Strategic Manipulation and Regular Decomposition of Fuzzy Preference Relations

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Abstract

Gibbard (1973) and Satterthwaite (1975) have shown independently that any non-dictatorial voting choice procedure is vulnerable to strategic manipulation, when individuals express their preferences trough weak relations on the set of alternatives. This paper extends their result to the case of fuzzy weak preference relations on the set of alternatives. For this purpose, the manipulability and the dictatorship properties of fuzzy social choice functions are stated in terms of a symmetric and regular component of individual fuzzy weak preference relations. The proof of the established result is done by induction on the number of individuals.

Key words: fuzzy preference relation, manipulability, fuzzy social choice functions.

1 Introduction

A voting choice procedure is known to be subject to strategic manipulation when an individual reveals a non-sincere preference relation in order to change the social choice in his favor. Gibbard (1973) and Satterthwaite (1975) (henceforth G-S) proved, independently, that any non-dictatorial voting choice procedure is subject to manipulability whenever the set of alternatives contains at least three candidates. It was assumed that individuals can express their preferences trough weak preference relations on the set of alternatives. It is also well-known that, in many situations individuals have some difficulties to express clearly their preferences on the set of alternatives but they can, however, specify a preference degree for each ordered pair of alternatives , *i.e.*, they can express their preferences trough fuzzy weak preference relations on the set of alternatives (Fodor and Roubens, 1994). Therefore, the choice of a single alternative requires the use of a fuzzy social choice function (*FSCF*).

In the literature (*e.g.* Barrett, et al. 1990; Garcia-Lapresta and Llamazares, 2000), there exist two ways to deal with an *FSCF*. The first one is based on the rule "aggregation-and-defuzzification". It consists of applying to a preference profile a fuzzy social welfare function that leads to a social fuzzy relation, and then of generating, from the comprehensive fuzzy relation, a collective choice by applying a choice function. The second one makes use of the rule "defuzzification-and-aggregation", and consists of applying a choice function that generates on the basis of each individual fuzzy relation his best alternative set, and then by using a voting choice procedure the social choice is obtained on the basis of the individual choices.

This paper deals with the strategic manipulation of *FSCF*s whenever individuals have fuzzy weak preference relations on the set of alternatives. The basic idea is to specify the asymmetric component of a fuzzy individual weak preference relation to define the manipulability and dictatorship of an *FSCF*. In fact, we explore the ways for the decomposition of a fuzzy weak preference relation into a symmetric component and an asymmetric component as in Dutta (1987), Richardson (1994) and Fono and Andjiga (2005). We are particularly interested in the more general one as it was proposed by Fono and Andjiga (2005). These authors worked on a certain type of fuzzy weak relations satisfying a max-$\star$-transitivity, where $\star$ is a t-norm. Given a fuzzy weak preference, we associate to each alternative a score equal to the cardinal of the subset of alternatives with a null strict preference degree over the considered alternative. Therefore, an individual can manipulate an *FSCF* if there exists a fuzzy relation securing him an outcome with a greater cardinal score than the one of the sincere social choice.

The paper is organized as follows. Section 2 presents the main concepts, definitions, and notation. Section 3 presents the different definitions of manipulability and dictatorship of an *FSCF*. Section 4 establishes the impossibility result regarding the strategy-proofness of fuzzy social choice functions. The last section provides concluding remarks.

2 Concepts: Definitions and notation

This section presents the basic concepts and properties of fuzzy operators and fuzzy relations. In addition, it provides a review of how to decompose a fuzzy relation into a symmetric component and an asymmetric one.

2.1 Mathematical preliminaries

Given a finite set of *alternatives*, $X = \{x, y, z, \ldots\}$ with $|X| \geq 3$, fuzzy binary relations can be introduced to model the vagueness or fuzzy aspects of preferences. These fuzzy relations can be defined as fuzzy sets in the two-dimensional cartesian product, $X^2 = X \times X$ with a membership function, R.

Consider a finite set of *individuals*, $N = \{1, 2, \ldots, i, \ldots, n\}$, the *social choice problem* consists of finding the best alternative in X according to the preferences of all individuals in N. The best alternative is also called the *social choice*. It is assumed here that individuals express their preferences as fuzzy relations on the set X. The formal definitions of fuzzy operators and fuzzy binary relations as well as some of their fundamental properties are introduced next as in Fono and Andijga (2005).

Definition 1 (t-norm) *A t-norm is a function* $\star : [0,1] \times [0,1] \to [0,1]$ *satisfying the following properties for all* $x, y, z, u \in [0,1]$*:*

1. $x \star 1 = x$,
2. $x \star y \leq u \star z$ *if* $x \leq u$ *and* $y \leq z$,
3. $x \star y = y \star x$,
4. $(x \star y) \star z = x \star (y \star z)$.

Definition 2 (t-conorm) *A t-conorm is a function* $\oplus : [0,1] \times [0,1] \to [0,1]$ *satisfying the following properties for all* $x, y, z, u \in [0,1]$*:*

1. $x \oplus 0 = x$,
2. $x \oplus y \leq u \oplus z$ *if* $x \leq u$ *and* $y \leq z$,
3. $x \oplus y = y \oplus x$,
4. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.

Definition 3 (quasi-inverse of a t-norm) *Let* $\star$ *be a continuous t-norm. The quasi-inverse of* $\star$ *is the internal composition law denoted by* $\|$ *and defined for all* $x, y \in [0,1]$ *as follows:*

$$x \| y = \max\{t \in [0,1], x \star t \leq y\}.$$

Definition 4 (quasi-subtraction of a t-conorm) *Let* $\oplus$ *be a continuous t-conorm. The quasi-subtraction of* $\oplus$ *is the internal composition law denoted by* $\ominus$ *and defined for all* $x, y \in [0,1]$ *as follows:*

$$x \ominus y = \min\{t \in [0,1], x \oplus t \geq y\}.$$

Definition 5 (strict t-conorm) *A t-conorm* $\oplus$ *is strict if for all* $x, y \in [0,1], \forall\, z \in [0,1[$*, with* $x < y$*, then* $x \oplus z < y \oplus z$.

Example 1

1. *Let* $\star_Z$ *and* $\oplus_Z$ *denote the Zadeh's min t-norm and the Zadeh's max t-conorm respectively, i.e., for all* $x, y \in [0,1], x \star_Z y = \min\{x, y\}$ *and* $x \oplus_Z y = \max\{x, y\}$*. The quasi-subtraction of* $\oplus_Z$ *is defined as follows:*

   for all $x, y \in [0,1], x \ominus_Z y = x$*, if* $x > y$*; and 0, otherwise.*

2. *Let* $\star_L$ *and* $\oplus_L$ *denote the Lukasiewicz's t-norm and the Lukasiewicz's t-conorm respectively, i.e., for all* $x, y \in [0,1], x \star_L y = \max\{0, (x + y - 1)\}$ *and* $x \oplus_L y = \min\{1, (x + y)\}$*. The quasi-subtraction of* $\oplus_L$*, denoted by* $\ominus_L$*, is defined as follows:*

   for all $x, y \in [0,1], x \ominus_L y = \max\{0, (y - x)\}$.

Definition 6 (fuzzy binary relation)

A fuzzy binary relation (FBR) on X is a function $R : X^2 \to [0,1]$.

1. *R is reflexive, if for all $x \in X, R(x,x) = 1$,*
2. *R is connected, if for all $x, y \in X, R(x,y) + R(y,x) \geq 1$,*
3. *R is a fuzzy weak preference relation (FWPR) if it is reflexive and connected,*
4. *R is max-$\star$-transitive if for all $x, y, z \in X, R(x,z) \geq R(x,y) \star R(y,z)$.*

Remark 1 *Let R be an FWPR and $\star$ a t-norm.*

1. *For all $(x,y) \in X^2, R(x,y)$ is the degree to which x is at least as good as y,*
2. *R is a crisp binary relation if for all $x, y \in X, R(x,y) \in \{0,1\}$.*

Example 2

1. *The max-$\star_Z$-transitivity is known as the min-transitivity, which definition is as follows,*
$$\forall\ x, y, z \in X, R(x,z) \geq \min\{R(x,y), R(y,z)\},$$
2. *The max-$\star_L$-transitivity is called the L-transitivity, which definition is as follows,*
$$\forall\ x, y, z \in X, R(x,z) \geq R(x,y) + R(y,z) - 1.$$

Definition 7 (symmetry and asymmetry) *Let R be a crisp preference relation. R is said to be*

1. *symmetric, if for all $x, y \in X, R(x,y) = R(y,x)$,*
2. *asymmetric, if for all $x, y \in X, R(x,y) \wedge R(y,x) = 0$.*

2.2 Decomposing fuzzy binary relations

A crisp weak preference relation R can be decomposed into a crisp indifference relation I and a crisp strict preference relation P, $R = P \cup I$, if and only if

$$\begin{cases} I \textit{ is symmetric}, \\ P \textit{ is asymmetric}, \\ I \cap P = \emptyset. \end{cases}$$

This decomposition is unique (*e.g.* De Beats *et al.*, 1995; Llamazares, 2005). When R is an $FWPR$, there are many decompositions of R into a symmetric component, I, and an asymmetric component, P. Let us recall two of them:

1. *Generic decomposition*: Richardson (1994) uses a generic t-conorm $\oplus$ to model a fuzzy union and replaces the condition "$I \cap P = \emptyset$" by the following one:

"P is simple, i.e., $\forall\, x, y \in X, R(x,y) = R(y,x) \Rightarrow P(x,y) = P(y,x)$."

The author states that if R, I, and P are fuzzy relations satisfying the following properties,

1. $\forall\, x, y \in X, R(x,y) = P(x,y) \oplus I(x,y)$,
2. P is asymmetric and I is symmetric,
3. P is simple,

then $\forall\, x, y \in X$,

$$\begin{cases} I(x,y) = R(x,y) \wedge R(y,x), \\ P \text{ is regular, i.e., } \forall\, x, y \in X, R(x,y) \leq R(y,x) \Rightarrow P(x,y) = 0. \end{cases}$$

But this decomposition does not stipulate how one can obtain the value of $P(x,y)$ when $P(x,y) > 0$. The answer to this question is given by Fono and Andjida (2005).

2. *Regular decomposition*: Fono and Andjida (2005), determine for a given t-conorm $\oplus$, a class of regular fuzzy strict components of a certain *FWPR*, R. Each class has a minimal element called the minimal regular strict preference P_R associated with $\oplus$. It is defined as follows:

Definition 8 (minimal regular fuzzy strict component) *Let $\oplus$ be a continuous t-conorm, $\ominus$ be its quasi-substraction, and R be an $FWPR$. The minimal regular fuzzy strict component P_R associated with $\oplus$ is defined as follows:*

$$\forall\, x, y \in X, P_R(x,y) = R(x,y) \ominus R(x,y)$$

Proposition 1 (Fono and Andjida, 2005)

If R be an $FWPR$, and I and P be two fuzzy binary relations. If $\oplus$ is a strict t-conorm or the Zadeh's max t-conorm, then the two following statements are equivalent:

1. *I and P are, "the fuzzy indifference of R" and "the fuzzy strict preference of R", respectively, i.e., R, I, and P verify Richardson's properties (a), (b), and (c).*

2. *for all* $x, y \in X$, $\begin{cases} (i)\ I(x,y) = R(x,y) \wedge R(y,x), \\ (ii)\ P(x,y) = P_R(x,y). \end{cases}$

Example 3 (Dutta, 1987; Richardson, 1998)

1. *If* $\oplus$ *is the Zadeh's max t-conorm, then for all* $x, y \in X$,

$$P_R(x,y) = \begin{cases} R(x,y), & \text{if } R(x,y) > R(y,x)\ ; \\ 0, & \text{otherwise.} \end{cases}$$

2. *If* $\oplus$ *is the Lukasiewicz's t-conorm, then for all* $x, y \in X$,

$$P_R(x,y) = \max\{0, (R(x,y) - R(y,x))\}.$$

Remark 2 (Fono and Andjida, 2005)

1. *If* R *is a crisp relation, then for any t-conorm* $\oplus$*, the minimal regular fuzzy strict preference* P_R *of* R *associated with* $\oplus$ *becomes the crisp preference of* R *defined by* $\forall\ x, y \in X, xP_Ry \Leftrightarrow (xRy \text{ and } not(yRx))$.
2. *For any* $x, y \in X$*, such that* $R(x,y) > R(y,x)$*, the real value* $P_R(x,y)$ *is the degree to which* x *is strictly preferred to* y.
3. *For any* $x, y \in X$*, such that* $R(x,y) = R(y,x)$*, then* x *is equivalent to* y *with degree* $I(x,y)$.

Now, a certain type of transitivity can be introduced as follows.

Definition 9 (pos-transitivity)

Let $\oplus$ *be a continuous t-conorm,* $\ominus$ *be its quasi-substraction and* R *be an* $FWPR$*. The minimal regular fuzzy strict component* P_R *associated with* $\oplus$ *is said to be pos-transitive, if*

$$\text{for all } x, y, z \in X, (P_R(x,y) > 0 \text{ and } P_R(y,z) > 0) \Rightarrow P_R(x,z) > 0.$$

The pos-transitivity means that: if x is strictly preferred to y and y is strictly preferred to z, then x is strictly preferred to z. Let now consider the following condition.

Condition (φ-condition)
Let $\star$ be a t-norm and R be a max-$\star$-transitive $FWPR$. Consider $\alpha^*(x,y,z) = R(x,y) \star R(y,z)$, and $\beta^*(x,y,z) = \min\{(R(y,z)\|R(y,x)), (R(x,y)\|R(z,y))\}$. R satisfies the φ-condition, if for all $x, y, z \in X$,

$$R(x,y) > R(y,x) \text{ and } R(y,z) > R(z,y) \text{ imply}$$
$$\left(\left\{ \begin{array}{l} R(x,z) \in [\alpha^*, \beta^*] \\ R(z,x) \in [\alpha^*, \beta^*] \end{array} \right\} \Rightarrow R(x,z) > R(z,x) \right).$$

The following proposition characterizes the set of *FWPR*s composed of pos-transitive minimal regular fuzzy strict components.

Proposition 2 (Fono and Andjida, 2005)

Let $\star$ be a t-norm and R be a max-$\star$-transitive $FWPR$.

$$P_R \textit{ is pos-positive} \Leftrightarrow R \textit{ satisfies the } \varphi\textit{-condition.}$$

In what follows, we will assume that a fuzzy strict preference of a given *FWPR* is defined by any regular P and $\oplus$ is a strict t-conorm or the Zadeh's max t-conorm.

3 Fuzzy social choice functions

This section introduces the fundamental definition of fuzzy social choice functions. A definition of the manipulation of fuzzy social choice functions and its dictatorship are also presented here.

Definition 10 (fuzzy social choice function)

Let $\mathcal{R}_N = (R_1, R_2, \ldots, R_i, \ldots, R_n)$ be a profile of individuals' preference relations. A fuzzy social choice function ($FSCF$) is a function that associates a single alternative (in X) to a profile of individuals' preference relations.

Consider the following additional notation,

- $H^\star$ is the set of *FWPR*s satisfying both max-$\star$-transitivity and the φ-condition.
- The elements of $H^\star$ are called $\star$-fuzzy orders and are denoted by R_i or $\overline{R}_i, i \in N$. R_i and $\overline{R}_i$ are considered to be distinct.
- $(\mathcal{R}_N \mid \overline{R}_i)$, is the profile of individuals' preference relations $(R_1, \ldots, R_{i-1}, \overline{R}_i, R_{i+1}, \ldots, R_n)$, where individual i declares the fuzzy preference relation $\overline{R}_i$ instead of R_i.
- P_{R_i} is the regular strict preference of R_i.

- $N_R(x)$ is the *cardinality* of the subset $\{y \in X \mid P_R(x,y) > 0\}$ and it is called the score of x on the basis of R.

- $g_R : X \longrightarrow \{0, \ldots, q-1\}$ is a mapping that associates to each alternative x its $N_R(x)$, where $q = |X|$.

- f_R is an alternative such that $\max_{y \in X}\{g_R(y)\} = g(f_R)$.

- s_R is an alternative such that $g_R(s_R)$ is the second greater value in $\{g_R(y), y \in X\}$.

Definition 11 ($\star$-fuzzy social choice function)

Let $\star$ be a t-norm. A $\star$-fuzzy social choice function ($\star$-FSCF) is an FSCF such that the profiles of individuals' preference relations belong to $(H^\star)^n$.

Example 4

Consider the following illustrative example with $X = \{a, b, c\}$ and $N = \{1, 2, 3\}$. The relations, R_i, for $i \in \{1, 2, 3\}$ belong to $H^\star$, where $\star$ is the Zadeh's min t-norm. They are presented in the following tables.

R_1	a	b	c
a	1	0.7	0.8
b	0.5	1	0.6
c	0.5	0.5	1

R_2	a	b	c
a	1	0.8	0.7
b	0.4	1	0.4
c	0.4	0.6	1

R_3	a	b	c
a	1	0.4	0.7
b	0.8	1	0.7
c	0.4	0.4	1

Consider the Zadeh max t-conorm. For $i \in \{1, 2, 3\}$, each R_i can be decomposable according to Proposition 1. One can observe that each P_i is pos-transitive, for $i \in \{1, 2, 3\}$. In addition, according to R_3, $N_{R_3}(a) = |\{c\}| = 1$, $N_{R_3}(b) = |\{a, c\}| = 2$ and $N_{R_3}(c) = |\emptyset| = 0$. Thus, $g_{R_3} : x \in X \mapsto N_{R_3}(x)$, $f_{R_3} = b$, and $s_{R_3} = a$. Now let us apply the arithmetic mean function as an $FSWF$ to obtain the fuzzy social relation R^s. Then, we consider $C(X, R^s) = \{x \in X : R^s(x,y) > R^s(y,x)\} = \{a\}$ as the social choice. Such an $FSCF$ can be viewed as a $\star$-FSCF, where $\star = \min$.

R^s	a	b	c
a	1	0.63	0.73
b	0.56	1	0.56
c	0.43	0.5	1

Consider the context where a profile of individuals' preference relations has to be expressed in $(H^\star)^n$, and the social choice is obtained by using a $\star$-*FSCF*, $\nu^\star$. Suppose that an individual $m \in N$, with a sincere preference relation, R_m, knows the *(n-1)* preference relations declared by the remaining individuals and the $\star$-*FSCF*, $\nu^\star$. Therefore, he can anticipate the outcome $\nu^\star(\mathcal{R}_N)$. The question to be asked is when individual m will be motivated to change the social choice $\nu(\mathcal{R}_N)$. The answer to this question is to check if there exists a social choice with a greater score than the one that provides the outcome $\nu(\mathcal{R}_N)$ according to his sincere *FWPR*. If there exists a binary relation, $\overline{R}_m$, in $H^\star$, such that the outcome $\nu(\mathcal{R}_N \mid \overline{R}_m)$ has a greater score than the one of the outcome $\nu(\mathcal{R}_N)$, the individual m can manipulate the $\star$-*FSCF*, $\nu^\star$, by revealing $\overline{R}_m$. Consequently, the manipulability of a $\star$-*FSCF* can formally be introduced as follows. We provide at the same time the definition of dictatorship and strategy-proofness.

Definition 12 (manipulability, dictatorship, and strategy-proofness)

Let $\star$ be t-norm and $\nu^\star$ be a $\star$-$FSCF$.

1. *The function $\nu^\star$ is said to be manipulable by the individual m at $\mathcal{R}_N \in (H^\star)^n$ via $\overline{R}_m \in H^\star$ if $N_{R_m}(\nu^\star(\mathcal{R}_N \mid \overline{R}_m))) > N_{R_m}(\nu^\star(\mathcal{R}_N))$.*
2. *The function $\nu^\star$ is said to be dictatorial if there exists $d \in N$ such that for every $\mathcal{R}_N \in (H^\star)^n$, if $\nu^\star(\mathcal{R}_N) = a$, then $N_{R_d}(a) \geq N_{R_d}(x), \forall\, x \in X$.*
3. *The function $\nu^\star$ is said to be strategy-proof, if $\nu^\star$ is not manipulable.*

Example 5

Let us go back again to the previous example. Individual 3 *can manipulate the* $\star$-FSCF. *Indeed, $N_{R_3}(b) > N_{R_3}(a)$. Therefore, he can reveal the non-sincere fuzzy relation R'_3 to obtain b as the social choice.*

R'_3	a	b	c
a	1	0.1	0.7
b	1	1	1
c	0.3	0.3	1

Finally, unanimity for $\star$-$FSCF$s can be introduced as follows.

Definition 13 (unanimity)

Let $\nu^\star$ be a $\star$-$FSCF$. $\nu^\star$ satisfies unanimity if for all outcomes x and profiles $\mathcal{R}_N$ such that all individuals i, $i \in N$ have the same alternative f_{R_i}, then $\nu^\star(\mathcal{R}_N) = x$.

4 Impossibility result

The following theorem presents an extension of *G-S* fundamental result.

Theorem (impossibility result)
Let $\star$ be t-norm and $\nu^\star$ be a $\star$-FSCF. If $\nu^\star$ is strategy-proof, then it is dictatorial.

The proof proceeds by induction on the number of individuals as in Sen (2001).

Step 1. This step consists of establishing the theorem in the case where $N = \{1, 2\}$. Consider a strategy-proof $\star$-*FSCF*, $\nu^\star$. Firstly, we will prove a first statement: for a given profile $\widetilde{\mathcal{R}}_N = (R_1, R_2)$ the outcome of $\nu^\star$ must be an element of the set $\{f_{R_1}, f_{R_2}\}$. Secondly, we will show that if the first statement holds for one profile, then it holds for any profile in $(H^\star)^2$.

1. Fix a profile $\widetilde{\mathcal{R}}_N = (R_1, R_2) \in (H^\star)^2$. We prove that if $f_{R_1} \neq \nu^\star(\widetilde{\mathcal{R}}_N)$, then $f_{R_2} = \nu^\star(\widetilde{\mathcal{R}}_N)$.

   Suppose that $\nu^\star(R_1, R_2) = c$, $f_{R_1} = a, f_{R_2} = b$, and c is distinct from a and b. Note that a and b must be distinct from each other, otherwise we immediately contradict unanimity. Let $\overline{R}_2$ be a $\star$-fuzzy order with $f_{\overline{R}_2}$ and $s_{\overline{R}_2}$ correspond to b and a, respectively.

   - Observe that $\nu^\star(R_1, \overline{R}_2)$ cannot be equal to b. In fact, if $\nu^\star(R_1, \overline{R}_2) = b$, then $N_{R_2}(\nu^\star(R_1, \overline{R}_2)) > N_{R_2}(\nu^\star(R_1, R_2))$, since $\nu^\star(R_1, \overline{R}_2) = b$ corresponds to $f_{\overline{R}_2}$. Thus, $\nu^\star$ is manipulable at $\mathcal{R}_N$ via $\overline{R}_2$. Therefore, $\nu^\star(R_1, \overline{R}_2)$ must be different from b.
   - Let $\nu^\star(R_1, \overline{R}_2) = x$. Consider that the alternative x is distinct form a and b. We have $N_{\overline{R}_2}(a) > N_{\overline{R}_2}(x)$ and $\nu^\star$ would manipulate at $(R_1, \overline{R}_2)$ via a relation R with N_R corresponding to alternative a. The outcome would then be a because of the unanimity of $\nu^\star$. Therefore, $\nu^\star(R_1, \overline{R}_2) = a$.
   - Let $\overline{R}_1$ be a $\star$-fuzzy order with $f_{\overline{R}_1}$ and $s_{\overline{R}_1}$ corresponding to a and b, respectively. We must have $\nu^\star(\overline{R}_1, \overline{R}_2) = a$, otherwise individual 1 manipulates at $(\overline{R}_1, \overline{R}_2)$ via $\overline{R}_1$.
   - Let $\nu^\star(\overline{R}_1, R_2) = x$. If $x = b$, then individual 2 manipulates at $(\overline{R}_1, \overline{R}_2)$ via R_2. If x is distinct from both a and b, then

$N_{\overline{R}_1}(x) < N_{\overline{R}_1}(b)$. Therefore, individual 1 will manipulate at $(\overline{R}_1, R_2)$ via a relation R with N_R corresponding to alternative b. Therefore, $x = a$. But, then individual 1 manipulates at $\mathcal{R}_N$ via $\overline{R}_1$.

2. Now, show that if $\nu^\star(\mathcal{R}_N) = x$, with $f_{R_1} = x$ or $f_{R_2} = x$, for a given $\widetilde{\mathcal{R}}_N$, then it is holds for any $\mathcal{R}_N$.

   Let $\mathcal{R}_N$ be a profile where $f_{R_1} = a$, $f_{R_2} = b$, and $a \neq b$.

   - Holding that individual's 2 preference relation fixed at R_2, observe that the outcome for all profiles where $s_{\overline{R}_1} = a$, must be a. Otherwise, individual 1 manipulates $(\overline{R}_1, R_2)$ via R_1.
   - Holding that 1's preference relation fixed at R_1, observe that the individual 2 can never obtain outcome b by varying R_2. According to the point 1, it follows that the outcome must be either a or b.
   - Consider an arbitrary outcome c distinct from both a and b. and c be s_{R_1}. Let $\overline{R}_1$ be a $\star$-fuzzy order where c and a are $f_{\overline{R}_1}$ and $s_{\overline{R}_1}$, respectively. According to the point 1., it follows that $\nu^\star(\overline{R}_1, R_2)$ is either b or c. However, if it is b, individual 1 would manipulate at $(\overline{R}_1, R_2)$ via R_1. Therefore, the outcome is $f_{\overline{R}_1}$.
   - The proof is completed by showing that the outcome is f_{R_1} irrespective of f_{R_2}. Pick an arbitrary outcome x distinct from b and c. Consider that $b = f_{R_2}$ and $x = s_{R_2}$. Let $\overline{R}_2$ be a $\star$-order, where $s_{\overline{R}_2} = x$ and $f_{\overline{R}_2} = b$. Note that $\nu^\star(\overline{R}_1, \overline{R}_2)$ must be either c or x. But if it is x then individual 2 will manipulate at $(\overline{R}_1, R_2)$ via $\overline{R}_2$. Since x and c were picked arbitrarily, 2. is established.

Step 2. Let $n \geq 3$. Consider the following two statements

Statement (a) : for all k with $k \leq n$, if $\nu^\star : (H^\star)^k \to X$ is strategy-proof, then f is dictatorial.

Statement (b) : if $\nu^\star : (H^\star)^n \to X$ is strategy-proof, then $\nu^\star$ is dictatorial.

We will show that statement (a) implies statement (b).

Assume that statement (a) holds. Let $\nu^\star$ be strategy-proof $\star$-FSCF $\nu^\star$: $(H^\star)^n \to X$. Define a $\star$-$FSCF$ $\mu : (H^\star)^{n-1} \to X$ as follows. For all

$(R_1, R_3, \ldots, R_n) \in (H^\star)^{n-1}, \mu(R_1, R_3, \ldots, R_n) = \nu(R_1, R_1, R_3, \ldots, R_n)$. Since ν satisfies unanimity, μ satisfies unanimity as well. Note that μ is strategy-proof. Otherwise, ν is manipulable. Pick an arbitrary $n-1$ individual profile $(R_1, R_1, R_3, \ldots, R_n)$ and let $\mu(R_1, R_3, \ldots, R_n) = \nu(R_1, R_1, R_3, \ldots, R_n) = a$. Let $\overline{R}_1$ be an arbitrary $\star$-fuzzy order. Let $\nu(\overline{R}_1, R_1, R_3, \ldots, R_n) = b$ and $\nu(\overline{R}_1, \overline{R}_1, R_3, \ldots, R_n) = \mu(\overline{R}_1, R_3, \ldots, R_n) = c$. Since ν is strategy-proof, $a \neq b$ implies $N_{R_1}(a) > N_{R_1}(b)$, $c \neq b$ implies $N_{R_1}(b) > N_{R_1}(c)$. Since P_{R_1} is pos-transitive, $a \neq c$, implies $N_{R_1}(a) > N_{R_1}(c)$. Therefore, μ cannot be manipulated by individual 1. Since μ satisfies unanimity and it is strategy-proof, statement (a) implies that μ is dictatorial. There are two cases to consider.

- Suppose that the dictator say, individual j, is one of the individuals from 3 through N. We will prove that j is a dictator for ν.

  Pick an arbitrary profile $(R_1, R_2, R_3, \ldots, R_n)$. Let a be s_{R_j} and let $\nu(R_1, R_2, R_3, \ldots, R_n) = b$. Since j dictates in μ, individual 1 can change the outcome from b in the profile $(R_1, R_2, R_3, \ldots, R_n)$ to a by announcing R_2. Since ν is strategy-proof, we must have $N_{R_1}(b) > N_{R_1}(a)$. Similarly, since $\nu(R_1, R_1, R_3, \ldots, R_n) = a$, we must have $N_{R_1}(a) > N_{R_1}(b)$, or else individual 2 will manipulate at $(R_1, R_1, R_3, \ldots, R_n)$ via R_2. Thus, we have $a = b$. Therefore, $\nu(R_1, R_2, R_3, \ldots, R_n) = a = s_{R_j}$. This returns that j dictates in ν.

- Finally, we need to consider the case where j is individual 1 in μ. Pick arbitrary $n-2$ individual profile $(R_3, R_4 \ldots, R_n)$. Now define a two individual $\star$-$FSCF$ λ as follows: for all pairs of $\star$-fuzzy orders R_1, R_2, $\lambda(R_1, R_2) = \nu(R_1, R_1, R_3, \ldots, R_n)$. Since individual 1 is a dictator in μ, it follows that λ satisfies unanimity. Moreover, since ν is strategy-proof, it follows immediately that λ is strategy-proof too. From step 1, we know that λ is strategy-proof, i.e., λ is dictatorial. In order to complete the proof, we need only to show that the identity of the dictator does not depend on the $n-2$ profile $(R_3, R_4, \ldots, R_n)$ while 2 is dictator for $(\overline{R}_3, \overline{R}_4, \ldots, \overline{R}_n)$. Now, progressively change preferences for each individual from 3 through n from the first profile to the second. There must be an individual j for $3 \leq j \leq n$ such that 1 is the dictator in $(\overline{R}_3, \ldots, \overline{R}_{j-1}, R_j, \ldots, R_n)$ while 2 dictates in $(\overline{R}_3, \ldots, \overline{R}_{j-1}, \overline{R}_j, R_{j+1}, \ldots, R_n)$. Let a and b be such that $N_{R_j}(a) > N_{R_j}(b)$. Pick R_1 and R_2 such that $b = f_{R_1}$ and $a = f_{R_2}$, respectively. Then, $\nu(R_1, R_2, \overline{R}_3, \ldots, \overline{R}_{j-1}, R_j, \ldots, R_n) = b$ while $\nu(R_1, R_2, \overline{R}_3, \ldots, \overline{R}_{j-1}, \overline{R}_j, R_{j+1}, \ldots, R_n) = a$. Clearly j will manipulate at $(R_1, R_2, \overline{R}_3, \ldots, \overline{R}_{j-1}, R_j, R_{j+1}, \ldots, R_n)$ via $\overline{R}_j$. This completes the proof of step 2.

Since the result is trivially true in the case of $n = 1$, steps 1 and 2 complete the proof of the theorem.

$\square$

Concluding Remarks

This paper generalizes to the fuzzy context the well-known result of *G-S* on the manipulability of crisp social choice functions. The paper shows how an individual can manipulate a social choice even if the preferences of the individuals are fuzzy. A new definition of the fuzzy manipulability and dictatorship of fuzzy social choice functions was given by considering the decomposition of weak fuzzy individual preference relations. A future research avenue is to consider other types of fuzzy relation decompositions (De Beats *et al*, 1995).

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