Fuzzy Strategy-Proofness and a Fuzzy Version of Arrow's Result

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Abstract

In many social decision-making contexts, it is frequent to encounter manipulators that attempt to change the social choice in their favor by misrepresenting preferences. Ben Abdelaziz *et al.* (2007) have shown that strategic manipulations can be observed when certain types of fuzzy social choice functions are applied. This paper restates the previous result for fuzzy preference orders by means of fuzzy game forms.

key words: Gibbard-Satterthwaite's Theorem, fuzzy preference order, Dutta's Theorem.

1 Introduction

In various social decision-making contexts, voting choice procedures can be used to aggregate crisp preference relations on a finite set of alternatives into a social choice. A voting choice procedure is known to be subject to strategic manipulation when an individual reveals a non-sincere preference relation in order to change the social choice in his favor. In this case, a non-sincere social choice is obtained. Gibbard (1973) and Satterthwaite (1975) (henceforth *G-S*) proved, independently, that any non-dictatorial voting choice procedure is manipulable when the set of alternatives contains at least three elements. However, voting choice procedures are strongly connected to social welfare functions. A social welfare function can be defined as an aggregation rule of individual preference orders into a social order on the set of alternatives. Arrow (1963) sets some conditions for an social welfare function as follows (see also Satterthwaite, 1975).

1. *Citizens' Sovereignty* (*CS*): for every two alternatives, there exists an individual's preference profile allowing the first to be socially better than the second one.

2. *Pareto Optimality* (*PO*): if all the individuals prefer an alternative to another, then the first is socially better than the second one.

3. *Independence of Irrelevant Alternatives* (*IIA*): the social preference over two given alternatives should not change, when the individual's preference profile changes for the remaining alternatives.

4. *Non-Negative Response* (*NNR*): consider an initial individual's preference profile on the set of alternatives. When some individual preference orders are changed to place a given alternative in a better position, it cannot have a worse social rank than the initial social one.

5. *Dictatorship* (*D*): there exists an individual (the dictator) whose preferences induce the social preferences.

Notice that *PO* implies *CS* (Arrow, 1963; Satterthwaite, 1975). Arrow (1963) proved that every social welfare function satisfying *CS*, *IIA*, and *NNR* conditions, is dictatorial. In addition, *G-S* showed that a procedure exists for constructing a social welfare function satisfying *CS*, *IIA*, and *NNR* conditions for every strategy-proof procedure. It should be noticed too that in Arrow and *G-S* works the individual preferences are presented by a (strict or weak) crisp order. But in many situations, individuals have some difficulties to express clearly their preferences on the set of alternatives. To deal with the preference vagueness, the fuzzy binary relation concept can be used (*e.g.* Barrett *et al.* 1986, Dutta, 1987). Therefore, the choice of a single alternative requires the use of a fuzzy social choice function . In the literature (*e.g.* Dutta, 1987; Fodor, 1994), there exist two ways to deal with an fuzzy social choice function. The first one is based on the rule "aggregation-and-defuzzification". It consists of applying to a preference profile a fuzzy social welfare function that leads to a social fuzzy relation, and then of generating, from the comprehensive fuzzy relation, a collective choice by applying a choice function. The second one makes use of the rule "defuzzification-and-aggregation", and consists of applying a choice function that generates on the basis of each individual fuzzy relation his best alternative set, and then by using a voting choice procedure the social choice is obtained on the basis of the individual choices.

When the attention is focused on weak individual preference relations, a single attempt to define the manipulability of an $FSCF$ has already been proposed by Ben Abdelaziz *et al.* (2007). On the other hand, Dutta (1987) generalized the Arrow's conditions for fuzzy orders. He stated that there exists no *FSWF* satisfying all fuzzy counterparts of Arrow's conditions when individuals express their preferences as fuzzy orders.

The purpose of this paper is to deal with the manipulability of an fuzzy social choice function and to extend Gibbard's framework to the context of fuzzy preferences. The basic idea is to consider the concept of *straightforward fuzzy game form.* The generalization of *G-S* result is then restated relying up on the impossibility Dutta's theorem.

This paper is organized as follows. Section 2 presents the elementary concepts on fuzzy binary relations; also a generalization of the game form concept to a fuzzy framework is provided. The main theorem is also provided with its proof. Finally, conclusions and future research avenues are provided.

2 Manipulability and Fuzzy Game Forms

After introducing elementary definitions, this section provides the concept of fuzzy game form. Then, the link of the three types of manipulability of fuzzy social choice functions and the three types of straightforward fuzzy game form are established. Nextly, a procedure will be introduced for constructing a fuzzy social welfare functions satisfying the Dutta's conditions, except non-dictatorship, from every straightforward fuzzy game form. Finally, the impossibility result on the ℓ-strategy-proofness of fuzzy social choice functions will be restated.

Given a non-empty set of *alternatives* $X = \{x_1, x_2, \ldots, x_j, \ldots, x_m\}$, with $m \geq 3$, fuzzy preference relation on X can be defined as a function $R : X^2 \longrightarrow [0,1]$. For all $(x_j, x_k) \in X^2, R(x_j, x_k)$ is the degree to which x_j is at least as good as x_k.

Suppose a finite set of *individuals* $N = \{1, 2, \ldots, i, \ldots, n\}$. Given that each individual i's preferences are expressed as a fuzzy preference relation R_i over X, a social choice can be obtained by means of a fuzzy social choice function $\nu : (R_1, R_2, \ldots, R_i, \ldots, R_n) \rightarrow x_k \in X$.

Let consider the additional notation.

- $\mathcal{D}_c$, is the set of all fuzzy preference relations satisfying reflexivity and connectedness, *i.e.* $\forall x_j \in X, R(x_j, x_j) = 1$ and $R(x_j, x_k) + R(x_k, x_j) \geq 1$.
- $\mathcal{D}$, is the set of all fuzzy preference relations in $\mathcal{D}_c$ satisfying max-min transitivity, *i.e.* $\forall x_j, x_k, x_\ell \in X, R(x_j, x_\ell) \geq \min\{R(x_j, x_k), R(x_k, x_\ell)\}$.
- $\mathfrak{R}_N = (R_1, R_2, \ldots, R_i, \ldots, R_n)$ is a profile of individuals' preference relations.
- $\mathfrak{S}_N = (s_1, s_2, \ldots, s_i, \ldots, s_n)$ is a profile individuals' strategies.
- $(\mathfrak{S}_N \mid R_i)$, is the profile of individuals' relations $(s_1, \ldots, s_{i-1}, R_i, s_{i+1}, \ldots, s_n)$, where individual i declares the fuzzy preference relation R_i instead of S_i.

Throughout this paper, we consider that R and R_i, for $i \in N$, are elements of the set $\mathcal{D}$. They are defined as fuzzy orders.

2.1 Fuzzy game form

We consider in this section the concept of fuzzy game form and the straightforwardness property.

In the social choice theory the concept of *game forms* is firstly introduced by Gibbard (1973) as follows.

Definition 1 (game form, Gibbard, 1973) *A game form is a system which allows each individual his choice among a set of strategies, and makes an outcome depend, in a determinate way, on the strategy each individual chooses.*

With the assumption that each individual has fuzzy preference relations over the set of possible alternatives, we define a special kind of game forms, called *fuzzy game forms*, as follows.

Definition 2 (fuzzy game form) *A game form is said to be fuzzy if the set of strategies of each individual is the set of fuzzy binary relations on X.*

Now, we generalize the concept of Gibbard's strategy dominance in three manner in the fuzzy context.

Definition 3 (ℓ-dominant strategy) *Let g be an game form and s be in $\mathcal{D}$, and $i \in N$.*

(i) A strategy s is 1-dominant for i and $R_i \in \mathcal{D}$, over X, if for every $\mathfrak{S}_N \in \mathcal{D}^n$, $R_i(g_1(\mathfrak{S}_N|s), g_1(\mathfrak{S}_N)) \geq R_i(g_1(\mathfrak{S}_N), g_1(\mathfrak{S}_N|s))$.

(ii) A strategy s is 2-dominant for i and $R_i \in \mathcal{D}$, over X, if for every $\mathfrak{S}_N \in \mathcal{D}^n$, $R_i(g_2(\mathfrak{S}_N|s), g_2(\mathfrak{S}_N)) \geq \alpha$.

(iii) A strategy s is 3-dominant for i and $R_i \in \mathcal{D}$, over X, if for every $\mathfrak{S}_N \in \mathcal{D}^n$, $d(R_i, X)(g_3(\mathfrak{S}_N|s)) \geq d(R_i, X)(g(\mathfrak{S}_N))$.

In other words, for $\ell \in \{1, 2, 3\}$, a strategy s is ℓ-dominant for individual i if for each assignment of strategies of other individuals, it produces for individual i an outcome at least as good as any other outcome on the basis of his preference relation, $R_i \in \mathcal{D}$ in a certain way. Therefore, the *ℓ-straightforwardness* of fuzzy game forms can be presented as follows.

Definition 4 (ℓ-straightforward fuzzy game form) *Let g be an fuzzy game form and $\ell \in \{1, 2, 3\}$. g is ℓ-straightforward if for every individual, i, and a given fuzzy preference relation, $R_i \in \mathcal{D}$, over X, there is a strategy which is ℓ-dominant for i.*

In the next section, we will establish the connection between fuzzy game forms and fuzzy social choice functions.

2.2 Fuzzy game forms and associated fuzzy social choice functions

In this paper, a fuzzy social choice functions is considered to be a mechanism for choosing a social choice from a finite set of possible alternatives X, on the basis of fuzzy individuals' sincere preferences. They are assumed to be fuzzy orders over X. Therefore, a fuzzy game form can correspond to a fuzzy social choice function when the strategy of each individual coincides with his sincere fuzzy preference relation. However, when a fuzzy social choice function is manipulable, there is a profile of individual sincere preferences that gives some individual an incentive to declare a non-sincere fuzzy preference relation. More specifically, the manipulation of a fuzzy social choice function can be expressed in terms of fuzzy game forms into two steps as follows:

1. Starting with a profile of individual sincere preferences, some individual $i \in N$ has an incentive to reveal a non-sincere fuzzy preference relation (see definition 5). It is chosen from the set of admissible fuzzy binary relations associated to the fuzzy social choice function and it depends closely on his sincere preference relation. Thus, for individual i, a mapping σ_i that associates to each sincere fuzzy preference relation the strategic one can be defined. The declared fuzzy relation can be viewed as the strategy of individual i.
2. When each individual strategy is given, a mechanism is applied to choose the social choice from the set of alternatives. Such a mechanism can viewed as a fuzzy game form. Of course, the selected alternative is considered to be better than the sincere social choice by individual i.

Moreover, for $\ell \in \{1, 2, 3\}$, an ℓ-fuzzy game form is considered to be associated to an ℓ-fuzzy social choice function, if they coincide in non-manipulation situations. Now, in order to present the link between the ℓ-straightforwardness fuzzy game form and the ℓ-manipulability of the associated ℓ-fuzzy social choice functions, for $\ell \in \{1, 2, 3\}$, we reintroduce the ℓ-manipulability concept of fuzzy social choice functions in terms of the associated fuzzy game form [Ben Abdelaziz *et al*, 2007].

Definition 5 (ℓ-manipulability) *Let ν be a fuzzy social choice functions and g be its associated fuzzy game form.*

i) The function ν is said to be 1-manipulable if there exists an $m \in N$, $\mathfrak{R}_N \in \mathcal{D}^n$, and $R'_m \in \mathcal{D}$ such that there exists $x \in X$, such that $R_m(x, g(\mathfrak{R}_N)) >$

$R_m(g(\mathfrak{R}_N), x)$, $g(\mathfrak{R}_N \mid R'_m) \neq g(\mathfrak{R}_N)$, *and* $R_m(g(\mathfrak{R}_N \mid R'_m), g(\mathfrak{R}_N)) \geq R_m(g(\mathfrak{R}_N), g(\mathfrak{R}_N \mid R'_m))$.

ii) *The function* ν *is said to be 2-manipulable if there exists an* $m \in N$, $\mathfrak{R}_N \in \mathcal{D}^n$, *and* $R'_m \in \mathcal{D}$ *such that for a fixed* $\alpha \in]0, 1/2]$, *there exists* $x \in X$, *such that* $R_m(g(\mathfrak{R}_N), x) < \alpha$, $g(\mathfrak{R}_N \mid R'_m) \neq g(\mathfrak{R}_N)$ *and* $R_m(g(\mathfrak{R}_N \mid R'_m), g(\mathfrak{R}_N)) \geq \alpha$.

iii) *The function* ν *is said to be 3-manipulable if there exists an* $m \in N$, $\mathfrak{R}_N \in \mathcal{D}^n$, *and* $R'_m \in \mathcal{D}$ *there exists* $x \in X$, *such that* $d(R_m, X)(x) > d(R_m, X)\ (g(\mathfrak{R}_N))$, $\nu(\mathfrak{R}_N \mid R'_m) \neq \nu(\mathfrak{R}_N)$, *and* $d(R_m, X)(g(\mathfrak{R}_N \mid R'_m)) \geq d(R_m, X)(g(\mathfrak{R}_N))$.

Lemma 1 (sufficient condition for ℓ-straightforwardness) *Let* g *be a fuzzy game form, and* ν *be its associated fuzzy social choice functions. If* f *is* ℓ*-strategy-proof, then* g *is* ℓ*-straightforward.*

Proof

- $\underline{\ell = 1}$
  Suppose that g is not 1-straightforward. Thus, there exists an individual i and $R_i \in \mathcal{D}_1$, such that there is no 1-dominant-strategy for i and R_i. Hence, for an $\mathfrak{S}_N \in \mathcal{D}^n$, any strategy $s \in \mathcal{D}$ is not 1-dominant for i and R_i. Then, in particular, R_i is not 1-dominant, *i.e.* $R_i(g(\mathfrak{S}_N), g(\mathfrak{S}_N|R_i)) > R_i(g(\mathfrak{S}_N|R_i), g(\mathfrak{S}_N))$. Thus, there a profile $(\mathfrak{S}_N|R_i)$ such that ν is 1-manipulable by individual i.

- $\underline{\ell = 2}$
  Suppose that g is not 2-straightforward. Thus, there exists an individual i and $R_i \in \mathcal{D}$, such that there is no 2-dominant-strategy for i and R_i. Hence, for an $\mathfrak{R}_N \in \mathcal{D}^n$, any strategy $s \in \mathcal{D}$ is not 2-dominant for i and R_i. Then, in particular, R_i is not 2-dominant, i.e. $R_i(g(\mathfrak{S}_N|R_i), g(\mathfrak{S}_N)) < \alpha$. Since, R_i is reflexive, $R_i(g(\mathfrak{S}_N), g(\mathfrak{S}_N|R_i)) > \alpha$. Thus, there a profile $(\mathfrak{S}_N|R_i)$ such that ν is 2-manipulable by individual i.

- $\underline{\ell = 3}$
  Suppose that g is not 3-straightforward. Thus, there exists an individual i and $R_i \in \mathcal{D}$, such that there is no 3-dominant-strategy for i and R_i. Hence, for an $\mathfrak{R}_N \in \mathcal{D}^n$, any strategy $s \in \mathcal{D}$ is not 3-dominant for i and R_i. Then, in particular, R_i is not 3-dominant, i.e. $d(X, R_i)(g(\mathfrak{S}_N) > d(X, R_i)(g(\mathfrak{S}_N|R_i))$. Thus, there a profile $(\mathfrak{S}_N|R_i)$ such that ν is 2-manipulable by individual i.

$\square$

Finally, the concept of dictatorship for a game form g can be defined in three manners and it corresponds to the dictatorship of the associated fuzzy social choice function.

Definition 6 (ℓ-dictatorship) *Let ν be a fuzzy social choice function and g be its associated fuzzy game form. g is said to be*

i) 1-dictatorial if there exists a $d \in N$ such that for every $\mathfrak{S}_N \in \mathcal{D}^n$, $\forall\ x \in X, s_d(g(\mathfrak{S}_N), x) \geq s_d(x, g(\mathfrak{S}_N))$.

ii) 2-dictatorial if there exists a $d \in N$ such that for every $\mathfrak{S}_N \in \mathcal{D}^n$, for a fixed $\alpha \in]0, 1/2]$, $\forall\ x \in X, s_d(g(\mathfrak{S}_N), x) \geq \alpha$.

iii) 3-dictatorial if there exists a $d \in N$ such that for every $\mathfrak{S}_N \in \mathcal{D}^n$, $\forall\ x \in X, d(s_d, X)(g(\mathfrak{S}_N)) \geq d(s_d, X)(x)$.

It should be remarked that when a fuzzy game form is ℓ-dictatorial, it coincides with the associated fuzzy social choice function. Indeed, the dictator does not need a strategy different to his sincere preference relations and the other individuals are not able to change the outcome of the fuzzy social choice function.

2.3 Dictatorship of straightforward fuzzy game forms

Starting with an ℓ-straightforward fuzzy game form, for $\ell \in \{1, 2, 3\}$, we prove, in this section, the existence of a fuzzy social welfare function that associates to each profile of individual fuzzy orders a fuzzy order and satisfies some conditions proposed by Dutta (1987). Then, the ℓ-dictatorship of the fuzzy game form is established relying up on the Dutta's impossibility result.

Let us begin with introducing Dutta's conditions and impossibility result. Since he focused attention in weak fuzzy preference relations (*i.e.* reflexive), he proposed a certain decomposition of a fuzzy order into a fuzzy indifference relation I and a fuzzy strict preference relation P.

Proposition 1 (fuzzy indifference and fuzzy strict preference) *Let R belong to $\mathcal{D}$. If R satisfies the following conditions for all $x_j, x_k \in X$,*

i) $R = P \cup I$, i.e. $(P \cup I)(x_j, x_k) = \max\{P(x_j, x_k), I(x_j, x_k)\}$;

ii) I is symmetric, i.e. $I(x_j, x_k) = I(x_k, x_j)$;

iii) P is antisymmetric, i.e. $P(x_j, x_k) > 0$, *then* $P(x_k, x_j) = 0$*;*

iv) If $R(x_j, x_k) = R(x_k, x_j)$, then $P(x_j, x_k) = P(x_k, x_j)$*;*

Then,

$$P(x_j, x_k) = \begin{cases} 0, & \textit{if } R(x_j, x_k) \leq R(x_k, x_j); \\ R(x_j, x_k), & \textit{otherwise.} \end{cases}$$

$$I(x_j, x_k) = \min\{R(x_j, x_k), R(x_k, x_j)\}.$$

Definition 7 (Dutta's conditions) *Let* $h : \mathfrak{R}_N \in \mathcal{D}^n \mapsto \widehat{R} \in \mathcal{D}$ *denote a fuzzy social welfare function. Let* P_i *and* $\widehat{P}$ *denote the strict preference relations corresponding to* R_i *and* $\widehat{R}$*, respectively.* h *satisfies the*

1. *Fuzzy Pareto Optimality(FPO) : If for all* $\mathfrak{R}_N \in \mathcal{D}^n$*, and for all distinct* $x_j, x_k \in X$*,*
$$\widehat{P}(x_j, x_k) \geq \min_{i \in N} \{P_i(x_j, x_k)\}.$$

2. *Fuzzy Independence of Irrelevant Alternatives (FIIA) : if for all* $i \in N$*, for all* $\mathfrak{R}_N, \mathfrak{R}'_N \in \mathcal{D}^n$*, and for all distinct* $x_j, x_k \in X$*,*
$$\begin{cases} R_i(x_j, x_k) = R'_i(x_j, x_k), \\ R_i(x_k, x_j) = R'_i(x_k, x_j), \end{cases} \quad \textit{then} \begin{cases} \widehat{R}(x_j, x_k) = \widehat{R}'(x_j, x_k), \\ \widehat{R}(x_k, x_j) = \widehat{R}'(x_k, x_j). \end{cases}$$

3. *Fuzzy Positive Responsiveness (FPR): If for all* $\mathfrak{R}_N, \mathfrak{R}'_N \in \mathcal{D}^n$*, and for all distinct* $x_j, x_k \in X$*,*
$$\begin{cases} R_i(x_k, x_j) = R'_i(x_k, x_j), \forall\, i \neq h, \\ \widehat{R}(x_k, x_j) = \widehat{R}(x_j, x_k), \\ (P_h(x_j, x_k) = 0 \textit{ and } P'_h(x_j, x_k) > 0), \\ \ \textit{or } (P_h(x_k, x_j) > 0 \textit{ and } P'_h(x_k, x_j) = 0), \end{cases} \quad \textit{then } \widehat{P}'(x_j, x_k) > 0.$$

4. *Fuzzy Non-Dictatorship (FND): Consider that there exists no individual* $i \in N$ *such that for all* $\mathfrak{R}_N \in \mathcal{D}^n$*, and for all distinct* $x_j, x_k \in X$*, if* $P_i(x_j, x_k) > 0$*, then* $\widehat{P}(x_j, x_k) > 0$.

Moreover, Dutta (1986) showed the following fuzzy version of the Arrow's impossibility theorem.

Theorem 1 (impossibility theorem of Dutta, 1986) *Let $h : \mathcal{D}^n \longrightarrow \mathcal{D}$ denote a fuzzy social welfare function. If h satisfies FPO, $FIIA$, and FPR, then h is fuzzy dictatorial.*

In the sequel of the paper, we consider only fuzzy social choice functions satisfying the following condition:

$$\text{if } \nu(\mathfrak{R}_N) = x \text{ is chosen from the set of possible alternative } X, \text{ then } x = \nu(\mathfrak{R}_N)$$
$$\text{when } \nu(\mathfrak{R}_N) \text{ is chosen from the } S \cup \{x\}, \text{ for any } S \text{ subset of } X.$$

Lemma 2 (connection with Dutta's conditions) *Let g be fuzzy game form and ℓ be in $\{1, 2, 3\}$. If g is ℓ-straightforward, then there exists a fuzzy social welfare function which satisfies FPO, FIIA, and PPR.*

Proof

Let g denote an ℓ-straightforward game form, for $\ell \in \{1, 2, 3\}$. The proof of this lemma can be outlined as follows.

1. Construct a fuzzy social welfare function h such that for each fuzzy profile $\mathfrak{R}_N \in \mathcal{D}^n$, a fuzzy relation $\widehat{R} = h(\mathfrak{R}_N)$ fulfills reflexivity, connectedness, and the max-min transitivity;

2. Prove that this function, h, satisfies all the impossibility conditions except non-dictatorship.

Let us start the proof according to the previous steps.

1. *Construct a fuzzy social welfare function*
   Let h denote a fuzzy social welfare function, $h : \mathcal{D}^n \longrightarrow \mathcal{D}$ with $\widehat{R} = h(\mathfrak{R}_N)$, such that

   - for all $x_j \in X, \widehat{R}(x_j, x_j) = 1$
   - for all distinct $x_j, x_k \in X$,

$$\widehat{R}(x_j, x_k) = \begin{cases} 1, & \text{if } \begin{cases} g(\mathfrak{S}_N, \{x_j, x_k\}) = x_j \\ R_i(x_j, x_k) > R_i(x_k, x_j), \forall i \ \in N \end{cases} \\ \gamma \in]\frac{1}{2}, 1[, & \text{if } g(\mathfrak{S}_N, \{x_j, x_k\}) = x_j; \\ \beta \in [0, \frac{1}{2}[, & \text{if } g(\mathfrak{S}_N, \{x_j, x_k\}) = x_k; \\ \widehat{R}(x_k, x_j) = \epsilon, & \text{if } R_i(x_j, x_k) = R_i(x_k, x_j), \forall i \ \in N; \end{cases}$$

   - $\gamma + \beta \geq 1, \alpha \geq \epsilon \geq \frac{1}{2}$

where $\mathfrak{S}_N = (\sigma_1(R_i), \ldots, \sigma_i(R_i), \ldots, \sigma_n(R_n))$ when g is associated to a fuzzy social choice function applied to $\mathfrak{R}_N$.

The fuzzy relation $\widehat{R}$ verifies:

(a) *reflexivity*: for all $x_j \in X, \widehat{R}(x_j, x_j) = 1$.

(b) *connectedness*: for all $x_j, x_k \in X, \widehat{R}(x_j, x_k) + \widehat{R}(x_k, x_j) \geq 1$

(c) *max-min transitivity*: let $x_j, x_k, x_t \in X$, we have to prove that

$$\widehat{R}(x_j, x_t) \geq \min(\widehat{R}(x_j, x_k), \widehat{R}(x_k, x_t)) \tag{2.1}$$

- If $\widehat{R}(x_j, x_t) = 1$, then Eq. (.1) is always verified.
- $\widehat{R}(x_j, x_t) = \gamma \in [\frac{1}{2}, 1[$
  - If $\widehat{R}(x_j, x_k) = \widehat{R}(x_k, x_t) = 1$, then for all $i \in N$, $R_i(x_j, x_k) > R_i(x_k, x_j)$ and $R_i(x_k, x_t) > R_i(x_t, x_k)$. Since for all $i \in N, R_i$ verifies the max-min transitivity, $R_i(x_j, x_t) > R_i(x_t, x_j)$, for all $i \in N$ (Ovchinnikov, 1981). Thus, $\widehat{R}(x_j, x_t) = 1$. This leads to a contradiction with the assumption that $\gamma \neq 1$.
  - If $\min(\widehat{R}(x_j, x_k), \widehat{R}(x_k, x_t)) \in \{\gamma, \beta, \epsilon\}$, then Eq. (.1) holds.
- $\widehat{R}(x_j, x_t) = \epsilon$
  - If $\min(\widehat{R}(x_j, x_k), \widehat{R}(x_k, x_t)) = \gamma$, then $g(\mathfrak{S}_N, \{x_j, x_k\}) = x_j$ and $g(\mathfrak{S}_N, \{x_k, x_t\}) = x_k$. Thus, $g(\mathfrak{S}_N, \{x_j, x_t\}) = x_j$. Consequently, ϵ must be equal to γ and Eq. (.1) holds.
  - If $\min(\widehat{R}(x_j, x_k), \widehat{R}(x_k, x_t)) = 1$, then the previous reasoning can be adopted.
  - If $\min(\widehat{R}(x_j, x_k), \widehat{R}(x_k, x_t)) = \beta$, then Eq. (.1) holds.
- $\widehat{R}(x_j, x_t) = \beta$
  - If $\min(\widehat{R}(x_j, x_k), \widehat{R}(x_k, x_t)) = \gamma$ or 1, then $g(\mathfrak{S}_N, \{x_j, x_t\}) = x_j$. This leads to a contradiction with the assumption that $\widehat{R}(x_j, x_t) = \beta$.
  - If $\min(\widehat{R}(x_j, x_k), \widehat{R}(x_k, x_t)) = \beta$, then Eq. (2) holds.

2. *Checking Dutta's impossibility conditions, except fuzzy dictatorship*

   (a) *FIIA*
   If for all $\mathfrak{R}_N, \mathfrak{R}'_N \in \mathcal{D}^n$, and for all distinct $x_j, x_k \in X$, for all $i \in N$,
   $\begin{cases} R_i(x_j, x_k) = R'_i(x_j, x_k), \\ R_i(x_k, x_j) = R'_i(x_k, x_j), \end{cases}$ then $g(\mathfrak{R}_N, \{x_j, x_k\}) = g(\mathfrak{R}'_N, \{x_j, x_k\})$.
   Thus, $\widehat{R}(x_j, x_k) = \widehat{R}'(x_j, x_k)$ and $\widehat{R}(x_k, x_j) = \widehat{R}'(x_k, x_j)$. Therefore, it can be concluded that IIA holds for function h.

   (b) *FPO*

   If for all $\mathfrak{R}_N \in \mathcal{D}^n$, and for all distinct $x_j, x_k \in X$, then

$$\widehat{P}(x_j, x_k) = \begin{cases} \widehat{R}(x_j, x_k), & \text{if } \widehat{R}(x_j, x_k) > \widehat{R}(x_k, x_j); \\ 0, & \text{otherwise.} \end{cases}$$

   - $\widehat{R}(x_j, x_k) > \widehat{R}(x_k, x_j)$ is verified only when $\widehat{R}(x_j, x_k) \in \{1, \gamma\}$. Therefore,
     i. if for all $i \in N$, $R_i(x_j, x_k) > R_i(x_k, x_j)$, then for all $i \in N$, $P_i(x_j, x_k) > 0$, and $\widehat{P}(x_j, x_k) = 1 \geq \min_{i \in N}\{P_i(x_j, x_k)\}$.
     ii. If there exists $i \in N$, such that $R_i(x_j, x_k) \leq R_i(x_k, x_j)$, Then $P_i(x_j, x_k) = 0$ and $\widehat{P}(x_j, x_k) = \gamma \geq \min_{i \in N}\{P_i(x_j, x_k)\}$.
   - $\widehat{P}(x_j, x_k) = 0$, if $\widehat{R}(x_j, x_k) \leq \widehat{R}(x_k, x_j)$
     – $\widehat{R}(x_j, x_k) = \widehat{R}(x_k, x_j)$. Thus, for all $i \in N$, $R_i(x_j, x_k) = R_i(x_k, x_j)$ and $P_i(x_j, x_k) = 0$.
     – $\widehat{R}(x_j, x_k) < \widehat{R}(x_k, x_j)$

       i. If $\widehat{R}(x_j, x_k) = 1$, then for all $i \in N$, $R_i(x_j, x_k) < R_i(x_k, x_j)$ and $P_i(x_j, x_k) = 0$.
       ii. If $\widehat{R}(x_j, x_k) = \gamma$, then there exists an $i \in N$, such that $R_i(x_j, x_k) < R_i(x_k, x_j)$ and $P_i(x_j, x_k) = 0$.

   Therefore, it can be concluded that FPO holds for the function h.

3. *FPR*
   Suppose that for all $\mathfrak{R}_N, \mathfrak{R}'_N \in \mathcal{D}^n$, and for all distinct $x_j, x_k \in X$,

- for all $i \in N - \{k\}, R_i(x_j, x_k) = R'_i(x_j, x_k)$, and,
- $\widehat{R}(x_j, x_k) = \widehat{R}(x_k, x_j)$, then
  Thus, for all $i \in N$, $R_i(x_j, x_k) = R_i(x_k, x_j)$. Therefore, $P_k(x_j, x_k) = 0$.
  Now, it must be shown that if $P'_k(x_j, x_k) > 0$, then $\widehat{P}'(x_j, x_k) > 0$.

  - $\ell = 1$
    Since g is supposed to be a 1-straightforward game form and $P'_k(x_j, x_k) > 0$, the outcome of g is x_j face to $\{x_j, x_k\}$. Then, $\widehat{R}'(x_j, x_k) = \gamma > \widehat{R}'(x_k, x_j)$ and $\widehat{P}'(x_j, x_k) = \widehat{R}'(x_k, x_j) > 0$.

  - $\ell = 2$
    Since g is supposed to be a 2-straightforward game form and $R'_k(x_j, x_k) > \alpha$, the outcome of g is x_j face to $\{x_j, x_k\}$. Then, $\widehat{R}'(x_j, x_k) = \gamma > \widehat{R}'(x_k, x_j)$ and $\widehat{P}'(x_j, x_k) = \widehat{R}'(x_k, x_j) > 0$.

  - $\ell = 3$
    Since g is supposed to be a 3-straightforward game form and $d(R'_k, \{x_j, x_k\}(x_j) > d(R'_k, \{x_j, x_k\}(x_k)$, the outcome of g is x_j face to $\{x_j, x_k\}$. Then, $\widehat{R}'(x_j, x_k) = \gamma > \widehat{R}'(x_k, x_j)$ and $\widehat{P}'(x_j, x_k) = \widehat{R}'(x_k, x_j) > 0$.

Therefore, it can be concluded that (PR) holds for the function h starting with any ℓ-straightforward fuzzy game form, for $\ell \in \{1, 2, 3\}$.

$\square$

With the help of the fuzzy social welfare function, h, introduced in the previous section, now we prove the ℓ-dictatorship of an ℓ-straightforward fuzzy game form.

Theorem 2 (dictatorship of straightforward fuzzy game form) *Let g be an fuzzy game form and $\ell \in \{1, 2, 3\}$. If g is ℓ-straightforward , then it is ℓ-dictatorial.*

Proof
The proof consists of showing that the ℓ-dictatorship of g corresponds to the ℓ-dictatorship of the associated fuzzy social welfare function.

According to Lemma 2, there exists a fuzzy social welfare function, h, satisfying $FIIA$, FPO, and FPR. Therefore, h is fuzzy dictatorial according to Dutta's impossibility theorem. Now, we establish that a dictator of h is an ℓ-dictator of g, for $\ell \in \{1, 2, 3\}$. Let $d \in N$ be the dictator of h, *i.e.* for all $\mathfrak{R}_N \in \mathcal{D}^n$, and for all distinct $x_j, x_k \in X$, $P_d(x_j, x_k) > 0$, then $\widehat{P}(x_j, x_k) > 0$.

- $\ell = 1$
  Consider that $g(R_N, X) = x_j$, we have to prove that for all $x_k \in X$, $R_d(x_j, x_k) \geq R_d(x_k, x_j)$. We have for all $x_k \in X$, $g(R_N, \{x_j, x_k\}) = x_j$, since $g(R_N, X) = x_j$. Thus, $\widehat{R}(x_j, x_k) \geq \widehat{R}(x_k, x_j)$.

  If $\widehat{R}(x_j, x_k) > \widehat{R}(x_k, x_j)$, then $\widehat{P}(x_j, x_k) > 0$. Thus, for all $x_k \in X, R_d(x_j, x_k) \geq R_d(x_k, x_j)$ and d is 1-dictator.

  If $\widehat{R}(x_j, x_k) = \widehat{R}(x_k, x_j)$, then for all $i \in N, R_i(x_j, x_k) = R_i(x_k, x_j)$. Thus, for all $x_k \in X, R_d(x_j, x_k) \geq R_d(x_k, x_j)$ and d is 1-dictator.

- $\ell = 2$
  Consider that $g(R_N, X) = x_j$, we have to prove that for all $x_k \in X$, $R_d(x_j, x_k) \geq \alpha$, for $\alpha \in]0, \frac{1}{2}]$.
  We have for all $x_k \in X, g(R_N, \{x_j, x_k\}) = x_j$, since $g(R_N, X) = x_j$. Thus, $\widehat{R}(x_j, x_k) \geq \widehat{R}(x_k, x_j)$. Therefore, for all $x_k \in X, \widehat{R}(x_j, x_k) \geq \alpha$, since $\alpha \leq \frac{1}{2}$.

- $\ell = 3$
  Consider that $g(R_N, X) = x_j$, we have to prove (by induction) that for all $x_k \in X$, $d(R_d, X)(x_j) \geq d(R_d, X)(x_k)$.
  It should be remarked that for all $x_k \in X, g(R_N, \{x_j, x_k\}) = x_j$, since $g(R_N, X) = x_j$. Thus, $\widehat{R}(x_j, x_k) \geq \widehat{R}(x_k, x_j)$.

  - Let $X = \{x_1, x_2, x_3\}$. Consider that x_1 is $g(\mathfrak{R}_N, X)$. We have to prove that $d(R_d, X)(x_1) \geq d(R_d, X)(x_3)$ and $d(R_d, X)(x_1) \geq d(R_d, X)(x_2)$.

    To prove the first inequality, three cases are considered. The proof is based on the fact that (see Dasgupta and Deb, 1996), if for all $x_j, x_t, x_q \in X$,

$$\begin{cases} \widehat{R}(x_j, x_k) > \widehat{R}(x_k, x_j), \\ \widehat{R}(x_k, x_t) > \widehat{R}(x_t, x_k), \end{cases}$$

then $R^s(x_j, x_t) \geq \min\{\widehat{R}(x_j, x_k), \widehat{R}(x_k, x_t)\}$ when $\widehat{R}$ satisfies the *max-min* transitivity.

1. $\widehat{R}(x_1, x_3) > \widehat{R}(x_3, x_2) > \widehat{R}(x_3, x_1)$

   Since
   $$\widehat{R}(x_1, x_2) \geq \min\{\widehat{R}(x_1, x_3), \widehat{R}(x_3, x_2)\} = \widehat{R}(x_3, x_2),$$
   $$d(X, \widehat{R})(x_1) = \min\{\widehat{R}(x_1, x_3), \widehat{R}(x_1, x_2)\} \geq \widehat{R}(x_3, x_2)$$
   $$> \widehat{R}(x_3, x_1) = d(X, \widehat{R})(x_3)$$
   Then,
   $$d(X, \widehat{R})(x_1) \geq d(X, \widehat{R})(x_3).$$

2. $\widehat{R}(x_3, x_2) > \widehat{R}(x_1, x_3) > \widehat{R}(x_3, x_1)$.
   Since
   $$\widehat{R}(x_1, x_2) \geq \min\{\widehat{R}(x_1, x_3), \widehat{R}(x_3, x_2)\} = \widehat{R}(x_1, x_3),$$
   $$d(X, \widehat{R})(x_1) = \widehat{R}(x_1, x_3) \geq \widehat{R}(x_3, x_2)$$
   and
   $$\widehat{R}(x_3, x_2) > \widehat{R}(x_3, x_1) = d(X, \widehat{R})(x_3).$$
   Then,
   $$d(X, \widehat{R})(x_1) \geq d(X, \widehat{R})(x_3).$$

3. $\widehat{R}(x_1, x_3) > \widehat{R}(x_3, x_1) > \widehat{R}(x_3, x_2)$

   Since
   $$\widehat{R}(x_1, x_2) \geq \min\{\widehat{R}(x_1, x_3), \widehat{R}(x_3, x_2)\} = \widehat{R}(x_3, x_2),$$
   $$d(X, \widehat{R})(x_1) = \widehat{R}(x_1, x_3) \geq \widehat{R}(x_3, x_2) = d(X, \widehat{R})(x_3).$$
   Then,
   $$d(X, \widehat{R})(x_1) \geq d(X, \widehat{R})(x_3).$$

The same reasoning is used to prove that $d(X, \widehat{R})(x_1) \geq d(X, \widehat{R})(x_2)$ and we obtain that the property is verified when X contains only three elements.

- We suppose that d is a 3-dictator for g when X contains p elements and we show that d remains 3-dictator when X contains $(p+1)$ elements. Let $X = \{x_1, x_2, \ldots, x_p, x_{p+1}\}$. If $g(\mathfrak{R}_N, X) = x_1$, then $d(R_d, X)(x_1) = \min_{j \in 1,\ldots,p+1}\{R_d(x_1, x_j)\}$=$\min(d(R_d, X - \{x_{p+1}\})(x_1), R(x_1, x_{p+1}))$

  - $d(R_d, X - \{x_{p+1}\})(x_1) \leq R(x_1, x_{p+1})$

    $$\forall j \in \{1, \ldots, p\}, d(R_d, X)(x_1) = d(R_d, X - \{x_{p+1}\})(x_1)$$

    Then, $d(R_d, X)(x_1) \geq d(R_d, X - \{x_{p+1}\})(x_j)$.

  - $d(R_d, X - \{x_{p+1}\})(x_1) \geq R(x_1, x_{p+1})$. Thus, $d(R_d, X)(x_1) = R_d(x_1, x_{p+1})$. Suppose that there exist j such that $R_d(x_1, x_{p+1}) < d(R_d, X)(x_j)$. Thus, there exists k such that $R_d(x_j, x_k) > R_d(x_1, x_{p+1})$. We have $R_d(x_1, x_{p+1}) \geq \min(R_d(x_1, x_j), R_d(x_j, x_{p+1}))$. Since $R_d(x_1, x_{p+1})$ is $\min_k(x_1, x_k)$, $R_d(x_1, x_{p+1}) \geq R_d(x_j, x_{p+1})$. Thus, $R_d(x_j, x_k) \geq R_d(x_j, x_{p+1}))$. This leads to the contradiction with the assumption that $R_d(x_j, x_k) > R_d(x_1, x_{p+1})$. Therefore, d is ℓ-dictator for g.

For $\ell \in \{1, 2, 3\}$, we can conclude that g is ℓ-dictatorial. This implies the ℓ-dictatorship of ν.

□

Finally, we can conclude the generalization Gibbard's result for fuzzy preference orders as follows.

Consequence 1 (dictatorship of an ℓ-strategy-proof fuzzy social choice function) *Let ν be a fuzzy social choice function and $\ell \in \{1, 2, 3\}$. If ν is ℓ-strategy-proof, then it is ℓ-dictatorial.*

Proof
According to Lemma 1, since ν is ℓ-strategy-proof, it can be associated to an ℓ-straightforward fuzzy game form g. Therefore, g is ℓ-dictatorial because of Theorem 2. Consequently, the associated fuzzy social choice function ν is ℓ-dictatorial.

3 Conclusions

In this paper the fuzzy counterpart of *G-S* manipulability theorem was established. The proof is strongly relied on Dutta's impossibility theorem established for fuzzy social welfare functions. The contribution of this paper can be viewed as an extension of Gibbard's work to the fuzzy framework. A possible avenue for future research is to deal with the extension of the G-S's theorem by the use of other impossibility fuzzy versions of Arrow's theorem.

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