= \left\lceil \frac{F^{-1}_{\chi^2_{(2(7+1))}}(1-0.05)}{2 \times 0.05} \right\rceil \\ &\overset{table}{=} \left\lceil \frac{26.30}{2 \times 0.05} \right\rceil \\ &= 263. \end{aligned}$$

- **How to proceed...**
  The lot should be accepted iff the number of defective components, in a sample of $n = 263$, does not exceed $c = 7$, and that is NOT the case.

Since the lot has been reject and rectifying inspection has been adopted, the quality engineer has to:

(i) screen/inspect the remaining $N - n = 5000 - 263 = 4737$ components;

(ii) rework and return to the lot all the defective components found, namely the 10 defective components found in the sample.

(c) Suppose incoming lots of size $N = 5000$ contain 1% defective components. **(2.0)**

Find the corresponding AOQ (average outgoing quality) and ATI (average total inspection) of a single sampling plan with $n = 263$, $c = 7$ and RECTIFYING INSPECTION.

Comment on the values of AOQ and ATI.

- **Single sampling plan for attributes with rectifying inspection**
  $N = 5000$ (batch size much larger than the sample size)
  $n = 263$ (sample size)
  $c = 7$ (acceptance number)

- **Auxiliary r.v. and its approximate distribution**
  D = number of defective defective components in the sample $\overset{a}{\sim}$ Binomial(n, p)

- **Requested probability of acceptance**
  $$\begin{aligned} P_a(p) &= P(D \leq c) \\ &\simeq F_{Binomial(n,p)}(c) \\ P_a(p_1) &\simeq F_{Binomial(263,0.01)}(7) \\ &\overset{Mathematica}{\simeq} 0.994572 \end{aligned}$$
  [Alternatively, we could have used the Poisson approximation to get $P_a(p_1) \simeq F_{Poisson(263\times 0.01)}(7) \simeq F_{Poisson(2.60)}(7) \overset{tables}{=} 0.9947$.]

- **Requested average outgoing quality (AOQ) / Comment**
  $$\begin{aligned} AOQ(p) &\overset{(13.14)}{=} \frac{p\,(N-n)\,P_a(p)}{N} \\ AOQ(p_1) &\overset{(a)}{\simeq} \frac{0.01 \times (5000 - 263) \times 0.994572}{5000} \\ &\simeq 0.009423. \end{aligned}$$
  Due to the rectifying inspection $AOQ(p_1)$ is smaller than $p = p_1 = 0.01$. However, the relative reduction in the percentage defective is insignificant,
  $$\begin{aligned} \left[1 - \frac{AOQ(p_1)}{p_1}\right] \times 100\% &\simeq \left(1 - \frac{0.009423}{0.01}\right) \times 100\% \\ &\simeq 5.77\%, \end{aligned}$$
  after all $p = p_1 = AQL$.

- **Requested average total inspection (ATI) / Comment**
  Following , we get
  $$\begin{aligned} ATI(p) &\overset{(13.15)}{=} n\,P_a(p) + N\,[1 - P_a(p)] \\ ATI(p_1) &\simeq 263 \times 0.994572 + 5000 \times (1 - 0.994572) \\ &\simeq 288.712. \end{aligned}$$
  Since $p = p_1 = AQL$, the probability of lot acceptance is larger than $1-\alpha = 0.99$. Consequently, we accept the lot more than 99% of the time and thus do not need to inspect the remaining $N - n = 5000 - 263$ items. Unsurprisingly, the average number of items we have to inspected is very close to the sample size, $n = 263$.