Distributing Multipartite Entanglement over Quantum Networks

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The quantum internet will enable quantum networked technologies, namely by distributing bipartite entanglement over large distances. However, some of its possible applications in areas such as communication, sensing and computation may benefit from multipartite entanglement being shared between several nodes. In our work, we address the problem of distributing optimally this type of entanglement over noisy quantum networks, where each link is an entangled pair. To do this, we describe the noise of the network with depolarising channels, verifying its effect on the distribution of GHZ multipartite entangled states. We also introduce tools from classical routing theory that are capable of creating a framework to address the optimality problem, allowing the inclusion of additional parameters. An algorithm for optimal distribution of a 3-qubit GHZ state maximising simultaneously the fidelity of the final state and the probability of success is presented and simulated in different models of quantum networks. We also derive approximations on the complexity scaling of our algorithms that corroborate the polynomial runtimes of the simulations. Furthermore, we determine the conditions yielding this simultaneous optimality for GHZ states with a higher number of qubits, and for other types of multipartite entanglement. This work paves the way to optimally generate multipartite quantum correlations over noisy quantum networks, an important resource for distributed quantum technologies.

Keywords: Quantum Networks, Routing, Multipartite Entanglement, Entanglement Distribution, GHZ Quantum States, Fidelity

I. INTRODUCTION

With an increasingly growing number of applications for quantum processing of information, more than ever, the need of quantum internet connecting these quantum devices capable of processing something that is fundamentally different from the classical counterparts, is also growing. There's an international effort that aims at delivering a quantum internet - Quantum Internet Alliance (QIA) - which envisions several intermediary steps characterised by their functionality [1] before arriving at a full-fledged quantum internet.

A quantum network is no more and no less that a set of terminals capable of connecting to each other using quantum communication, in the same way a network is a set of terminals connected to each other through classical communication. Quantum communication relies on encoding the information in qubits which are the fundamental unit of quantum information, analogously as bits are the classical unit of information. Using these qubits, applications can be developed, for example protocols for ensuring secure and private communications and access to quantum computers and quantum metrology networks.

Because of the nature of the physical processes and technologies that are the ground base for quantum communications, a quantum network will be inherently distinct when it comes to describing the parameters that affect the communications, *e.g* fidelities of shared quantum states, decoherence times of quantum memories and probabilistic behaviour of quantum communication. All this requires finding an approach to quantum networks that is compatible with their description and current state of the art of the concealed quantum technologies. Moreover, the current stratification of the model of the quantum internet [1, 2] involves various stages of functionalities according to the current developments in the underlying engineering of devices capable of quantum communication and processing of information.

The upcoming Quantum internet development relies on connecting two points through a quantum link, relying on bipartite entanglement which has been in particular focus lately. All in all, multipartite entanglement, which is able to connect more than two points in a quantum internet, comes as the natural extension when regarding the types of states that we might consider. Moreover, some applications like quantum sensor networks [3–6], some quantum communication protocols [7–9] and different forms of performing distributed quantum computation [10, 11] all require the distribution of multipartite entangled states across a quantum network. Finding the optimal way to distribute this multipartite states is therefore necessary for the applications built on top of it, specially considering that some parameters of the final state can render the state useless, *e.g* the fidelity of a quantum state usually has a threshold that guarantees the presence of entanglement in the state [12]. Furthermore, understanding how the noise of the network, present in each individual quantum link, affects the final state is key to finding the optimal way to distribute such state.

For the bipartite case, the protocols for extending the range of entanglement are well established [13], and so is the characterisation of the parameters that affect the

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final quality of the entangled pair and its distribution. The problem of routing when it comes to bipartite entanglement is regarded in [14–18], mostly by maximising the probability and rate of success using varying schemes, including on-demand generation of entanglement and also considering continuous models, with continuous entanglement generation in the background.

When considering the case of multipartite entangled state, some schemes for distribution have already been studied [19–21], but still, most work makes some simplifications such as by considering pure states or homogeneous networks. The past work mainly focused on finding distribution schemes rather than implementing them on a quantum network and finding the best way to distribute given such scheme, which introduces one of the main goals of this present work.

II. OBJECTIVES AND OVERVIEW

Given the developments in the schemes for distributing multipartite entanglement, the main goal of this project is create a systematic approach, *i.e* a framework, to distribute multipartite states across a network, taking into account a given scheme for distribution. We should be able to provide an algorithm that takes into account heterogeneous networks and the possibility of mixed states arriving from errors in the network protocols like entanglement generation and entanglement swapping. This is done by introducing different parameters that affect the quality of the final entanglement, namely the fidelity (a measure of quality of a quantum state), probability of success, communication times and quantum memory decoherence factor.

More than just including this parameters, to make the framework complete, the method should leave space and considerations in case more parameters are needed or some constraints in the quantum network exist. This is crucial for a developing quantum internet with changing underlying technologies.

This was done dividing the framework in three main components that depend on each other, but making the approach systematical.

The first component was borrowed from algebraic theory of classical routing. It's an important concept which essentially groups all the characteristics of each parameter in one mathematical object called an algebra for routing. In this object the values of the parameters of each link in the network are taken into account, as well as how the weights of each path are calculated and how the best paths are found by ordering relations. These algebras have well-defined properties that can provide advantages when trying to prove if the algorithms provides the optimal solutions for the routing problem, making it a simple yet effective approach to characterise the multiple parameters needed. In a language closer to the usual in this subject, the algebras define a metric for how paths are created and weight, for this reason we use both terms equivalently.

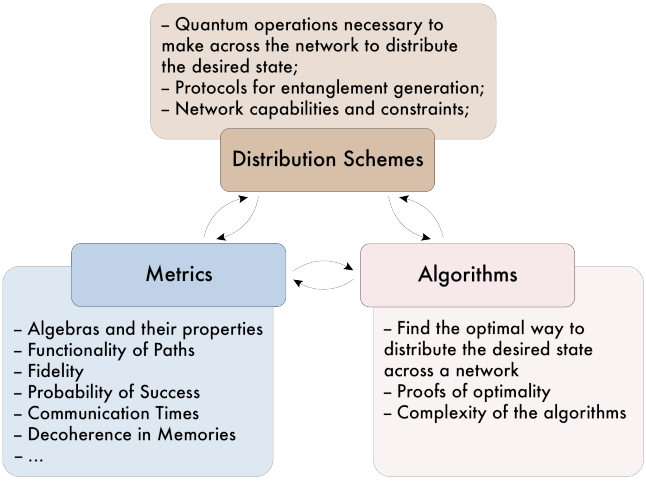


FIG. 1: Overview of the constructed framework in this project.

The second component is the distribution scheme. This distribution scheme states which operations and protocols must be realised to achieve the desired state distributed across the terminal nodes of the network, *i.e* the nodes that in the end share the desired state. This distribution scheme is an input, as much as the parameters of the network are, and will affect the metrics and how to find the algorithm that optimizes the state distribution. In this project, we use one distribution scheme for Greenberger–Horne–Zeilinger (GHZ) and graph states from [19] and introduce one trivial scheme for any arbitrary state.

The third and last component is the algorithm which finds the optimal way to distribute the desired state given a distribution scheme. Together with the algebra properties, this algorithms can be proven if they are exact or not, and their complexities are analysed.

This three components and their relations are depicted in *Figure 1.*

III. DISTRIBUTION SCHEMES

A distribution scheme is essentially a set of instructions, where each instruction can be either a quantum operation on qubits or a protocol with well-defined metrics for its effect on the parameters of distribution. After the scheme is applied, the result is the distribution of the desired state among the terminal nodes. Examples of quantum operations are quantum gates and measurements. One example of a quantum protocol is entanglement swapping that takes two different entangled pairs sharing one common node and extends the range of the entanglement creating an entangled pair between the furthest away nodes. This is done by performing a set of measurements in the qubits of the shared node and communicating the outcome, sending its information to one node to perform corrections and retrieve the desired

state.

A. GHZ Distribution Scheme

This first distribution scheme is capable of distributing GHZ states and any graph state, which follows from the fact that the GHZ state is a complete K_n graph state and from this complete graph, any other type of graph state can be obtained through stochastic local operations assisted with classical communications (SLOCC).

Starting from a tree configuration of links in a network, where each link is an entangled pair, a set of LOCC operations that consist of successive applications of the star expansion protocol [19] can be made to obtain in the end a star-graph, which is also LU (local unitary) equivalent to a GHZ state.

For this reason, finding the optimal way to implement this scheme in a quantum network starts by first solving the Steiner tree problem, *i.e* finding the shortest tree connecting the set of terminals, regarding the necessary metrics.

To perform the necessary calculations, we adapted this protocol to an identical one that can be understood as merging star-graph states (or GHZ states) across a tree. Since every tree can be modified through SLOCC to set of star-graph states the protocol merges every GHZ state into another GHZ state connecting every node.

From the way the states are distributed using this scheme, we will use the name tree scheme when referring to it.

B. Arbitrary State Distribution Scheme

Unlike [19], where all states considered for distribution have necessarily a correspondent graph state (up to LU), we want to generalise for any state possible. Taking advantage of the symmetry of the entangled bipartite state $|\phi^+\rangle = (|00\rangle+|11\rangle)/\sqrt{2}$, to distribute any multipartite state would pass through first establishing entanglement between every terminal and one center node and then projecting the desired state in the center node. This would result in the desired state distributed across the terminal nodes. Because of this, finding the best way to implement this scheme is equivalent to finding the shortest-star connecting the terminal nodes under the necessary metrics.

From the way the states are distributed using this scheme, we will use the name star scheme when referring to it.

IV. METRICS IN A QUANTUM NETWORK

To fully describe the metrics in a quantum network, we took advantage of tools in the classical theory of algebraic routing, namely the algebras for routing. They are a simple yet complete way of fully describing each parameter important for the entanglement distribution in a quantum network.

A. Bipartite Entanglement

Before jumping into distributing multipartite entanglement, since most schemes of distribution start from constructing paths and merging those paths, it is important to first introduce the metrics for distributing end-to-end bipartite entanglement. There are already protocols for distributing bipartite entanglement across a chain of entangled pairs: entanglement generation and entanglement swapping. There are several crucial parameters for deciding the best way to distribute bipartite entanglement, namely:

1. *Fidelity* - the quality of the entangled state. It is a measure of how close a distributed state is to the desired state. This parameter has a threshold, meaning that for values inferior to, in this case, $1/2$ the entanglement present in the state vanishes, rendering the state useless.

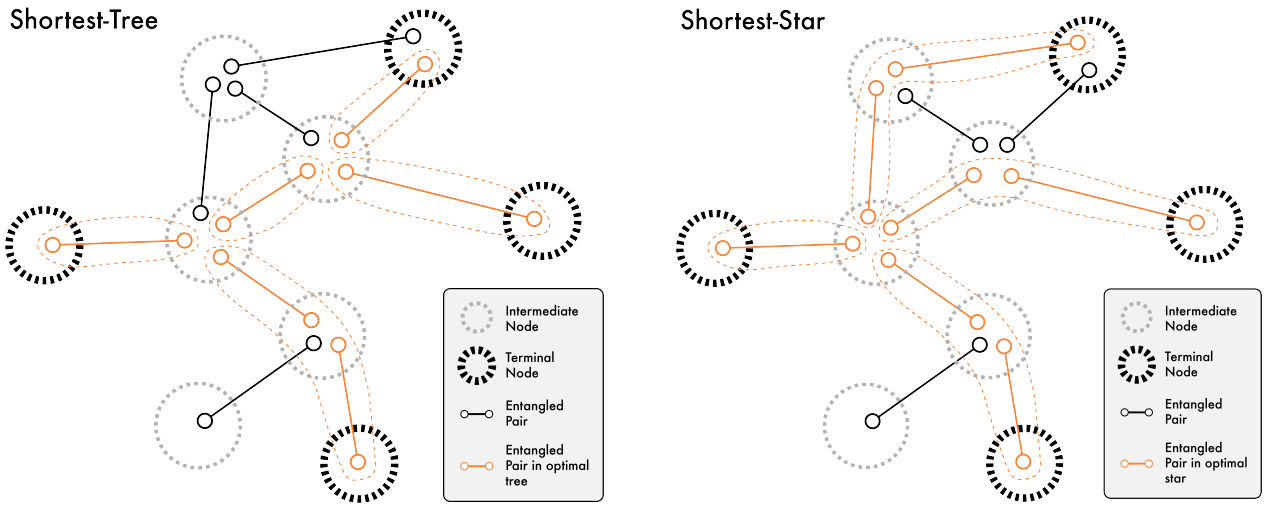
2. *Waiting time* - the time it takes between starting the protocol and signalling its completion. Since this is not always deterministic, this time must be considered part of a metric that takes also into account the probability of success and becomes the waiting time for each try.

3. *Quantum memories* - it is known [1] that in some of the early stages of the quantum internet, the capabilities of each node to preserve a given quantum state will depend on the quantum memories used. This memories, while being useful for storing the qubits for longer periods of time, will introduce an error that can be quantised and therefore minded across a network, given a distribution scheme.

4. *Probability of Success* - since some of the steps in distributing entanglement are not necessarily deterministic (entanglement generation, entanglement swapping,...), it is important to introduce a metric for the probability of success in generating end-to-end entanglement. This metric will depend on the nodes characteristics.

B. Multipartite Entanglement

As we cross to multipartite entanglement, a few things become different. In our work we analyse in detail the effect on the fidelity for both schemes and refrain the other parameters to a simpler approach. We derive expressions for the fidelity of the final state when a generalisation to n qubits of a GHZ is considered, using properties of the depolarising channel correspondent to



(a) Shortest tree connecting a set of 4 terminals in a network. (b) Shortest star connecting a set of 4 terminals in a network.

FIG. 2: In theses figures an example of what a shortest tree and a shortest star are, highlighted in orange. These examples correspond to the optimal solutions for distributing a GHZ state and an arbitrary state respectively, assuming we want to minimize the number of entangled pairs used. Keep in mind that groups surrounded by a traced orange line are LOCC equivalent to an entangled pair.

undergoing equally probable bit-flip, phase-flip and bit-phase-flip errors.

For both schemes, the expression for a GHZ state of 3 qubits is identical. This comes from the fact that the shortest-tree connecting three terminals is always equivalent to the shortest-star. However, when more qubits are considered the tree scheme becomes different since some shortest-trees might include something called Steiner nodes, which are auxiliary nodes necessary to connect the terminal nodes that are not part of the terminal nodes and need to be measured in the end. This measurement will induce a depolarising channel on one of the terminals, creating a more complex expression.

In here we present the completely-mixed GHZ state with n qubits, which is the result of the star scheme:

$$f = \frac{\prod_{i=1}^{n} \frac{1+\gamma_i}{2} + \prod_{i=1}^{n} \frac{1-\gamma_i}{2} + \prod_{i=1}^{n} \gamma_i}{2} \qquad (1)$$

Where $\gamma_i = \frac{4F_i - 1}{3}$ is the fidelity of each path between terminal i and the center node. This transformation (between F_i and γ_i) was made to simplify calculations of the fidelities of a path, since using γ results in only multiplying the γ values of each entangled pair along the path.

For the tree scheme, we would have to add for every Steiner node, a depolarising channel on the center node

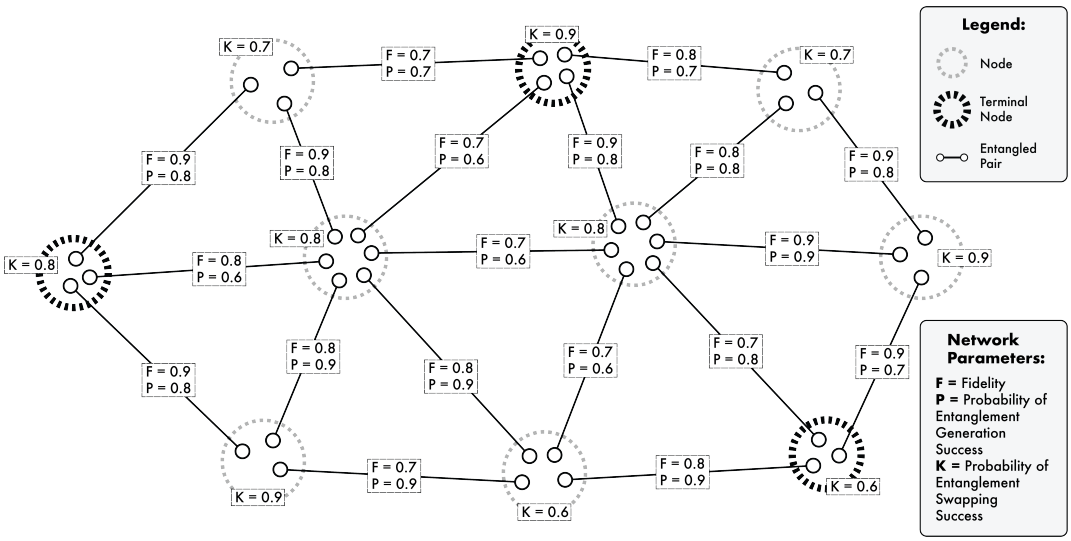
V. ALGORITHMS AND SIMULATIONS

In this project we adapted an algorithm for the multi-objective shortest-path (MOSP) problem, created an algorithm to solve the Steiner tree problem, which finds the best tree to distribute the desired states, and an algorithm to find the best star to distribute an arbitrary state. Along with extra properties for the metrics, we proved the exactness of the star-algorithm when distributing GHZ states.

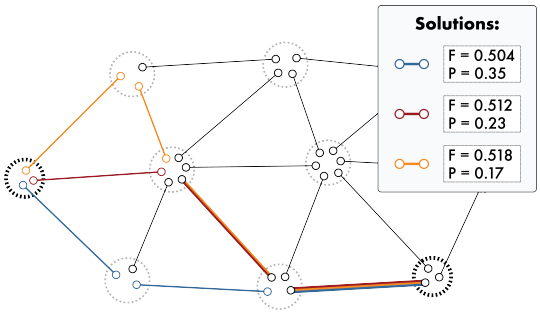
A. MOSP Algorithm

The MOSP algorithm provides the optimal paths between one node and every other node in a network. It has been thoroughly studied in literature [22, 23] and implemented in this thesis, providing some slight alterations to the data structure that facilitated the transition between finding the best way to distribute bipartite entanglement and to distribute multipartite entanglement.

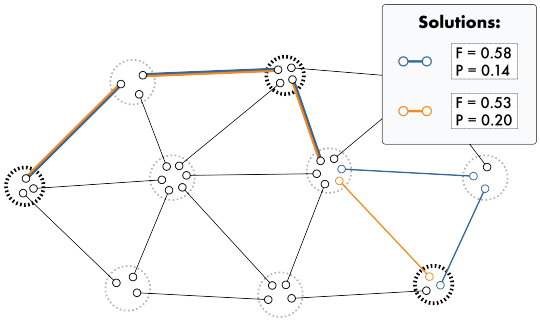
The algorithm goes from one node to its neighbours, finding the set of non-dominated paths for each node. It is easy to understand that for only one objective, if two paths are equally optimal, they must have the same weight. When we consider more objectives, equally important, the optimality of a path becomes more complicated since one path can be better in one way while the other can be better in another way. This requires a relation to be established - the dominance relation - and the notion of optimality becomes the set of non-dominated paths. For a path to dominate another it must be better than the other in every sense (or for every objective).



(a) Example of a simple quantum network with correspondent parameters. The final path fidelity depends on the independent entangled pairs fidelity and the probability of success depends on the probability of entanglement generation success and of entanglement swapping success.



(b) Solutions for the optimal paths problem



(c) Solution for the shortest tree connecting three terminals, equivalent to the distribution of 3-GHZ state

FIG. 3: Examples of shortest-paths and shortest-trees (which are equivalent to shortest-star for 3 qubits GHZ states, which is the case) in a simple network. The parameters used are the fidelity and the probability of success.

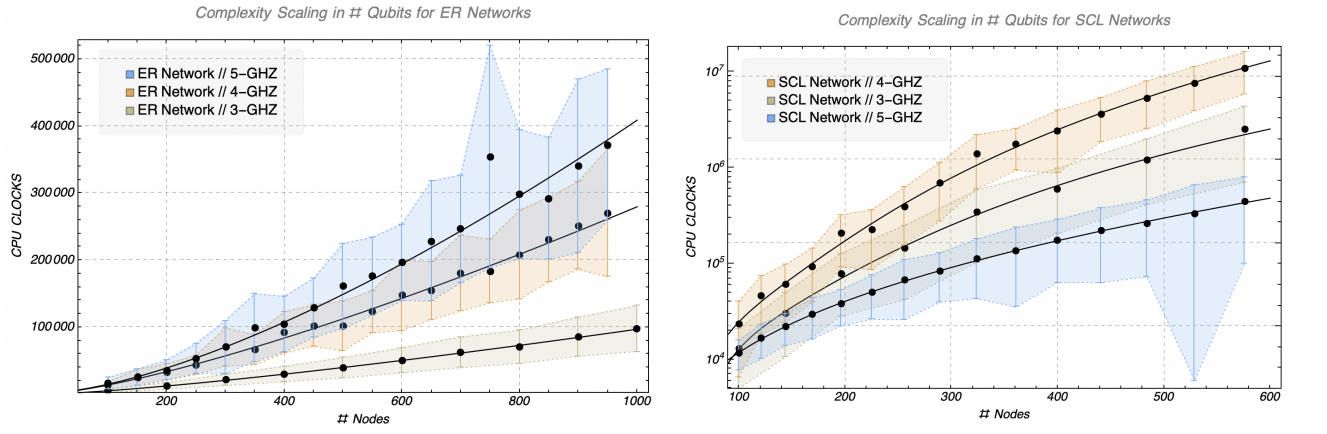
B. Steiner Tree Algorithm

The implementation of this multi-objective Steiner tree algorithm, which finds the shortest tree connecting the required nodes (called terminal nodes), was made using the MOSP algorithm as a starting point. Its structure takes a lot from the structure in our MOSP algorithm, but instead of only saving paths connecting two nodes, we allow the possibility of creating trees, comparing trees only and only if they connect the same set of terminal nodes. From there, the previous dominance relations can be implemented for the rest of the set of objectives.

The algorithm starts at every terminal by finding the best paths to the neighbouring nodes and when it finds another path from another node, besides adding the path, it also considers the tree constituted by both paths and adds it to the neighbours list. By doing this an approximated version of this algorithm (note that this problem is computationally extensive) can also be considered by implementing an ordering in the list of the to visit nodes, together with a stop condition of the first found tree connecting all terminal nodes, *i.e* as soon as a tree is found connecting all nodes, the algorithm stops.

C. Star Algorithm

The star-algorithm is very easy to understand: first it finds the shortest-path between each terminal and every other node using the MOSP algorithm and then creates all possible stars, only choosing the non-dominated ones for the set of solutions and constantly updating the set of solutions if any new solution is found and a previous one must be discarded. Some speed-ups are performed in several stages of the algorithm, derived from properties of the algebras.



(a) (b)

FIG. 4: Simulations for the Star-algorithm varying the number of terminals.

VI. COMPLEXITY

The star-algorithm needs the multi-objective shortest-path algorithm since the first step in finding the shortest is finding the shortest-paths from each terminal to all other nodes of the network. From the shortest-path algorithm structure, we can decompose its complexity in the following elements:

1. How many nodes are visited or revisited, which depends on the number of nodes N, and how many times each nodes is visited $h_{visit} = h_{visit}(N, \lambda, ...)$

2. Each time a node is visited, how many optimal paths $h_{paths} = h_{paths}(N, \lambda, k, ...)$ it adds to its neighbours

From these quantities we can draft a complexity for the algorithm, taking into account that before adding a path from the visited node to its neighbour, the dominance relation must be verified for, at most, all paths on the neighbour:

$$\mathcal{O}\Big(MOSP\Big) = \mathcal{O}\Big(N \cdot h_{visit}(N,\lambda)\Big)\cdot \cdot \mathcal{O}\Big(\lambda \cdot h_{paths}(N,\lambda,k)^2\Big) \tag{2}$$

The quantities h_{visit} and h_{paths} will rely on the type of network and the parameters distribution across the network. Using uniform distributions for the parameters of the fidelity and the probability of success, we are able to calculate the complexity of the algorithm by deriving expressions for the quantities h_{visit} and h_{paths} that depend on the network type and parameter distributions, and therefore compare these results with the simulations.

Considering that the star-algorithm structure, its complexity will depend on two different things: the complexity of the multi-objective shortest-path algorithm and the number of possible choices for trees that depend on the number of optimal paths from each terminal in each node. Therefore, the complexity of the star-algorithm is given by:

$$\mathcal{O}\Big(Star\Big) = \mathcal{O}\Big(T\Big)\cdot\mathcal{O}\Big(MOSP\Big) + + \mathcal{O}\Big(N \cdot h_{paths}(N,\lambda,k)^T\Big) \tag{3}$$

Where T is the number of terminals.

In the case of an ER network, the quantity h_{visit} should grow with the number of neighbours, *i.e* the average degree of the network. However, due to the priority queue ordering, this quantity is minimised to an average of $h_{visit} = const$. For an SCL network, the quantity $h_{visit} = const$, since the number of neighbours is always the same, from the network construction, and also from the priority queue ordering.

As for the quantity h_{paths} the calculated results, that take into account the threshold value for the fidelity and its distribution, are the following:

1. ER networks with uniform distributions of the parameters fidelity and probability of success and appropriate scaling of fidelity parameters

$$h_{paths} = 1 + \eta\frac{\log N}{\log\langle\lambda\rangle} \tag{4}$$

2. SCL networks with uniform distributions of the parameters fidelity and probability of success and appropriate scaling of fidelity parameters

$$h_{paths} = \sqrt{N} + \eta N \tag{5}$$

Using this, the complexity of the MOSP algorithm and the Star-algorithm can be calculated to be:

For ER networks:

MOSP Algorithm:

$$\begin{aligned}\mathcal{O}\Big(MOSP\Big) &= \mathcal{O}\Big(N\cdot\lambda\cdot h_{paths}(N,\lambda,k)^2\Big)\\ &= \mathcal{O}\Big(N\cdot\lambda\cdot\Big(1+\eta\frac{\log N}{\log\lambda}\Big)^2\Big)\end{aligned} \tag{6}$$

Star Algorithm:

$$\begin{aligned}\mathcal{O}\Big(Star\Big) = \mathcal{O}\Big(T\cdot N\cdot\lambda\cdot\Big(1+\eta\frac{\log N}{\log\lambda}\Big)^2\Big)+\\ \mathcal{O}\Big(N\cdot\Big(1+\eta\frac{\log N}{\log\lambda}\Big)^T\Big)\end{aligned} \tag{7}$$

For SCL networks:

MOSP Algorithm:

$$\begin{aligned}\mathcal{O}\Big(MOSP\Big) &= \mathcal{O}\Big(N\cdot h_{paths}(N,k)^2\Big)\\ &= \mathcal{O}\Big(N\cdot\Big(\sqrt{N}+\eta N\Big)^2\Big)\end{aligned} \tag{8}$$

Star Algorithm:

$$\begin{aligned}\mathcal{O}\Big(Star\Big) = \mathcal{O}\Big(T\cdot N\cdot\Big(\sqrt{N}+\eta N\Big)^2\Big)+\\ +\mathcal{O}\Big(N\cdot\Big(\sqrt{N}+\eta N\Big)^T\Big)\end{aligned} \tag{9}$$

From these complexities for the different algorithms, we can observe that for ER networks, both the MOSP and the Star algorithm are almost linear in the number of nodes of a network, making it a suitable option for a future quantum network with a large number of quantum nodes. For the case of the SCL, the complexity is not linear in the number of nodes, but it also is not exponential, growing polynomially with the number of nodes. These derivations come in line with the numerical simulations made in the previous section, corroborating the results.

VII. CONCLUDING REMARKS

In the beginning of this thesis, we proposed to introduce a framework in which we gather the several parts of the problem - the network parameters, the entanglement distribution schemes and the routing algorithms - and assemble them in order to solve the routing problem when dealing with multipartite entanglement distribution.

We did this by taking some concepts from classical routing theory - the algebras for routing and multi-objective routing - that became how we would describe each of the parameters taken into account, when finding the optimal way to distributing multipartite entanglement. We also used something very important from quantum channels and operators theory that would allow us to derive simple expressions for the fidelity of the final state. As we said in the beginning, the fidelity is a very important metric regarding the functionality of a quantum state, which is why, having a useful description of the depolarising channel and a complete form of calculating the resulting state with depolarising channels, is of utmost importance in calculating the fidelity of the state. This was achieved taking into account the individual fidelities of each quantum link of a network and the distribution schemes. Moreover, we introduced from the literature a few quantum network metrics capable of modelling the entanglement distribution in the case of bipartite, such as communication times, memory decoherence times and the probability of success. This was also extended for the multipartite case.

For both distribution schemes, the tree scheme and the star scheme, we implemented two different algorithms, targeting the two different problems (shortest-tree and shortest-star). This algorithms were built on top of the foundations laid by the classical routing theory, gathering the previously calculated metrics, with their underlying properties, and the distribution schemes themselves, into an efficient and adaptable approach on the problem of finding the optimal way of distributing multipartite entanglement in a quantum network. This was fundamental to ensure that our algorithm for the star scheme provided the optimal solution.

While simulating our algorithm on random networks, we stumbled upon a dilemma that foresighted a new problem to ensure that a quantum network is connected - the scaling problem. This problem rose from the fact that the fidelity is a parameter with a threshold of functionality. We solved it by distributing the parameters across the network such that the largest distance in a network remains connected. By doing this in a statistical manner, it shed some light on the problem of deriving the complexity of our algorithm, which we accomplished by taking into consideration the structure of the algorithm and defining some quantities that depended on the statistical distribution of the parameters. This resulted in the fact that this approach on the complexity is broader for other types of parameters and respective distributions. We presented several results from the simulations, which came in agreement with the calculated values, namely with the star algorithm scaling almost linearly (apart from a poly-logarithmic function) with the number of nodes of the network. In detail, for the case of the ER network, the scaling was in $\mathcal{O}\Big(N\cdot(1+\eta\log N)^T\Big)$ which is specially important for a network with an increasing number of nodes. For the case of the SCL network, the scaling was in $\mathcal{O}\Big(N\cdot(\sqrt{N}+\eta N)^T\Big)$.

While this framework is used to solve the problem taking into account these two distinct distribution schemes, it is adaptable to new schemes and other parameters capable of characterising the entanglement distribution, for example the rate of distribution [24].

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