Probabilistic approach to ship operational safety

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Probabilistic approach to ship operational safety accounting for uncertainties
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My loved ones, for being the root of my growth.
ABSTRACT

In ship operations it is essential to evaluation of the safety of the operating vessel, particularly when hazardous environmental conditions arise, and those aspect is commonly tackled by decision support system which must considered the epistemic and aleatory uncertainties arising from the input data.

In this thesis work it is presented the framework of a ship mission performance assessment for giving guidance to the Decision Maker both for sudden onboard decision and for planning the mission on land. The assessment is performed through the estimation of the probability of exceedance of given operational limits with numerical (Monte Carlo, Direct Integration) or approximated (FOSM, FORM, Maximum Log-Likelihood) methods, the main differences due to the implementation of the different methods are investigated. Furthermore, three main application of the methods are presented: it is illustrated the assessment of the failure probability of an operating ship for different scenarios with an increasing number of uncertain variables and it is performed the choice both of the safest route for a containership given some alternatives, and of the safest area to operate for a fishing vessel given the forecasted data for a whole ocean area.

Keywords:
Probability theory, Reliability Methods, Seakeeping hazards, Ship operations, Uncertainties.
RESUMO

Em operações com navios, é essencial avaliar a segurança das operação envolvidas, particularmente quando condições ambientais adversas surgem e esses aspectos são comumente tratados por sistemas de suporte à decisão, que precisam considerar as incertezas provenientes da coleta de dados.

Nesta tese, é apresentado a estruturação da avaliação de performance de uma missão naval para prover diretrizes aos tomadores de decisão, seja em situações excepcionais à bordo ou situações de planejamento da missão. Essa avaliação é feita através da estimativa de probabilidade de certos limites operacionais serem excedidos, através de métodos numéricos (Monte Carlo, Direct Integration) e métodos aproximados (FOSM, FORM, Maximum Log-Likelihood). As principais diferenças devido à implementação dos diferentes métodos são investigadas. Além disso, três principais aplicações do método desenvolvido são apresentadas: avaliações da probabilidade de falha de um navio em operação para diferentes cenários com um número crescente de variáveis aleatórias, análise de rota mais segura para um navio contentor e também análise de região mais segura de operação de um navio pesqueiro, dados a previsão do tempo para uma área geográfica.

**Palavras-Chaves:**

Incertezas, Métodos de confiabilidade, Operações navais, Teoria de probabilidade.
Declaração / Statement

Declaro que o presente documento é um trabalho original da minha autoria e que cumpre todos os requisitos do Código de Conduta e Boas Práticas da Universidade de Lisboa.

I declare that this document is an original work of my own and that it fulfils all the requirements of the Code of Conduct and Good Practices of the Universidade de Lisboa.

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<tr>
<th>Acronym</th>
<th>Full Form</th>
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<tr>
<td>CENTEC</td>
<td>Centre for Marine Technology and Ocean Engineering</td>
</tr>
<tr>
<td>FMEA</td>
<td>Failure Mode and Effect Analysis</td>
</tr>
<tr>
<td>FORM</td>
<td>First Order Reliability Method</td>
</tr>
<tr>
<td>FOSM</td>
<td>First Order Second Moment Method</td>
</tr>
<tr>
<td>ISO</td>
<td>International Organization for Standardization</td>
</tr>
<tr>
<td>RAO</td>
<td>Response Amplitude Operator</td>
</tr>
<tr>
<td>TANA</td>
<td>Two-point Adaptive Nonlinear Approximations</td>
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SYMBOLOGY:

\( H_s \)  Significant wave height

\( H_s^* \)  Significant wave height at the checking point

\( P_f \)  Failure probability

\( S_R \)  Response spectrum

\( T_m \)  Waves mean period

\( T_m^* \)  Waves mean period at the checking point

\( T_p \)  Wave peak period

\( X_i \)  Random variable

\( a_{ij} \)  Nataf transformation coefficients

\( e_{p_i} \)  Elasticity of a generic parameter \( p_i \)

\( g_{\text{acc}} \)  Acceleration of gravity

\( s_i \)  Mechanical sensitivity

\( z_j \)  Complex notation of the ship harmonic motion

\( \alpha_i \)  Random variable directional cosines

\( \beta_{\text{generalized}} \)  Generalized reliability index

\( \beta_{\text{geometric}} \)  Geometric reliability index

\( \beta_{\text{simple}} \)  Simple reliability index

\( \theta_i \)  Motion phase angle respect to the wave motion

\( \mu_{X_i} \)  Random variable mean

\( \xi_i^a \)  Amplitude of the ship motion

\( \rho_{ji} \)  Correlation coefficient between the random variables \( X_i \) and \( X_j \)

\( \rho'_{ji} \)  Correlation coefficient between the random variables \( Y_i \) and \( Y_j \)

\( \sigma_{X_i} \)  Random variable standard deviation

\( \omega_e \)  Encounter frequency

\( \alpha_y \)  Vector of factors of importance of distribution parameters
\( \Delta T \) \quad \text{trim}

\( \Delta T^* \) \quad \text{Trim at the checking point}

\( \Phi \) \quad \text{Gaussian joint cumulative distribution function}

\( D \) \quad \text{Spreading function}

\( F \) \quad \text{Joint cumulative distribution function}

\( GM_T \) \quad \text{Transversal metacentric height}

\( GM_T^* \) \quad \text{Transversal metacentric height at the checking point}

\( I() \) \quad \text{Indicator function}

\( L \) \quad \text{Ship length}

\( N \) \quad \text{Number of simulations}

\( S \) \quad \text{Wave spectrum}

\( T \) \quad \text{Draft}

\( T^* \) \quad \text{Draft at the checking point}

\( f \) \quad \text{Joint probability density function}

\( g() \) \quad \text{Margin function}

\( o \) \quad \text{Omission factor}

\( v \) \quad \text{Ship speed}

\( v^* \) \quad \text{Ship speed at the checking point}

\( x, y, z \) \quad \text{Coordinates of the point of interest}

\( H \) \quad \text{Hessian matrix of } \ln f_X(X)

\( I \) \quad \text{Identity matrix}

\( J \) \quad \text{Jacobian matrix}

\( P' \) \quad \text{Correlation matrix of } Y

\( U \) \quad \text{Vector of gaussian, standardized and uncorrelated random variables}

\( X \) \quad \text{Vector of random variables}

\( Y \) \quad \text{Vector of gaussian, standardized and correlated random variables}

\( v \) \quad \text{Normal to the surface of } \ln f_X(X) \text{ at one point}
\( \chi \) Relative wave direction

\( \chi^* \) Relative wave direction at the checking point

\( \omega \) Wave frequency

\( \epsilon \) Shooman’s error

\( \phi \) Gaussian joint probability density function

\( \gamma \) Vector of factors of physical importance


1 INTRODUCTION

1.1 Background and Motivation

The challenging role of conducting a vessel along its mission can be assisted by a decision support system, relaying on numerical models, able to predict the operational profile of the ship and to return valuable information to the decision maker. On one hand, before the beginning of the mission the decision maker must define which predetermined path, characterized by potential environmental and ship service conditions, the vessel should take. On the other hand, also during the mission the decision maker may need to alter the operational profile of the vessel, for instance to avoid inclement weather.

The process of giving support to the decision maker is very often funded on a risk-based approach, which usually considers economic and social criteria while complying with safety requirements related to operational hazards predicted by the model. Within the scope of this thesis, navigational hazards related with seakeeping events will be considered, while other risks, e.g. of collision or grounding, will not be taken into account. Typical seakeeping events are related to excessive roll motion, green water (deck wetness), local acceleration and slamming. Such seakeeping events can be referred as “hazards”, since its consequences are considered not acceptable, and their evaluation is carried out through risk-based approaches. Within this context, the concept of risk is connected to accidental events of probabilistic nature; ISO31000 (ISO - The International Organization for Standardization, 2009) defines the term risk as “effect of uncertainties on objects”. This probabilistic risk approach provides a measure with respect to the probability of experiencing the specified adverse consequences, associated to a probability of failure, within a specified time interval. For operational purposes the time interval considered is the duration of a mission, while the failure is represented by the ship responses exceeding predefined thresholds, which can be expressed as limit states violation.

Since the models involved in supporting the decision making generally receive as input the ship characteristics and the environmental conditions, returning as an output the best alternative according to the imposed criteria, inevitable randomness is introduced by the input data and propagated in the final assessment. Variability, inaccuracy, lack of knowledge, degree of belief, are different connotation of uncertainties, but it is possible to classify them into two macro-categories: aleatory, when they are inherent uncertain, and epistemic, when they originate from lack of knowledge and/or data. The variable sea states along the ship mission are the most evident random source in the probabilistic analysis, their irregularity originates by the irregular natural process in which the solar energy is transformed into wave energy by winds. The oceans absorb a different net heat flux from the solar radiation according to several factors like the latitude, atmospheric composition and time of the day, hence, the related pressure gradients which generate waves are inevitably random. This physically inherent random fluctuations of the sea state variables represent the aleatory uncertainties of the assessment. Another source of uncertainties lies in the measurement methods to define physical quantities that express the ship characteristics. Moreover, the statistical descriptors involved in the assessment are subjected to a statistical uncertainty related to the incomplete information due to the sample size. Furthermore, model uncertainties arise when physical variables are neglected from the formulation of a relevant limit state because not known, when an idealized mathematical model is implemented or when the interplay with other variables is not considered or poorly modelled.
By considering the uncertainties, the stochastic approach provides more detailed information and a broader perspective of the system than the deterministic one. Even if the probability theory has gained the interest of the engineering community, it is still challenging to explicitly evaluate uncertainties in complex systems. Thus, thanks to computer calculations, numerical solutions are found for realistic problems of high complexity/large scale. Since another important requirement of an engineering assortment is to balance realism against operability, approximations are often introduced.

Different methods for the probabilistic assessment of the safety of a vessel involved in its mission, meant to give support to the decision maker, are introduced within this thesis work and their application is investigated in different scenarios.

1.2 Objectives

• Perform a Literature Review deepening how uncertainties are accounted in the risk assessment and in the decision support in ship operations.
• Study and identification of hazards in ship operations (seakeeping events), mathematical formulation of hazards through limit state functions and threshold values.
• Development of a procedure which returns the probability of exceedance of given operability limits, starting from the weather condition forecasted and the relevant ship characteristics, accounting for the sources of uncertainties in ship operations
• Implementation of the methodology defined previously for a given route and for its alternatives, identification of the best solution.
• Implementation of the methodology defined previously for a given ocean area meant for ranking the safety of a given ship within that geographical zone.

1.3 Structure of the Thesis

This thesis work is structured in five main chapters and two appendices. Chapter 1 introduces the topic investigated together with its background, it also describes the goals and organization of the work. Chapter 2 reports the literature review for understanding the origin of the procedures implemented within this thesis work. Chapter 3 illustrates in detail the methodology implemented to reach the goals and Chapter 4 analyses the results obtained by the implementation of the methods proposed in some characteristic cases. Finally, Chapter 5 shows the main conclusions of this thesis and offers recommendations for further development of this work.
2 LITERATURE REVIEW

2.1 Historical background

The study of ship operations relies on the assessment of the effects of the environment on ship performance. The continuation of the theoretical studies of the ship motions in regular waves to confused seas was extensively tackled in 1953 by the work of St. Denis and Pierson (Denis and Pierson, 1953) through the statistical modelling of the sea state characterized by an energy spectrum. Moreover, early applications in ship operations where based on empirical curves in function of wave height and direction (James, 1957) which made possible to describe and quantify the involuntary speed reduction due to the action of waves, leading to the use of the successive time fronts method to assist the route planning process. These computations were performed manually until 1962, when they were adapted to computer based routines able to deterministically identify the routing minimum time considering stationary wave fields (Haltiner, Hamilton and ‘Arnason, 1962). Eight years later it was presented a numerical method based on the Strip Theory (Salvesen, Tuck and Faltinsen, 1970) which assumed an arbitrary heading, constant speed and sinusoidal regular waves for predicting heave, pitch, sway roll and yaw motions, as well as shear forces, bending and torsional moments. This approach improved the process of determining the seaworthiness of the ship starting from the geometric description of the hull, the distribution of weights and information about the sea environment. In 1980 the focus shifted from the use of empirical performance curves to the application of a dynamic programming framework computing the ship responses amplitude operators which made possible to account for more detailed physics aspects of the operating ship, the ship resistance and power in calm water and in waves (Frankel and Chen, 1980). Moreover, standard wave spectral forms were associated to sea state conditions with the Brettschneider formulation. In 1989 the need for considering uncertainties introduced by environmental forecasts was noticed and treated by Hagiwara (1989) who considered the routing problem also from a stochastic point of view. In addition, he also computed the probability of Green Water at the bow identified as a measure of the damage in rough seas. The rapid increase of computational power of the past two decades made possible to implement always more sophisticated computational method up to genetic algorithms (Hinnenthal, 2008) to produce a set of different routes by perturbing a parent route while considering slamming and vertical and lateral accelerations as safety constrains, achieving the simultaneous minimization of estimated time of arrival and fuel consumption. Moreover, the same new approach implemented ensemble forecast in numerical weather prediction which consists in a set of forecast obtained perturbing the input of a high-resolution deterministic forecast, quantitatively accounting the forecast uncertainties (Saetra et al., 2002) measured by the ensemble spread. More recent studies applied evolutionary multi-objective optimization to optimize up to three criteria accounting for the decision maker preference to speed up the selection of the best solution (Vettor and Guedes Soares, 2016; Szlapczynska and Szlapczynski, 2019).

The list of past research presented above is only partial but emphasizes the significant role of the assessment of ship operability, accounting for its related uncertainties, in the evolution of the tools applied for giving support to the decision maker. In particular, the operability of a ship can be defined as the “ability to carry out her mission safely” (Fonseca and Guedes Soares, 2002) which decreases with the effects of waves, and it can be measured through operability index thanks to seakeeping criteria which embody the limits acceptable of operation.
2.2 Concepts of structural reliability

It is emphasized that within this thesis work the approach to assess ship operation develops from tools and concepts implemented in structural reliability theory\(^1\), commonly applied to rationally assess uncertainties with both marine and land-based structures in stage of structural design and analysis.

For illustrating the fundamental aspects of reliability analysis, the following general example is given by many authors (Mansour, 1990; Ditlevsen and Madsen, 2005; Choi, Canfield and Grandhi, 2007; Lemaire, 2009; Robert E. Melchers, 2018): let's consider a simple beam affected by a loading caused by the environment, as the wave load for marine structures. The strength of such beam and the load acting, which were traditionally treated deterministically through fixed values, are connotated with probabilities distributions expressing the uncertainties related to those parameters. Hence, the strength and the load are treated as random variables which does not only assume a deterministic value but variate over certain intervals. It is possible to express the margin of safety between the strength of the considered structure, and the load acting on it, by the function expressing the difference of the two terms. In this elementary case the margin function is considered linear and, if both strength and resistance follow a gaussian distribution, then a reliability index can be identified as the ratio of the mean and standard deviation of the margin function considered at the mean point. If the margin function is not linear more rigorous definition of the reliability are introduced and it can be computed geometrically as illustrated in the Hasofer-Lind method (Hasofer and Lind, 1974). Moreover, when applying such iterative algorithm, the search of the locally most central point can be performed through the use of advanced adaptive linear regressions, such as TANA or TANA2 (Choi, Canfield and Grandhi, 2007) in order to circumvent the lack of convergence that may arises from strongly not linear limit state function. These algorithms are usually applied when the margin function equation is not known analytically, in the other cases the Lagrange multiplier method can be implemented (Choi, Canfield and Grandhi, 2007; Lemaire, 2009). Additional computational difficulties arises when the random variables are not gaussian distributed hence more sophisticated methods are implemented (Rackwitz and Flessler, 1978) basing on transformation (Rosenblatt, 1952; Nataf, 1962; Zhao and Ang, 2012) for reconducting the problem into a gaussian distributed one.

In particular the FOSM method (Choi, Canfield and Grandhi, 2007; Lemaire, 2009), the FORM method (Ditlevsen and Madsen, 2005; Lemaire, 2009; Robert E. Melchers, 2018), the Maximum Log-likelihood method (Breitung, 1991; Breitung and Faravelli, 1994), and the Monte Carlo Simulation Method (Lemaire, 2009; Robert E. Melchers, 2018) will be investigated within this thesis work.

2.3 Probabilistic approaches to marine safety

The probabilistic approach developed within this thesis is based on the computation of the ship motions for time intervals representing the duration of the mission, on the other hand long term predictions of the most probable maximum values are needed at design stage, and can be obtained with the methodology illustrated

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\(^1\) For readers just introduced to the application of structural reliability theory it is advised to start with the very clear and simple book “Reliability-based Structural Design” (Choi, Canfield and Grandhi, 2007), then to widen and deepen the possible applications with the books “Structural Reliability” (Lemaire, 2009) and “Structural Reliability Analysis and Prediction” (Robert E. Melchers, 2018), knowing that a more mathematically rigorous vision is given by the book “Structural Reliability Methods” (Ditlevsen and Madsen, 2005), while the specific application to ship structure is well presented in "An introduction to Structural Reliability Theory” (Mansour, 1990).
by Guedes Soares and Trovao (1992). Furthermore, long term distributions are usually related to the assessment of wave induced loads, and the effect of the modelling of the wave climate on such assessment was analyzed by Guedes Soares and Trovao (1991), who also quantified sensitivity of the wave modelling on the ship responses, which resulted more relevant than the uncertainty in the computation on the transfer function. A sensitivity study of wave data on marine structures responses was performed also by Guedes Soares and Viana (1988).

Moreover, it must be mentioned that advanced decision support system can also be exploited in design phase (long term probability assessment) and in real time during the ship mission (short term probability assessment). Since in the two different contexts different time spans are considered, the main difference arises from the fact that in design phase substantially more random variables are supposed to be take into account and the decision taken consists in the mitigation of risk in ship life by the implementation of passive risk measures, while during operation the ship can even be considered as a time invariant system hence the random variables can be assumed composed only by environmental parameters and the mitigation of risk is effected through active risk measures, such as voluntary speed reduction or course modification (Spanos et al., 2008). In particular, Guedes Soares (1990a) proposed decision rules about the modification of the course and observed that heavy weather manoeuvres can increase the probability of structural failure due to wave induced bending moments.

Tillig et al. (2018) carried out a study of the influence on the of uncertainties in different stages of the design process, affecting the accuracy of the characterization of the ship’s wetted surface, the computation of calm water resistance and effective wake, wind/air resistance, effective wake, hull and propeller efficiency, added resistance in waves a specific fuel oil consumption of the main engine. In particular, it was shown how more detailed information, models and methods in the different stages reduce the uncertainty on the fuel consumption prediction, analysing also the interplay between the reduction of uncertainty of each design steps with the uncertainty of the generic ship energy system model used for the power prediction (Tillig et al., 2017). The sources of uncertainty accounted in this research were: the variability of the input data of the models such as the hull shape, which can differ from the designed one, the structural model inadequacy in representing true physics, observation errors in experimental measures and, lastly, interpolated data substituting precise computations or measured experimental data.

However, a modern decision support system should identify the best solution ensuring the safety of the ship, the optimality of its performance and the minimization of the journey cost predicting the ship motions according to reliable environmental data. Such prediction is typically performed with seakeeping code implementing methods which can display different level of sophistication and Guedes Soares (1991), mainly focusing on wave loads and tackled the model uncertainties of the response of the intact ship by quantifying a model uncertainty factor of different type of linear theories, and also the uncertainty of linear effects was evaluated. The effect of the uncertainty on ship responses, associated with the spectral shape, on short term ship response was also investigated by Guedes Soares (1990). A study of the same author about the model uncertainties in long term distributions of wave loads can be found in Guedes Soares and Moan (1991) focusing on fatigue design, while a modelling uncertain factor was introduced in long term formulation of wave induced vertical bending moment by Guedes Soares (1993) to represent the non-linearity of the ship structure. More detailed studies were carried out for a containership by Guedes Soares and Schellin (1996). and for a
tanker by Guedes Soares and Schellin (1998). One year later, different organizations performed a comparative study on the procure of computing the long term wave induced loads bending moments for the two type of ship previously mentioned, evaluating the model uncertainties (Guedes Soares, 1999).

Furthermore, a specific study on roll was performed by Vasconcellos and Oliveira (2011) who analysed the effect of the uncertainty of the vessel displacement and centre of gravity to evaluate the randomness of the motion. The model uncertainties of seakeeping computations of damaged ships has been recently evaluated in a broad benchmark study by Parunov et al. (2020).

Natskår, Moan and Alvær (2015) studied the uncertainty affecting environmental data predicted with an ensemble forecast, outlining a method for assessing its quality. They treated the significant wave height, the wave period, and the wind speed as random variables, considering the significant wave height lognormally distributed and the wave period also lognormally distributed conditional on the significant wave height, while the wind speed was described by a two parameter Weibull distribution, also conditional to the significant wave height. Furthermore, the impact of weather variability and uncertainty on the implementation of speed strategies to reduce fuel consumption and emissions was tackled by Norlund and Gribkovskaia (2017) thanks to the development of a simulation-optimization tool.

Moreover, an alternative framework (Perez, 2015) of the traditional computation of ship operability is given by a full probabilistic approach which, given an envelope of possible operation conditions, considers the logical proposition of desirable vessel performance "O", and through the Bayesian network assess the ship operability computing \( P(O|D, I) \), or rather the conditional probability of the logical proposition of being true, given the set of data "D" about the ship behaviour and the background information "I". In such approach the uncertainties related to the environment, the sailing conditions, and the vessel handling condition (associated with mass distribution, vessel design, and motion control systems) are modelled through probabilities distribution. It is just mentioned that the uncertainty related to the sailing conditions is described by the product of the marginal probability of ship speed and wave encounter angles, assumed independent. The first one is ruled by the type of mission, such as transit or station keeping, while the second one is assumed uniform distributed with some geographical exceptions.

Focusing on onboard decision support system, an improvement of the traditional applications is given by a better estimate of uncertainties through real-time correction of weather predictions made thanks the joint use of on board monitoring system and ship-buoy analogy (Vettor et al., 2020). Furthermore, two different, but not opposing, approaches in defining a rational risk-based decision support framework, are discussed in the following paragraphs.

The first approach (Spanos et al., 2008; Papatzanakis, Papanikolaou and Liu, 2012) relay on a probabilistic core, which considers the hazardous events related to the seakeeping performance of the vessel and numerically estimates their probability. Hence considering the respective consequences (about the safety and economical points of view) the corresponding risk is assessed and alternative operative conditions, implying lower risk, are proposed. In such procedure it is assumed that the ship will operate mainly within safety operational boundaries and that hazardous situations with low probability and high consequences are addressed in design phase or by qualitative guidance to the master and/or his training; consequently, events less probable than \(1E^{-3}\) are neglected. Five seakeeping events are identified either as exceedance of
threshold values or number of occurrences, they were green water at the deck, propeller racing, bow slamming, total acceleration at the bridge and bow vertical acceleration. The estimation of the probability of failure is carried out with both a simulation method (Monte Carlo) and other approximated method commonly applied in structural reliability (FORM and SORM). The authors after noticing a convergence of the methods for lower probability level concluded that when high probability of occurrence is faced the FORM method must be verified by the use the simulation method, while in other cases when low probability levels are expected representing the area of interest of an onboard decision support system, the result given by FORM method are satisfactory. It is stressed that this observations are valid for bidimensional limit states $g(H_s, \tau_p)$ and when “multiple variables need to be taken into consideration, studies were not conclusive” (Spanos et al., 2008).

The second approach (Dong, Frangopol and Sabatino, 2016) concerns more structural reliability rather than the seakeeping performance, the failure events are associated to fatigue damage and hull girder collapse and the uncertain data is related to still water and wave induced bending moment model prediction, as well as the determination of hull resistance. The probability of flexural failure is estimated either with simulation technique and-or the first order reliability method (FORM), while the spectral based cumulative fatigue damage is considered as a performance indicator. In particular, repair loss due to flexural failure, the fatigue damage, the total travel time and the CO$_2$ emission are criteria included in a multi-attribute decision support system. Through the use of multi attribute utility function the alternative that ensures the highest multi-attribute utility value is proposed to the decision maker.
3 METHODOLOGY

The first part of this chapter, composed by the first two subchapters, describes how to evaluate the safety of an operating ship given certain input conditions. In the second part of this chapter different approaches to assess the ship operational safety in a probabilistic perspective are presented. The methods described in this chapter are applied in the next chapter where the differences, already mentioned within these pages, can be observed with practical case studies.

The method to estimate the probability of exceeding the operational limits can be divided in two steps. Firstly, the margin function and the joint probability density function are computed for each point of the space of the random variables, which are selected from the ship operational characteristics and/or the environmental conditions. Secondly, the probability of failure is computed according to what illustrated in this chapter. The schematization of the framework of the first step is illustrated in the figure below:

![Figure 1: schematization of the computation of the margin function and the joint probability density function](image)

From a practical point of view the output of the seakeeping code is given as text files which are read with a separated MATLAB code, made for this purpose. Such MATLAB code is also extended to obtain all the results presented in this thesis work.

3.1 Deterministic seakeeping assessment

All the computations and observations performed in the next sections are based on the response amplitude operator of the ship motions, obtained using a code developed at CENTEC. Such code, hereafter referred as “seakeeping code”, allows to compute, among the others the ship motions in frequency domain according with the linear strip theory (Salvesen, Tuck and Faltinsen, 1970) for a predefined combination of wave directions, ship displacements, trims, Froude numbers and metacentric heights.

3.1.1 Input variables

The ship characteristics are introduced in the seakeeping code through the input variables which, as “variables”, can be in principle changed for every implementation of the seakeeping code, but, for the sake of this thesis work, they can be divided in the two categories “Ship Performance fixed data” and “Ship Performance varying data”, as it follows:

| Ship Performance fixed data | Ship Performance varying data |
The fixed data are established with the selection of the ship to be analysed while, for the same ship, different operational conditions are investigated by changing the input varying data. Those data are represented by vectors of variables and ship motions are computed for each combination of such variables. The fixed input data are composed by the information necessary to characterize the ship, the domain of frequencies in which the motions are computed and the locations of the ship of which responses are significant to the seakeeping assessment. Within this thesis work, is has been considered the locations of the fore castle and the bridge, which were estimated from the other ship data by comparison with similar ships. Moreover, the seakeeping code can consider by default an adequate range of frequencies, corresponding to wave lengths comparable with the ship length, according to other data provided.

### 3.1.2 Seakeeping code output

It is recalled that, according to the strip theory linear approach (Guedes Soares, Fonseca and Centeno, 1995), the solutions of the equation of motions of a rigid body in regular waves are of the harmonic and linear type:

\[
\xi_i(t) = \xi_i^a \cos(\omega_e t + \theta_i) \quad i = 1, ..., 6 \quad \text{with} \quad \omega_e = \omega + \frac{\omega^2 v}{g_{acc}} \cos(\chi) 
\]

Eq. 1

Where \( \xi_i^a \) is the motion amplitude, \( \theta_i \) is the motion phase angle respect to the wave motion, \( \omega_e \) is the encounter frequency, \( \omega \) is the wave frequency, \( v \) is the ship speed, \( g_{acc} \) the acceleration of gravity, and \( \chi \) is the heading angle, or rather the relative wave direction.

In the output of the seakeeping code any operational condition is identified by mean draft, trim, ship speed, transversal metacentric height, and longitudinal position of the centre of gravity and flotation. Hence, for each
condition, heading angles and waves frequency the seakeeping code provides the ratios $\frac{\xi^2}{\zeta^2}, \frac{\xi^3}{\zeta^3}, \frac{\xi^4}{\zeta^4}, \frac{\xi^5}{\zeta^5}, \frac{\xi^6}{\zeta^6}$, where $\zeta^a$ is the amplitude of the encountered wave and $k$ the wave number, together with the phase angles $\theta_i$, with $i = 2, ..., 6$. From those ratios it is possible to define the response amplitude operator (RAO) of each motion, which express the as the ratio between the motion amplitude and the amplitude of the encountered wave, or rather the amplitude of the ship motion caused by a regular wave of unit amplitude.

### 3.1.3 Derived responses RAOs

It is once again recalled that, according to the strip theory linear approach (Guedes Soares, Fonseca and Centeno, 1995) vertical and lateral motion induced by regular harmonic waves of any point $(x, y, z)$ are harmonic function of the ship motions $\xi_i(t)$, with $i = 2, ..., 6$, expressed by:

$$\xi_{VM}(t) = \xi_3(t) - x\xi_5(t) + y\xi_4(t) = \xi_{VM}^a \cos(\omega_c t + \theta_{VM})$$

for absolute vertical motion

$$\xi_{LM}(t) = \xi_2(t) + x\xi_6(t) - z\xi_4(t) = \xi_{LM}^a \cos(\omega_c t + \theta_{LM})$$

for absolute lateral motion

While the relative vertical motion is obtained through the regular wave motion $\xi(t)$ as:

$$\xi_{RVM}(t) = \xi_{VM}(t) - \xi(t) = \xi_{RVM}^a \cos(\omega_c t + \theta_{RVM})$$

The corresponding velocities and accelerations are:

$$\xi_{VV}(t) = \dot{\xi}_{VM}(t) = -\omega_c \xi_{VM}^a \sin(\omega_c t + \theta_{VM})$$

for absolute vertical velocity

$$\xi_{LV}(t) = \dot{\xi}_{LM}(t) = -\omega_c \xi_{LM}^a \sin(\omega_c t + \theta_{LM})$$

for absolute lateral velocity

$$\xi_{RVV}(t) = \dot{\xi}_{RVM}(t) = -\omega_c \xi_{RVM}^a \sin(\omega_c t + \theta_{VM})$$

for relative vertical velocity

$$\xi_{VLA}(t) = \ddot{\xi}_{VM}(t) = -\omega_c^2 \xi_{VM}^a \cos(\omega_c t + \theta_{VM})$$

for absolute vertical acceleration

$$\xi_{LDA}(t) = \ddot{\xi}_{LM}(t) = -\omega_c^2 \xi_{LM}^a \cos(\omega_c t + \theta_{LM})$$

for absolute lateral acceleration

Hence, it is possible to express the harmonic functions $\xi(t)$ and $\xi_i(t)$, with $j = 2, 3, 4, 5, 6$ of the formulations above in complex notation $x_1, x_2 \in \mathbb{C}$, such that $x = \xi^a (\cos \theta + i \sin \theta)$, to simplify the sum of the trigonometric functions in Eq. 2 and Eq. 3. The absolute value of the resulting complex numbers represents the amplitude of the absolute, or relative, motions of the ship at any of the locations of interest. Thus, from the RAOs and phase angles of the ship motions given as an output of the seakeeping code it was directly obtained the complex transfer functions of the absolute vertical and lateral acceleration at bridge, and the relative motions and velocity at the forecastle, according to the formulation below:

$$z_{VM} = z_3 - xz_5 + yz_4 \rightarrow z_{VV} = -\omega_c z_{VM} \rightarrow z_{VA} = -\omega_c^2 z_{VV}$$

$$z_{LM} = z_2 + xz_6 - zz_4 \rightarrow z_{LV} = -\omega_c z_{LM} \rightarrow z_{LA} = -\omega_c^2 z_{LV}$$

$$z_{RVM} = z_{VM} - z_\zeta \rightarrow z_{RVV} = -\omega_c z_{RVM}$$

With $z_{VM}, z_{VV}, z_{VA}, z_{LM}, z_{LV}, z_{LA}, z_{RVM}, z_{RVV} \in \mathbb{C}$ and $x, y, z$ coordinates of the point of interest.
3.1.4 Ship responses

In the previous pages it has been introduced how to determine the response of the ship to regular harmonic waves of unit amplitude thanks to the output produced by the seakeeping code.

In this section it is described how to obtain the response to more realistic environmental condition characterised by irregular short-crested waves. The sea-state is assumed to be fully developed and multidirectional, characterized by statistically stationary conditions which describe the random wave elevation.

The irregular wave elevation is described as the superposition of theoretically infinite independent regular waves which result is described through the integral parameters significant wave height $H_s$ and mean period $T_m$ (or also different period as the peak period $T_p$). Under the assumption of Gaussian distributed wave elevations, for a given wave direction it is possible to exploit formulations which describe, using the wave parameters, how the variance of the sea state is distributed over the frequency domain. Moreover, the distribution of the variance of the sea state is proportional to the distribution of its energy and it is possible to express the wave spectrum in function of the set of wave frequency $\omega$ belonging to the domain defined as input of the seakeeping code. In particular it is adopted the formulation proposed by the International Ship Structures Congress (ISSC, 1964), such that:

$$S(\omega) = 0.1107 \frac{H_s^2 \omega_m^5}{\omega^5} \exp\left\{-0.4427 \left( \frac{\omega}{\omega_m} \right)^{-4}\right\}$$

Eq. 6

with $\omega_m = 1.296\omega_p$

Where the input wave parameters are $H_s$ and the mean frequency $\omega_m$ or the peak frequency $\omega_p$.

This model provides a one-dimensional frequency spectrum and does not describe the energy distribution in different direction. Thus, it is introduced the directional spectrum as the product of the frequency spectrum and the dimensionless spreading function $D(\theta)$, assumed to be independent on the wave frequency and defined in the domain of the wave incident directions $\theta$ such that:

$$\int_0^{2\pi} D(\theta) \, d\theta = 1$$

Eq. 7

Moreover, given a predominant wave direction of the sea state $\theta_{\text{max}}$, and the spread parameter $s = 1.7$ is considered (Lucas and Guedes Soares, 2009):

$$D(\theta | s, \theta_{\text{max}}) = \frac{2^{2s-1} \Gamma(s+1) \Gamma(s)}{\pi \Gamma(2s)} \cos^{2s}(\theta - \theta_{\text{max}}) \text{ for } |\theta - \theta_{\text{max}}| < \frac{\pi}{2}$$

Eq. 8

Hence, the resulting directional spectrum is given by $S(\omega, \theta) = S(\omega)D(\theta)$. Then, ship responses are computed as a linear transformation of the wave elevation by the motion transfer functions given as outputs of the seakeeping code, the response spectrum is computed as:

$$S_R(\omega, \theta) = RAO(\omega, \theta)^2 S(\omega, \theta)$$

Eq. 9

According to the Parseval theorem, the variance of the considered response is computed as its integral of the response spectrum over the frequency and directional domains. Note that the ship motions, velocities and acceleration at the point of interest were expressed in the encounter frequency domain in a reference moving
with the ship, while the wave spectrum were expressed in the wave frequency in an earth-fixed reference.

From the energy conservation principle is it possible to derive the following relation

\[ S(\omega_e) = \frac{S(\omega)}{1 - 2\omega \frac{U}{g} \cos(\chi)} \]  

Eq. 10

With \( \chi \) the relative wave direction. Hence, it is possible to convert the response spectrum between encounter frequency and wave frequency domains. It is worth noticing that the integrations over the encounter frequency domain, necessary to compute the moments of the spectrum, can result in numerical errors due to the distortion of the spectrum when waves approach from direction aft of beam seas (Journee and Massie, 2001). Hence it is preferable to perform the integration over the wave frequency domain as follows:

\[ m_{0,\text{mot}} = \int_0^{2\pi} \int_0^\infty S_R(\omega) d\omega d\theta \quad \text{variance of the response spectrum of ship motions} \]

\[ m_{0,\text{vel}} = \int_0^{2\pi} \int_0^\infty S_R(\omega) d\omega d\theta \quad \text{variance of the response spectrum of ship velocities} \]  

Eq. 11

\[ m_{0,\text{acc}} = \int_0^{2\pi} \int_0^\infty S_R(\omega) d\omega d\theta \quad \text{variance of the response spectrum of ship accelerations} \]

3.1.5 Margin function

From the knowledge of the ship responses it is possible to evaluate the hazards of the mission performed by the ship. Within this context hazards are defined as adverse seakeeping events of the intact ship which are associated to quantifiable undesired consequences and, thus, must be prevented in advance (Spanos et al., 2008). In the following sections the hazard of vertical and lateral acceleration at the bridge, slamming and deck submergence will be translated into a mathematical formulation using threshold values and the variance of the response spectra. In the last part of the chapters the seakeeping events are associated to a limit state violation.

In certain cases, hazards are defined using probability of exceedance; thus, it is recalled that the ship responses were considered as linear transformations of a random, stationary, Gaussian process (wave elevation), consequently the ship responses are a stationary, zero mean and Gaussian process too. Moreover, if the response spectrum is narrow banded, than the probability distribution of the maxima is a Rayleigh distribution (Faltinsen, 1990). Hence, the probability that a maximum is larger than a certain value \( X \) is described by:

\[ P(X > X) = \exp \left( -\frac{X^2}{2m_0} \right) \]  

Eq. 12

In particular, for green water it is conservatively considered the probability of the relative vertical motion at the forecastle to exceed the freeboard, while the swell up phenomena is neglected. For slamming it is considered the joint probability of the keel emergence (relative motion at the bow) and of exceeding the critical velocity \( V_{cr} = 0.093(g_{acc} L)^{0.5} \) (Ochi, 1964) (relative velocity at the bow), considered as independent events.

In other cases, such as roll and vertical and lateral accelerations at the bridge, hazards are defined by means of root mean square of the response spectra which, because of zero-mean process, it is equal to the standard deviation.
By identifying the hazards, the limiting criteria listed in Table 2 are considered, depending on the type of ship and work that must be done onboard.

<table>
<thead>
<tr>
<th>Working/living condition onboard</th>
<th>Light manual work</th>
<th>Heavy manual work</th>
<th>Intellectual work</th>
<th>Transit passenger</th>
<th>Cruise liner</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMS lateral acceleration [g]</td>
<td>0.1</td>
<td>0.07</td>
<td>0.05</td>
<td>0.04</td>
<td>0.03</td>
</tr>
<tr>
<td>RMS vertical acceleration [g]</td>
<td>0.2</td>
<td>0.15</td>
<td>0.1</td>
<td>0.05</td>
<td>0.02</td>
</tr>
<tr>
<td>RMS roll [°]</td>
<td>6</td>
<td>4</td>
<td>3</td>
<td>2.5</td>
<td>2</td>
</tr>
<tr>
<td>Slamming probability [%]</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>Green water probability [%]</td>
<td>5</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 2: Operational limits

Where RMS stands for root mean square, or rather $\sqrt{m_0}$. Hence, to measure how much the ship performance comply with the selected limits it is possible to compute the difference between the imposed limit and the value of the response analysed. It comes that if such difference is negative, then the ship does not fulfil the operational criteria while if it is positive it does. Moreover, to facilitate the comparison of results of different operational criteria, typically characterized by different unit of measure, such differences are normalized over the respective limits imposed, outlining the following non-dimensional formulation:

$$g_i = \frac{\text{Limit}_i - \text{Response}_i}{\text{Limit}_i}$$

with $i = 1, ..., n$ hazards

Eq. 13

The scalar function above is also referred as margin function which expresses the difference from the limit state $g = 0$. It is worth noticing that different margin function can be referred to the same limit state. The concept of limit state will be further explored in the following section.

Let us consider a generic system which, subjected to some actions, produces responses of some kind, as illustrated in the Figure 2.

![Figure 2: schematization of a generic system](image)

Such system is characterized by a vector of actions variables $I$, by the vector of variables $C$, which describes the system characteristics and by the variables representing the output response $O$. In general, $\{I, C, O\}$ are not independent and among those variables can be identified the vector $X = \{x_1, ..., x_n\}$ that defines the state of the system and typically concerns with geometry, strength properties and actions. It is worth noticing that the state variables can be chosen freely and independently within a certain subset of the n-dimensional space $\mathbb{R}^n$. This subset represents the domain of definition of the model, thus to each choice of variables values corresponds a uniquely defined structure (ship) which satisfies, or not, given requirements.

Let us consider a piecewise differentiable function $g(X)$ defined everywhere in the domain of definition of the model, which subdivides the space of the state variables in two subsets where $g$ is positive and negative, respectively corresponding to the safe states and the unsafe states (also called safe region and failure region). Usually in the two regions the system is characterized by significant differences in performance. Hence, the
circumstance of a system, including the loads, at which the system performance is just not as required, is called limit state. If the safe set is simply connected the limit state is called regular (Ditlevsen and Madsen, 2005) and is identifiable by the set of state variables values at which the function \( g(X) \) is null. Consequently, it is also possible to define the limit state equation:

\[
g(X) = 0
\]

Eq. 14

Considering a generic bidimensional case having \( X = \{X_1, X_2\} \), the graphical representation of the limit state equation is illustrated in Figure 3.

Figure 3: margin function and limit state (dotted line) of a generic bidimensional case.

It must be noticed that the choice of \( g(X) \) is not unique, for instance also \( g_2(X) = g_3(X) \) is valid, in some applications this will lead to problems of formulation invariance which will be further investigated in the section “Lack of formulation invariance”.

Since ships and offshore structures undergo a wide range of operational situations from frequent to extreme, in general, Ultimate Limit States, Fatigue Limit States and Serviceability Limit States are considered for those structures. At design stages, the main objective is to define a structure able to withstand all the operational scenarios with adequate resistance associated to limited undesirable consequences (Kaminski and Rigo, 2018) thus the focus is typically on Ultimate Limit States, and Fatigue Limit States. On the other hand, for the sake of assessing the operational safety, Serviceability Limit States will be considered in order to check the adequacy of the structure during normal operation. It is clear that in the two implementations just mentioned different time interval are considered, corresponding on the ship life in one case, and the duration of the ship mission in the other. This thesis work is focused on the latter case.

### 3.2 Assessment of the probability of failure

In this section it is introduced the concept of failure of the system as disfunction of certain operational conditions. Generally, the study of the system failure implies an extensive analysis of the system, which is essential before judging its safety or, more precisely, reliability, which defines the ability of a system to function properly at a certain operational condition, in a given time interval (ISO/IEC, 1997). Typical procedures are based on Failure Mode and Effect Analysis (FMEA) which leads to the construction of Fault Tree representing
graphically the logical relationship between failure events or, alternatively to Reliability Block diagram (Kumamoto and Henley, 2010). Those procedures are not described within this thesis work, but it is assumed that they were implemented resulting in the identification of the hazards previously listed in Table 2, each corresponding to a failure mode. The logical relationship between the failure modes is investigated in subchapter 3.8.

3.2.1 Probabilistic nature of the limit state

This whole thesis work is based on the fact that, since the limit state problem definition in ship operations is stochastic and the state variables are uncertain, the state of the system is aleatory thus the performance of the system should be judged in accordance with its probability of failure. The stochastic model of state variables is represented by a joint probability distribution $f_X(\mathbf{X})$ which defines the probability of any set of values of $\mathbf{X}$. One simple case is if the state variables are independent, thus the stochastic model is fully represented by the set of marginal probability distributions $f_{X_i}(X_i)$, with $f_X(\mathbf{X}) = \prod_{i=1}^{n} f_{X_i}(X_i)$.

Anyway, in general it is not always possible to have full probabilistic information and, for instance, it can be very challenging to evaluate the joint probability distribution of the state variables, which is necessary for the computation of the system probability of failure:

$$P_f = P[g(\mathbf{X}) \leq 0] = \int ... \int_{g(\mathbf{X}) \leq 0} f_X(\mathbf{X})d\mathbf{X} \quad \text{Eq. 15}$$

with Eq. 15 typically called “reliability integral”. Therefore, different methods are implemented according to the information available and, based on more or less strict hypothesis, they produce analysis of different depths, as it will be illustrated in the following sections. Generally, the integration of joint probability density function over the failure domain, where $g(\mathbf{X}) \leq 0$, could be made easier to handle by simplification and/or numerical treatment in the definition of the failure domain, the joint probability density function, and the integration process itself. Among the many different approaches proposed in the literature, two dominant approaches prevail.

One consists in solving the integration, either numerically or applying the Monte Carlo method that, through simulations, performs the multidimensional integration of $f_X(\mathbf{X})$ required to find $P_f$ in an approximated manner; that is, instead of solving the integration analytically the result is taken as the sample mean of the simulated distribution.

The other approach circumvents the entire integration process transforming $f_X(\mathbf{X})$ into a multi-Normal probability density function and results in an approximate estimation of $P_f$. The FOSM method and the FORM method, illustrated in the next chapters, belongs to this class of approaches.

3.2.2 Levels of Reliability assessment

A complete risk assessment supposes the deterministic response modelling, the evaluation of the margin function and the definition of the limit state, as well as the statistical description of actions and system characteristics, the computation of the probability of failure and, finally, the risk evaluation.
Consequently, it is possible to define a hierarchy of the measurement of the limit state violation (Mansour, 1990; Robert E. Melchers, 2018) basing to which “level” of the risk assessment is achieved depending also on the quality of data available.

- Level 1 is related to the partial safety factor approach, where every component of the limit state formulation is multiplied by partial factors which embodies the uncertainties of the problem. In the elementary case for which \( g = R - S \) (Mansour, 1990), in Level 1 methods the limit state is evaluated as \( \phi_R R \leq \gamma_S S \), with \( \phi_R \) and \( \gamma_S \) factors of safety selected according to experiments, experience, economic and (sometimes) political considerations. It is highlighted that no probability distributions are accounted in this approach.

- Level 2 procedures consider only Gaussian distributions and simplified mathematical formulation of the margin function, as it will be further investigated within the FOSM method.

- Level 3 procedures compute a better estimate by considering complete stochastic characterization of the state variables, but usually still approximate, as it will be shown, for instance, in the FORM method.

- Level 4 correspond to the complete risk assessment consisting in any previous level plus the final risk quantification through economic data. This approach is typically performed in decision problems.

Note that the classification proposed can variate with different authors even if the spirit is unchanged (Lemaire, 2009), and that in the list is not illustrated the Level 0 which corresponds to a complete deterministic analysis.

The following table rapidly compares the four Reliability Levels:

<table>
<thead>
<tr>
<th>Level</th>
<th>Calculation methods</th>
<th>Probability distribution</th>
<th>Limit state</th>
<th>Uncertainties</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: partial safety factors methods</td>
<td>simple</td>
<td>not necessary</td>
<td>usually linear function</td>
<td>arbitrary</td>
<td>partial factors</td>
</tr>
<tr>
<td>2: second moment methods</td>
<td>second moment algebra</td>
<td>assumed Gaussian</td>
<td>linear / approximated as linear</td>
<td>included as second moment of the Gaussian variables</td>
<td>&quot;Nominal failure probability &quot;</td>
</tr>
<tr>
<td>3: &quot;exact&quot; methods</td>
<td>numerical integration / simulations</td>
<td>necessary</td>
<td>any type</td>
<td>included as second moment of the random variables</td>
<td>Approximated Failure probability</td>
</tr>
<tr>
<td></td>
<td>transformations</td>
<td>necessary, transformed into gaussian</td>
<td>linear/ approximated as linear, sometimes at the second order</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| 4: decision method | any of above, including economic data | Maximum benefit / minimum cost |

Table 3: hierarchy levels of reliability assessment
3.3 FOSM

The First Order Second Moment method corresponding at the Level two Reliability assessment as reported in Table 3, is introduced in this section. This method is usually applied when the random variables $X$ are Gaussian distributed or when solely an incomplete probabilistic information is available, i.e., the joint probability distribution $f_X(X)$ is unknown. In the latter case, the random variables are assumed Gaussian distributed hence $f_X(X) = \phi_n(X)$ can be fully represented through the means, variances and covariances of the random variables since $\phi_n(X) = \phi_n(X, \mu, \Sigma)$ with $\mu$ vector of the means and $\Sigma$ covariances matrix. Indeed, it is:

$$\phi_n(X) = \frac{1}{\sqrt{2\pi}^n \det \Sigma} \exp \left( -\frac{1}{2} (X - \mu)^T \Sigma^{-1} (X - \mu) \right) \quad \text{Eq. 16}$$

The method is called Mean Value First Order Second Moment method (MVFOSM) (Choi, Canfield and Grandhi, 2007) or just FOSM and represents the limit state function $g(X)$ as the first order Taylor series expansion at the mean value point $\mu = E[X]$. As previously mentioned, the name “Second Moment” implies that both outputs and input of this method are described in terms of mean value and standard deviation, higher moments are not accounted.

It is clear that this situation meets most practical uncertainty assessment problems when it can be difficult to select the type of distribution basing on reliable data. Avoiding the choice or investigation of the probability distributions is one of the most appealing reason for opting for the implementation of this method (Ditlevsen and Madsen, 2005). Another further simplification is performed when the correlation between the random variables is unknown. In such case, i.e. unknown probability distribution and unknown correlation between random variables, the FOSM method gives a rapid solution as it follows.

Considering the means vector $\mu_X = \{\mu_{x_1}, ..., \mu_{x_n}\}^T$ of the random variables $X$, and the gradient of the limit state $\nabla g(\mu_X) = \left\{ \frac{\delta g(\mu_X)}{\delta x_1}, ..., \frac{\delta g(\mu_X)}{\delta x_n} \right\}$ computed at $\mu_X$ is, the approximate limit-state function at the mean point is written as:

$$\bar{g}(X) \approx g(\mu_X) + \nabla g(\mu_X)^T (X - \mu_X) \quad \text{Eq. 17}$$

Hence, it is considered the general case of correlated random variables $X$ with correlation coefficients $\rho_{ij} = Corr[X_i, X_j] = \frac{Corr[X_i, X_j]}{\sigma_{i} \sigma_{j}}$, with the expected value and the variance of the approximated margin function computed through the second moment algebra (Choi, Canfield and Grandhi, 2007), it follow:

$$E[g] = g(\mu_{x_1}, ..., \mu_{x_n}) = \mu_g \quad \text{Eq. 18}$$

$$Var[g] = D[g]^2 = \sum_{i=1}^{n} \left( \frac{\delta g(\mu_{x_i})}{\delta X_i} \right)^2 \sigma_{x_i}^2 + \sum_{i=1}^{n} \sum_{j\neq i}^{n} \left( \frac{\delta g(\mu_{x_i})}{\delta X_i} \right) \left( \frac{\delta g(\mu_{x_j})}{\delta X_j} \right) \rho_{ij} \sigma_{x_i} \sigma_{x_j} = \sigma_g^2 \quad \text{Eq. 19}$$

3.3.1 Simple/Cornell reliability index

Considering that the margin function is linear, or it has been linearized, the same is true also for the limit state. Hence, the limit state is represented by a hyperplane, while if $n = 3$ by a plane or, if $n = 2$ by a line. To provide
a graphical representation, a generic bidimensional case is illustrated in Figure 4, which also highlights the point P on the limit state, which will be useful for the FORM method illustrated in the next subchapter.

Figure 4: space of the random variables $\mathbf{X}$ and their joint density functions, the marginal distributions, and the linearization of the limit state.

In this context is it possible to introduce the invariant number $\beta_{\text{simple}}$, called simple reliability index (Ditlevsen and Madsen, 2005), defined as

$$
\beta_{\text{simple}} = \frac{E[g]}{D[g]} \tag{Eq. 20}
$$

which can be understood as a measure of safety with respect to exceeding the limit state. Given the probability density function of $g(\mathbf{X})$, it represents the distance from the equation $g(\mathbf{X}) = 0$ to the mean value $\mu_g$ by unit of standard deviation $\sigma_g$, as shown in Figure 5.

Figure 5: probability density function of $g(\mathbf{X})$

Hence, once applied the FOSM it is always possible to compute $\beta_{\mu,\text{simple}} = \frac{\mu_g}{\sigma_g}$. Furthermore, since the probability distribution of the random variables $\mathbf{X}$ is Gaussian, the same applies to $g(\mathbf{X})$ as a linear combination of Gaussian variables. Hence in this case $\beta$ assumes one further peculiar meaning, it represents the distance from the origin to the limit state hyperplane in the standardized space where $\sigma_g = 1$, as shown in the bottom
Note that the shape of the gaussian probability distribution $\phi_n(X)$ is well known and given in Eq. 16, thus it is possible to compute the probability of failure:

$$P_f = \int_{-\infty}^{0} \phi_n(g(X)) \, dg(X) \quad \text{Eq. 21}$$

Moreover, by standardizing the Gaussian distribution through a change of coordinates:

$$Z = \frac{(g(X) - \mu_g)}{\sigma_g} \quad \text{when } g(X) = 0 \rightarrow Z = \frac{(0 - \mu_g)}{\sigma_g} = -\frac{\mu_g}{\sigma_g} = -\beta \mu$$

Thus

$$P_f = \int_{-\infty}^{-\beta} \varphi_Z(z) \, dz = \Phi_z(-\beta \mu) \quad \text{Eq. 23}$$

With $\varphi_Z$ standard gaussian probability density function and $\Phi_z$ standard cumulative gaussian distribution function. It is worth noticing that if the standard deviation of the limit state function increases or if the mean of the limit state function decreases, then $\beta$ decreases and $P_f$ increases. A graphical description of this behaviour is represented in the figure below:

![Figure 6: gaussian probability density function and Standard Gaussian probability density function](image)

It is emphasized that, since the probability distribution is Gaussian and the Z-Space is standardized, the cumulative probability distribution $\Phi_z$ has a specific formulation. Hence, in this case an analytical relationship is established between the reliability index $\beta$ and $P_f$. It is now clear that whenever $X$ is actually not gaussian distributed, and/or $g(X)$ not linear at the mean point, $P_f$ computed in Eq. 23 is just an approximation, that is the probability of failure that it would have if $X$ were gaussian distributed and $g(X)$ linear at the mean point.
3.3.2 Lack of formulation invariance

If, on one hand, the $\beta_u$ is a reasonable approximation of $\beta$ in many practical cases, namely when $g(X)$ is nearly-linear and the uncertainties are small, on the other hand a different choice of the formulation regarding of the definition of the failure criteria is associated to different $g$ functions which identify the same limit state (Ditlevsen and Madsen, 2005). For instance, taken $g_2(X) = g_3(X)$ the computation of $\beta_{\mu,\text{simple}} = \frac{\mu_2}{\sigma_2}$ leads to a different reliability index, thus a different failure probability estimate. More in general, equivalent but different margin function leads to different $\beta_{\text{simple}}$. This should be avoided since, ideally, any safety measure should not be affected by the formulation of the failure criteria (Robert E. Melchers, 2018).

3.4 FORM

In this chapter other methods able to estimate the probability of failure through approximations of $g(X)$, also accounting for non-gaussian distributed $X$, will be illustrated. Moreover, a new measure of reliability index applicable within this method will be presented, solving the problem of formulation invariance of the simple reliability index previously introduced.

Eq. 23 gives the probability $P[g(X) \leq 0]$ with $g(X)$ linear and $X$ Gaussian distributed. On one hand, in cases of not gaussian distributed variables $\Phi_x(-\beta_\mu)$ is a “nominal” probability, with $\beta_\mu$ being a direct measure of the safety of the problem: that is if $\beta_\mu$ increases the safety increases, if it decreases the safety decreases. On the other hand, if $g(X)$ is not linear the first two moments of $g(X)$ could be not readily obtainable, moreover, non-linear combination of Gaussian distributed random variables do not necessarily lead to a Gaussian distributed $g(X)$. Hence, as done in the FOSM also in this method an approximation to linearize $g(X)$ is performed at some point $X^*$ which, in general, is no longer the mean point, as done within the FOSM method, but is referred as “checking point”, which characteristics will be illustrated soon. It is just emphasized that changing the linearization point will change the estimate of the reliability index.

The method is called more generally First Order Reliability Method (FORM) since $f_x(X)$ is assumed to be known and it is no longer supposed Gaussian a-priori but based on a description of the moments of $X$. This implies to have probabilistic information more detailed than the second moment descriptions of the state variables and this could be practically challenging. In addition, if the distribution of the random variables is not gaussian, a transformation of the variables $X$ is performed in order to make them Gaussian. Moreover, in order to define an invariant formulation, as illustrated in the following section, it is required that the transformation produces standardized and uncorrelated variables, in addition to Gaussian distributed.

Thus, the procedure to be applied requires two steps: first to transform the variables $X$ in uncorrelated standardized gaussian variables, which techniques are described in section 3.4.2 and second to compute with such variables the reliability index in an invariant form as defined in section 3.4.1.

3.4.1 Geometric/ Hasofer and Lind reliability index

Considering Gaussian distributed, standardized and uncorrelated random variables $U$ the geometric reliability index is defined as the distance in the $U$ space from the origin to the limit state surface, resulting:
where the minimum of the distance $\sqrt{\mathbf{U}^T \mathbf{U}}$ is obtained by investigating over the entire limit state surface $g(\mathbf{U}) = 0$ domain. The point on $g(\mathbf{U}) = 0$ having $\beta = \min \sqrt{\mathbf{U}^T \mathbf{U}}$ is referred as “globally most central limit-state” point or more rapidly as “checking point”. Note that in this context $\beta$ is a distance hence always positive, however, by convention it is considered negative if the origin of the $\mathbf{U}$ space is in the failure domain (Lemaire, 2009). A bidimensional case of a generic $g(U_1, U_2)$ is depicted below:

![Figure 7: geometric reliability index identification](image)

**Figure 7:** geometric reliability index identification

Once $\beta_{\text{geometric}}$ has been identified the probability of failure is approximated by the FORM method analogously as done by the FOSM method (see Figure 4 and Figure 6) with: $P_f = \Phi(-\beta_{\text{geometric}})$ with $\Phi$ gaussian standard cumulative distribution function $N(0,1)$. This entails of substituting the limit state surface $g(\mathbf{U}) = 0$ by a hyperplane passing through the checking point $\mathbf{U}^*$ which verifies Eq. 24. In particular, the equation of the hyperplane is given by $\tilde{g}(\mathbf{U}) = \sum_{i=1}^{n} \alpha_i U_i + \beta = 0$ which is orthogonal to the vector $\mathbf{U}^* \mathbf{O}$ having directional cosines $\alpha_i$. A graphical description is illustrated in the figure below:

![Figure 8: FORM approximation for a generic bidimensional space](image)

**Figure 8:** FORM approximation for a generic bidimensional space

Given a certain problem based on the random variables $\mathbf{X}$, it can be referred to the variables $\mathbf{U}$ according to the following scenarios: if $\mathbf{X}$ is constituted by gaussian independent variables $U_i$ can be obtained from $X_i$ by standardizing analogously as done in Eq. 22, if $\mathbf{X}$ is made by correlated gaussian variables an intermediate step of decorrelation must be added, while in all the other cases a non-trivial transformation must be considered, as illustrated in the following sections. It is observed that the transformation in the latter case is
not linear and even if \( g(X) = 0 \) is linear, after being mapped in the U-space, \( g(U) = 0 \) can result as not linear, as depicted in the figure below.

![Figure 9: the transformation does not lead to linear \( g(U) \)](image)

Figure 9 clearly shows that the accuracy of the FORM estimation is related to the shape of the limit state or, more precisely, to the shape of the limit state at \( U^* \). Hence, \( \Phi(-\beta_{\text{geometric}}) \) will be the more accurate the more the limit state is linear in the U-space. Such scenario happens if the following two sufficient, but not necessary, conditions are both verified: \( X \) are gaussian distributed hence the transformation \( X \rightarrow U \) is linear, and \( g(X) \) is linear. Furthermore, the FORM approximation can be better judged in the transformed space through the curvature of \( g(U) \); when the concavity is turned toward the origin the failure domain the probability of failure is underestimated, while in the convex case they are overestimated, as shown in the figure below.

![Figure 10: FORM approximation for concave or convex limit state](image)

Hence according to the scenario \( \Phi(-\beta_{\text{geometric}}) \) represents a lower or upper bound of the actual probability of failure.

Note that, as previously discussed, in order to define a formulation invariant measure of reliability it is common to evaluate the problem not in the physical space of the random variables, but into a standardized normal uncorrelated space. Hence, the next sections list some methods for performing such transformation, which can be applied in different circumstances, but all leading from a non-gaussian non-standardized correlated space to a gaussian standardized uncorrelated one. The different circumstances are represented either by the degree of knowledge of the probabilistic information or by the degree of simplification of a complex problem into a more treatable one. Furthermore, all these methods rely on transformations called “iso-probabilistic”,

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since it is required that the probability content in the physical space is the same of the one resulting in the transformed space.

### 3.4.2 Transforming dependent non-gaussian random variables

When dealing with dependent random variables it is no longer possible to transform random variable from the original physical non-gaussian space into the uncorrelated gaussian space variable by variable, as done for independent variable obtaining an equivalent normal distribution (Choi, Canfield and Grandhi, 2007; Robert E. Melchers, 2018). Furthermore, two different approaches can be applied, one needs the joint conditional density function as input (Rosenblatt, 1952), while the other needs the conditional marginal distributions and the covariances (Nataf, 1962). However, even if from a practical point of view those methods are applied in different conditions, i.e., according to the probabilistic information available, it has been observed an equivalence of the two methods in some scenarios (Lebrun and Dutfoy, 2009). These two methods are described in the following paragraphs and then a remark about the process of uncorrelation is made, lastly the influence of the ordering on the outcome of the transformations is outlined.

#### 3.4.2.1 Rosenblatt transformation

Let us first introduce the Rosenblatt transformation by considering a vector $R$ of uniformly distributed random variables, the vector $U$ of Standardized Normal variables and the vector $X$ of random variables in the physical space. The Rosenblatt transformation in the n-dimensional space is defined as:

<table>
<thead>
<tr>
<th>Principle</th>
<th>Direct transformation</th>
<th>Inverse transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi(U_1) = R_1 = F_1(X_1)$</td>
<td>$U_1 = \Phi^{-1}[F_{X_1}(X_1)]$</td>
<td>$X_1 = F_{X_1}^{-1}\Phi(U_1)$</td>
</tr>
<tr>
<td>$\Phi(U_2) = R_2 = F_2(X_2</td>
<td>X_1)$</td>
<td>$U_2 = \Phi^{-1}[F_{X_2}(X_2</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$\Phi(U_n) = R_n = F_n(X_n</td>
<td>X_1, ..., X_{n-1})$</td>
<td>$U_n = \Phi^{-1}[F_{X_n}(X_n</td>
</tr>
</tbody>
</table>

**Table 4: Summary of Rosenblatt transformation**

Where $\Phi(\cdot)$ is the univariate standard normal cumulative distribution function, and $F_n(X_n|X_1, ..., X_{n-1})$ the conditional marginal distribution of $X_n$. Note that Eq. 25 represents the direct Rosenblatt transformation which offers a general solution for performing the transformation from the physical space $X$ into the gaussian decorrelated and standardized space $U$. Although even if its formulation is quite straight forward to apply, it presents some weak points summarized in the following paragraph.

As already mentioned, the Rosenblatt transformation is a mathematically exact method which requires the knowledge of $F_{X_i}(X_i|X_j, ..., X_n)$ or the joint density $f_X(X)$ from which the conditional cumulative distributions are derived. Note that this rich information is not available in many practical cases; moreover, it is observed that the computation of the conditional cumulative distributions $F_{X_i}(X_i|X_j, ..., X_n) = \int_{-\infty}^{x_i} f_{X_i}(t_i|X_j, ..., X_n) dt_i$ can be challenging if no closed form of the solution is readily obtainable. This is due to the nature of the distributions of the random variables, hence, it is also highlighted that if $X = \{X_1, ..., X_n\}$ is composed by gaussian and/or
lognormal distributed random variables that computation can be easily performed, while if the distribution type of the variables is mixed, e.g. lognormal and Gumbel or exponential, the joint probability density function cannot be expressed in an analytical formulation (Choi, Noh and Du, 2007). Another practical complication of not having simple form of \( F_s(X_i|X_1, \ldots, X_n) \) arises when implementing the inverse Rosenblatt transformation presented in Eq. 26 since the inversion will be carried out numerically, hence if the numerical method is chosen poorly some inconsistencies may occur.

3.4.2.2 Nataf transformation

An alternative to the Rosenblatt transformation is the Nataf transformation, which belongs to the Copula family of transformations, and it is summarized as follows:

### Principles

1. **Gaussian copula leads to:**
   
   \[
   f_X(X) = \phi_n(Y, P) f(X_1) \phi(Y_1) \cdots f(X_n) \phi(Y_n)
   \]

   With \( Y_i = \Phi^{-1}[F_s(X_i)] \), \( Y \sim N(0, I) \) where \( I \) is the identity matrix and \( Y \) is correlated with the matrix \( P \) made by the correlation coefficients \( \rho_{ij} \).

2. **Decorrelation**

   \[
   a_{ij} = \begin{cases} 
   1 - \sum_{k=1}^{i-1} a_{ik} & \text{if } i = j \\
   \rho_{ij} - \sum_{k=1}^{i-1} a_{ik} a_{jk} & \text{if } i > j 
   \end{cases}
   \]

   Eq. 29

### Direct transformation

Eq. 27

\[
U_1 = \frac{\Phi^{-1}[F_s(X_1)]}{a_{11}} \quad U_2 = \frac{\Phi^{-1}[F_s(X_2)] - a_{12}U_1}{a_{22}} \\
\vdots \\
U_n = \frac{1}{a_{nn}} \left( \Phi^{-1}[F_s(X_n)] - a_{1n}U_1 - \cdots - a_{n-1,n}U_{n-1} \right)
\]

### Inverse transformation

Eq. 28

\[
X_1 = F_{X_1}^{-1}[\Phi(a_{11}U_1)] \\
X_2 = F_{X_2}^{-1}[\Phi(a_{12}U_1 + a_{22}U_2)] \\
\vdots \\
X_n = F_{X_n}^{-1}[\Phi(a_{1n}U_1 + a_{2n}U_2 + \cdots + a_{nn}U_n)]
\]

<table>
<thead>
<tr>
<th>Principles</th>
<th>Direct transformation</th>
<th>Inverse transformation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Eq. 27</td>
<td>Eq. 28</td>
</tr>
</tbody>
</table>
| 1. Gaussian copula leads to: | \[
   f_X(X) = \phi_n(Y, P) f(X_1) \phi(Y_1) \cdots f(X_n) \phi(Y_n)
   \] | \[
   U_1 = \frac{\Phi^{-1}[F_s(X_1)]}{a_{11}} \\
   U_2 = \frac{\Phi^{-1}[F_s(X_2)] - a_{12}U_1}{a_{22}} \\
   \vdots \\
   U_n = \frac{1}{a_{nn}} \left( \Phi^{-1}[F_s(X_n)] - a_{1n}U_1 - \cdots - a_{n-1,n}U_{n-1} \right)
   \] |
| 2. Decorrelation | \[
   a_{ij} = \begin{cases} 
   1 - \sum_{k=1}^{i-1} a_{ik} & \text{if } i = j \\
   \rho_{ij} - \sum_{k=1}^{i-1} a_{ik} a_{jk} & \text{if } i > j 
   \end{cases}
   \] | \[
   X_1 = F_{X_1}^{-1}[\Phi(a_{11}U_1)] \\
   X_2 = F_{X_2}^{-1}[\Phi(a_{12}U_1 + a_{22}U_2)] \\
   \vdots \\
   X_n = F_{X_n}^{-1}[\Phi(a_{1n}U_1 + a_{2n}U_2 + \cdots + a_{nn}U_n)]
   \] |

Table 5: Summary of Nataf transformation

Note that the step of decorrelation can always be performed (Gupta, Móri and Székely, 2000) and it is the analogous of determining the eigenvalues and eigenvectors of \( P' \) (Ditlevsen and Madsen, 2005; Noh, Choi and Du, 2009). However, if the correlation coefficient \( \rho < 0.2 \) the variables can be approximated as independent while if \( \rho > 0.8 \) they can be treated as fully dependent hence one of the correlated variables can replace the others (Robert E. Melchers, 2018). In particular, since the matrix \( P' \) is always positive it can be decomposed into upper and lower triangular matrices according to the Cholesky factorization (Choi, Noh and Du, 2007; Lemaire, 2009), which leads to the definition and implementation of the coefficients \( a_{ij} \) computed from the correlation coefficients \( \rho_{ij} \) (Noh, Choi and Du, 2009). Furthermore, it is stressed the Nataf transformation is based on exploiting approximate empirical expression (Ditlevsen and Madsen, 2005; Lemaire, 2009; Robert E. Melchers, 2018) of \( r_{ij} = \rho_{ij}/\rho_{ij} \) for computing \( \rho_{ij} \), from \( \rho_{ij} \) (the correlation coefficients of \( X \)).
3.4.2.3 Remarks about transforming into the U-Space

In short, to reparametrize a vector of random, non-gaussian distributed, correlated variables $X$ into a vector of random gaussian distributed, standardized, decorrelated variables $U$ the Rosenblatt (Eq. 25) or Nataf (Eq. 26) transformation can be implemented if the joint probability distribution or the just the marginal distributions are known. It is noticed that the Rosenblatt is an exact mathematical method while the Nataf approximate empirically $\rho_{ij}$. But, when dealing solely with lognormal and gaussian variables, $\rho_{ij}$ can be computed analytically hence in this case there is no difference in the outcomes of the two method.

Another aspect that must be underlined is that, as the Rosenblatt transformation, also the Nataf transformation does not have a pre-established order when computing $U$, but all $n!$ conditional ordering possibilities are legitimate. It has been showed that when $f_X(X)$ is a symmetric function, that is it remains unchanged for any permutation of the variables, the outcome of a different ordering can be obtained by permutation of the variables (Ditlevsen and Madsen, 2005). Furthermore, it is possible to link the concept of permutation to the ordering chosen in the transformation since the change of the order of conditioning is equivalent to introduce an additional orthogonal transformations, through a permutation matrix, that changes the coordinates of the checking point but not its norm nor the curvatures of the limit state surface at such point (Lebrun and Dutfoy, 2009). Therefore, theoretically the geometric reliability index (which is the norm of the checking point), the FORM approximation (which depends only on the norm of the checking points) do not depend on the ordering. Hence, if $X$ is composed solely with lognormal and gaussian variables the choice of the conditional ordering should not affect the estimate of the failure probability. Anyway, since in most practical cases the checking point is not known in advance but is identified through the implementation of a numeric method, the outcome of the search of the checking point for different ordering can differ according to the precision of the search method.

Furthermore, the term “transformation” can result misleading since no variation to the structure of the problem is made (Breitung, 1991), but rather a reparameterization is performed. Keeping that in mind, it is possible to mathematically define the reparameterization that map the function $g(X)$ in $g(U)$. Indeed, if the relationship $U_i = U_i(X)$ between $U$ and $X$ has unique inverse, then $f_X(X) = f_U(U)|J|$, where $|J|$ is the determinant of the Jacobian matrix $J$ defined as $j_{ij} = \partial U_i/\partial X_i$. Moreover, the same transformation is valid for any relationship between any continuous function (Robert E. Melchers, 2018), thus we can similarly express $g(X)$ in function of $g(U)$ as $g(X) = g(Y)|J|$. Hence, even if no explicit formulation of $g(U)$ is available, it is possible to evaluate the partial derivates of $g(U)$ from the derivates of $g(X)$ (Robert E. Melchers, 2018).

Thus, it is noticed that $U \sim N(0, I)$ is the standard gaussian decorrelated vector of random variable, thus $f_U(U) = \phi_n(U)$ hence $f_X(X) = \phi_n(U)|J|$. From this expression it is clear that there isn’t any inherit reason for the two functions on the opposite sides of the equation to have a maximum in the same point, if the Jacobian is not constant (Lemaire, 2009). In particular, by identifying the point $U^* = \max \phi_n(U)$ and considering the inverse reparameterization $U_i \rightarrow X_i$, the associated point in the physical space is $X^* = \max f_X(X)|J^{-1}|$ or, thanks to the rules of matrix inversion $X^* = \max \frac{f_X(X)}{|J|}$. For stressing what just highlighted, within this thesis work the point $U^*$ is not referred as Most Probable Point (MPP), as done in many cases in the literature, but simply as "checking point" (Robert E. Melchers, 2018). Typically, when dealing with reliability-based design problems such point is
called “design point” (Lemaire, 2009). Another possible proper denomination of $U^*$ is given by the term “most central point” (Ditlevsen and Madsen, 2005).

A final remark about transforming variables from the physical space $X$ to the standardized gaussian uncorrelated space $U$ is made. Considering two generic random variables $Z_1, Z_2$ they are decorrelated if $E[Z_1Z_2] = E[Z_1]E[Z_2]$ and independent if $f_{Z_1Z_2}(Z_1, Z_2) = f_{Z_1}(Z_1)f_{Z_2}(Z_2)$. It results that independence implies decorrelation while decorrelation, in general, doesn’t implies independence. However, in case of jointly gaussian random variables, being uncorrelated is equivalent to being independent, hence $U$ is a vector of standard gaussian uncorrelated, and thus independent, random variables.

### 3.5 MAXIMUM LOG-LIKELIHOOD

The limitation of the FORM method is to be implemented only in the standard gaussian decorrelated space thus proper transformation are essential and must be preliminary implemented. However, this can be avoided if a different method is applied, which requires solving the maximum log-likelihood constrained problem (Breitung, 1991; Breitung and Faravelli, 1994). For introducing such method, referred as Maximum Log-Likelihood, the meaning of minimizing the distance in the $U$ space proposed in Eq. 24 must be firstly properly analysed. In particular, if we consider the gaussian probability density function expressed as function of the random vector $U$:

$$
\phi_n(U) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{1}{2} \sum_{i=1}^{n} U_i^2\right) = (2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|U|^2}{2}\right)
$$

Eq. 30

with $|U| = \sqrt{\sum_{i=1}^{n} U_i^2}$ the Euclidean norm of the random vector $U$. Hence it is evaluated

$$
\ln[\phi_n(U)] = \ln\left[(2\pi)^{-\frac{n}{2}} \exp\left(-\frac{|U|^2}{2}\right)\right] = -\frac{n}{2} \ln(2\pi) - \frac{|U|^2}{2}
$$

Eq. 31

Since $-\frac{n}{2} \ln(2\pi)$ is a constant term, from Eq. 31 it is clear that when $-\frac{|U|^2}{2}$ increases, or rather $|U|$ decreases, also $\ln[\phi_n(U)]$ increases.

As a consequence, the minimization of the Euclidean norm (distance) $|U|$, done within the FORM method, implies the maximization of the natural logarithm of the probability density function $\ln \phi_n(U)$, which is also referred as log-likelihood function.

Keeping this concept in mind, it is recalled that when applying a reparameterization (such as the Rosenblatt or Nataf transformations illustrated in section 3.4.2) for mapping a generic physical space of the random variables $X$ into the $U$ space, no change of the structure of the problem is performed, just the shape of the failure domain is modified. This means that the maximization of log-likelihood function in the $U$ space is the same as in the physical space (Breitung, 1991). Thus, asymptotic methods can be implemented directly in the physical space for computing the probability of failure, the difference is that no geometric interpretation of the procedure is clearly identifiable, just the probabilistic one, which reflect the fact that the problem itself is probabilistic, not geometric.
In particular, it is possible to obtain approximated evaluation of the probability of failure also for non-gaussian distributed random vector \( X \) if it is known \( f_X(X) \). Considering \( U_i = \Phi^{-1}(F_{X_i}(X_i)) = \Phi^{-1}(\int_{-\infty}^{X_i} \exp \ln f_X(X_i)) \), through the Laplace method is possible to derive the following approximated relationship (Breitung, 1991):

\[
|U| = \sum_{i=1}^{n} U_i^2 \sim -2 \sum_{i=1}^{n} \ln \left( f_X(X_i) \right) = \sqrt{-2 \ln f_X(X)}
\]

Defining the log-likelihood function \( l(X) = \ln f_X(X) \) it is identified the checking point \( X^* \) such that

\[
l(X^*) = \max_{g(X)=0} l(X)
\]

A bidimensional example of the procedure just illustrated is shown in the following figure:

![Figure 11: bidimensional generic example of log likelihood maximization](image)

Hence, the probability of failure is asymptotically approximated by second-order Taylor expansions at \( X^* \) by (Breitung and Faravelli, 1994):

\[
P_f = (2n)^{n-1} \frac{f_X(X^*)}{|\nabla l(X^*)| |\det H^*|}
\]

Where the matrix \( H^* \) is composed by:

\[
H^* = PHP^T - vv^T
\]

Characterized as it follows:

1. \( v = \frac{\nabla l(X^*)}{|\nabla l(X^*)|} \) normal to the surface of \( \ln f_X(X) \) at \( X^* \)
2. \( P = I_n - vv^T \) with \( I_n \) n-dimensional unity matrix
3. \( \lambda_l = \frac{|\nabla l(X^*)|}{|\nabla g(X^*)|} \)
4. \( H = \frac{\delta^2 l(X^*)}{\delta X \delta X} - \lambda_l \frac{\delta^2 g(X^*)}{\delta X \delta X} \) \( i, j = 1, \ldots, n \) Hessian of \( \ln f_X(X) \) in local coordinates
From the formulations just introduced it is clear that no difficulties arise when dealing with correlated variables, since no conditional distribution function, as required in the Rosenblatt transformation, must be calculated. When \( f_X(X) \) is known, the point of maximum likelihood is firstly identified, then the first and second derivatives of both the limit state and the log-likelihood are evaluated at such point and insert in Eq. 35, so that Eq. 34 can be implemented and \( P_f \) evaluated. However, it is stressed that \( P_f \) is assessed just in an approximated manner, which is accurate only asymptotically. The appropriateness of such computation depends on the shape of the limit state function and on the log-like likelihood function. If the maximum \( X^* \) is a sharp peak, the approximation should be reasonable, while if it is a flat peak it can be not sufficiently precise (Breitung, 1991).

### 3.6 MONTE CARLO SIMULATION METHOD

This section is based on the principle that instead of solving a certain integration by numerical or analytical mathematical methods, it is possible to simulate a large number of experiments evaluating the limit state violations and computing the probability of failure accordingly (Ditlevsen and Madsen, 2005; Lemaire, 2009; Robert E. Melchers, 2018). The Monte Carlo Simulation Method can be a useful tool to implement in the assessment of the probability of failure of the system, particularly when other methods are inefficient, i.e. when dealing with strongly not linear \( g \) functions (as FOSM and FORM). Furthermore, the accuracy of the calculation can be assessed statistically.

The Monte Carlo Simulation Method is a probabilistic numerical procedure which assumes \( f_X(X) \) to be known and consist of the random sampling a set of values of \( X \) from \( f_X(X) \) and evaluating if \( g(X) \leq 0 \) for each simulated point. If it is true, the limit state is violated and thus simulation outcome is that the system has failed for such simulated point. A sketch of this process is illustrated in Figure 12:
Figure 12: Monte Carlo simulation of random points. The green area is the safe domain while the white area is the failure domain, on the axis the marginal distribution of the simulated random variables are displayed.

If \( N \) simulations are performed it is possible to approximate the probability of failure of the system \( P[g(X) \leq 0] \) given in Eq. 15 as:

\[
P_f = P[g(X) \leq 0] = \int \ldots \int_{g(X) \leq 0} f_X(X) dX = \frac{n(g(X) \leq 0)}{N} = \frac{N_F}{N}
\]

Eq. 36

With \( N_F \) is the number of system failures. It is clear that the number of simulations \( N \) are related to the accuracy given by the method in estimating \( P_f \). In the following figure it can be appreciate how the marginal distribution of the simulated points becomes more similar to the theoretical ones with increasing of the simulated sample size.
Furthermore, it is highlighted that the Monte Carlo Simulation Method is worth being implementing if the number of simulations $N$ is less than the number of integration point required by a numerical integration (Choi, Canfield and Grandhi, 2007).

The simplest Monte Carlo approach is illustrated in this paragraph, which expresses the probability of failure given in Eq. 36 as:

$$P_f \approx \frac{N_F}{N} = \frac{1}{N} \sum_{i=1}^{N} I[g(X_i)]$$

Eq. 37

With $N_F$ is the number of system failures, $N$ is the total number of simulations, $X_i$ is the $i^{th}$ simulated random variable from $f_X(X)$, and $I()$ is the indicator function which identifies the ingratiation domain being:

$$I = \begin{cases} 1 & \text{if } g(X_i) \leq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Eq. 38

In addition, an evaluation of the number of simulations required for a certain confidence considering a confidence interval of 95% it is obtained as (Lemaire, 2009):

$$\overline{P_f} \left(1 - 1.96 \sqrt{\frac{1 - \overline{P_f}}{N \overline{P_f}}}\right) \leq P_f \leq \overline{P_f} \left(1 - 1.96 \sqrt{\frac{1 - \overline{P_f}}{NP_f}}\right)$$

Eq. 39

With $\overline{P_f}$ estimate of $P_f$ and $N$ number of simulations. From Eq. 39 it is possible to derive the Shooman’s error (Shooman, 1968) assuming $1.96 \approx 2$, hence:
\[ \epsilon = 2 \sqrt{\frac{1 - \bar{P}_f}{N \bar{P}_f}} \]

Eq. 40

Where \( \epsilon \) represents the probability of 95\% that the actual value of \( P_f \) belongs to the interval given \( \bar{P}_f (1 - \epsilon) \leq P_f \leq \bar{P}_f (1 + \epsilon) \). As a consequence, to evaluate properly a probability of failure equal to \( 10^{-n} \) the associated number of simulations must be chosen between the interval \([10^{n+2}, 10^{n+3}]\). Hence, now is clear that when dealing with low failure probabilities the suitable number of simulations for obtaining a good result can be impractically great while, in the opposite case, this method can be implemented easily. In conclusion, despite the computational time, Monte Carlo Simulation Method is a powerful tool for controlling the accuracy of the other faster approximative methods illustrated previously.

### 3.7 DIRECT NUMERICAL INTEGRATION METHOD

The analytical integration of the reliability integral given in Eq. 15 is possible only for some special cases which can result in restricted practical interest, while a numerical solution can be always performed numerically, assuming that \( f_x(X) \) is well known, and standard routines for many types of numerical integration are commonly available of most computer systems. However, it is highlighted that when the probability of failure is very small, that is when \( P_f \) has the same order of magnitude of the integration error, the computation may result in significant errors (Lemaire, 2009). Furthermore, another practical limit is due to the rapidly increasing computational time as the number \( n \) of dimensions of the problem increases. Recalling the indicator function defined in Eq. 38 let’s rewrite Eq. 15 as:

\[
P_f = P[g(X) \leq 0] = \int \ldots \int _{g(X) \leq 0} f_X(X) dX = \int \ldots \int _{\Omega} I[g(X)] f_X(X) dX = \int \ldots \int _{\Omega} f_i(X) dX
\]

Eq. 41

With \( f_i(X) = I[g(X)] f_X(X) \). As illustrated in Figure 14 computing \( f_i(X) \) means evaluating \( f_X(X) \) just in the failure domain.

![Figure 14: bidimensional example of the integrand function for the numerical integration (described by points)](image)

Then the trapezoidal rule method, among many other methods, can be applied, and the integral evaluated by the sum of each trapezoidal component (in Figure 14 the vertices of the trapezoids are defined by the dots).

### 3.8 PROBABILITY OF FAILURE OF THE SYSTEM

In section Margin function it was introduced the concept of failure scenario related to the occurrence of an event and moreover, a limit state function was defined in order to separate failure domain and safe domain.
Since in ship operations the failure does not depend on one single event, it is possible to consider the system introduced in Figure 2 and associate a “component” to each failure scenario and limit state. Note that in such sense it is not meant necessarily to be referred to an actual physical component of the system. Let’s consider several components contributing to the failure in a combined manner according to the characteristics of the system, causing the failure or success of the resulting system, hence the probability of failure is referred to this outcome. Furthermore, systems can be idealized into the two elementary categories of series systems and parallel systems, while combination of these leads to more complex models which can describe also conditional aspects. Details of the two elementary categories are well known (Ditlevsen and Madsen, 2005; Lemaire, 2009; Robert E. Melchers, 2018) and the aspects concerning the applications of this thesis work is presented in the following paragraph.

Considering a set of failure events $E_i$ having probability $P(E_i)$, if the happening of an event implies the failure of the system the system of events is called series system, while if only the occurrence of all events causes the failure of the system the system of events is called parallel system. The probability of failure of the series system is:

$$P_{f,sys} = P(E_1 \cup \ldots \cup E_n)$$  

Eq. 42

Which can be expressed as:

$$P_{f,sys} = P(E_1) + P(E_2) - P(E_1 \cap E_2) + P(E_3) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) + P(E_4) - \ldots$$  

Eq. 43

And can be rewritten (Robert E. Melchers, 2018) as:

$$P_{f,sys} = P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i) - \sum_{i<j}^{n} P(E_i \cap E_j) + \sum_{i<j<k}^{n} P(E_i \cap E_j \cap E_k) - \ldots$$  

Eq. 44

Note that the signs are alternating with the increasing of the order of the terms. A series system can be visualized graphically as the reliability block diagram depicted in the Figure 15.

![Figure 15: reliability block diagram for a series system](image)

Recalling what illustrated in subchapter 3.1.5, within this thesis work the operational limits are defined as scenarios in which the ship operations are judged as not allowable, hence reaching of just one of the limit states represents an adverse scenario, or rather a failure of the system. It follows that the system can be
modelled as a series system, also referred as “weakest link” system, as illustrated in the example given in the figure below:

![Series System Diagram]

Figure 16: series system composed by seakeeping hazards

The computation of the probability of failure implementing Eq. 44 can be generally not straightforward since it necessitates the knowledge of the terms expressing the probability of failure of intersection of many failure event. Hence, lower and upper bounds are defined in order to avoid the computation of the exact system probability of failure. Typically, two categories of bounds can be identified: first order bounds are based on the knowledge only on the term \( P(E_i) \) while second order bounds are based also on \( P(E_i \cap E_j) \). It is possible also the implement higher orders bounds, i.e. third order bounds based on \( P(E_i \cap E_j \cap E_k) \), but their evaluation could be quite complex. The implementation of first order bounds is investigated within this thesis work and thus they are introduced in the following paragraph.

Generally, the probability of the system is bounded by 0 and 1, but a classic approximation, considering two events, is given by:

\[
P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \leq P(E_1) + P(E_2)
\]  
Eq. 45

Hence the sum of the probabilities of the events will always be an upper bound of the probability of a series system

\[
P(E_1 \cup \ldots \cup E_n) \leq \min \left( \sum_{i=1}^{n} P(E_i) ; 1 \right)
\]  
Eq. 46

Note that if the probability of intersections of the events are neglectable \( P(E_1 \cup \ldots \cup E_n) \approx \sum_{i=1}^{n} P(E_i) \). However, a more precise upper bound can be obtained assuming statistically independent failure events, which lead to the following expression:

\[
P_{\text{f,sys}} \leq P_{\text{f,indip,sys}} = P \left( \bigcup_{i=1}^{n} E_i \right) = 1 - P \left( \bigcap_{i=1}^{n} S_i \right) = 1 - \prod_{i=1}^{n} P(S_i)
\]  
Eq. 47

Where \( S_i \) represents the survival event. Note that, as mentioned, \( 1 - \prod_{i=1}^{n} \left( 1 - P(E_i) \right) \leq \min \left( \sum_{i=1}^{n} P(E_i) ; 1 \right) \).

On the other hand, a lower bound can be imposed assuming perfectly correlated failure events, for which:

\[
P_{\text{f,perfect cor,sys}} = \max P(E_i) \leq P_{f,\text{sys}}
\]  
Eq. 48

Summarizing, for a series system it is valid:
\[
\max P(E_i) \leq P_{f,\text{sys}} \leq 1 - \prod_{i=1}^{n}(1 - P(E_i))
\]

Eq. 49

The formulation above can be used to impose some crude bounds on the failure probability of any series system being characterized by failure modes which are somewhere between fully dependent and completely independent (Cornell, 1967). Moreover, it is noticed that those bounds are the narrower the more a failure event is dominant, i.e., its probability is significantly bigger compared to the ones of the other events in series.

3.9 SENSITIVITY ANALYSIS

This section introduces a procedure to assess the importance of a random variable with respect to the safety of the system in order to identify the most and least significant variables.

Firstly it is introduced the mechanical sensitivity \( s_i \), which is simply the derivative of the margin function \( g(X) \) with respect to the random variable \( X_i \). Note that \( X_i \) can have a small mechanical importance on \( g(X) \), but a wide probabilistic dispersion \( \sigma_i \), thus, even if \( s_i \) is small, it can still affect the failure of the system, anyway such computation permits to identify the "stress" variables, which increment has a negative effect on \( g(X) \), or "resistance" variables, which increment has a positive effect on \( g(X) \), by the sign of \( s_i \).

However, in order to include also the importance of the uncertainty linked to the random variable \( X_i \) it is possible to introduce a more complete measure of sensitivity, as described in the following subchapter. Moreover, the omission factor is also introduced at the end of this subchapter. Within this thesis work, the implementation of such sensitivity parameters is specifically referred to the FORM method.

3.9.1 FORM sensitivity measures

Once the checking point is identified with the FORM method it is possible to identify the directional cosines already introduced in Figure 8, they represent the sensitivity of the reliability index \( \beta_{\text{geom}} \) in the \( U \) space to the gaussian standardized uncorrelated variables \( U_i \) (Ditlevsen and Madsen, 2005; Choi, Canfield and Grandhi, 2007; Lemaire, 2009; Robert E. Melchers, 2018), the geometrical meaning is illustrated in Figure 17.

![Figure 17: geometric meaning of the directional cosines in the bidimensional case](image-url)
Please note that the negative sign of $\alpha_i$ is due to the fact that they are the component of the unit outward normal at the checking point, which can be mathematically defined by:

$$\alpha_i = \frac{\delta \beta}{\delta U_i} = -\frac{\delta g(U)}{\|\nabla g(U)\|}$$

Eq. 50

Hence since a negative sign is introduced with Eq. 50 a positive $\alpha_i$ means that $g(\cdot)$ decreases with an increment of the random variables, while a negative $\alpha_i$ means that $g(\cdot)$ decreases as the random variable increases. Moreover, even if, due to transformation in the uncorrelated space, there is not a rigorous physical meaning, $\alpha_i$ have a relevant practical application since if they are very high it means that there is the necessity of being very accurate about the determination of $U_i$, while if they are very low it means that $U_i$ does not significantly contributes to $\beta$ and thus $P_f$, hence it can be treated deterministically, reducing the dimension of the $U$, and thus $X$, space (Robert E. Melchers, 2018).

### 3.9.2 Omission factors

The omission factors (Ditlevsen and Madsen, 2005; Lemaire, 2009) represents the relative error of reliability index that occurs when a certain random variables $X_i$ is assumed to be deterministic, thus $\sigma_{X_i} = 0$, and $X_i$ is equal to a characteristic value $x_i$, which can differ from $\mu_{X_i}$. It can be expressed as:

$$o_{X_i=x_i} = \frac{\beta_{X_i=x_i}}{\beta}$$

Eq. 51

Note that when the FORM (or FOSM) method is applied, $\beta$ is the geometric (or simple) reliability index. Anyway it can be useful to introduce the generalized reliability index (Ditlevsen and Madsen, 2005) defined as

$$\beta_{\text{generalized}} = -\Phi^{-1}(P_f)$$

Eq. 52

Where $P_f$ is assumed to be computed with enough accuracy to represent the exact solution of the reliability integral given in Eq. 15. Hence the omission factor can be evaluated even if different methods from the FORM or FOSM are implemented.
4 CASE STUDY

In this chapter the calculations described in chapter 3 are implemented in different scenarios. Firstly, it is investigated the application of the methods according to an increasing number of uncertain data among the operational variables of a ship considering a given instant of its mission and, furthermore, a sensitivity analysis is carried out. Secondly, those methods are extended for assessing the safety of some alternative routes according to the environmental conditions given by forecasts and some assumed service conditions. Thirdly, a whole ocean area is considered to assess the operational safety of an operating fishing vessel.

4.1 Instantaneous probability of failure

It is possible to distinguish the random variables of the reliability integral (Eq. 15), associated to an operating ship, according to two categories: environmental and service variables. Thus, the vector of random variables \( \mathbf{X} \) can be expressed as \( \mathbf{X} = \{ \mathbf{X}_{\text{environmental}}, \mathbf{X}_{\text{service}} \} \). Hence, it is considered that the first category includes the significant wave height \( H_s \), the mean wave period \( T_m \) and the relative wave direction \( \chi \), while the second category comprises the ship draft \( T \), trim \( \Delta T \), speed \( V \) and the transverse metacentric height \( GM_T \). It follows that \( \mathbf{X} = \{ H_s, T_m, \chi, T, \Delta T, V, GM_T \} \). Recalling what illustrated in section 3.1, it is clear that the environmental random variables affects the wave spectrum \( S \), while the service random variables influences the response amplitude operators \( RAO_s \) hence, from Eq. 9, it is evident that together they affects the response spectrums \( S_R \) and so its moments. As a consequence, the margin functions based on the normalized difference of the imposed limits and the response spectral moments (Eq. 13) is driven by \( \mathbf{X} \) for fixed limits. Note that also all the other parameters listed in Table 1 affects the margin function, but they are considered deterministic within this thesis work, thus are not included in \( \mathbf{X} \). Nevertheless, if there is a reason to believe that any of those parameters is a random variable of known probabilistic distribution, mean and standard deviation, it can also be included in the vector \( \mathbf{X} \). Furthermore, it is observed that the nature of the random variables affects their joint distribution \( f_X \) and so do their mean \( \mu_X \) and standard deviation \( \sigma_X \). Consequently, the reliability integral will be influenced also by the statistical characterization of \( \mathbf{X} \) other than the margin function \( g(X) \).

Keeping those principles in mind, in this section it is illustrated the solution of the reliability integral according to the methods illustrated in the chapter 3. At a first stage, only the significant wave height and the mean period are considered as random variables, thus \( \mathbf{X} = \{ H_s, T_m \} \), similarly to what done by Spanos (2008). Then the relative wave direction is also included, leading to \( \mathbf{X} = \{ H_s, T_m, \chi \} \). Finally, the general case for which \( \mathbf{X} = \{ H_s, T_m, \chi, T, \Delta T, V, GM_T \} \) is analysed.

For the computations performed within this subchapter it is considered the containership “S175”, with \( L_{bp} = 175 \text{ m}, B = 25.4, T_{\max} = 9.5 \text{ m} \). More detailed information is given in ANNEX 1.

Furthermore, for the best understandig it is strongly advised to read the following subchapters chapter after the chapter 3 and according to the proposed order.

4.1.1 Statistical characterization of \( \mathbf{X} \)

Within this thesis work the variables \( H_s \) and \( T_m \) are assumed to be lognormally distributed to prevent physically inconsistent negative values which can be caused, for instance, by a gaussian distribution with mean close to zero and a wide standard deviation, analogously to what done by Natskår, Moan and Alvær(2015). All the
other random variables $X, T, \Delta T, V, GMT$ are assumed to be gaussian distributed. It is mentioned that the statistical characterization of both $H_s$ and $T_m$ is distinguished from the other random variables since their standard deviations are derived from forecasted data while in the other cases it was estimated with the formulation presented later in Eq. 57.

Apart from what already illustrated in section 3.4.2.3, another useful consequence of this assumption is that the joint distribution of lognormal and gaussian distributed variables $q$ and $p$, $n = q + p$, is readily obtained (Fletcher and Zupanski, 2006) by:

$$f_{p,q}(X) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \left( \prod_{i=1}^{q} \frac{1}{X_i} \right) \exp \left( -\frac{1}{2} (\mathbf{X} - \mu_X)^T \Sigma^{-1} (\mathbf{X} - \mu_X) \right)$$  \hspace{1cm} \text{Eq. 53}

Where $\mathbf{X}$ is the vector of gaussian distributed variables computed from $\mathbf{X}$ such that $\mathbf{X}^T = \{\ln X_q, X_p\}$, $\mu_{X}$ is the related vector of means and $\Sigma$ is the covariance matrix, defined as:

$$\Sigma = \begin{pmatrix}
\sigma_{X_1}^2 & \rho_{12}\sigma_{X_1}\sigma_{X_2} & \rho_{13}\sigma_{X_1}\sigma_{X_3} & \cdots & \rho_{1n}\sigma_{X_1}\sigma_{X_n} \\
\rho_{12}\sigma_{X_2}\sigma_{X_1} & \sigma_{X_2}^2 & \rho_{23}\sigma_{X_2}\sigma_{X_3} & \cdots & \rho_{2n}\sigma_{X_2}\sigma_{X_n} \\
\rho_{13}\sigma_{X_3}\sigma_{X_1} & \rho_{23}\sigma_{X_3}\sigma_{X_2} & \sigma_{X_3}^2 & \cdots & \rho_{3n}\sigma_{X_3}\sigma_{X_n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho_{1n}\sigma_{X_n}\sigma_{X_1} & \rho_{2n}\sigma_{X_n}\sigma_{X_2} & \rho_{3n}\sigma_{X_n}\sigma_{X_3} & \cdots & \sigma_{X_n}^2
\end{pmatrix}$$  \hspace{1cm} \text{Eq. 54}

Note that, according to the distribution assumed, $X_q^T = (H_s, T_m)$ and $X_p^T = (X, T, \Delta T, V, GMT)$. In addition, the means and standard deviations of $\ln X_q$ are defined (Choi, Canfield and Grandhi, 2007; Lemaire, 2009; Robert E. Melchers, 2018) by:

$$\sigma_{q_i} = \sqrt{\left( \frac{\sigma_{X_{q_i}}}{\mu_{X_{q_i}}} \right)^2 + 1} \quad \text{and} \quad \mu_{q_i} = \ln \mu_{X_{q_i}} - \frac{1}{2} \sigma_{q_i}^2$$  \hspace{1cm} \text{Eq. 55}

Moreover, in Eq. 54 the correlation coefficients $\rho_{ij}$ between the gaussian distributed variables $\bar{X}$ are defined from the correlation coefficients $\rho_{X_iX_j}$ of $X$ according to (Ditlevsen and Madsen, 2005; Noh, Choi and Du, 2009):

$$\rho_{ij} = \frac{\ln \left( 1 + \rho_{X_iX_j} \kappa_{X_iX_j} \right)}{\sqrt{\ln(1+\kappa_{X_i}^2)\ln(1+\kappa_{X_j}^2)}} \quad \text{if} \ X_i, X_j \text{ lognormal distributed} \hspace{1cm} \text{Eq. 56}$$

$$\rho_{ij} = \frac{\kappa_{X_j}}{\sqrt{\ln(1+\kappa_{X_i}^2)}} \rho_{X_iX_j} \quad \text{if} \ X_i, X_j \text{ lognormal and gaussian distributed}$$

With $\kappa_{X_i} = \frac{\sigma_{X_i}}{\mu_{X_i}}$.

Moreover, as mentioned previously, while the means and standard deviations of $H_s, T_m$ can be obtained directly from forecasted data, the standard deviations of $X, T, \Delta T, V$ and $GMT$ can be computed from their means with:

$$\sigma_{X_i} = \Phi^{-1} \left[ 1 - 1 - P(\mu_{X_i} - c \leq X \leq \mu_{X_i} + c) \right]$$  \hspace{1cm} \text{Eq. 57}
Where, fixed the limit c, \( P(\mu_{X_i} - c \leq X \leq \mu_{X_i} + c) \) is the assumed probability that \( X_i \) belongs to the interval \([\mu_{X_i} - c, \mu_{X_i} + c]\), and for all the cases it is considered \( P(\mu_{X_i} - c \leq X \leq \mu_{X_i} + c) = 0.9 \), hence \( \sigma_{X_i} = (c - \mu_{X_i})/1.6449 \). Furthermore, the following numeric values were considered:

<table>
<thead>
<tr>
<th>( \mu_{X_i} ) [m]</th>
<th>( T_m ) [s]</th>
<th>( \chi ) [°]</th>
<th>( T ) [m]</th>
<th>( \Delta T ) [m]</th>
<th>( V ) [kn]</th>
<th>( GM_T ) [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.5</td>
<td>10</td>
<td>110</td>
<td>9</td>
<td>0</td>
<td>24</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 6: statistical characterization of X

It is observed that when, within this chapter, a random variable is treated deterministically, it means that it is assumed to be constant and equal to the respective value reported in the row \( \mu_{X_i} \) of Table 6, while the row \( \sigma_{X_i} \) is neglected. It is observed that the means are chosen comparing with values typically presented in environmental forecasts for the North Atlantic Ocean and examples of route planning involving the ship “S175”.

Furthermore, note that as shown in the last row of Table 6, it is assumed that the correlation variables are just \( H_s \) and \( T_m \), which correlation coefficient is estimated from the Scatter Diagram for North Atlantic Ocean (DNV GL, 2018). It is stressed that this is done just to introduce realistic data meaning that in other applications if specific statistical information is available the data presented in Table 6 should be changed also adding more statistical descriptors or more random variable.

With that being said, to compute \( \rho_{H_s,T_m} \) firstly the joint and marginal density distributions of \( H_s \) and \( T_m \), depicted in Figure 18, are obtained from the Scatter Diagram.

![marginal pdf of Tz](image1)

![marginal pdf of Hs](image2)

Figure 18: marginal probability density of \( T_z \) and \( H_s \)

Secondly, from the marginal distributions the means \( \mu \) and standard deviations \( \sigma \) of \( H_s \) and \( T_z \) are computed with \( \mu = \sum f(x_i)x_i \) and \( \sigma = \sqrt{\sum f(x_i)x_i^2 - \mu^2} \). Thirdly, from the joint distribution given in the Scatter Diagram the covariance \( \sigma_{H_s,T_z} \) is evaluated as \( \sigma_{H_s,T_z} = \sum ((H_s - \mu_{H_s})(T_z - \mu_{T_z}) f_{H_s,T_z} \). Hence, the correlation coefficient \( \rho_{H_s,T_z} \) between \( H_s \) and the zero up-crossing period \( T_z \) is estimated by \( \rho_{H_s,T_z} = \frac{\sigma_{H_s,T_z}}{\sigma_{H_s}\sigma_{T_z}} \). It is then assumed that \( T_z \) and \( T_m \) are related by a constant as depicted by the relationship \( \frac{T_m}{1.924} = \frac{T_z}{1.771} \) suggested by the ITTC (Ittc, 2002) and thus the correlation coefficient \( \sigma_{H_s,T_m} \) is assumed equal to the correlation coefficient \( \sigma_{H_s,T_z} \).
Moreover, considering the values reported in the first row $\mu_x$ of Table 6, the ISSC wave spectrum $S(\omega)$ and the spreading function $D(\theta)$ are defined respectively according to Eq. 6 and Eq. 9. Hence $S(\omega, \theta)$ is obtained by $S(\omega, \theta) = S(\omega)D(\theta)$. The result of such procedure is reported in Figure 19.

\[ S(\omega) = \text{ISSC (1964)} \]
\[ D(\theta) = \text{Lucas and Guedes Soares (2009)} \]

Figure 19: wave spectrum for $H_s = 4.5 \text{ m}, T_m = 10 \text{ s}$ and $\chi = 110^\circ$

4.1.2 1st case: $X = \{H_s, T_m\}$

Before illustrating the results obtained from the different methods described in Chapter 3, it is meaningful to understand the behavior of $g(X)$ over the analyzed domain. Indeed, since it is considered $X = \{H_s, T_m\}$, then $g(H_s, T_m)$ is a bi-dimensional function which can be fully visualized graphically. Hence, in the following figures (Figure 20 and Figure 21) it is reported how the margin function of the seakeeping hazards, identified with $g_i = \frac{\text{Limit}_{i-\text{Response}} - \text{Limit}_i}{\text{Limit}_i}$, varies over the domain of $H_s$ and $T_m$, for $\chi = 110^\circ$, $T = 9 \text{ m}$, $\Delta T = 0$, $v = 24 \text{ kn}$ and $G_M T = 1 \text{ m}$. It is stressed that the evaluation of the $g$ functions and the subsequent computations is performed numerically considering $[1 \text{ m}; 15 \text{ m}]$ with a step of 0.25 m for the domain of $H_s$ and $[4 \text{ s}; 18 \text{ s}]$ with a step of 0.25 s for the domain of $T_m$, and computing the margin function for each point, all the others computation starting from such points will also be numeric.

Moreover, particular attention should be given to the limit state curves, identifying $g(H_s, T_m) = 0$, which are highlighted in black, they show that the biggest failure domain is outlined for Roll while the smallest is given by Green Water, it is stressed that such results are obtained considering $\chi = 110^\circ$, $T = 9 \text{ m}$, $\Delta T = 0$, $v = 24 \text{ kn}$ and $G_M T = 1 \text{ m}$. It is observed that limit state curves reported in the following figures express the same concepts of the Maximum Allowed Significant Wave Heights curves typically used to compute the operational index assessing the seakeeping performance of a ship involved in a given mission (Guedes Soares, Fonseca and Centeno, 1995), since they both identify the environmental conditions for which the operational limit is exceed.

Furthermore, Figure 20 and Figure 21 also illustrate the mean point in the yellow circle, corresponding to $H_s = 4.5 \text{ m}$ and $T_m = 10 \text{ s}$, which can help to predict the quality of the result obtained with the FOSM method, in particular for the Slamming case. Indeed, in such case the linearization of the margin function at the mean point clearly induces a big approximation and, consequently, a big error is expected on the evaluation of $P_f$.

It is also mentioned that since all the limit state curves are always convex, the FORM method will overestimate the probability of failure, as will be further explained in section 4.1.2.2.
Figure 20: bidimensional g(X) of vertical and lateral acceleration at the bridge, the black line is the limit state curve while the yellow circle is the mean point.
Figure 21: bidimensional g(X) of Roll, Green Water, and Slamming, the black line is the limit state curve while the yellow circle is the mean point.
However, it is observed that identification of the failure domain itself presented in Figure 20 and Figure 21 is not sufficient itself to get an idea on the solution of the integral $P_r = P[g(X) \leq 0] = \int \ldots \int g(X) \leq 0 \mathcal{f}_X(X) dX$, since the value assumed by the integrand function $f_X(X)$ inside the failure domain play also a key role. Hence, $f_X(X)$ is also represented graphically in Figure 22, which shows that in a restricted area the joint pdf as a peak and then decreases rapidly. Figure 23 depicts some contours of low joint probability included in the failure domains. In particular, it is considered the bidimensional joint lognormal probability distribution obtained either from Eq. 53 or by the following more compact formulation (Noh, Choi and Du, 2009) given by Eq. 58.

$$f_X(H_s, T_m) = \frac{1}{2\pi \sigma_{q_1} \sigma_{q_2} \sqrt{1 - \rho_{12}^2}} \exp\left(-\frac{1}{2} \left(\frac{\ln H_s - \mu_{q_1}}{\sigma_{q_1}}\right)^2 - 2 \rho_{12} \frac{(\ln H_s - \mu_{q_1})(\ln T_m - \mu_{q_2})}{\sigma_{q_1} \sigma_{q_2}} + \left(\frac{\ln T_m - \mu_{q_2}}{\sigma_{q_2}}\right)^2\right)$$

Eq. 58

With $\rho_{12} = 0.5798, \sigma_{q_1} = 0.2195, \sigma_{q_2} = 0.0898, \mu_{q_1} = 1.4800$ and $\mu_{q_2} = 2.2986$.

Figure 22: joint bi-dimensional probability distribution of $H_s$ and $T_m$

Figure 23: Visualization of the joint probability of $f_X(X)$ inside the failure domain (above the limit states)
Considering Figure 23 it is possible to expect that the integration (Eq. 15) of \(f_X(X)\) over the outlined failure domains will define low \(P_f\) being the integrand function always very low (smaller than 0.01) in the integration domain. In addition, it is possible also to suppose that the biggest probability of failure is due to Roll, while the lowest is given by Green Water. Those hypotheses will be confirmed at the end of the applications of the methods, presented in the following subsections:

### 4.1.2.1 FOSM:

The following table illustrates the outcome of the FOSM method, described in subchapter 3.3, which can be summarized in computing \(E[g(X)] = g(4.5\ m, 10\ s)\), evaluating \(\delta g(X)/\delta H_s\) and \(\delta g(X)/\delta T_m\) at \(H_s = 4.5\ m\) and \(T_m = 10\ s\) (which was done numerically with the method of central difference) for computing \(\text{STD}[g(X)]\) with Eq. 19, and then computing \(P_f = \Phi(-E[g(X)]/\text{STD}[g(X)])\). With \(\Phi\) standard gaussian C.D.F. The results obtained are shown in Table 7, reported below:

<table>
<thead>
<tr>
<th>Margin Function</th>
<th>(E[g(X)])</th>
<th>(\delta g(X)/\delta H_s)</th>
<th>(\delta g(X)/\delta T_m)</th>
<th>(\text{STD}[g(X)])</th>
<th>(P_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA</td>
<td>0.51471</td>
<td>-1.08E-01</td>
<td>4.35E-02</td>
<td>9.11E-02</td>
<td>5.647</td>
</tr>
<tr>
<td>LA</td>
<td>0.68014</td>
<td>-7.11E-02</td>
<td>2.37E-02</td>
<td>6.14E-02</td>
<td>11.09</td>
</tr>
<tr>
<td>RL</td>
<td>0.51394</td>
<td>-1.08E-01</td>
<td>-2.87E-02</td>
<td>1.25E-01</td>
<td>4.123</td>
</tr>
<tr>
<td>GW</td>
<td>0.99989</td>
<td>-5.81E-04</td>
<td>2.11E-04</td>
<td>4.97E-04</td>
<td>2012</td>
</tr>
<tr>
<td>SL</td>
<td>0.99989</td>
<td>0</td>
<td>0</td>
<td>Inf</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 7: 1st case FOSM results

Table 7 shows that for all the considered seakeeping hazards the estimated failure probability is quite low, moreover in order to give a reference it is noticed that to obtain at least \(P_f \geq 0.01\) it should be \(\beta_{\text{simple}} = \frac{E[g(X)]}{\text{STD}[g(X)]} \leq -\Phi^{-1}(0.01) \approx 2.33\), but the combinations of relatively high \(E[g(X)]\) and low \(\text{STD}[g(X)]\) leads to higher \(\beta_{\text{simple}}\) and thus lower \(P_f\). Increasing \(\sigma_{X_i}\) would increase \(\text{STD}[g(X)]\) and thus increase \(P_f\). Different margin function with lower \(g(4.5\ m, 10\ s)\) or higher derivate at such point would also increase \(P_f\). However, as previously mentioned the value obtained for Slamming seems underestimated since the linearization of the Slamming margin function at the mean point surely is a rough approximation of its actual shape (see Figure 21). It is anticipated that this observation will be confirmed by the application of the Monte Carlo Simulation method.

### 4.1.2.2 FORM:

This subsection describes the outcome of the implementation of the FORM method described in subchapter 3.4, which can be summarized in the following steps: the limit state curve outlined in Figure 20 and Figure 21 is transformed in the \(U\) Space with Eq. 27, then the checking point is defined by minimizing \(\beta_{\text{geom}} = \sqrt{U_1^2 + U_2^2}\), or rather the distance from the origin to the transformed limit state curve (see Figure 24 on the left). The minimization is carried out evaluating the transformed limit state point by point, lastly the probability of failure is computed by \(P_f = \Phi(-\beta_{\text{geom}})\). Such procedure is applied for each margin functions considering the two alternative conditional ordering possible with Eq. 27. It is stressed that the notation \(X_i\) indicates a coordinate of the checking point in the original space while the notation \(U_i\) identify the coordinates of the checking point in the transformed space.
The following Table 8 and Figure 24 respectively illustrates the results obtained and depict the geometrical meaning of the implemented procedure. Note that Table 8 also illustrates the coordinates of the checking points $H_s^*$ and $T_m^*$ transformed from the $U$ space back to the original space with Eq. 28.

<table>
<thead>
<tr>
<th></th>
<th>$U_1^*$</th>
<th>$U_2^*$</th>
<th>$H_s^*$ [m]</th>
<th>$T_m^*$ [s]</th>
<th>$\beta_{\text{geom}}$</th>
<th>$P[g(X)&lt;0]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA</td>
<td>Canonical conditional ordering</td>
<td>3.67</td>
<td>-1.74</td>
<td>9.84</td>
<td>10.62</td>
<td>4.06</td>
</tr>
<tr>
<td></td>
<td>Alternative conditional ordering</td>
<td>0.71</td>
<td>4.00</td>
<td>9.83</td>
<td>10.62</td>
<td>4.06</td>
</tr>
<tr>
<td>LA</td>
<td>Canonical conditional ordering</td>
<td>5.59</td>
<td>-2.84</td>
<td>15.00</td>
<td>10.82</td>
<td>6.27</td>
</tr>
<tr>
<td></td>
<td>Alternative conditional ordering</td>
<td>0.93</td>
<td>6.21</td>
<td>15.00</td>
<td>10.82</td>
<td>6.27</td>
</tr>
<tr>
<td>RL</td>
<td>Canonical conditional ordering</td>
<td>3.15</td>
<td>0.13</td>
<td>8.77</td>
<td>11.85</td>
<td>3.15</td>
</tr>
<tr>
<td></td>
<td>Alternative conditional ordering</td>
<td>1.94</td>
<td>2.48</td>
<td>8.77</td>
<td>11.85</td>
<td>3.15</td>
</tr>
<tr>
<td>GW</td>
<td>Canonical conditional ordering</td>
<td>5.59</td>
<td>-4.84</td>
<td>15.00</td>
<td>9.35</td>
<td>7.40</td>
</tr>
<tr>
<td></td>
<td>Alternative conditional ordering</td>
<td>-0.70</td>
<td>7.36</td>
<td>15.00</td>
<td>9.35</td>
<td>7.40</td>
</tr>
<tr>
<td>SL</td>
<td>Canonical conditional ordering</td>
<td>3.57</td>
<td>-1.59</td>
<td>9.61</td>
<td>10.67</td>
<td>3.91</td>
</tr>
<tr>
<td></td>
<td>Alternative conditional ordering</td>
<td>0.77</td>
<td>3.83</td>
<td>9.61</td>
<td>10.67</td>
<td>3.91</td>
</tr>
</tbody>
</table>

Table 8: 1st case FORM results

Figure 24: on the right: limit states and joint P.D.F in $X$ space, on the left: limit states and joint P.D.F in $U$ space. Table 8 shows that different conditional orderings lead to the same checking point in the original space and the same $\beta_{\text{geom}}$ (see the result obtained with the dotted and continues lines on Figure 24) thus same $P_f$. Thus, the good quality of the numeric minimization procedure is confirmed since methods not precise enough lead to different results for different conditional orderings (see section 3.4.2.3). On the other hand, Figure 24 depicts that considering Lateral Acceleration and Green water limit states, a better minimization solution can be found widening the domain but doing so means a further decrement of the associate failure probability, being the corresponding checking point characterized with a great distance from the origin in the transformed space; hence the true solution corresponds to an even smaller probability of failure which is of little interest.

Moreover, the $P_f$ associated to Roll is the biggest and dominates the other failure probabilities by at least one order of magnitude, while the smallest probability of failure is given by Green Water, coherently to what previously supposed from Figure 23. Furthermore, it must be noticed that since the transformed limit state curves are always convex, the linearization at the checking point implied by the FORM method induces an overestimation of the failure domain and, consequently, $P_f$ (see Figure 10). However, since the estimated $P_f$
are low for all the seakeeping hazards considered the FORM approximation is not very significant (Spanos et al., 2008), as will be also confirmed by the Monte Carlo simulation method.

4.1.2.3 MAXIMUM LOG-LIKELIHOOD:

This subsection illustrates the results of the application of the Maximum Log-Likelihood method, described in subchapter 3.5, which can be summarized in the following steps: firstly, a checking point \( X^* \) is identified for every margin function in the physical space through the maximization of the natural logarithm of \( f_X(X) \) (also referred as log-likelihood function \( l(X) \)) among each points of the limit states. Secondly, the first and second derivates of both the margin and log likelihood function are evaluated at the checking point. Lastly the probability of failure is approximated from the Hessian and Gradient of both the margin function and the log-likelihood function through the asymptotic formulation given by Eq. 34. The first three columns of Table 9 lists the outcome of the checking points search, also depicted in Figure 25, while the 4th, 5th and 6th columns reports the three parameters needed for the computation of Eq. 34. It is recalled that the matrix \( H^* \) is obtained with Eq. 35 from the Hessian and the gradient of both the margin and log-likelihood function, more detailed mathematical aspects exceed the aim of this thesis work but can be found in the appendix of Breitung and Faravelli, (1994).

![Figure 25: Log-Likelihood Maximization over the limit state curves, the checking points are highlighted with the black marker.](image)

| Hs [m] | Tm [s] | l(X*) | fX(Hs, Tm) | |det H*| | Pf[|g(X)|<0] |
|---|---|---|---|---|---|---|
| VA | 9.72 | 10.50 | -10.59 | 2.51E-05 | 3.35 | 182.38 | 1.97E-06 |
| LA | 15.00 | 10.83 | -22.48 | 1.73E-10 | 4.23 | 1524.08 | 3.72E-12 |
| RL | 8.79 | 11.75 | -7.33 | 6.75E-04 | 1.75 | 35.52 | 2.23E-04 |
| GW | 15 | 9.35424 | -29.999 | 9.37E-14 | 7.50707 | 1220.71 | 1.27E-15 |
| SL | 9.5 | 10.5495 | -9.9547 | 4.75E-05 | 3.18714 | 40.2159 | 8.33E-06 |

Table 9: 1st case result for the Log-Likelihood method

As reported in the last column of Table 9 overall similar results to the FORM method are obtained, the most relevant probability of failure is given by Roll, followed by Slamming, Vertical and Lateral Acceleration at the bridge and Green Water. The Monte Carlo solutions will confirm the order of magnitude of \( P_f \) for Roll but will also show that the FORM results will be slightly more accurate than the one reported in Table 9. An increment
of the quality of the derivatives computational method, which in this case were obtained with the method of central difference, can be performed to improve the quality of such results.

4.1.2.4 DIRECT NUMERICAL INTEGRATION METHOD:

This subsection illustrates the results of the application of the Direct Numerical integration, described in subchapter 3.7, which can be simply performed starting from the discrete domain considered at the beginning of section 4.1.2, or rather \([1m, ..., 15m] \times [4s, ..., 18s]\) with a step of \(0.25m\) and \(0.25s\). Hence, just the points such that \(g(H_s, T_m) \leq 0\), and the associated values of \(f_X(H_s, T_m)\) are considered (see Figure 23) for each margin functions. The integral \(P_f = P[g(H_s, T_m) \leq 0] = \int \int_{g(H_s, T_m)\leq 0} f_X(H_s, T_m) \, dH_s \, dT_m\) is then solved performing a trapezoidal numerical integration. The following table illustrated the results obtained with the implementation of such procedure for each seakeeping hazards.

<table>
<thead>
<tr>
<th>(P[g(X) &lt; 0])</th>
<th>VA</th>
<th>LA</th>
<th>RL</th>
<th>GW</th>
<th>SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.13E-05</td>
<td>2.53E-11</td>
<td>7.75E-04</td>
<td>1.75E-15</td>
<td>4.2E-05</td>
<td></td>
</tr>
</tbody>
</table>

Table 10: 1st case, results of the direct numerical integrations

Table 10 generally confirms the results previously obtained with the FORM and Maximum Log-Likelihood method and the Monte Carlo simulation method, as shown in the next section, will highlights that the direct integration will give the most accurate result, proving the degree of discretization of the domain was accurate enough. Hence, it is stressed that if wider integration steps are considered, a decrement of the accuracy of this method is expected, particularly when the solution is a low probability, since \(P_f\) could have the same order of magnitude of the integration error (Lemaire, 2009).

4.1.2.5 MONTE CARLO SIMULATION METHOD:

This subsection describes the outcome of the implementation of the Monte Carlo Simulation method, described in subchapter 3.6, and its application is divided in the following main steps: firstly, the estimation of the number of required simulations \(N\) is performed from the FORM results, which are known to be accurate for low probabilities (Spanos et al., 2008), secondly the simulation of a set of random points identified by couples of \(H_s\) and \(T_m\), is performed with a built-in MATLAB function starting from the formulation of \(f_X(X)\) previously given in Eq. 58 and by the estimated required number of simulations. Thirdly the probability of occurrence of \(g(X_{simulated}) \leq 0\) is evaluated for the simulated sample and for all margin function, checking the convergence with the progressive increment of the simulated sample size. If it is true, the last value represents the estimated failure probability \(\bar{P}_f\). Note that, within practical limitations, \(X_{simulated}\) is computed considering the maximum number of required simulations \(N\) given by the FORM results. Moreover, it is stressed that with the term “required simulations” it is implied “required simulations for having a good accuracy of the results”, it is recalled that this is achieved fixing a low value of the error \(\epsilon\) for which the estimated probability of failure \(\bar{P}_f\) verifies \(\bar{P}_f(1-\epsilon) \leq P_f \leq \bar{P}_f(1+\epsilon)\), and thus is possible to approximate \(P_f \approx \bar{P}_f\). Within this thesis work it is fixed \(\epsilon = 0.05\) and thus from Eq. 40 it comes \(N = 4 \frac{1-P_f}{\epsilon^2 \bar{P}_f} = 4 \frac{1-P_f}{0.05^2 \bar{P}_f}\) resulting:

<table>
<thead>
<tr>
<th>(P_{f, FORM})</th>
<th>VA</th>
<th>LA</th>
<th>RL</th>
<th>GW</th>
<th>SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.43E-05</td>
<td>1.76E-10</td>
<td>8.04E-04</td>
<td>6.96E-14</td>
<td>4.70E-05</td>
<td></td>
</tr>
<tr>
<td>(1-P_f)</td>
<td>6.58E+07</td>
<td>9.09E+12</td>
<td>1.99E+06</td>
<td>2.27E+16</td>
<td>3.40E+07</td>
</tr>
</tbody>
</table>

Table 11: 1st case, Required Simulations for having \(\epsilon = 0.05\)
Table 11 show that \( N \) increases with decrement of probability, as explained in subchapter 3.6, and it is chosen to apply the Monte Carlo Method only to the Roll which is the one requiring the more contained computational effort. Note that applying such choice means also evaluating the \( g \) function with the more relevant failure probability (since it is meaningful to spend the computational power just on the significant cases, while neglecting the less important). Hence it is chosen \( N = N_{RL} = 1.99E + 06 \), and the random points sketched in Figure 26 are simulated.

Figure 26: 1st case, simulated points. On the axis the simulated marginal distributions are also shown. The simulated probability of failure computed from the random sample shown in Figure 26 for Roll is 7.50E-04. From such results the actual error committed is computed with \( \epsilon = 2\left[(1 - P_f)/(N P_f)\right]^{0.5} \) (Eq. 40) resulting in \( \epsilon_{RL} = 0.0517 \). The convergence of the progressive mean failure probability with the increasing simulation sample size is reported below:

Figure 27: 1st case, computation of \( P_f \) with the increasing of the simulated sample size.

As already anticipated, the Monte Carlo results confirmed the results obtained with the FORM, Maximum Log-Likelihood and Direct Integration method for Roll, while questioning the accuracy of the FOSM method which assigned lower probability. The overall comparison can be found in Table 12 in the next section.

4.1.2.6 SYSTEM PROBABILITY OF FAILURE:

This subsection summarizes the results obtained so far and evaluate the system failure probability as illustrated in the subchapter 3.8. Firstly, the selection of the best estimation of \( P_f \) is performed considering both the outcome of the FORM method and the Monte Carlo simulation method, which replace the FORM green water and roll probability of failure, as illustrated in the last column of the following table.
Table 12: summary of the estimations of the failure probability for the 1st case

<table>
<thead>
<tr>
<th>seakeeping hazard</th>
<th>estimated $P[g(X)&lt;0]$</th>
<th>Monte Carlo Simulation</th>
<th>FORM + Monte Carlo</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FOSM</td>
<td>LOG-L.</td>
<td>DIRECT INT.</td>
</tr>
<tr>
<td>VA</td>
<td>8.16E-09</td>
<td>1.97E-06</td>
<td>2.13E-05</td>
</tr>
<tr>
<td>LA</td>
<td>7.38E-29</td>
<td>3.72E-12</td>
<td>2.53E-11</td>
</tr>
<tr>
<td>RL</td>
<td>1.87E-05</td>
<td>2.23E-04</td>
<td>7.75E-04</td>
</tr>
<tr>
<td>GW</td>
<td>0</td>
<td>1.27E-15</td>
<td>1.75E-15</td>
</tr>
<tr>
<td>SL</td>
<td>0</td>
<td>8.33E-06</td>
<td>4.25E-05</td>
</tr>
<tr>
<td>worst</td>
<td>1.87E-05</td>
<td>2.23E-04</td>
<td>7.75E-04</td>
</tr>
</tbody>
</table>

Hence, the first order bounds are considered, and it is recalled that those bounds converge the more a failure event dominates over the other. Such consideration also justifies the interest in increasing the accuracy of the estimation of the probabilities for roll and green water with the Monte Carlo simulation method, Indeed, if just the FORM method is applied, it follows:

$$\max P_f^{\text{FORM}} = 8.04E - 04 \leq P_{f,\text{system}} \leq 1 - \prod_{i=1}^{n} (1 - P_f^{\text{FORM}}) = 8.75E - 04$$

While considering the improved first order bounds obtained by the joint use of FORM and Monte Carlo simulated results:

$$\max P_f^{\text{FORM+MC}} = 7.53E - 04 \leq P_{f,\text{system}} \leq 1 - \prod_{i=1}^{n} (1 - P_f^{\text{FORM+MC}}) = 8.25E - 04$$

Furthermore, recalling the first part of Eq. 44, as the probability of failure of the series system is given by the union of the failure domain, it comes that the direct extension of Eq. 15 is the integration of $f_X(X)$ over the domain represented by the union of the failure domains identified by each seakeeping hazards. Hence, the implementation of the direct numerical integration leads to $P_{f,\text{system}} = 7.80E - 04$, which falls between the bounds previously identified by the joint use of FORM and Monte Carlo method. It is stressed that since the Monte Carlo simulation did not give information about the accuracy in computing $P_{f,VA}, P_{f,LA}, P_{f,GW}, P_{f,SL}$ of both the direct integration and FORM method, the latter was chosen in place of the direct integration because while it is known that good accuracy is obtained for low probability with the FORM, in the other case having the magnitude of the integration step bigger than the magnitude of the integration results can lead to errors (Lemaire, 2009). However, the choosing the direct integration instead of the FORM does not significantly change the estimate of the first order bounds made also with the Monte Carlo results for Green Water and Roll.

4.1.3 2nd case: $X = \{H_s, T_m, \chi\}$

In this subchapter also the relative wave direction is considered random variable, but it is stressed again that the evaluation of the $g$ functions $g = \frac{\text{Limit}_{-\text{Response}}}{\text{Limit}_{+}}$ and the subsequent computations is performed numerically point by point considering the discrete intervals already introduced at the beginning of section 4.1.2, in addition to $[0^\circ; 180^\circ]$ with a step of $10^\circ$ for the domain of $\chi$. Hence, it is no longer possible to visualize graphically the complete behavior of $g(H_s, T_m, \chi)$ over the domain, anyway, it is still possible to show the variation of the limit states curves for fixed $\chi$, as illustrated in Figure 28, which help visualize the relevance of the seakeeping
events according to the operational conditions. Figure 28 shows what a variation on $\chi$ (implied by its uncertainty) cause on the ship seakeeping behaviour; in particular it shows that if the ship encounters a sea state with $\chi > \mu_X = 110^\circ$ the Roll and Lateral Acceleration become less significant (decrement of failure domains) while Slamming, Green Water and Vertical Acceleration become more relevant (increment of the failure domains). The opposite happens for $\chi < \mu_X = 110^\circ$, as expected. Note that $\chi \ll 110^\circ$ or $\chi \gg 110^\circ$ are not considered since associated with very low probabilities and plotting also the associated limit states would decrease the readability of Figure 28. Moreover, it is clear that the variability of $\chi$ has its biggest effect on Roll.

Since the visualization of the probability content given by the shape of $f_X(H_s, T_m, \chi)$ in the domain $[1m, ..., 15m] \times [4s, ..., 18s] \times [0^\circ, ... 180^\circ]$ is difficult, the visualization of its probability content in the failure domains would be chaotic, and no direct preliminary consideration about what to expect on the solution of $P_f = P[g(X) \leq 0] = \int ... \int_{g(X)\leq0} f_X(X) dX$ can be done. On the other hand, since it has been noticed that the Slamming limit states curves always include the Green Water one, and the same is true for Roll and Lateral Acceleration, it is reasonable to expect that the biggest failure probability is associated to Slamming for the seakeeping hazards related to vertical motions and to Roll for the ones related to horizontal motions. Furthermore, it is recalled that the joint probability distribution is obtained from Eq. 53 as:

$$f_X(H_s, T_m, \chi) = \frac{1}{(2\pi)^{3} 0.0015^2 H_s T_m} \exp \left( -\frac{1}{2} \begin{pmatrix} \ln(H_s) - 1.480 \\ \ln(T_m) - 2.298 \\ \chi - 110 \end{pmatrix}^T \begin{pmatrix} 0.0482 & 0.0114 & 0 \\ 0.0114 & 0.0081 & 0 \\ 0 & 0 & 83.17 \end{pmatrix}^{-1} \begin{pmatrix} \ln(H_s) - 1.480 \\ \ln(T_m) - 2.298 \\ \chi - 110 \end{pmatrix} \right)$$

Eq. 59

The outcomes of the methods described in the previous chapter are illustrated in the following paragraphs.
4.1.3.1 FOSM

The following table illustrates the outcome of the FOSM method, described in subchapter 3.3, which can be summarized in computing $E[g(X)] = g(\{4.5 \, m, 10 \, s, 110^\circ\},$ evaluating $\delta g(X)$ $\delta Hs,$ $\delta g(X)$ $\delta Tm$ and $\delta g(X)$ $\delta \chi$ at $X = \{4.5m, 10s, 110^\circ\}$ (which was done numerically with the method of central difference) for computing $STD[g(X)]$ with Eq. 19, and then computing $P_f = \Phi(- E[g(X)] / STD[g(X)])$. With $\Phi$ standard gaussian C.D.F. The results obtained are shown in Table 13, reported below:

<table>
<thead>
<tr>
<th></th>
<th>$E[g(X)]$</th>
<th>$\delta g(X)/\delta Hs$</th>
<th>$\delta g(X)/\delta Tm$</th>
<th>$\delta g(X)/\delta \chi$</th>
<th>$STD[g(X)]$</th>
<th>$\beta_{simple}$</th>
<th>$P_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA</td>
<td>0.515</td>
<td>-1.08E-01</td>
<td>4.35E-02</td>
<td>-5.07E-03</td>
<td>1.02E-01</td>
<td>5.04</td>
<td>2.38E-07</td>
</tr>
<tr>
<td>LA</td>
<td>0.680</td>
<td>-7.11E-02</td>
<td>2.37E-02</td>
<td>4.30E-04</td>
<td>6.15E-02</td>
<td>11.06</td>
<td>9.49E-29</td>
</tr>
<tr>
<td>RL</td>
<td>0.514</td>
<td>-1.08E-01</td>
<td>-2.87E-02</td>
<td>1.61E-02</td>
<td>1.93E-01</td>
<td>2.66</td>
<td>3.86E-03</td>
</tr>
<tr>
<td>GW</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Inf</td>
<td>0</td>
</tr>
<tr>
<td>SL</td>
<td>0.99999</td>
<td>-5.81E-04</td>
<td>2.11E-04</td>
<td>-3.58E-05</td>
<td>5.94E-04</td>
<td>1682</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 13: 2nd case FOSM results

Table 13 shows that because the deterministic value of $\chi$ for $X = \{Hs, Tm\}$, was equal to the mean point considered in this section for $X = \{Hs, Tm, \chi\}$, the values in the first column are equal to the values of $E[g(X)]$ obtained for $X = \{Hs, Tm\}$. On the other hand, the inclusion of both $\delta g(X)$ $\delta \chi$ and $\sigma_\chi$ slightly increases $STD[g(X)]$ (see Eq. 19), and consequently $P_f$. Note that such increment is more significant in the case of Roll for which $\delta g(X)$ $\delta \chi$ is maximum. Even if the big effect of the variability of $\chi$ on Roll was also noticed with Figure 28, the Slamming failure probability seems underestimated since, as previously observed, Figure 28 also suggests that $P_{f,SL}$ should be bigger than the one obtained for Vertical Acceleration. The following computations will confirm this observation.

4.1.3.2 FORM

This subsection describes the outcome of the implementation of the FORM method illustrated in subchapter 3.4, and the same procedure already summarized in section 4.1.2.2 is applied, firstly all the limit states are mapped from the domain $[1m, ..., 15m] \times [4s, ..., 18s] \times [0^\circ, ..., 180^\circ]$ to the $U$ space with Eq. 27, where the checking point is identified for all the margin functions such that the distance from the origin $\beta_{geom} = \sqrt{U_1^2 + U_2^2 + U_3^2}$ is minimum, and lastly the failure probability is computed as $P_f = \Phi(- \beta_{geom})$. The results and also the coordinates of the checking point in the physical space, obtained from the coordinates in the $U$ space with Eq. 28, are shown in the table below:

<table>
<thead>
<tr>
<th></th>
<th>$U_1'$</th>
<th>$U_2'$</th>
<th>$U_3'$</th>
<th>$H_0$ [m]</th>
<th>$T_0$ [s]</th>
<th>$\chi'$ [deg]</th>
<th>$\beta_{geom}$</th>
<th>$P[g(X)&lt;0]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA</td>
<td>3.23</td>
<td>-1.43</td>
<td>1.10</td>
<td>8.94</td>
<td>10.62</td>
<td>120</td>
<td>3.70</td>
<td>1.07E-04</td>
</tr>
<tr>
<td>LA</td>
<td>5.59</td>
<td>-2.84</td>
<td>0</td>
<td>15.00</td>
<td>10.82</td>
<td>110</td>
<td>6.27</td>
<td>1.76E-10</td>
</tr>
<tr>
<td>RL</td>
<td>1.95</td>
<td>0.01</td>
<td>-1.10</td>
<td>6.74</td>
<td>11.03</td>
<td>100</td>
<td>2.24</td>
<td>1.26E-02</td>
</tr>
<tr>
<td>GW</td>
<td>5.39</td>
<td>-1.78</td>
<td>2.19</td>
<td>14.35</td>
<td>11.58</td>
<td>130</td>
<td>6.09</td>
<td>5.72E-10</td>
</tr>
<tr>
<td>SL</td>
<td>2.96</td>
<td>-1.19</td>
<td>1.10</td>
<td>8.42</td>
<td>10.65</td>
<td>120</td>
<td>3.38</td>
<td>3.66E-04</td>
</tr>
</tbody>
</table>

Table 14: 2nd case FORM results

Table 14 shows that for Lateral Acceleration the probability of failure is very low and its computation is not affected by the uncertainty introduced on $\chi$, indeed the first two coordinates of the checking point in the physical
space are the same that the one obtained in the previous section for $X = (H_s, T_m)$, while $U_3^* = 0$, thus from Eq. 28 $\chi^* = \mu_x = 110^\circ$, hence the same $\beta_{geom}$ is identified, which causes the same $P_f$.

On the other cases the estimated probability of failures increases, and the most significant change is found for Roll, which is the most dominant failure probability. Recalling that the FORM method has good accuracy in case of low probabilities, this suggests that the probability content inside the surface defined by the Roll limit state curves is the biggest, as will be confirmed by the application of the direct integration method.

Figure 29 and Figure 30 illustrate by points the surfaces outlined by the limit state curves both in the physical and the transformed space; the identification of the checking point is also shown on the right and the checking points are highlighted by the yellow circles. It is noticed that since all those surfaces are convex the linearization at the checking point implied by the FORM method induces an overestimation of the failure domain, and thus of $P_f$. Note that both in the charts in the middle and on the right, the colored straight lines representing $\beta$ connects the checking points $U = \{U_1^*, U_2^*, U_3^*\}$ to the origin of the transformed space $\{0,0,0\}$.

Figure 29: evaluation of the geometric reliability index for the limit state surface of vertical acceleration at the bridge (on the top in red) and the of lateral acceleration at the bridge (on the bottom in green)
Figure 30: Evaluation of the geometric reliability index for the limit state surface of roll (on the top in pink) and green water (on the bottom in light blue)
4.1.3.3 MAXIMUM LOG-LIKELIHOOD

This subsection illustrates the results of the application of the Maximum Log-Likelihood method, described in subchapter 3.5. Firstly, the checking points $X^*$ are identified directly in the physical space through the maximization of $l(X) = \ln f_X(X)$ on the points of the limit state surface of each margin function. Secondly, the first and second derivatives of both the margin and log likelihood function are evaluated at such checking points with the method of central difference. Lastly the probability of failure is approximated through the asymptotic formulation given by Eq. 34, which requires $f_X(H_s^*,T_m^*,\chi^*)$, the absolute value of $\nabla l(X^*)$, and the absolute value of the determinant of the matrix $H^*$, which is obtained with Eq. 35 from the Hessian and Gradient of both the margin and log-likelihood function. The following table (Table 15) illustrates the results of the maximization of the log likelihood function, the three parameters needed for the computation of Eq. 34, and the associated estimated probability of failure.

|     | $H_s^*$ [m] | $T_m^*$ [s] | $\chi^*$ [deg] | $l(X^*)$ | $f_X(H_s^*,T_m^*,\chi^*)$ | $|\nabla l(X^*)|$ | $|\det H^*|$ | $P[g(X)<0]$ |
|-----|-------------|-------------|-----------------|----------|---------------------------|-----------------|--------------|--------------|
| VA  | 8.84        | 10.50       | 120             | -12.23   | 4.88E-06                  | 3.01            | 7.42E+07    | 1.18E-09    |
| LA  | 15.00       | 10.83       | 110             | -25.61   | 7.57E-12                  | 4.23            | 4.72E+04    | 5.18E-14    |
| RL  | 6.74        | 11.00       | 100             | -7.65    | 4.75E-04                  | 1.49            | 1.67E+02    | 1.55E-04    |
| GW  | 14.28       | 11.5        | 130             | -24.47   | 2.35E-11                  | 3.07            | 7.26E+01    | 5.66E-12    |
| SL  | 8.31        | 10.5        | 120             | -11.03   | 1.62E-05                  | 2.79            | 2.42E+07    | 7.45E-09    |

Table 15: 2nd case result for the Log-Likelihood method

Table 15 shows that the maximum log-likelihood method results define the same seakeeping hazards order of severity of the FORM even if different order of magnitude are obtained. It is anticipated that the Monte Carlo method will confirm the result obtained with FORM method, rather than the one reported in this Table 15. It is observed that decreasing the discretization step of $\chi$, or improving the method of computation of the derivates, can reduce possible numerical errors. Another reason of the inaccuracy of the results is given by the asymptotic nature of Eq. 34, which is highly conditioned by the behaviour of $\ln f_X(X)$ at the checking point, it is stressed that: "If the maximum is a sharp peak, the approximations should be good. If it is flat, they won’t be as good" (Breitung, 1991).Figure 31 shows that, considering the limit state surface of Vertical Accelerations, the point of maximum log likelihood is not a sharp peak, since the colour variation in the neighbourhood of the point with maximum $\ln f_X(X)$ is not strong. A similar behaviour is observed, but not reported, for the other seakeeping hazards.

Figure 31: maximum of the log likelihood function computed on the limit state surface points
4.1.3.4 DIRECT NUMERICAL INTEGRATION METHOD

This subsection illustrates the results of the application of the Direct Numerical integration, described in subchapter 3.7, which can be simply performed starting from the discrete domain \([1m, ..., 15m] \times [4s, ..., 18s] \times [0^\circ, ..., 180^\circ]\) with steps of 0.25m, 0.25s and 10°. Hence, the integral \(P_f = \int \int \int g(H_s, T_m, \chi) f_X(H_s, T_m, \chi) dH_s dT_m d\chi\) is then solved exploiting a trapezoidal numerical integration. The following table illustrates the results obtained with the implementation of such procedure for each seakeeping hazards:

<table>
<thead>
<tr>
<th></th>
<th>VA</th>
<th>LA</th>
<th>RL</th>
<th>GW</th>
<th>SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P[g(X)&lt;0])</td>
<td>7.61E-05</td>
<td>1.92E-11</td>
<td>1.17E-02</td>
<td>1.74E-10</td>
<td>3.39E-04</td>
</tr>
</tbody>
</table>

Table 16: 2nd case, result for the direct numerical integration method.

Table 6 shows a global accordance with the FORM results and differences with both the FOSM and Maximum Log-Likelihood results. And, it is anticipated the Monte Carlo method will confirm the results obtained with the direct integration, which are the closest to the ones obtained with the simulation procedure, proving that the 3D degree of discretization of the domain was accurate enough for the proper implementation of this method.

4.1.3.5 MONTE CARLO SIMULATION METHOD:

This subsection describes the outcome of the implementation of the Monte Carlo Simulation method, described in subchapter 3.6. Firstly, it is estimated \(N\), or rather, the required simulations for such that it is valid \(\bar{P}_f (1 − 0.05) \leq P_f \leq \bar{P}_f (1 + 0.05)\), with \(\bar{P}_f\) the estimated probability of failure. The computation is done from Eq. 40 by \(N = 4 \frac{1-P_f}{0.05P_f}\), and the FORM results are exploited, the results listed in the following table are obtained:

<table>
<thead>
<tr>
<th></th>
<th>VA</th>
<th>LA</th>
<th>RL</th>
<th>GW</th>
<th>SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_f,\text{FORM})</td>
<td>1.07E-04</td>
<td>1.76E-10</td>
<td>1.26E-02</td>
<td>5.72E-10</td>
<td>3.66E-04</td>
</tr>
<tr>
<td>(N=4 \frac{1-P_f}{0.05P_f})</td>
<td>1.49E+07</td>
<td>9.09E+12</td>
<td>1.26E+05</td>
<td>2.80E+12</td>
<td>4.37E+06</td>
</tr>
</tbody>
</table>

Table 17: 2nd case, Required Simulations

Table 17 shows that the required simulations for Lateral Acceleration at the bridge are impractically high, thus the method is not applied in such case, while it is fixed \(N = N_{VA} = 1.49E + 07\). Note that it implies also reaching good approximation also for the cases where \(N_i < N_{VA}\). Then the simulation of the set of random points is performed with a built-in MATLAB function starting from the formulation of \(f_X(H_s, T_m, \chi)\) (Eq. 59) and \(N\). The following figure shows the simulated values of \(H_s, T_m\) and \(\chi\), and the associate marginal joint probability density function. As expected, the simulated density functions coincide with the analytical curves of \(f_{H_s}(H_s), f_{T_m}(T_m)\) and \(f_{\chi}(\chi)\).
Figure 32: 2nd case. On the top left: Simulated points, the color bar refers to $\chi$ and its units are in degree. On the other corners: marginal pdf from of the simulated sample. Hence, the probability of occurrence of $g(X_{\text{simulated}}) \leq 0$ is computed for the simulated sample, together with its mean value, which is plotted in function of the increasing sample size in the figure below. Note that the Vertical Acceleration failure probability, since very low, is referred to the axis on the right.

Figure 33: 2nd case, computation of $P_f$ with the increasing of the simulated sample size. Figure 33 shows that the estimated probabilities converge being $\epsilon$ the actual error computed from Eq. 40 respectively 0.058, 0.00474 and 0.029 for Vertical Acceleration, Roll and Slamming, the associated failure probabilities are $P_{f,VA} = 7.94E - 05$, $P_{f,RL} = 1.17E - 02$ and $P_{f,SL} = 3.17E - 04$. A direct comparison with the results obtained with the other method is show by Table 18 in the following subsection.

4.1.3.6 SYSTEM PROBABILITY OF FAILURE:
This subsection summarizes the results obtained so far and evaluate the system failure probability as illustrated in the subchapter 3.8. Firstly, the selection of the best estimation of $P_f$ is performed considering both
the outcome of the FORM method and the Monte Carlo simulation method, which replace the FORM Vertical Acceleration, Green Water and Roll probability of failure, as illustrated in the last column of the following table.

<table>
<thead>
<tr>
<th>seakeeping hazard</th>
<th>estimated P[g(X)&lt;0]</th>
<th>Monte Carlo Simulation</th>
<th>FORM+ Monte Carlo P[g(X)&lt;0]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FOSM</td>
<td>LOG-L.</td>
<td>DIRECT INT.</td>
</tr>
<tr>
<td>VA</td>
<td>2.38E-07</td>
<td>1.18E-09</td>
<td>7.61E-05</td>
</tr>
<tr>
<td>LA</td>
<td>9.49E-29</td>
<td>5.18E-14</td>
<td>1.92E-11</td>
</tr>
<tr>
<td>RL</td>
<td>3.86E-03</td>
<td>1.55E-04</td>
<td>1.17E-02</td>
</tr>
<tr>
<td>GW</td>
<td>0.00E+00</td>
<td>5.66E-12</td>
<td>1.74E-10</td>
</tr>
<tr>
<td>SL</td>
<td>0</td>
<td>3.39E-04</td>
<td>3.66E-04</td>
</tr>
<tr>
<td>worst</td>
<td>3.86E-03</td>
<td>1.55E-04</td>
<td>1.17E-02</td>
</tr>
</tbody>
</table>

Table 18: summary of the estimations of the failure probability for the 2nd case

From Table 18 it is noticed that most hazardous event is due to the roll motion for all methods. Comparing with the previous results for X = \( \{H_s, T_m\} \), the FOSM, FORM, numerical integration and Monte Carlo method outlines a global increment of the probability of failure with exception of lateral acceleration which, for the considered operational conditions, is not significantly affected by inclusion of uncertainty about \( \chi \), coherently with what initially noticed in Figure 28.

Secondly, the joint use of the outcome of the Monte Carlo Simulation method and the FORM method, reported in the last column of Table 18, is implemented to evaluate the failure probability of the system. In particular, considering the first order bounds computed only from the FORM results, it follows:

\[
\max P_f^{ FORM } = 0.0126 \leq P_f^{ system } \leq 1 - \prod_{i=1}^{n} (1 - P_f^{ FORM }) = 0.0130
\]

While considering the first order bounds obtained by the joint use of FORM and Monte Carlo simulated results in:

\[
\max P_f^{ FORM+MC } = 0.0117 \leq P_f^{ system } \leq 1 - \prod_{i=1}^{n} (1 - P_f^{ FORM+MC }) = 0.0121
\]

Moreover, the system probability of failure, evaluated with the direct numerical integration of \( f_X(X) \) over the union of the failure domains, results 0.0121. Hence it can be said that the implementation of the Monte Carlo method for the most significant seakeeping events, aimed at reducing the overestimation performed by the FORM method, both shifts and narrows the first order probability bounds and accordance was found between the joint use of these two methods and the results produced with the direct integration method.

4.1.4 3rd case: \( X = \{H_s, T_m, \chi, T, \Delta T, v, GM_T\} \)

The generic case, where all the operational and service variables identified at the beginning of this chapter are treated as random variables, leading to \( X = \{H_s, T_m, \chi, T, \Delta T, v, GM_T\} \) is investigated within this subsection. It is observed that for merely practical reasons, due to the high computational time of the RAOs with the seakeeping code and the associated limit state numeric identification, the domain of \( T, \Delta T, v, \) and \( GM_T \) was characterized by a rational selection of a finite number of points while the same discrete domain of the previous case is considered for \( H_s, T_m \) and \( \chi \). Thus, different levels of discretization of the domain are introduced for the new variables when applying the different methods, considering the best practical way to obtain significant results.
according to the peculiarities of each method. In particular the standard characterisation of the new random variables is done according to the following discretization: \( T \in \{7.5\, m, 8\, m, 9\, m\}, \Delta T \in \{-0.75\, m, 0\, m, 0.25\, m\}, \nu \in \{15\, kn, 22.5\, kn, 25\, kn\}, \, G M \in \{0.5\, m, 1\, m, 1.2\, m\}. \) Hence additional points are considered when required to improve the quality of the numerical solution.

Differently from the previous cases with \( X = \{H, T\} \) or \( X = \{H, T, \nu\} \), its no longer possible to directly visualize the behaviour of \( g(H, T, H, T, \nu, G M) \) for identifying the failure domains, and also the representation of \( f_x(H, T, H, T, \nu, G M) \) inside the 7-dimensional space is difficult. Hence, no rigorous preliminary consideration is done about what expect from the solution of the integral \( P_f = P[g(X) \leq 0] = \int_{\gamma} f_{g(X) = 0} f_X(X) dX \).

Moreover, from Eq. 53 the joint probability density function \( f_X(H, T, H, T, \Delta T, H, G M) \) is obtained as:

\[
f_X(H, T, H, T, \Delta T, H, G M) = \frac{1}{(2\pi)^3 \sqrt{1.3563E - 06}} \exp \left( -\frac{1}{2} \left( \begin{array}{c}
\ln(H) - 1.480 \\
\ln(T) - 2.298 \\
\chi - 110 \\
T - 9 \\
\Delta T - 0 \\
v - 24 \\
G M - 1 \\
\end{array} \right)^T \left( \begin{array}{c}
\ln(H) - 1.480 \\
\ln(T) - 2.298 \\
\chi - 110 \\
T - 9 \\
\Delta T - 0 \\
v - 24 \\
G M - 1 \\
\end{array} \right) \right) \tag{60}
\]

With

\[
\Sigma = \begin{pmatrix}
0.0482 & 0.0114 & 0 & 0 & 0 & 0 & 0 \\
0.0114 & 0.0081 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 83.17 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0037 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0024 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0924 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0037
\end{pmatrix}
\]

The outcome of the methods described in chapter 3 are illustrated in the following subsections.

### 4.1.4.1 FOSM

The following table illustrates the outcome of the FOSM method, described in subchapter 3.3. Firstly, it is computed \( E[g(X)] = g(\mu_X) = g(4.5\, m, 10\, s, 110^\circ, 9\, m, 0.24\, kn, 1\, m) \), and then with the method of central difference it is numerically evaluated, \( \delta g(X) / \delta X_i \) at \( X = \mu_X \). Hence, using Eq. 19 \( \text{STD}[g(X)] \) is obtained from the derivatives just computed, the standard deviations and the correlation coefficients. Finally, it is obtained \( P_f = \Phi(- \beta_{simple}) = \Phi(- E[g(X)] / \text{STD}[g(X)] \) with \( \Phi \) standard gaussian C.D.F. The results obtained are shown in Table 13, reported below:

<table>
<thead>
<tr>
<th>( E[g(X)] )</th>
<th>( \delta g(X) / \delta H_s )</th>
<th>( \delta g(X) / \delta T )</th>
<th>( \delta g(X) / \delta \nu )</th>
<th>( \delta g(X) / \delta G M )</th>
<th>( \delta g(X) / \delta \nu )</th>
<th>( \delta g(X) / \delta G M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA 0.515</td>
<td>-1.08E-01</td>
<td>4.35E-02</td>
<td>-5.07E-03</td>
<td>8.22E-03</td>
<td>-7.24E-03</td>
<td>-1.36E-02</td>
</tr>
<tr>
<td>LA 0.680</td>
<td>-7.11E-02</td>
<td>2.37E-02</td>
<td>4.30E-04</td>
<td>-8.08E-04</td>
<td>2.75E-03</td>
<td>-1.38E-03</td>
</tr>
<tr>
<td>RL 0.514</td>
<td>-1.08E-01</td>
<td>2.87E-02</td>
<td>1.61E-02</td>
<td>-1.26E-01</td>
<td>-2.61E-03</td>
<td>-5.20E-02</td>
</tr>
<tr>
<td>GW 1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>SL 0.9999</td>
<td>-5.86E-04</td>
<td>2.13E-04</td>
<td>-3.60E-05</td>
<td>2.87E-04</td>
<td>6.35E-05</td>
<td>-2.15E-05</td>
</tr>
<tr>
<td>( \text{STD}[g(X)] )</td>
<td>( \beta_{simple} )</td>
<td>( P_f )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VA 1.02E-01</td>
<td>5.03</td>
<td>2.43E-07</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LA 6.15E-02</td>
<td>11.06</td>
<td>1.03E-28</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>RL 1.95E-01</td>
<td>2.64</td>
<td>4.14E-03</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 19: 3rd case, FOSM results

Table 19 shows results very similar to the ones obtained with the same method considering less random variables. Indeed, the same operational conditions identify the same mean point for all the cases while the additional terms \( (\delta g(\mu_{\chi_i})/\delta x_i)^2 \sigma_{\chi_i}^2 \) considered in Eq. 19 have low magnitude, thus \( E[g(X)] \) is the same and \( \text{STD}(g[X]) \) is almost identical, leading to \( P_f \) similar to the ones obtained for \( X = \{H_s, T_m\} \) and \( X = \{H_s, T_m, \chi\} \). Hence, as will be confirmed by the application of other methods in the next subchapter, the probability of failures appears underestimated. No additional points were considered in the domain previously introduced when applying this method and, since the derivate at the mean point are computed numerically with the method of central difference, it is mentioned that incrementing the degree of discretization of domain would reduce possible numeric errors in the computation of \( \delta g(\mu_{\chi_i})/\delta x_i \) and thus \( P_f \).

4.1.4.2 FORM

The following table illustrate the outcome of the FORM method, described in subchapter 3.4. Firstly all the limit states are mapped from the domain in the \( \chi \) space to \( U \) space with Eq. 27, where the checking point is identified for all the margin functions such that the distance from the origin \( \beta_{\text{geom}} = \sqrt{\sum(U_i^*)^2} \) is minimum. Lastly the failure probability is computed as \( P_f = \Phi(-\beta_{\text{geom}}) \). It is highlighted that, in the process of minimization of the checking point (minimize \( \beta_{\text{geom}} \)), the domains of \( T, \Delta T, v, GM_T \), were progressively widened adding more points close to the solutions obtained at different iteration steps, until convergence was found. At the end of such process the domain characterized by \( T[m] \in \{8, 8.5, .9\}, \Delta T[m] \in \{-0.95, -0.9, -0.8, -0.25, 0, 0.35, 0.4, 0.45\} \), \( v[\text{kn}] \in \{23.5, 24, 24.5, 25\} \) and \( GM_T[m] \in \{0.95, 1, 1.1, 1.5\} \) is outlined. The computed probability of failure is listed in the last column of Table 20 and the coordinates of the checking point in the physical space, obtained from the coordinates in the \( U \) space with Eq. 28, are also reported.

<table>
<thead>
<tr>
<th>VA</th>
<th>LA</th>
<th>RL</th>
<th>GW</th>
<th>SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U_1^* )</td>
<td>( U_2^* )</td>
<td>( U_3^* )</td>
<td>( U_4^* )</td>
<td>( U_5^* )</td>
</tr>
<tr>
<td>3.3</td>
<td>-1.3</td>
<td>1.1</td>
<td>0</td>
<td>8.2</td>
</tr>
<tr>
<td>5.6</td>
<td>-2.9</td>
<td>0</td>
<td>0</td>
<td>-19.5</td>
</tr>
<tr>
<td>1.4</td>
<td>-0.1</td>
<td>-1.1</td>
<td>0</td>
<td>7.2</td>
</tr>
<tr>
<td>5.20</td>
<td>-1.72</td>
<td>2.19</td>
<td>0</td>
<td>-19.55</td>
</tr>
<tr>
<td>2.87</td>
<td>-1.08</td>
<td>1.10</td>
<td>0</td>
<td>-19.55</td>
</tr>
</tbody>
</table>

Table 20: 3rd case, FORM results

Table 20 shows that the highest probability of failure is due to Roll which means that for the associate transformed limit state the has the smallest distance from the origin of the 7-dimensional \( \chi \) space. Comparing with the result obtained with the same method for \( X = \{H_s, T_m, \chi\} \) it is observed an slight increment of the probability of failure due to slamming and a decrement of the Roll failure probability. It is stressed that it is not directly clear if the linearization at the checking point in the 7-dimensional space induces an overestimation or an underestimation of the failure domain, thus of \( P_f \), but a comparison with the results obtained with other method will be carried out at the end of the section.

4.1.4.3 MAXIMUM LOG-LIKELIHOOD

This subsection illustrates the results of the application of the Maximum Log-Likelihood method, described in subchapter 3.5. Firstly, the checking points \( \chi^* \) are identified directly in the physical space through the
maximization of \( l(X) = \ln f_X(X) \) on the points of the limit state surface of each margin function. It is recalled that computation of limit state is based on the definition of the ship response is made through the RAOs obtained from the seakeeping code. Thus, it is observed that, from a mere practical point of view, obtaining the RAOs associated to many different operational conditions identified by many different values of \( T, \Delta T, v, G.M_T \) is very time consuming. Hence, the maximization of \( \ln f_X(X) \) is done with the points analyzed for the minimization of \( \beta_{geom} \) required by the FORM method and not separately. However, it is observed that in the previous applications the checking point identified by the two different methods were similar. Once the checking points are identified, the first and second derivatives of both the margin and log likelihood function are evaluated at such points with the method of central difference. Lastly the probability of failure is approximated through the asymptotic formulation given by Eq. 34, which requires \( f_X(H^*_m, T^*_m, X^*, T^*, \Delta T^*, v^*, G.M^*_T) \), the absolute value of \( \nabla l(X^*) \), and the absolute value of the determinant of the matrix \( H' \), which is obtained with Eq. 35 from the the Hessian and Gradient of both the margin and log-likelihood function. Table 21 reports the results of the maximization of the log likelihood function, the parameters needed for the computation of Eq. 34, and the associated estimated probability of failure.

|        | \( H^*_m \) | \( T^*_m \) | \( \chi^* \) | \( T^* \) | \( \Delta T^* \) | \( V^* \) | \( G.M^*_T \) | \( l(X^*) \) | \( f_X(X^*) \) | \( |\nabla l(X^*)| \) | \( |\text{det } H'| \) | \( P[g(X)<0] \) |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|----------------|----------------|-----------------|
| VA     | 8.83        | 10.50       | 120         | 9           | 0           | 24          | 1           | -6.1        | 2.30E-03     | 3.01           | 1.19E+16        | 4.31E-10        |
| LA     | 15.00       | 10.66       | 110         | 9           | 0           | 24          | 1           | -20.1       | 1.95E-09     | 4.5            | 1.23E+11        | 7.63E-14        |
| RL     | 5.98        | 10.50       | 100         | 9           | 0           | 24          | 1.1         | -1.8        | 1.62E-01     | 27.1           | 2.68E+11        | 7.15E-07        |
| GW     | 14.3        | 11.5        | 130         | 9           | 0           | 24          | 1           | -18.32      | 1.10E-08     | 3.07           | 7.58E+11        | 2.56E-13        |
| SL     | 8.3         | 10.5        | 120         | 9           | 0           | 24          | 1           | -4.88       | 7.56E-03     | 2.78           | 3.73E+14        | 8.72E-09        |

Table 21: 3\(^{rd}\) case Maximum Log-Likelihood results

Table 21 shows different results from the FOSM and FORM method, even if the ranking of \( P_f \) severity is the same of the one obtained previously for the FORM method. However, it is recognized that a more detailed search of the 7-dimensional checking point of maximum log-likelihood, which was not performed for problem of computational time, should be done, and that incrementing the degree of discretization of domain would reduce possible numeric errors in the computation of the first and second order derivatives of \( g(X) \) and \( l(X) \) thus \( P_f \).

4.1.4.4 DIRECT NUMERICAL INTEGRATION METHOD

This subsection illustrates the results of the application of the Direct Numerical integration, described in subchapter 3.7. Firstly it is considered the domain outlined in the application of the FORM method, identified by \( \Delta T \in \{-0.95 \text{ m}; -0.9 \text{ m}; -0.8 \text{ m}; -0.25 \text{ m}; 0 \text{ m}; 0.35 \text{ m}; 0.4 \text{ m}; 0.45 \text{ m}\} \), \( v \in \{23.5 \text{ kn}; 24 \text{ kn}; 24.5 \text{ kn}; 25 \text{ kn}\} \), \( G.M_T \in \{0.95 \text{ m}; 1 \text{ m}; 1.1 \text{ m}; 1.5 \text{ m}\} \), \( H_s \in \{1 \text{ m}; \ldots; 15 \text{ m}\} \) with step of 0.25 m, \( T_m \in \{4 \text{ s}; \ldots; 18 \text{ s}\} \) with a step of 0.25 s and \( \chi \in \{0^\circ; \ldots; 180^\circ\} \) with 10\(^{th}\) step. Hence, the integral \( P_f = \int \int \int \int \int g(X) f_X(H_s, T_m, X, T, \Delta T, v, G.M_T) \, dX \) is then solved exploiting a trapezoidal numerical integration. Table 22 illustrates the results obtained with the implementation of such procedure for each seakeeping hazards.

<table>
<thead>
<tr>
<th></th>
<th>VA</th>
<th>LA</th>
<th>RL</th>
<th>GW</th>
<th>SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P[g(X)&lt;0] )</td>
<td>3.09E-04</td>
<td>1.03E-10</td>
<td>5.68E-02</td>
<td>7.38E-10</td>
<td>1.28E-03</td>
</tr>
</tbody>
</table>

Table 22: 3\(^{rd}\) case, result for the direct numerical integration method.

Table 22 shows that results obtained are generally in accordance with the FORM method, with exception of the probability of failure due to Roll. A more detailed comparison is reported later in subsection 4.1.4.6.
This subsection describes the application of the Monte Carlo Simulation method, illustrated in subchapter 3.6. The first step of the method is estimating \( N \), the required simulations for such that it is valid \( \bar{P}_f (1 - 0.05) \leq P_f \leq \bar{P}_f (1 + 0.05) \), with \( \bar{P}_f \) the estimated probability of failure. Such estimation is done from Eq. 40 by \( N = 4 \frac{1-P_f}{0.05P_f} \), and the FORM results are exploited, the results listed in Table 23 are obtained:

<table>
<thead>
<tr>
<th>( P_f^{FORM} )</th>
<th>VA</th>
<th>LA</th>
<th>RL</th>
<th>GW</th>
<th>SL</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>1.20E-04</td>
<td>1.40E-10</td>
<td>7.46E-03</td>
<td>2.56E-13</td>
<td>8.72E-09</td>
</tr>
<tr>
<td>1.33E+07</td>
<td>1.15E+13</td>
<td>2.13E+05</td>
<td>8.65E+11</td>
<td>2.85E+06</td>
<td></td>
</tr>
</tbody>
</table>

Table 23: 3rd case, Required Simulations.

The next step of the method should be simulating the margin function \( N \) times but it is observed that the computational effort required to compute the seven-dimensional margin functions at one point significantly increases, when compared with the computation done by less random variables (thus less dimensions). Consequently, even evaluating the \( g \) function for the number of simulations required by Roll, which is the most probable event, thus characterized by the lowest \( N \), is impractical. It is true also if the higher Roll probability of failure estimated with the direct numerical integration since the computational time is still too demanding (about 90 hours with a processor Intel® Core™ i5-82500 CPU @ 1.60 GHz). Hence, just for practical reasons, the Monte Carlo Simulation method is not implemented for this case.

4.1.4.6 SYSTEM PROBABILITY OF FAILURE:
This subsection summarizes the results obtained so far and evaluate the system failure probability as illustrated in the subchapter 3.8. Firstly, the best results should be selected and thus a comparison of all the value estimated with the previous method is made thanks to Table 24.

<table>
<thead>
<tr>
<th>seakeeping hazard</th>
<th>estimated ( P[g(X)&lt;0] )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FOSM</td>
</tr>
<tr>
<td>VA</td>
<td>2.43E-07</td>
</tr>
<tr>
<td>LA</td>
<td>1.03E-28</td>
</tr>
<tr>
<td>RL</td>
<td>4.14E-03</td>
</tr>
<tr>
<td>GW</td>
<td>0</td>
</tr>
<tr>
<td>SL</td>
<td>0</td>
</tr>
<tr>
<td>worst</td>
<td>4.14E-03</td>
</tr>
</tbody>
</table>

Table 24: summary of the estimations of the failure probability for the 3rd case.

Table 24 shows that, generally, accordance is found only between the FORM and the direct integration, with exception on Roll. It is observed that also for \( X = \{H_s, T_m, \chi\} \) and \( X = \{H_s, T_m\} \) good accordance was found. However, since it is not known if the FORM method overestimates or underestimates \( P_f \), since the domain of integration is not accurately discretized, and since it was not implemented the Monte Carlo simulation method, it is not possible to judge if the FORM or the direct integration results are the most accurate. On the other hand, it is reasonable to exclude the FOSM results since it returns always the same underestimated \( P_f \). Moreover, it is observed that it would be recommendable to implement the Monte Carlo Simulation in order to judge the quality of estimates given by the different methods, if high computational power and/or an analytical formulation of the \( g \) function would be available. However, analogously to what done in two previous
subchapters, the system probability of failure is evaluated considering the first order bounds computed from the FORM results, resulting:

$$\max P_f^{\text{FORM}} = 0.0075 \leq P_f^{\text{system}} \leq 1 - \prod_{i=1}^{n}(1 - P_f^{\text{FORM}}) = 0.0081$$

In addition, the system probability of failure, evaluated with the direct numerical integration of $f_X(X)$ over the union of the failure domains, results 0.0013 exceeding the interval identified by the probability bounds just computed. Thus, it is also observed that an increase in the resolution of the discretization of the domain of $X$ would be recommendable in order improve the quality of such integration.

### 4.1.5 Comparison of the FORM result for the three cases

The comparison of the different results obtained with the application of the FORM method for a different number of random variables, reported in the three previous subchapters is illustrated in this section and depicted in Figure 34. The analysis is limited to the FORM method to comply with document length requirement and because good accordance was found with Monte Carlo simulation and/or the numerical integration, being also faster than the two numerical methods, thus more practical.

![Figure 34: Comparison of the outcomes of the FORM method for different cases and seakeeping event. 1st case refers to $X = \{H_s, T_m\}$, 2nd case refers to $X = \{H_s, T_m, \mathcal{X}\}$, and 3rd case refers to $X = \{H_s, T_m, \mathcal{X}, T, \Delta T, \nu, GM_T\}$.

Figure 34 show that, given the input data described in Table 6 and considered the ship S175 operating in uncertain environmental conditions described by $H_s$, $T_m$, $\mathcal{X}$, if a simplified approach considering uncertain variable $X = \{H_s, T_m\}$ is applied, then the probability of failure associated to the seakeeping events is globally underestimated. Moreover, if $T, \Delta T, \nu, GM_T$ are also uncertain then considering $X = \{H_s, T_m, \mathcal{X}\}$ induces a slight underestimation of the probability of failure of Slamming and Vertical Accelerations. However, it is stressed that a Monte Carlo simulation should be carried out to evaluate the appropriateness of the FORM result, which for the first two cases are satisfactory. These observations cannot be generalized because different operational conditions characterized by, e.g., different means of the variables may lead to different results.
It is also stressed that the results of the FORM method rely on an accurate estimation of the limit-state function at the checking points often characterized by severe sea-states (see for instance Table 20). The adoption of a seakeeping code based on linear strip theory can thus affect the accuracy of the method. In such context it would be meaningful to evaluate the difference in the results when applying non-linear seakeeping software, though such objective is out of the scope of this work. It is also noticed that such necessity does not arise when applying the FOSM method, for which it is sufficient a good estimation of ship motions at the mean point, generally defined by more permissive operational conditions than the checking points.

**4.1.6 Sensitivity analysis**

In this section the sensitivity analysis is performed with the formulation introduced in subchapter 3.9, it is stressed that the notation $X_1$ indicates a coordinate of the checking point in the original space while the notation $U_1$ identify the coordinates of the checking point in the transformed space. It is recalled that the sensitivity of the checking point is not only dependent by the shape of the margin function and on the transformation implemented to map the points from the $X$ to the $U$ space, but also on numerical aspects such as the accuracy of the seakeeping code and the discretization of the domain.

**4.1.6.1 FORM, 1st case**

Firstly, it is recalled that considering $X = \{H_s, T_m\}$, such vector is mapped in $U$ space with Eq. 27 as $U_1 = \Phi^{-1}[F_{H_s}(H_s)]$ and $U_2 = (\Phi^{-1}[F_{T_m}(T_m)] - \rho_{12}\Phi^{-1}[F_{H_s}(H_s)]) / \sqrt{1-\rho_{12}^2}$ with $F_X$ marginal cumulative distribution, $\Phi^{-1}$ inverse of the normal cumulative distribution function and $\rho_{12}$ defined with Eq. 56.

Secondly, it is recalled that different ways of evaluating $\delta g(X)/\delta U_1$ and $\delta g(X)/\delta U_2$ can be implemented. The most straightforward is computing it with the method of central difference directly in the $U$ space while the other approach is based on the computation of $\delta g(X)/\delta X_1$ in the physical space, for then computing the derivates in the transformed space thanks to the Jacobian matrix $J$ of the transformation (Lemaire, 2009; Robert E. Melchers, 2018). Even if it is not reported, both ways were applied, and the same results were found.

Hence, form the derivates in the transformed space at the checking point is it possible to define the directional cosines with Eq. 50, visualized in Figure 8, as $\alpha_i = -\frac{\delta g(U)}{\delta U_i} / ||\nabla g(U)||$, which measures the sensitivity of $\beta$ to $U$ and if it results very low, then it means the associated variables can be treated deterministically (Robert E. Melchers, 2018). Table 25 list the results obtained.

<table>
<thead>
<tr>
<th></th>
<th>$\delta g(X)/\delta H_s$</th>
<th>$\delta g(X)/\delta T_m$</th>
<th>$\delta g(X)/\delta U_1$</th>
<th>$\delta g(X)/\delta U_2$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA</td>
<td>0.102</td>
<td>0.100</td>
<td>-0.220</td>
<td>0.078</td>
<td>0.943</td>
<td>-0.334</td>
</tr>
<tr>
<td>LA</td>
<td>-0.067</td>
<td>0.080</td>
<td>-0.218</td>
<td>0.064</td>
<td>0.960</td>
<td>-0.280</td>
</tr>
<tr>
<td>RL</td>
<td>-0.114</td>
<td>-0.012</td>
<td>-0.220</td>
<td>-0.010</td>
<td>0.999</td>
<td>0.046</td>
</tr>
<tr>
<td>GW</td>
<td>-0.400</td>
<td>0.265</td>
<td>-1.320</td>
<td>0.182</td>
<td>0.991</td>
<td>-0.136</td>
</tr>
<tr>
<td>SL</td>
<td>-0.623</td>
<td>0.572</td>
<td>-1.315</td>
<td>0.447</td>
<td>0.947</td>
<td>-0.322</td>
</tr>
</tbody>
</table>

Table 25: FORM Sensitivity factors for the 1st case $X = \{H_s, T_m\}$

The first four column from the left of Table 25 shows that the margin function slope at the checking point in original and transformed space the same sign and the comparison with in Figure 35 and Figure 36 with Figure 20 and Figure 21 confirm that's there is no radical change in the shape of the margin function. On the other hand, the derivates at the checking point respect to $H_s$ becomes more relevant in the $U$ space. Consequently, the computation of $\alpha_i$ ranked $U_1$, and thus $H_s$ as the most relevant random variable for all the margin functions.
while $U_2$ is less important. This can be visualized in Figure 35 and Figure 36 which show that the effect of a little variation of $U_1$ has a bigger influence than $U_2$ on the failure probability, since it is associated to a more relevant shift in the safe (or failure) domain. In Figure 35 and Figure 36 it is also possible to visualize the role of the sign of Table 25 since the increment of $U_2^*$ or decrement of $U_1^*$ to the coordinate of the checking points means moving from the limit states to the safe domain, with exception of Roll. It must be noticed that such consideration cannot be generalized; in fact, for instance, considering the same ship (that is fixing $g(X)$), a different joint probability distribution can variate the results obtained.

Please note that in Figure 35 and Figure 36 on the right it is reported a pie charts reporting $\alpha_i^2$ (as advised by Lemaire (2009)) while on the left it is represented the margin function mapped in the $U$ space. On the bottom sections of the margin function at the checking point are illustrated to visualize the role of $\delta g(X)/\delta U_i^*$. 
Figure 35: sensitivity factors in the U space, V.A. and L.A, 1st case
Figure 36: sensitivity factors in the U space, R.L. and GW, 1st case
4.1.6.2 FORM, 2nd case

Considering the formulation introduced in subchapter 3.9 and \( X = \{ H_m, T, \chi \} \), the sensitivity factors \( a_i = -\frac{\delta g(U)/\delta U_i}{\|\nabla g(U)\|} \), are defined in this section. The procedure illustrated in the previous section is still valid with the only addition of the new term \( U_3 = \Phi^{-1}[F_3(\chi)] \). The derivates are computed with the method of central difference, and the values of \( \delta g(U)/\delta X, \delta g(U)/\delta U_i \) and the associated sensitivity factor are listed in Table 26.

<table>
<thead>
<tr>
<th>( \frac{\delta g(X)}{\delta H_1} )</th>
<th>( \frac{\delta g(X)}{\delta T_m} )</th>
<th>( \frac{\delta g(X)}{\delta \chi} )</th>
<th>( \frac{\delta g(X)}{\delta U_1} )</th>
<th>( \frac{\delta g(X)}{\delta U_2} )</th>
<th>( \frac{\delta g(X)}{\delta U_3} )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA</td>
<td>-0.112</td>
<td>0.094</td>
<td>-0.008</td>
<td>-0.220</td>
<td>0.073</td>
<td>-0.076</td>
<td>0.901</td>
<td>-0.301</td>
</tr>
<tr>
<td>LA</td>
<td>-0.067</td>
<td>0.080</td>
<td>0.001</td>
<td>-0.221</td>
<td>0.064</td>
<td>0.012</td>
<td>0.959</td>
<td>-0.277</td>
</tr>
<tr>
<td>RL</td>
<td>-0.148</td>
<td>0.008</td>
<td>-0.024</td>
<td>-0.220</td>
<td>-0.007</td>
<td>0.216</td>
<td>0.713</td>
<td>0.022</td>
</tr>
<tr>
<td>GW</td>
<td>-0.417</td>
<td>0.412</td>
<td>-0.048</td>
<td>-1.312</td>
<td>0.349</td>
<td>-0.434</td>
<td>0.920</td>
<td>-0.245</td>
</tr>
<tr>
<td>SL</td>
<td>-0.711</td>
<td>0.520</td>
<td>0.070</td>
<td>-1.315</td>
<td>0.405</td>
<td>-0.642</td>
<td>0.866</td>
<td>-0.267</td>
</tr>
</tbody>
</table>

Table 26: FORM sensitivity factors, 2nd case

In this case no graphical interpretation of the result of Table 26 is available, however Table 26. shows that, similarly to the previous subsection, the derivates at the checking point respect to \( H_1^* \) becomes more relevant in the \( U \) space, the same happens also for \( \chi^* \) and \( U_1 \). In particular the computation of \( a_1 \) ranked again \( U_1 \), and thus \( H_1 \) as the most relevant random variable for all the margin functions while \( U_2 \) is less important. \( U_3 \), thus \( \chi \), is also relevant for the Roll case, coherently to what already observed in Figure 28.

4.1.6.3 FORM, 3rd case

Considering the formulation introduced in subchapter 3.9 and \( X = \{ H_m, T_m, \chi, T, \Delta T, v, GM_T \} \), the sensitivity factors \( a_i = -\frac{\delta g(U)/\delta U_i}{\|\nabla g(U)\|} \), are defined in this section. The same procedure described in subsection 4.1.6.1 is applied, with the inclusion of the terms \( U_4 = \Phi^{-1}[F_4(T)], U_5 = \Phi^{-1}[F_5(\Delta T)], U_6 = \Phi^{-1}[F_6(v)], U_7 = \Phi^{-1}[F_{GM_T}(GM_T)] \). The values of \( \delta g(U)/\delta X, \delta g(U)/\delta U_i \) computed with the central difference method, and the associated sensitivity factor are listed in Table 27.

<table>
<thead>
<tr>
<th>( \frac{\delta g(X)}{\delta H_1} )</th>
<th>( \frac{\delta g(X)}{\delta T_m} )</th>
<th>( \frac{\delta g(X)}{\delta \chi} )</th>
<th>( \frac{\delta g(X)}{\delta T} )</th>
<th>( \frac{\delta g(X)}{\delta \Delta T} )</th>
<th>( \frac{\delta g(X)}{\delta v} )</th>
<th>( \frac{\delta g(X)}{\delta GM_T} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA</td>
<td>-2.20E-01</td>
<td>7.55E-02</td>
<td>-7.68E-02</td>
<td>1.65E-03</td>
<td>0</td>
<td>-9.42E-03</td>
</tr>
<tr>
<td>LA</td>
<td>-2.21E-01</td>
<td>6.45E-02</td>
<td>6.39E-03</td>
<td>2.43E-03</td>
<td>-4.25E-05</td>
<td>-3.72E-03</td>
</tr>
<tr>
<td>RL</td>
<td>-2.20E-01</td>
<td>2.22E-02</td>
<td>2.17E-01</td>
<td>3.00E-03</td>
<td>0</td>
<td>-2.58E-02</td>
</tr>
<tr>
<td>GW</td>
<td>-1.32E+00</td>
<td>3.38E-01</td>
<td>-4.27E-01</td>
<td>-3.82E-02</td>
<td>1.27E-02</td>
<td>-8.11E-03</td>
</tr>
<tr>
<td>SL</td>
<td>-3.23E+00</td>
<td>1.16E+00</td>
<td>-1.32E+00</td>
<td>9.54E-02</td>
<td>2.21E-02</td>
<td>-4.54E-02</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
<th>( a_5 )</th>
<th>( a_6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>VA</td>
<td>8.97E-01</td>
<td>-3.08E-01</td>
<td>3.14E-01</td>
<td>-6.76E-03</td>
<td>0</td>
</tr>
<tr>
<td>LA</td>
<td>9.58E-01</td>
<td>-2.80E-01</td>
<td>-2.77E-02</td>
<td>-1.05E-02</td>
<td>1.85E-04</td>
</tr>
<tr>
<td>RL</td>
<td>6.65E-01</td>
<td>-6.71E-02</td>
<td>-6.59E-01</td>
<td>-9.11E-03</td>
<td>0</td>
</tr>
<tr>
<td>GW</td>
<td>9.24E-01</td>
<td>-2.37E-01</td>
<td>3.00E-01</td>
<td>2.68E-02</td>
<td>-8.90E-03</td>
</tr>
</tbody>
</table>

66
As shown in Table 27, \( \alpha_i \) ranks again \( U_1 \), and thus \( H_s \) as the general most important variable. Moreover, \( U_3 \) (thus \( \chi \)) play an important role also for Slamming and is even more relevant for Roll, as illustrated by \( \alpha_i \). On the other hand, generally, the lowest sensitivity is due to \( U_3 \) since it is associated with the smallest effect on \( \beta_{\text{geom}} \) due to a little variation of \( U_3 \). It is observed that considering \( T, \Delta T, v, GM_T \) and the associated \( U_3, U_5, U_\phi, U_\gamma \), the numerical computation of \( \delta g(U)/\delta U_i \), on which the computation of \( \alpha_i \) is based, is not performed in a highly discretize domain for problems of computational time. Hence, it would be meaningful to compare the results obtained with Table 27 with the one obtained with a more accurate method, i.e., based on an analytical description of the 7-dimensional margin function.

### 4.1.6.4 Omission Factor

It is recalled that given different outcomes of the same problem obtained accounting a different number of input data uncertainty, it is possible to express the relative error committed assuming a given random variable \( X_i \) equal to a deterministic value \( x_i \), this is measured by the omission factor \( o_{x_i=x_i} = \beta_{x_i=x_i}/\beta \). Note that considering the FORM method it can be also directly computed from \( P_{f_{X_i=x_i}} \) and \( P_f \) by \( o_{x_i=x_i} = \frac{\Phi^{-1}(P_{f_{X_i=x_i}})}{\Phi^{-1}(P_f)} \).

In particular, the omission factor defined by the inclusion of the random variable \( \chi \), which represents the transition from the first case with \( X = \{H_\phi, T_m\} \) (section 4.1.2) and the second case with \( X = \{H_s, T_m, \chi\} \) (section 4.1.3), and the one defined by the inclusion of the other random variables \( T, \Delta T, v, GM_T \), expressing the transition to the third case with \( X = \{H_s, T_m, \chi, T, \Delta T, v, GM_T \} \) (section 4.1.4), are reported in Table 28. In addition, a graphical representation is given in Figure 37. The comparison considers the result obtained with the FORM method.

<table>
<thead>
<tr>
<th>FORM P[( g(X)&lt;0 )]</th>
<th>omission factor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( o_{1^{st} \rightarrow 2^{nd}} )</td>
</tr>
<tr>
<td>1(^{st}) case</td>
<td>2(^{nd}) case</td>
</tr>
<tr>
<td>VA</td>
<td>2.43E-05</td>
</tr>
<tr>
<td>LA</td>
<td>1.76E-10</td>
</tr>
<tr>
<td>RL</td>
<td>8.04E-04</td>
</tr>
<tr>
<td>GW</td>
<td>7.03E-14</td>
</tr>
<tr>
<td>SL</td>
<td>4.70E-05</td>
</tr>
</tbody>
</table>

Table 28: omission factors computed with the FORM method

![Figure 37: omission factors]
Figure 37 and Table 28 show in a more direct way all the observation already made in subsection 4.1.5. It is here more evident that accounting for more random variables mostly have an influence on Green Water and Roll. Furthermore, considering the cases with \( X = \{H_s, T_m\} \) or \( X = \{H_s, T_m, X\} \) it was observed that due to convex shape of the limit state curves (or surfaces) the linearization at the checking point implied by the FORM leads always of a slight overestimation of the failure domain and the \( P_f \) while in the case for \( X = \{H_s, T_m, X, T, \Delta T, v, GM_T\} \) this is not visualizable and it was not possible to compare the result obtained with exact solutions. Hence, if possible, also to results reported in Table 28 should be compared with the ones obtained with a more accurate computation of \( P_f \).

### 4.1.7 Observations

A list of additional significant observation is reported below.

→ Firstly, when selecting the numeric values to be assigned in Table 6 different combinations of values has been tried, and not reported within this thesis work for the sake of brevity, but a significant set of data was selected in order to show not trivial cases with probability of failure too low, as they are expected to be in normal conditions, while still depicting a realistic scenario.

→ Secondly, studying the variation of the FORM results with an increased number of random variables it is found that the probability of failure generally increases. However, as highlighted in Mansour, (1990):“it should be emphasized that inclusion of the distribution information does not always yield larger safety index”. Thus, this result cannot be directly generalized, in fact an increase or reduction of the risk associated with the extension the domain of the variables on which the hazards depend, is subjected to the operational conditions in which the risk is evaluated, and whether these lay on a peak, pit, or slope of the considered hazard.

→ Thirdly, in all the three cases, general similarities can be noticed. Table 12, Table 18 and Table 24, show quite low probabilities of violation of the limit state for all the respective seakeeping hazards analyzed, with a minor exception for the 2\(^{nd}\) case. Consequently, also the first order bounds of the probability of series of the system gives low probabilities. Furthermore, it is recalled that as the first order bounds are the narrower the more dominant is the probability of failure of a seakeeping event over the others, and since the range identified by the bounds is never superior to 0.01 in any cases, a good and conservative approximation of the system probability of failure can be represented by the first order upper bound.

→ Fourthly, in the 1\(^{st}\) and 2\(^{nd}\) case, where the limit state can be easily visualized, it is clear that, as explained in subchapter 3.4, approximating the limit state as done with the FORM method implies overestimating the solution of the reliability integral. Anyway, such overestimation is negligible when the exact solution of the reliability integral is a low probability, as mentioned in subsection 4.1.2.6. In particular when applying the Monte Carlo simulation, consistently assumed as the exact solution of the reliability integral since a small value of the Shooman Error was imposed, the FORM method gave a better estimate than the FOSM and Maximum Log-likelihood method. In the 1\(^{st}\) and 2\(^{nd}\) cases, the numerical direct integration also gives good approximation of the exact solutions, but it is mentioned that the quality of such approximation decreases if the magnitude of the failure probability lowers and/or the resolution of the discretization decreases. Furthermore it was confirmed that the FOSM method can be inaccurate with low probabilities of failure or nonlinear margin function (Choi, Canfield and Grandhi, 2007)
Lastly, in the 1st case the application of the FORM method for the alternative conditional ordering gave identical results, confirming what discussed in subsection 3.4.2.3 and proving the good quality of the search procedure. Moreover, recalling that, since $X$ is composed by lognormal and gaussian distributed variables, the Nataf transformation is equivalent to the Rosenblatt transformation, and the reparameterization to the $U$ space can be performed analytically, and just for one conditional ordering. As a general consequence the FORM method results faster than the FOSM and Maximum Log-Likelihood method which are based on the computation of derivates, and also, as expectable, than the direct numerical integration method.

As a consequence, in the next applications reported in the following subchapters the estimation of the failure probability will be done with the application of the first order bounds obtained with the joint use of FORM and Monte Carlo method, considering $X = \{H_s, T_m, \chi\}$ since when considering $X = \{H_s, T_m, \chi, T, AT, v, GM_T\}$ the implementation of the Monte Carlo method was not possible.
4.2 NAVIGATIONAL SAFETY OF ALTERNATIVE ROUTES

This subchapter illustrates an application of the methods described in subchapter 4.1, for evaluating the failure probability of containership “S175”, with $L_{bp} = 175\ m, B = 25.4, T_{\max} = 9.5\ m$. considering a given mission. More detailed information about the ship can be found in ANNEX 1. Two main mission alternatives are considered, characterized by routes with different course and speed profile. Furthermore, for each route two more alternatives characterized by a different starting time are evaluated: the first option has a standard departure at midnight of the 23rd of September 2020, while in the second option the ship departure is delayed of twenty hours, and a perturbation moving from north-east affects more significantly the environmental conditions. Hence the total of mission alternatives assessed within this subchapter is four. A sketch of the routes waypoints is reported in Figure 38 together with an example of forecasted data:

![Routes analyzed](image)

Figure 38: From left: sketch of the routes, forecasted mean Hs, forecasted Hs standard deviation

Note that in Figure 38 the word “westward” implies that both routes are directed from Europe to the North America, and the term “fast” indicates the routes with an overall higher speed profile, as shown in Figure 39; hence from now on they will be referred in such way to distinguish them.

The information about the route waypoints, the ship heading, and the ship speed is taken from past routing studies about the same ship. However, it is stressed that within this context such information is considered as input to have realistic values, and it is not related to an optimization problem. On the other hand, uncertainties will be associates to some data, and the probability of exceeding the operational limits described in the first column of Table 2 will be computed accordingly.

![expected speed](image)

Figure 39: speed profile of the two routes
In particular, since the application of a maximum number of three random variables with \( X = \{ H_s, T_m, \chi \} \) was completely successful in the previous subchapter, also in this subchapter the significant wave height and the mean wave period and the relative wave direction are taken as uncertain and will be characterized for each waypoint as described in section 4.2.1. Hence, while \( v \) is considered varying as described in Figure 39, it will be considered \( T = 9m \), \( \Delta T = 0m \), \( GM_T = 1m \) and all these four parameters will be treated deterministically.

Hence, while \( T, \Delta T, GM_T \) are constant in all waypoint, \( v, H_s, T_m, \chi \) varies in each waypoint and consequently also the limit state surface \( g(H_s, T_m, \chi) = 0 \) identified by each margin function at different \( v \). Moreover, since \( H_s, T_m, \chi \) are random variables, different values of \( \mu_{H_s}, \mu_{T_m}, \sigma_{H_s}, \sigma_{T_m}, \sigma_{\chi} \) are associated through forecasted data to the waypoint corresponding at different instants of the mission. Hence also \( f_{H_s,T_m,\chi}(H_s, T_m, \chi) \) equation varies in each waypoint. Moreover, even if in realistic applications also \( \rho_{H_s,T_m} \) should vary in different geographical locations, within this subchapter it is assumed that \( \rho_{H_s,T_m} = \text{cost} \) and equal to the value estimated in Table 6. It follows that since both \( g(H_s, T_m, \chi) = 0 \) and \( f_{H_s,T_m,\chi}(H_s, T_m, \chi) \) varies during the mission \( P_f = \int \int g(H_s, T_m, \chi) = 0 \) and \( f_{H_s,T_m,\chi}(H_s, T_m, \chi) \) is solved considering each time a different integrand function and a different integration domain.

A sketch representing a generic bidimensional example of the variation of the same limit state and joint density function in different moment of the missions is given in Figure 40. Note that in such sketch also the correlation coefficient of the bidimensional PDF varies while in the application within this subchapter it is always constant.

Hence, the procedure applied in this subchapter can be summarized solving \( P_f = \int \int f(X) \leq 0 f_X(X) dX \) for each discrete waypoint through the methods applied in the previous subchapter considering the four different mission scenarios introduced. After such computation, at the end of this chapter all the results will be exploited to associate unique \( P_f \) to each of the four mission alternatives. Moreover, even if for each waypoint the \( P_f \) related to each seakeeping hazard will be defined with the joint use of the FORM (or in few cases FOSM) and Monte Carlo simulation method, also the FOSM and direct numerical integration method results are computed and reported in ANNEX 1. It is observed that the assessment performed within this subchapter is analogous.
to the approach given by Spanos et al. (2008) and Papatzanakis et al. (2012), but improved by successfully considering also the relative wave direction $\chi$ as a random variable.

4.2.1 Statistical characterization of $X$ with forecasted data

As mentioned, the significant wave height and the mean wave period and the relative wave direction are taken as uncertain, hence $f_{H_s, T_m, \chi}(H_s, T_m, \chi)$ is defined with $\chi$ gaussian distributed and $H_s, T_m$ lognormally distributed and correlated as illustrated in subsection 4.1.4. Moreover, the assignment of mean values $(\mu_{H_s}, \mu_{T_m}, \mu_{\chi})$ and standard deviations $(\sigma_{H_s}, \sigma_{T_m}, \sigma_{\chi})$ at each waypoint of the routes is based on the forecasted data provided by the NOAA Operational Model Archive and Distribution System (NOMADS) (Rutledge, Alpert and Ebisuzaki, 2006). In particular, NOMADS associates forecasted data to a fixed grid of geographical locations, with 0.5° of space resolution for both longitude and latitude, with a time step of three hours. Hence, knowing in which waypoint the ship is located at a given instant of the mission, the characterization of the environmental variables for each waypoint is performed interpolating the data offered by NOMADS both in time and space.

The forecasted data is summarized in the following subsections, describing the sea conditions encountered by the ship during its mission according to the different scenario. Each set of value defines different shapes of $f_X(H_s, T_m, \chi)$ in different waypoint analogously to what show in Figure 40 for a generic bidimensional case.

4.2.1.1 NO DELAY

Figure 41 illustrates the variation of $\mu_{H_s}, \mu_{T_m}, \mu_{\chi}$ during the mission time, the effect of $\sigma_{H_s}, \sigma_{T_m}, \sigma_{\chi}$ is expressed by the lines $\mu_{i} \pm \sigma_{i}$.

![Figure 41: random environmental data in case of no delay : $\mu_i$ and $\mu_i \pm \sigma_i$.](image)

Figure 41 shows uncertainties, represented by the lines $\mu_{i} \pm \sigma_{i}$, that globally increases with the passing of time, as expected, since the forecasted data became less precise. However, because the most severe condition is encountered in the first part of the mission for both the faster and slower route, it is also expected that this phenom does not affect significantly the assessment of the failure probability.
4.2.1.2 DELAY

Figure 42 reports the forecasted $\mu_{H_s}$, $\mu_{T_m}$, $\mu_\chi$ and $\mu_{\sigma_i}$ predicted in the case of delay.

![Figure 42: random environmental data in case of delay: $\mu_\chi$ and $\mu_{\sigma_i}$](image)

Figure 42 shows a slight global increment of the uncertainties being to the forecasted data more distant in time because of the delay. Moreover, it is also observed that the delay which force the ship to face more hazardous seas states in the first part of the mission, since a perturbation moves from the northwest Atlantic Ocean affecting more the European waters. Hence it is expected that this phenom does negatively affects the assessment of the failure probability.

4.2.2 Results

In this section the probability of exceeding the operational limit of Roll, Green Water and Vertical and Lateral Acceleration is illustrated for each discretized instant of the ship mission. Moreover, at each instant the probability of failure due to the union of those seakeeping events is computed, assuming it equal to the first order upper bounds by: $P_{f,\text{system}} \approx 1 - \prod_{i=1}^{n} (1 - P_{f_i}^{\text{FORM+MC}})$. The results of $P_{f_i}$ obtained with different methods are reported in ANNEX 1.

Two following subsection gives a graphical description of the results obtained for the case of delay and no delay for the faster and slower route alternatives.

4.2.2.1 NO DELAY

Figure 43 compares the estimated value of $P_{f,\text{system}}$ for the faster and slower route alternatives, note that the axis are in a logarithmic scale. In such comparison results smaller than $10^{-20}$ are not represented. Figure 43 shows that in the case of no delay the safest route is the “slower” which is practically safe ($P_{f,\text{system}} \ll 0.01$), and so is the “faster” route which shows a slight increment of $P_f$. Note that $P_{f,\text{system}}$ is also reported as dashed line.
Figure 43: NO delay, failure probability due to each seakeeping hazard

Figure 43 shows that $P_f$ due to Slamming or Roll always dominates the others, often by more than one order of magnitude. Consequently, the approximation introduced considering the first order bounds equal to $P_{f,system}$ is negligible. Furthermore, the most hazardous event happens always in the very first part of the mission, while in the other cases the failure probabilities are quite low (smaller than 0.01). Moreover, since the last part of the mission is characterized by greater uncertainty about $\chi$ as illustrated in Figure 41, other seakeeping hazards becomes slightly more relevant, particularly for the faster route.

4.2.2.2 DELAY

Figure 44 compares the estimated value of $P_{f,system}$ for the faster and slower route alternatives in the case of delay and shows that the delay has a undesirable effects on the ship operability since in the two cases the probability of failure at the beginning of the mission is incremented. The safer route is still the “slower” alternative, while the “faster” alternative is the most hazardous option. $P_{f,system}$ is also reported both as dashed lines and once again the probability of exceedance of a operational limit (Slamming or Roll) globally dominates other failure modes. Hence the use of the first order bounds to approximate $P_{f,system}$ is again satisfactory. Moreover, the global increment of the uncertainty in the last part of the mission, described in Figure 42, increases the probability of failure of failure of other seakeeping hazards.
In this subsection the failure probability associated with an entire route is assessed considering, firstly, the failure events at each waypoint of the route as a series of independent events and, secondly, computing through $P_{f,route} = 1 - \prod_{i=1}^{n}(1 - P_{f,i=t_{i}})$ (Eq. 47) the probability of failure associated of the system of waypoint, or rather, the route. Moreover, it is possible to associate to each route a generalized reliability index $\beta_{gen}$ given by $\beta_{gen} = -\Phi^{-1}(P_f)$ (Eq. 52), as shown in Table 29:

<table>
<thead>
<tr>
<th>ROUTE</th>
<th>Slower</th>
<th>Faster</th>
</tr>
</thead>
<tbody>
<tr>
<td>case</td>
<td>no delay</td>
<td>delay</td>
</tr>
<tr>
<td>$P_{f,route}$</td>
<td>3.3592E-11</td>
<td>0.0496</td>
</tr>
<tr>
<td>$\beta_{gen}$</td>
<td>6.527</td>
<td>1.649</td>
</tr>
</tbody>
</table>

Table 29: Generalized reliability index for each route and scenario

Table 29 allows to rank the routes from a safety point of view, and, if the consequences of the failure are known, it is possible to associate at each route and condition the relative risk. Furthermore, from Table 29 is also clear that the slower route is the best option for the criteria considered within this thesis work, and that in the case of delay the faster route is the most hazardous option. However, since such probability of failure is related to a small part of the mission, an appropriate modification in just some waypoints the route may have a huge positive effect on $P_{f,route}$.

4.2.3 Observations

As already mentioned, the FORM method is not reliable when high probabilities of failure are involved, as can be seen in the graphs of ANNEX 1, but on the other hand the number of required Monte Carlo simulations to achieve a good estimate of the reliability integral significantly decreases, hence the application of the joint use of the Monte Carlo method and the FORM method is in principle still a valid procedure. Anyway, in the cases
where the probability of failure is supposed to be high (represented by a negative expected value of the margin function, for instance), within this subchapter the FORM method is replaced by the FOSM method for estimating the number of required Monte Carlo simulations. Hence, the first order bounds are computed with the combined use of the Monte Carlo and FORM or FOSM methods according to the magnitude of the failure probability estimated with the approximated methods.

4.3 NAVIGATIONAL SAFETY OF AN OCEAN AREA

In this subchapter the navigational safety of a fishing vessel is evaluated in a wide geographical area at a given instant. Hence this subchapter can be interpreted as the joint application of the approaches presented in the previous two subchapters 4.1 and 4.2. Indeed, from a practical point of view, a set of point is chosen to represent discretely the ocean area, similarly to the waypoints representing the route, and in each point the solution of the integral \( P_f = \int \int \int f_X(H_s, T_m, \chi) \, dH_s \, dT_m \, d\chi \) considering a certain instant is computed. In this way it is possible to evaluate the effect on the failure probability of the ship operations caused by the different ship positions within the area of interest at the same instant. In the following subsections firstly the ocean area is defined, then it is presented the statistical characterization of \( X \), and lastly the outcome of the methods introduced in chapter 3 is presented. The ship considered within this subchapter a fishing vessel characterized by \( L_{bp} = 41.28 \, m \), \( B = 10.32 \, m \), \( T_{max} = 4.35 \, m \). More detailed ship data is given in ANNEX 2.

Differently from the previous Route application, in this subchapter \( v \) is considered constant and the ship heading is estimated with respect to the North with a straight line connecting the starting point (Azores or Lisbon) to the point of the ocean area in which the probability of failure is computed solving \( P_f = \int \int \int f_X(H_s, T_m, \chi) \, dH_s \, dT_m \, d\chi \).

4.3.1 The Ocean Area

Two different categories of mission are assumed in order to define the ocean area of interest. Firstly, it is considered that the fishing vessel operates in the Major Fishing Area within the Portuguese Waters, as outlined by the Food and Agriculture Organization of the United Nations (FAO, 2017), with two possible alternatives. Firstly, it can leave from Lisbon reaching the zone 27.9b.1 or, secondly, it can leave from the Azores for reaching the zone 27.10a.1. A sketch depicting such zones is given in Figure 45, reporting also the mean significant wave height forecasted by the NOMAD for the instant considered, or rather the 2\(^{nd} \) of October 2020 at 3 PM.

The representation of the Ocean Area by waypoints is described in Table 32 given in ANNEX 2. Such waypoints are also illustrated in Figure 45.
4.3.2 Statistical characterization of $X$ with forecasted data

As mentioned, it is considered $X = \{H_s, T_m, \chi\}$ with $H_s$ and $T_m$ are lognormal distributed and correlated with the correlation coefficient reported in Table 6, while $\chi$ is assumed gaussian distributed and not correlated with $H_s$ and $T_m$. The remaining variables $T$, $\Delta T$, $v$ and $GM_T$ are hypnotized deterministic and constant for every point of the ocean area with $T = 3.5 m$, $\Delta T = 0 m$, $v = 12 kn$ and $GM_T = 1 m$.

For each point of the Ocean Area the means of the environmental variables $\mu_X = \{\mu_{H_s}, \mu_{T_m}, \mu_{\chi}\}$ are obtained from the NOMADS and so are their respective standard deviations $\sigma_X = \{\sigma_{H_s}, \sigma_{T_m}, \sigma_{\chi}\}$, allowing $f_{H_s,T_m,\chi}(H_s, T_m, \chi)$ to be defined for each discrete waypoint. Note that, as mentioned in the previous subchapter while in this application $\rho_{H_s,T_m} = \text{const}$, in more realistic applications the variation of $\rho_{H_s,T_m}$ between the different geographical locations should be included if such information is available.

The representation of the forecasted data for each waypoint is illustrated in Figure 46, note that on the x-axis the waypoint identifier is reported.

Figure 46 shows that the area 27.9.b.1 is affected by the highest forecasted $\mu_{H_s}$, as illustrated also in Figure 45, moreover the uncertainties associated to the random variables are quite high. This is due to the fact that the date selected for this application is close to the date of end of the weather forecast, when the perturbation mentioned in the previous chapter, which was coming from North America, arrives in the European waters. Such choice was performed to not achieve trivial results characterized by very low probabilities.
4.3.3 Results

From the input data defined the probability of exceeding the operational limit of Roll, Green Water and Vertical and Lateral Acceleration is computed for all waypoints. The probability of failure due to the union of those seakeeping events is computed: \( P_{f, \text{system}} \approx 1 - \prod_{i=1}^{n} (1 - P_{f_i}^\text{FORM+MC}) \). In this section just the system probability of failure will be illustrated for each waypoint, while the results of \( P_{f_i} \) obtained with different methods for all the single seakeeping hazards are reported in ANNEX 2.

In order to better comprehend the results it is observed that the considered fishing ship being a completely different kind of ship than the containership S175, presents different limit state curves at \( \chi = \text{cost} \), as show in Figure 48.
Figure 47: contour of \( g(H_s, T_m, \chi) = 0 \) with \( \chi = \cos \) for the analyzed seakeeping hazards and the fishing vessel

Figure 47 shows that also seas states with significant wave height between 2m and 4m can imply the limit state violation for a relatively wide interval of wave mean period. Hence some quite hazardous operability conditions are identified for many waypoints. Figure 48 reports a map illustrating \( P_{f,system} \) or rather the probability of failure due to the union of the seakeeping events computed with the joint use of the Monte Carlo Simulation method and the FORM method.

Figure 48: navigational safety map of the investigated ocean area

Figure 48 shows that the estimated probability of failure in the west points of the fishing area 27.9 is up to 50%, while the west of the area 27.10.a1m, which presents low failure probabilities, represents the best alternative from a safety point of view. As it can be better appreciated in the more detailed results reported in ANNEX 2, the probability of failure associated to the seakeeping event are dominated by the probability of exceedance of the Roll operability limit, given in Table 2. Therefore, the first order bounds converge in such cases. This outcome confirmed what suggested by Figure 47 which depicted for Roll the biggest failure domain. This section presented a first simple step for an interesting application of the approach developed in this work, aimed at offering decision support tools for an increased navigational risk awareness when planning activities offshore.
5 CONCLUSIONS

Many probabilistic methods for assessing the failure probability of a ship operating during its mission have been introduced. Such methods outline a common framework which can be incorporated into a decision support system in order to give assistance to the decision maker, and can be implemented both for onboard assessment, to respond to specific operational conditions, and for route planning, to choose the best alternative from forecasted information. Moreover, the methods make possible to account for uncertain data in order to evaluate the ship safety accordingly. The fundamental aspects of the methods have been discussed and practical applications have been highlighted for judging instantaneous operational safety, the safest route alternative, and the safest area of a whole geographical ocean area of interest. Furthermore, a detailed comparison of the differences of the outcome obtained by applying the considered methods has been illustrated through an example, together with a sensitivity analysis for the FORM method. Moreover, since the framework illustrated has the ability to easily incorporate more than two random variables, an evolution of the approach given by Spanos and Papanikolaou (Spanos et al., 2008; Papatzanakis, Papanikolaou and Liu, 2012)., which only considers the significant wave height and the wave peak period, has been successfully applied for the cases just described. When dealing with many random variables linked to the direct input of the seakeeping program, it has been recognized the importance of building a detailed database with the RAOs computed from the seakeeping code for a wide and highly discretized domain. Such a detailed characterization of the inputs has not been always possible within this work due to purely computational limitations, nevertheless this does not invalidate the framework identified within this thesis to carry out the assessment of the operational failure probability, which is one of the main goals of the thesis itself.

Further development of this thesis work can include the consideration of uncertainties related to other possibly relevant parameters, e.g., the spread parameter involved in the definition of the directional wave spectrum. In addition, obtaining an accurate analytical formulation of the margin function and of the limit state (e.g. through response surface methodology), can significantly improve the computational time leading to the implementation of the Monte Carlo Method also for problems with more than three random variables. Moreover, since the reliability integral depends on the formulation of the limit state, which also comprises severe operational scenarios for which the assumption of linear strip theory loose validity, the obtained results should be confirmed by the application of a more accurate theory for assessing ship motions. The FOSM method is less affected by such problem, while for the FORM method, which is based on checking points comprised by limit states, a greater influence can be expected. Furthermore, because a crucial role is played by the statistical characterization of the random variables, a deeper knowledge about their correlations and/or their probabilistic distributions is of profound interest and can be directly implemented in the framework outlined within this thesis. Finally, the integration of such approach on existing or new decision support systems can increase the situation awareness, and lead to more effective response to operational hazards.
REFERENCES


Naval Hydrodynamics, pp. 544–596.


ANNEX 1

Ship data

The input data to the seakeeping program regarding both subchapter 4.1 and 4.2 are summarized in the following table:

<table>
<thead>
<tr>
<th>Ship data</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length between perpendiculars</td>
<td>175 m</td>
</tr>
<tr>
<td>Breadth</td>
<td>25.4 m</td>
</tr>
<tr>
<td>Maximum draft</td>
<td>9.5 m</td>
</tr>
<tr>
<td>Maximum speed</td>
<td>30 kn</td>
</tr>
<tr>
<td>Structural radius of gyration about the x axis</td>
<td>9 m</td>
</tr>
<tr>
<td>Structural radius of gyration about the y axis</td>
<td>42 m</td>
</tr>
<tr>
<td>Structural radius of gyration about the z axis</td>
<td>42 m</td>
</tr>
<tr>
<td>Structural product of inertia</td>
<td>0 kg*m²</td>
</tr>
<tr>
<td>Propeller diameter</td>
<td>6.15 m</td>
</tr>
<tr>
<td>Propeller expanded area ratio</td>
<td>0.9</td>
</tr>
<tr>
<td>Bulb transverse area</td>
<td>0 m²</td>
</tr>
<tr>
<td>Bulb vertical centre of transverse area</td>
<td>4.84 m</td>
</tr>
<tr>
<td>Breadth of bilge keels</td>
<td>0 m</td>
</tr>
<tr>
<td>Length of bilge keels</td>
<td>0 m</td>
</tr>
<tr>
<td>Wetted area of the appendages</td>
<td>60 m²</td>
</tr>
<tr>
<td>$C_{stem}$, Holtrop &amp; Mennen (1982) coefficient</td>
<td>0</td>
</tr>
<tr>
<td>Appendages resistance factor $1+k_2$, Holtrop &amp; Mennen (1982) coefficient</td>
<td>1.4</td>
</tr>
<tr>
<td>$C_{BTO}$, Holtrop &amp; Mennen (1982) coefficient</td>
<td>0.01</td>
</tr>
<tr>
<td>Bow thruster(s) tunnel diameter</td>
<td>2.6 m</td>
</tr>
<tr>
<td>Coordinates (x, y, z) of the bridge referred to Lbp/2 abd B/2 on the freeboard, positive towards fore perpendicular and upwards</td>
<td>-61.25 m 0 m 8.95 m</td>
</tr>
<tr>
<td>Coordinates (x, y, z) of the deck referred to Lbp/2 abd B/2 on the freeboard, positive towards fore perpendicular and upwards</td>
<td>83.12 m 0 m 10 m</td>
</tr>
<tr>
<td>Coordinates (x, y, z) of the bow keel referred to Lbp/2 abd B/2 on the freeboard, positive towards fore perpendicular and upwards</td>
<td>83.12 m 0 m -9 m</td>
</tr>
</tbody>
</table>

Table 30: S175 data

Furthermore, a sketch of the body plan by the offset table of the ship is depicted in the figure below
**Figure 49:** Ship S175, Sketch of the Body Plan, units in meters

**Speed profiles of the two alternative routes**

The comparison of the different speed profile of the two different routes considered in section 4.2 is given below:

**Figure 50:** Ship speed profile for the two routes
**Detailed results**

The comparison of the different failure probability estimations obtained with different methods is presented in this section for each scenario. The variation in time of \( g(\mu_X) \) and the probability of failure, is also reported to give indications about the deterministic distance from the failure domain. Just the seakeeping hazards with not excessively low failure probability, compared to the other seakeeping hazards, are reported.

**NO DELAY, SLOWER ROUTE**

![Graph](image1.png)

*Figure 51: \( g(E(X)) \) during the ship mission time*

![Graph](image2.png)

*Figure 52: vertical acceleration at the bridge probability of failure during the ship mission time*
Figure 53 Roll probability of failure during the ship mission time

Figure 54: Slamming probability of failure during the mission time
Figure 55: $g(E(X))$ during the ship mission

Figure 56: Vertical acceleration at the bridge failure probability during the ship mission time
Figure 57: Lateral acceleration at the bridge failure probability during the ship mission time

Figure 58: Roll failure probability during the ship mission time

Figure 59: Slamming failure probability during the ship mission time
Figure 60: \( g(E(X)) \) during the ship mission time

Figure 61: vertical acceleration at the bridge failure probability during the ship mission time
Figure 62: lateral acceleration at the bridge failure probability during the ship mission time

Figure 63: Roll probability of failure during the mission time

Figure 64: Slamming probability of failure during the mission time
Figure 65: $g(E(X))$ during the ship mission time

Figure 66: Vertical acceleration at the bridge failure during the ship mission time
Figure 67: Roll and green water, first order probability upper bound during the ship mission time

Figure 68: Roll failure probability during the mission time

Figure 69: Slamming failure probability during the mission time
Annex 2

Ship data

The input data to the seakeeping program regarding subchapter 4.3 is summarized in the following table.

<table>
<thead>
<tr>
<th>Ship data</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Length between perpendiculars</td>
<td>41.28 m</td>
</tr>
<tr>
<td>Breadth</td>
<td>10.32 m</td>
</tr>
<tr>
<td>Maximum draft</td>
<td>4.35 m</td>
</tr>
<tr>
<td>Maximum speed</td>
<td>15 kn</td>
</tr>
<tr>
<td>Structural radius of gyration about the x axis</td>
<td>4.10 m</td>
</tr>
<tr>
<td>Structural radius of gyration about the y axis</td>
<td>10 m</td>
</tr>
<tr>
<td>Structural radius of gyration about the z axis</td>
<td>10 m</td>
</tr>
<tr>
<td>Structural product of inertia</td>
<td>0 kg*m²</td>
</tr>
<tr>
<td>Propeller diameter</td>
<td>3.2 m</td>
</tr>
<tr>
<td>Propeller expanded area ratio</td>
<td>0.339</td>
</tr>
<tr>
<td>Bulb transverse area</td>
<td>0 m²</td>
</tr>
<tr>
<td>Bulb vertical centre of transverse area</td>
<td>0 m</td>
</tr>
<tr>
<td>Breadth of bilge keels</td>
<td>0 m</td>
</tr>
<tr>
<td>Length of bilge keels</td>
<td>0 m</td>
</tr>
<tr>
<td>Wetted area of the appendages</td>
<td>(1.795; 5.387) m²</td>
</tr>
<tr>
<td>$C_{stem}$, Holtrop &amp; Mennen (1982) coefficient</td>
<td>0</td>
</tr>
<tr>
<td>Appendages resistance factor 1+kO, Holtrop &amp; Mennen (1982) coefficient</td>
<td>(1.75; 1.75)</td>
</tr>
<tr>
<td>$C_{BTO}$, Holtrop &amp; Mennen (1982) coefficient</td>
<td>(0.004; 0.004)</td>
</tr>
<tr>
<td>Bow thruster(s) tunnel diameter</td>
<td>(1.16; 1.16) m</td>
</tr>
<tr>
<td>Coordinates (x, y, z) of the bridge referred to Lbp/2 abd B/2 on the freeboard, positive towards fore perpendicular and upwards</td>
<td>10.36 m</td>
</tr>
<tr>
<td>Coordinates (x, y, z) of the deck referred to Lbp/2 abd B/2 on the freeboard, positive towards fore perpendicular and upwards</td>
<td>0 m</td>
</tr>
<tr>
<td>Coordinates (x, y, z) of the bow keel referred to Lbp/2 abd B/2 on the freeboard, positive towards fore perpendicular and upwards</td>
<td>18.17 m</td>
</tr>
</tbody>
</table>

Table 31: Noruega data

Furthermore, a sketch of the body plan by the offset table of the ship is depicted in the figure below:
5.1.1 **Detailed results**

The following nomenclature is implemented to identify in the graphs the different locations of the two areas:

<table>
<thead>
<tr>
<th>Area 27.9b.1</th>
<th>Area 27.10a.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>43</td>
<td>342</td>
</tr>
<tr>
<td>43</td>
<td>346</td>
</tr>
<tr>
<td>43</td>
<td>346</td>
</tr>
<tr>
<td>42</td>
<td>346.5</td>
</tr>
<tr>
<td>41.5</td>
<td>347</td>
</tr>
<tr>
<td>41</td>
<td>347</td>
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<tr>
<td>40</td>
<td>346.5</td>
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<tr>
<td>39</td>
<td>346</td>
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<tr>
<td>38</td>
<td>346</td>
</tr>
<tr>
<td>36</td>
<td>347</td>
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<td>36</td>
<td>345</td>
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<td>36</td>
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<td>43</td>
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<td>39.5</td>
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<td>39.5</td>
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<td>36</td>
<td>342</td>
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<td>39.5</td>
<td>318</td>
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<tr>
<td>39.5</td>
<td>321</td>
</tr>
<tr>
<td>43</td>
<td>323</td>
</tr>
<tr>
<td>41.5</td>
<td>327.5</td>
</tr>
<tr>
<td>41.5</td>
<td>337.5</td>
</tr>
<tr>
<td>39.5</td>
<td>342</td>
</tr>
</tbody>
</table>

Table 32: nomenclature of the location investigated

**SUMMARY OF RESULTS**
Figure 71: $g(E(X))$ (above) and first order probability upper bound (below), for the two areas.

Figure 72: Slamming probability of failure (upper bound) for the two areas.
Figure 73: probability of failure for vertical and lateral acceleration at the deck roll, green water, and slamming, first order probability upper bound in the area 27.9b.1.
Figure 74: probability of failure for vertical and lateral acceleration at the deck roll, green water, and slamming, first order probability upper bound in the area 27.10a.1.