Reliability and Quality Control
MMA / LMAC

TEST 1 ("Recurso")
2nd. Semester — **2017/18**
2018/07/05 — **8:00 AM**

- Duration: **1h30m**
- **Please justify your answers.**
- This test has **one page** and **three questions**. The total of points is **20.0**.

1. Assume that a nonrepairable system has 6 components and structure function given by

$$\begin{aligned}\phi(\underline{X}) &= 1-(1-X_1X_2X_4X_5)\times(1-X_1X_2X_4X_6)\times(1-X_3X_4X_5)\times(1-X_3X_4X_6)\\ &= [1-(1-X_4)]\times[1-(1-X_1)(1-X_3)]\times[1-(1-X_2)(1-X_3)]\times[1-(1-X_5)(1-X_6)].\end{aligned}$$

(a) Identify the minimal path sets and minimal cut sets. **(2.0)**
Draw a reliability block diagram as close as possible of the system.

- **Structure function**
By considering $X_i \sim \text{Bernoulli}(p_i)$, $i=1,\ldots,6$ and applying results (1.13) and (1.14), we can conclude that the structure function of this system equals

$$\begin{aligned}\phi(\underline{X}) &\overset{(1.13)}{=} 1-\prod_{j=1}^{p^*}\left(1-\prod_{i\in\mathscr{P}_j} X_i\right)\\ &\overset{(1.14)}{=} \prod_{j=1}^{q}\left[1-\prod_{i\in\mathscr{K}_j}(1-X_i)\right],\end{aligned}$$

where $\mathscr{P}_j$ $(j=1,\ldots,p^*)$ and $\mathscr{K}_j$ $(j=1,\ldots,q)$ represent the p^* minimal path sets and the q minimal cut sets, respectively.

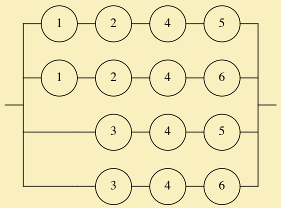
- **Minimal path sets**

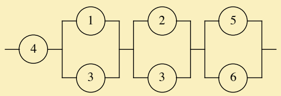
$$\begin{aligned}\mathscr{P}_1 &= \{1,2,4,5\}\\ \mathscr{P}_2 &= \{1,2,4,6\}\\ \mathscr{P}_3 &= \{3,4,5\}\\ \mathscr{P}_4 &= \{3,4,6\}\\ p^* &= 4 \text{ minimal path sets}\end{aligned}$$

- **Minimal cut sets**

$$\begin{aligned}\mathscr{K}_1 &= \{4\}\\ \mathscr{K}_2 &= \{1,3\}\\ \mathscr{K}_3 &= \{2,3\}\\ \mathscr{K}_4 &= \{5,6\}\\ q &= 4 \text{ minimal cut sets}\end{aligned}$$

- **Reliability block diagram** (in terms of minimal path/cut sets)
By capitalizing on Theorem 1.30 and on the minimal path/cut sets, we can provide two representations of the system:

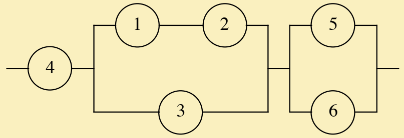




Since
- the first reliability block diagram in terms of minimal path sets has repeated components in the different series sub-systems and
- the reliability block diagram in terms of minimal cut sets has no repeated components in the different parallel sub-systems,

this last representation seems to the closest to the original system.

- **Obs. — Reliability block diagram** (the original system!)



(b) Now suppose that those 6 components operate independently and have reliabilities $p_i = p = 0.95$, $i=1,\ldots,6$. **(2.0)**
Calculate the reliability of the system.

- **Reliabilities of the components**
$p_i = p = 0.95$, $i=1,\ldots,6$
$\underline{p} = (p_1,\ldots,p_6)$

- **Reliability of the system**
Taking into account
  - the reliabilities of the components,
  - the fact that they operate in an independent fashion,
  - the second expression of the structure function, where $X_i \overset{i.i.d.}{\sim} \text{Bernoulli}(p_i = p = 0.95)$, $i=1,\ldots,6$,

we get the reliability of the system:

$$\begin{aligned}r(\underline{p}) &= E\left[\phi(\underline{X})\right]\\ &= E\{[1-(1-X_4)]\times[1-(1-X_1)(1-X_3)]\times[1-(1-X_2)(1-X_3)]\\ &\quad \times[1-(1-X_5)(1-X_6)]\}\\ &\overset{X_i\,indep}{=} E(X_4)\times E[(X_1+X_3-X_1X_3)\times(X_2+X_3-X_2X_3)]\times E(X_5+X_6-X_5X_6)]\\ &= E(X_4)\times E[(X_1X_2+X_1X_3-X_1X_2X_3)+(X_2X_3+X_3^2-X_3-X_2X_3^2)\\ &\quad -(X_1X_2X_3+X_1X_3^2+X_1X_2X_3^2)]\times E(X_5+X_6-X_5X_6)]\\ &\overset{X_i^k\sim X_i,\,k\in\mathbb{N}}{=} E(X_4)\times E[(X_1X_2+X_1X_3-X_1X_2X_3)+(X_2X_3+X_3-X_2X_3)\\ &\quad -(X_1X_2X_3+X_1X_3+X_1X_2X_3)]\times E(X_5+X_6-X_5X_6)]\\ &\overset{X_i\,indep;\,E(X_i)=p_i=p}{=} p\times(p^2+p^2-p^3+p^2+p-p^2-p^3-p^2+p^3)\times(p+p-p^2)\\ &= p\times(p+p^2-p^3)\times(2p-p^2)\\ &\overset{p=0.95}{\simeq} 0.943005.\end{aligned}$$

(c) Let T_i be the time to failure (in 10^3 days) of component i $(i=1,\ldots,6)$ and admit that T_i, $i=1,\ldots,6$, are independent r.v. with common failure function $\lambda(t)=2t$, for $t\geq 0$. **(3.5)**
Obtain the common reliability function $R(t)$ and expected value $\mu^\star$.
Capitalize on the stochastic ageing character of T_i to provide a lower limit for:
(i) the reliability function of the system at $t=0.5$;
(ii) the expected time to failure of the system.

- **Individual times and common hazard function**
  T_i = time to failure of component i, $i = 1, \ldots, 5$
  T_i are i.i.d. r.v. with common hazard function $\lambda(t) = 2t$, $t \geq 0$

- **Common reliability function**
  Prop. 3.3 leads to

$$R(t) \;=\; \exp\left[-\int_0^t \lambda(u)\,du\right] = \exp\left[-\int_0^t 2u)\,du\right] = e^{-t^2},\; t \geq 0.$$

- **Common expected value**
  According to (4.22), $R(t)$ coincides with the reliability function of a Weibull distribution with scale and shape parameters equal to $\delta = 1$ and $\alpha = 2$, respectively. Thus:

$$\begin{aligned}
\mu^\star &= E(T_i) \\
&= \delta \times \Gamma\left(1 + \frac{1}{\alpha}\right) \\
&= \Gamma\left(\frac{3}{2}\right) \\
&\overset{form.}{=} \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \\
&\overset{form.}{=} \frac{\sqrt{\pi}}{2} \\
&\simeq 0.886227.
\end{aligned}$$

- **Lower bound for $R_T(0.5)$**
  Let T represent the time to failure of the system.
  We are dealing with a coherent system characterized as follows:
  - $T_i \overset{i.i.d.}{\sim} IHR, i = 1, \ldots, 6$, because the times to failure T_i are i.i.d. r.v. with Weibull dist. with a shape parameter α larger than 1;
  - $\mu^* = E(T_i) = \mu_i = \frac{\sqrt{\pi}}{2} \simeq 0.886227$, $i = 1, \ldots, 5$.

  Consequently, we can apply Theorem 3.58 for $t = 0.5 < \min_{i=1,\ldots,6} \mu_i = \mu^\star = 0.886227$ and obtain the following lower bound for $R_T(t)$:

$$\begin{aligned}
R_T(t) &\geq r\left(e^{-t/\mu_1}, \ldots, e^{-t/\mu_6}\right) \\
&= r\left(e^{-t/\mu^\star}, \ldots, e^{-t/\mu^\star}\right) \\
&\overset{(b)}{=} e^{-t/\mu^\star} \times \left(e^{-t/\mu^\star} + e^{-2t/\mu^\star} - e^{-3t/\mu^\star}\right) \times \left(2e^{-t/\mu^\star} - e^{-2t/\mu^\star}\right) \\
&\simeq 0.328006.
\end{aligned}$$

- **Lower bound for $E(T)$**
  We are dealing with a coherent system involving:
  - $T_i \overset{i.i.d.}{\sim} IHR \overset{Prop. 3.36}{\Rightarrow} T_i \overset{i.i.d.}{\sim} IHRA \overset{Prop. 3.36}{\Rightarrow} T_i \overset{i.i.d.}{\sim} NBUE$;
  - $p^* = 4$ minimal path sets, with cardinals 3 or 4;

  Thus, we can apply Theorem 3.65 (or 3.69) and obtain the following lower bound for $\mu = E(T)$:

$$\begin{aligned}
\mu &\geq \max_{j=1,\ldots,p^*} \left\{ \left(\sum_{i \in \mathscr{P}_j} \mu_i^{-1} \right)^{-1} \right\} \\
&\overset{\mu_i = \mu^*}{=} \max_{j=1,\ldots,p} \left\{ \left(\frac{\#\mathscr{P}_j}{\mu^*} \right)^{-1} \right\} \\
&= \frac{\mu^*}{\min_{j=1,\ldots,p} \{\#\mathscr{P}_j\}} \\
&= \frac{\mu^*}{3} \\
&\simeq 0.295409.
\end{aligned}$$

2. A process plant needs a regular supply of high-pressure steam. The steam producing system crucially depends on a filter set in series with a 3−out−of−4 sub-system of burners. Admit the times to failure of the filter and the burners are independent r.v. with constant hazard rate λ.

(a) Derive the reliability function and the expected time to failure of the system described above. **(2.5)**

- **R.v.**
  T_i = time to failure of burner i, $i = 1, \ldots, 4$
  T_5 = time to failure of filter

- **Distribution**
  $T_i \overset{i.i.d.}{\sim}$ Exponential(λ), $i = 1, \ldots, 5$, because constant hazard rate means having an exponential distribution.
  $R(t) = P(T_i > t) = e^{-\lambda t}$, $\quad t > 0, \quad i = 1, \ldots, 5$

- **Relevant r.v.**
  T = duration of the steam producing system
  $T = \min\{T_{(3:4)}, T_5\}$, where $T_{(3:4)}$ represents the time to failure of the 3−out−of−4 subsystem with burners

- **Reliability function of T**

$$\begin{aligned}
R_T(t) &\overset{Exerc/Ex.(2.10)}{=} R(t) \times R_{T_{(3:4)}}(t) \\
&\overset{(2.8)}{=} R(t) \times F_{binomial(4,1-R(t))}(4-3) \\
&= R(t) \times \sum_{j=0}^{1} \binom{4}{j} [1 - R(t)]^j \, [R(t)]^{4-j} \\
&= R(t) \times \left\{[R(t)]^4 + 4\,[1 - R(t)]\,[R(t)]^3\right\} \\
&= R(t) \times \left\{4\,[R(t)]^3 - 3\,[R(t)]^4\right\} \\
&= 4\,[R(t)]^4 - 3\,[R(t)]^5 \\
&= 4\,e^{-4\lambda t} - 3\,e^{-5\lambda t},\; t > 0.
\end{aligned}$$

- **Expected value of T**

$$\begin{aligned}
E(T) &\overset{(2.10)}{=} \int_0^{+\infty} R_T(t)\,dt \\
&= \int_0^{+\infty} \left(4\,e^{-4\lambda t} - 3\,e^{-5\lambda t}\right) dt \\
&= \frac{1}{\lambda} \int_0^{+\infty} f_{Exp(4\lambda)}(t)\,dt - \frac{3}{5\lambda} \int_0^{+\infty} f_{Exp(5\lambda)}(t)\,dt \\
&= \frac{1 - 3/5}{\lambda} \\
&= \frac{2}{5\lambda}.
\end{aligned}$$

(b) Obtain the hazard rate function of the time to failure of this system. **(2.5)**
Devise the associated stochastic ageing character.
Determine the limit of the hazard rate function when $t \to +\infty$. Comment.
Note: If you did not solve (a), admit that the reliability function is given by $4\,e^{-4\lambda t} - 3\,e^{-5\lambda t},\; t > 0$.

- **Reliability function of T**
  $R(t) = 4\,e^{-4\lambda t} - 3\,e^{-5\lambda t},\; t > 0$

- **P.d.f. of T**

$$\begin{aligned}
f(t) &= -\frac{d\,R(t)}{dt} \\
&= -\frac{d\,(4\,e^{-4\lambda t} - 3\,e^{-5\lambda t})}{dt} \\
&= 16\lambda\,e^{-4\lambda t} - 15\lambda\,e^{-5\lambda t},\; t > 0
\end{aligned}$$

- **Hazard rate function of T**

$$\lambda(t) = \frac{f(t)}{R(t)} = \frac{16\lambda e^{-4\lambda t} - 15\lambda e^{-5\lambda t}}{4e^{-4\lambda t} - 3e^{-5\lambda t}} = \lambda \frac{16 - 15e^{-\lambda t}}{4 - 3e^{-\lambda t}}, \; t > 0.$$

- **Devising the stochastic ageing character of T**

Since

$$\lambda(t) = \lambda \frac{16 - 15e^{-\lambda t}}{4 - 3e^{-\lambda t}} = \lambda \times \left\{4 - \left[\frac{4e^{\lambda t}}{3} - 1\right]^{-1}\right\},$$

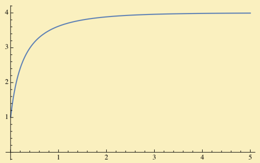
we can immediately conclude that $\lambda(t) \uparrow_t$, therefore $T \in IHR$.

- **Requested limit**

$$\lim_{t\to+\infty} \lambda(t) = \lim_{t\to+\infty} \lambda \frac{16 - 15e^{-\lambda t}}{4 - 3e^{-\lambda t}} = 4\lambda.$$

- **Comment**

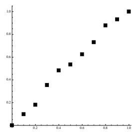
The hazard rate function increases and "soon" gets closer to its limiting value 4λ, as shown by the following plot for $\lambda = 1$:



This means that the time to failure has a "sort of memoryless character" for large values of t.

3. A microprocessor is to be used in a biomedical device and its time to failure is under study.

(a) The supplier collected 10 times to failure, obtained the ordered sample $(0.18, 0.35, 0.75, 1.09, 1.25, 1.58, 2.07, 2.98, 3.47, 4.73)$, and drew the TTT plot shown below. **(2.0)**



Exemplify the obtention of the first 4 points of such plot. Comment.

- **Failure times**

T_i = time to failure of microprocessor i, $i = 1, \ldots, 10$

- **Ordered sample**

$(0.18, 0.35, 0.75, 1.09, 1.25, 1.58, 2.07, 2.98, 3.47, 4.73)$

- **Total time on test up to time $t_{(i)}$**

$$\tau\left(t_{(0)}\right) = 0$$

$$\tau\left(t_{(i)}\right) = \sum_{j=1}^{i} (n-j+1)\left[t_{(j)} - t_{(j-1)}\right], \; i = 1, \ldots, n$$

$$\tau\left(t_{(10)}\right) = (10-1+1)\times(0.18-0) + (10-2+1)\times(0.35-0.18) + (10-3+1)\times(0.75-0.35) + \cdots + (10-10+1)\times(4.73-3.47) = 18.45$$

- **Abcissae of the TTT plot**

$\frac{i}{n}$, $i = 0, 1, \ldots, n$

- **Ordinates of the TTT plot**

$\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$, $i = 1, \ldots, n$.

- **First 4 points of the TTT plot**

i	$\frac{i}{n}$	$\tau\left(t_{(i)}\right) = \sum_{j=1}^{i}(n-j+1)\left[t_{(j)} - t_{(j-1)}\right]$	$\frac{\tau(t_{(i)})}{\tau(t_{(n)})}$
0	0	0	0
1	$\frac{1}{10} = 0.1$	$(10-1+1)\times(0.18-0) = 1.8$	$\frac{1.8}{18.45} \simeq 0.097561$
2	$\frac{2}{10} = 0.2$	$1.8 + (10-2+1)\times(0.35-0.18) = 3.33$	$\frac{3.33}{18.45} \simeq 0.180488$
3	$\frac{3}{10} = 0.3$	$3.33 + (10-3+1)\times(0.75-0.35) = 6.53$	$\frac{6.53}{18.45} \simeq 0.342953$

- **Comment on the TTT plot**

The points of the TTT plot are roughly around a 45^o line and, according to Note 5.5, this suggests that the data should be modelled by an Exponential distribution.

(b) The supplier of the microprocessors was also requested to perform a Type II/item censored test without replacement involving 10 microprocessors and to run this life test until three of them fail; the recorded failure times were $0.46, 0.54, 3.09$. **(2.5)**

Is the exponential model reasonable in light of this data? Obtain the p-value of an appropriated hypotheses test.

- **Life test**

Type II/item censored testing without replacement

- **Censored data**

$n = 10$

$r = 3$

$(t_{(1)}, \ldots, t_{(r)}) = (0.46, 0.54, 3.09)$

- **Cumulative total time in test**

According to Definition 5.17, the cumulative total time in test is given by:

$$\tilde{t} = \sum_{i=1}^{r} t_{(i)} + (n-r)\times t_{(r)} = (0.46 + 0.54 + 3.09) + (10-3)\times 3.09 = 25.72$$

- **Hypotheses**

$H_0 : T \sim \text{Exponential}(\lambda)$

$H_1 : T \sim \text{Weibull}(\lambda^{-1}, \alpha)$, $\alpha \neq 1$

- **Test statistic** (Bartlett's test)

$$B_r \overset{(5.18)}{=} \frac{2r}{1 + \frac{r+1}{6r}} \left(\ln\left(\frac{\mathcal{T}}{r}\right) - \frac{1}{r}\sum_{i=1}^{r} \ln\left\{(n-i+1)\times\left[T_{(i)} - T_{(i-1)}\right]\right\} \right) \overset{a}{\sim}_{H_0} \chi^2_{(r-1)}$$

where $T_{(i)} - T_{(i-1)}$'s represent the times between consecutive failure times.

- **Decision** (based on the p-value)

  The observed value of the test statistic is

$$\begin{aligned} b_r &= \frac{2r}{1+\frac{r+1}{6r}}\left[\ln\left(\frac{\sum_{i=1}^r z_i}{r}\right) - \frac{1}{r}\sum_{i=1}^r \ln(z_i)\right] \\ &\simeq \frac{2\times 3}{1+\frac{3+1}{6\times 3}} \times \left\{\ln\left(\frac{25.72}{3}\right)\right. \\ &\quad \left. -\frac{1}{3}\times[\ln(10\times 0.46)+\ln(9\times(0.54-0.46))+\ln(8\times(3.09-0.54))]\right\} \\ &\simeq 4.(90)\times(2.148657-1.404362) \\ &\simeq 3.653811. \end{aligned}$$

  Since the rejection region of the Bartlett's test is two-sided,

$$p-value = 2\times\min\{p^-,p^+\},$$

  where

$$\begin{aligned} p^- &= F_{B_r|H_0}(b_r) \\ &\simeq F_{\chi^2_{(r-1)}}(3.653811) \\ &= F_{\chi^2_{(2)}}(3.653811) \\ &\overset{Mathematica}{\simeq} 0.839089 \\ p^+ &= 1-F_{B_r|H_0}(b_r) \\ &= 1-p^- \\ &\simeq 1-0.839089 \\ &= 0.160911. \end{aligned}$$

  Thus,

$$p-value = 2\times\min\{0.839089, 0.160911\} = 0.321822$$

  and we should:
  - not reject H_0 for any significance level $\alpha_0 \le 32.1822\%$, namely at all the usual significance levels $(1\%, 5\%, 10\%)$;
  - reject H_0 for any significance level $\alpha > 32.1822\%$.

  [The result of this test agrees with the TTT plot in (a).]

  [Alternatively, we could have used the table with the c.d.f. values of chi-square distribution and conclude that: $F^{-1}_{\chi^2_{(2)}}(0.80) = 3.219 < 3.653811 < 3.794 = F^{-1}_{\chi^2_{(2)}}(0.85)$; $p^- \in (0.80, 0.85)$; $p^+ \in (0.15, 0.20)$; $p-value = 2\times\min\{p^-,p^+\} \in (0.30, 0.40)$. Thus, we should: not reject H_0 for any significance level $\alpha_0 \le 30\%$, namely at all the usual significance levels $(1\%, 5\%, 10\%)$; reject H_0 for any significance level $\alpha \ge 40\%$.]

(c) Compute the ML estimate of the median time to failure and obtain a 95% confidence interval for the expected duration of the life test described in (b). **(3.0)**

- **Distribution assumption**

  The assumption $T_i \overset{i.i.d.}{\sim} \text{Exponential}(\lambda)$, $i = 1,\dots,n$, is fairly reasonable since we did not reject H_0 for all the usual significance levels $(1\%, 5\%, 10\%)$ [and in light of the TTT plot].

- **ML estimate of $F_T^{-1}(0.5)$**

  According to tables 5.10 and 5.13, the ML estimate of $F_T^{-1}(0.5) = -\frac{1}{\lambda}\ln(1-0.5)$, under Type II/item censored testing without replacement, is given by

$$\begin{aligned} \hat{F}_T^{-1}(0.5) &= -\frac{1}{\hat\lambda}\ln(1-0.5) \\ &= -\frac{1}{r/\tilde t}\ln(1-0.5) \end{aligned}$$

$$\begin{aligned} \hat{F}_T^{-1}(0.5) &\overset{(b)}{=} -\frac{25.72}{3}\ln(1-0.5) \\ &\simeq 5.94258. \end{aligned}$$

- **Confidence interval for λ**

  According to Table 5.16,

$$\begin{aligned} CI_{(1-\alpha)\times 100\%}(\lambda) &= [\lambda_L;\lambda_U] \\ &= \left[\frac{F_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde t};\frac{F_{\chi^2_{(2r)}}(1-\alpha/2)}{2\times\tilde t}\right]. \end{aligned}$$

  If we take into account that
  - $n = 10$, $r = 3$,
  - the censored data is $(t_{(1)},\dots,t_{(r)}) = (0.46, 0.54, 3.09)$,
  - the cumulative total time in a Type II/item censored test without replacement is given by $\tilde t = 25.72$,

  we get

$$\begin{aligned} CI_{95\%}(\lambda) &= \left[\frac{F_{\chi^2_{(6)}}(0.025)}{2\times 25.72};\frac{F_{\chi^2_{(6)}}(0.975)}{2\times 25.72}\right] \\ &\overset{tables}{=} \left[\frac{1.237}{51.44};\frac{14.45}{51.44}\right] \\ &\simeq [0.024047; 0.280910] \end{aligned}$$

- **Expected duration of the life test**

  According to (4.12), this expectation duration of Type II/item censored test without replacement is given by

$$\begin{aligned} E(T_{r:n}) &= \sum_{i=1}^r \frac{1}{(n-i+1)\lambda} \\ &\overset{r=3,n=10}{\simeq} 0.336111\times\frac{1}{\lambda}, \end{aligned}$$

- **Confidence interval for $E(T_{r:n})$**

  Since $E(T_{3:10}) \simeq 0.336111 \times \frac{1}{\lambda}$ is a decreasing function of $\lambda > 0$, we obtain

$$\begin{aligned} CI_{95\%}(E(T_{3:10})) &= \left[0.336111\times\frac{1}{\lambda_U}, 0.336111\times\frac{1}{\lambda_L}\right] \\ &\simeq [1.196508; 13.977253]. \end{aligned}$$