Quantum solitonic turbulence in low-dimensional Bose-Einstein condensates

Clara Pereira
clara.pereira@tecnico.ulisboa.pt

Instituto Superior Técnico, Lisboa, Portugal

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Abstract

Soliton hydrodynamics is an appealing tool to describe strong turbulence in low-dimensional systems. Strong turbulence in quasi-one dimensional superfluids, such as Bose-Einstein condensates, involves the dynamics of dark solitons and, therefore, the description of a statistical ensemble of dark-solitons, i.e. soliton gases, is necessary. In turn, the evolution of this superfluid is described by the Gross-Pitaevskii equation, of which solitons are stationary solutions. In this work, a phase-space (kinetic) description of dark-soliton gases is proposed, introducing a kinetic equation that is formally similar to the Vlasov equation in plasma physics. Besides this, a code which numerically solves the Gross-Pitaevskii equation is also developed to provide a basis to compare and interpret the results to. It is shown that the proposed kinetic theory can capture the dynamical features of soliton gases and that the latter sustain an acoustic mode, a fact that is corroborated with the direct numerical simulations. This work's findings motivate the investigation of the microscopic structure of out-of-equilibrium and turbulent regimes in low-dimensional superfluids.

Keywords: Bose-Einstein Condensation, Solitons, Quantum Fluids, Kinetic Theory, Quantum Turbulence

1. Introduction

Unlike its classical analogue, which finds a comprehensive model in the Navier-Stokes equation, quantum turbulence (QT) does not fit in an unified and comprehensive description [1]. The difficulties in establishing a suitable framework are rooted not only in the coexistence of normal and superfluid phases, but most relevantly in the topological nature of turbulent structures: while in classical systems they assume arbitrary shapes and sizes, with lengths spanning over several orders of magnitude, in the quantum regime vorticity is quantised [2, 1]. As such, QT exhibits vortex tangling, as envisioned by Feynman, resulting in a vorticity distribution that is quite distinct from the continuous vorticity present in classical fluids, making the dynamics of QT rather complicated [3]. Both theoretical and experimental methods have been developed to produce and investigate vortex tangling in Bose-Einstein condensates (BEC) [4, 5].

The features of quantum turbulence are significantly affected by dimensional constraints. This is especially true for one-dimensional (1D) systems, where quantum fluctuations may play quite a significant role [6, 7, 8]. Moreover, angular momentum quantisation is not possible in one-dimensional systems, and the role of vortices — major turbulence manifestations in two and three dimensions — is played by dark solitons (DS), topological defects created by a phase jump in the order parameter [9, 10]. Another interesting aspect of dark-solitons is the fact of being fermionic [11, 12], being intimately related to the type-II excitations on top of a one-dimensional Bose gas as predicted by the Lieb-Liniger theory [13]. Moreover, the concept of "solitonic turbulence" is also present in some conditions of strong turbulence in classical systems [14, 15], which increases the interest around the development of statistical methods for solitons. Previous studies in 1D QT indeed exist, but are mostly (if not exclusively) performed in the weak turbulence regime [16], and it is still not clear what is supposed to happen in a strong turbulence situation. What sort of behaviour does one expect to observe for dark solitons in 1D? And how will their fermionic statistics work [17, 18]?

With the aim of understanding the microscopic processes leading to strong turbulence in 1D superfluids, in this work a kinetic theory of dark-soliton gases is developed based on the Klimontovich approach [19, 20], well-established in the context of plasma physics but recently applied to atomic systems [21]. The starting point is a reduction of DS to particle-like objects of effective negative mass,

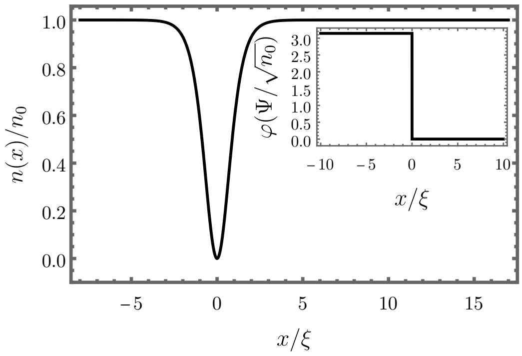

Figure 1: Density $n(x) = |\Psi(x,t)|^2$ profile of a dark soliton. Inset graph represents the corresponding phase.

and the determination of the Hamiltonian and corresponding canonical structure for a dark-soliton pair. Then, a distribution function describing a collection of DS in the phase space is constructed, and its corresponding transport equation obtained. Finally, by performing ensemble averages, the mean-field dynamics of a DS gas and respective excitation spectrum are determined. A gas of dark solitons is found to sustain a collective excitation, which is acoustic-like (massless) in the long-wavelength limit, in agreement with the Bethe ansatz solution of the Lieb-Liniger model. These results open a venue towards a theoretical framework able to capture the spectral properties of strong turbulence in 1D systems.

2. Background

2.1. Dark-Soliton Hamiltonian

A homogeneous, one-dimensional superfluid at zero temperature, is governed by the Gross-Pitaevskii(GP) equation [22]

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + g|\Psi(x,t)|^2\right)\Psi(x,t), \tag{1}$$

with $\Psi(x,t)$ being the superfluid order parameter, associated to the following Hamiltonian density

$$\mathcal{H}\{\Psi\} = -\frac{\hbar^2}{2m}\left|\frac{\partial\Psi}{\partial x}\right|^2 + \frac{g}{2}|\Psi|^4. \tag{2}$$

Dark solitons constitute exact solutions to Eq. (1) parametrised by s and v, standing for the centroid position and velocity, respectively, $\Psi(x,t;s,v) = e^{-i\mu t/\hbar}\psi_0[x;s,v]$, where $\mu = gn_0$ is the chemical potential, and [22]

$$\psi_0[x;s,v] = \sqrt{n_0}\left[i\beta + \gamma^{-1}\tanh\left(\frac{x-s}{\gamma\xi}\right)\right]. \tag{3}$$

Here, $\beta = v/c$, $\gamma = (1-\beta^2)^{-1/2}$, $c = \sqrt{gn_0/m}$ is the sound speed and $\xi = \hbar/\sqrt{gmn_0}$ is the healing length. Because of the translational invariance of the solution, the soliton Hamiltonian is a function of the DS velocity only,

$$\begin{aligned} H(v) &= \int\left(\mathcal{H}\{\Psi_0\} - \mu n_0\right)dx \\ &= \tfrac{4}{3}mc^2 n_0\xi\left(1-\tfrac{v^2}{c^2}\right)^{3/2} \end{aligned} \tag{4}$$

For small velocities, $v \ll c$, $H(v) \simeq |M_*|c^2 + M_* v^2/2$, where $M_* = -2mn_0\xi$ is the effective mass of the soliton [23]. Therefore, Eq. (4) suggests that DS may be regarded as relativistic hole-like particles, with c playing here the role of the speed of light. The canonical momentum may be obtained via the relation $v = \partial_p H(v)$, which can be readily inverted to provide

$$p = \int_0^v \frac{1}{u}\frac{\partial H(u)}{\partial u}du = M_* c\left(\frac{\beta}{\gamma} + \delta\right), \tag{5}$$

with $\delta = \arcsin(\beta)$ being the phase jump in the order parameter from $x - s = -\infty$ to $x - s = +\infty$. This shows that the parameter v is, indeed, the kinematic DS velocity, $v = \dot{s}$, since s and p constitute a canonical pair obeying the Poisson bracket $\{s,p\} = 1$. As such, Eq. (5) allows one to change the functional dependence on the Hamiltonian, $H(v) \to H(p)$.

With the establishment of the canonical equations for a dark soliton, it becomes possible to generalize to a set of N solitons. In order to avoid phase singularities within the order parameter, the DS gas is considered to be composed of an array kink−anti−kink pairs

$$\psi_{\text{gas}}[\{s_j;v_j\}] = \prod_{j=1}^{N}\psi_0[x_j,(-1)^j v_j]. \tag{6}$$

We can repeat the procedure of the single DS to formally obtain the equations of motion,

$$\dot{s}_j = \frac{\partial H_{\text{gas}}}{\partial p_j} = v_j, \qquad \dot{p}_j = -\frac{\partial H_{\text{gas}}}{\partial s_j}, \tag{7}$$

with H_{gas} obtained from Eq. (4) as

$$H_{\text{gas}}(s_j,p_j) = \sum_{k\neq j} H(s_k,p_k). \tag{8}$$

Notice that the Hamiltonian is now a function of the soliton positions s_j and their momenta p_j, as a consequence of the breakdown of translational invariance for the case of randomly distributed solitons. Of course, this results in $2N$ coupled equations, which are of little use. Crucially, Eq. (7) now encodes the motion that a single soliton undergoes due to its interaction with all the others. In what follows, a Klimontovich procedure based on the dynamics of Eq. (7) is implemented.

3. Implementation

3.1. Microscopic Phase-Space Distribution Function

In order to construct a statistical description of DS gases, the phase-space distribution function of the canonical variables s and p are defined as [24, 25]

$$\rho(s,p,t) = \sum_{j=1}^{N} \delta(s - s_j(t))\delta(p - p_j(t)), \quad (9)$$

satisfying the following relation with the total number of solitons in the gas

$$N = \iint \rho(s,p,t)\ ds dp. \quad (10)$$

Computing the time derivative, one has

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \sum_{j=1}^{N} \dot{s}_j \frac{\partial \delta(s-s_j)}{\partial s_j} \delta(p - p_j) \\ &+ \dot{p}_j \frac{\partial \delta(p-p_j)}{\partial p_j} \delta(s - s_j). \end{aligned} \quad (11)$$

By using the property $\partial_x \delta(x-y) = \partial_y \delta(y-x)$, one obtains

$$\frac{\partial \rho}{\partial t} + \dot{s}\frac{\partial \rho}{\partial s} + \dot{p}\frac{\partial \rho}{\partial p} = 0, \quad (12)$$

which means that the DS phase-space distribution function may be regarded as an incompressible fluid, $\dot{\rho} = 0$, in agreement with Liouville's theorem. The key point now is to understand that one can go from the discrete dynamics to a continuous description in the phase space by making use of $\rho(s,p,t)$, and writing Eq. (7) as $\dot{p} = -\partial_s V$, where V is obtained from Eq. (8) as

$$V(s,p,t) = \iint H_{\rm gas}(s - s', p - p')\rho(s', p', t) ds' dp'. \quad (13)$$

Notice that Eq. (13) defines a pseudo-potential depending on both s and p (or v), which is a consequence of the relativistic nature of DS: their mass depends on their velocity. Together with Eq. (5), establishing a relation between p and $v = \dot{s}$, the Klimontovich equation may be recast as

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial s} - \frac{\partial V}{\partial s}\frac{\partial \rho}{\partial p} = 0. \quad (14)$$

Eq. (14) is very useful for numerical simulations [26, 27, 28], but quite complicated to handle analytically. To describe the mean-field behaviour of DS gases, the ensemble averages $f \equiv \langle \rho \rangle$ and $\langle V \rangle$ are introduced, along with the corresponding deviations as

$$\delta\rho = \rho - f, \qquad \delta V = V - \langle V \rangle, \quad (15)$$

with $\langle V \rangle$ obtained from Eq. (13) via the replacement $\rho \to f$. Inserting in Eq. (14), one obtains a

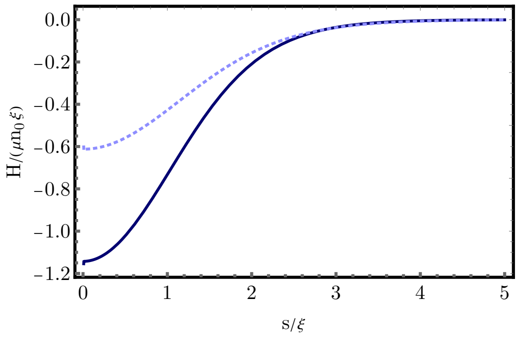

Figure 2: The effective pairwise pseudo-potential as a function of the soliton separation for $v = 0$ (solid line) and $v = 0.5c$ (dashed line).

kinetic equation for the smooth distribution function $f(s,p,t)$,

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial s}\right) f - \frac{\partial \langle V \rangle}{\partial s}\frac{\partial f}{\partial p} = \left\langle \frac{\partial \delta V}{\partial s}\frac{\partial \delta \rho}{\partial p} \right\rangle. \quad (16)$$

The r.h.s of the latter defines the collision integral, which depends on the details of the short-range nature of DS collisions. It can be constructed, at different levels of approximations, by making use of the BBGKY hierarchy [24], thus yielding different kinetic equations. In what follows, dilute soliton gases, $N_0 \xi \ll 1$, are considered, where $N_0 = 1/\langle s \rangle$ is the gas density determined by the averaged soliton separation $\langle s \rangle$. In that regime, the correlations between the multi-particle distributions are neglected and the collision integral in Eq. (16) is set to zero. As such, the collisionless kinetic equation for the single-particle distribution function is obtained,

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial s}\right) f - \frac{\partial \langle V \rangle}{\partial s}\frac{\partial f}{\partial p} = 0. \quad (17)$$

The latter is formally equivalent to the Vlasov equation widely used to describe fully ionised plasmas [25, 29]. The Vlasov equation has also been employed to describe photon-quasiparticles, where long-range interactions are absent [30]. As this work will testify briefly, the important difference stems from the fact that the mean-field soliton-soliton interaction is short-ranged, contrary to the case of electrons and ions in plasmas interacting through the Coulomb potential.

In order to model the interaction term in Eq. (17), soliton iteraction is considered to happen via an averaged two-body potential. This is justified since dark solitons are localised objects of size $\sim \xi$, making their interaction to be short ranged [9]. As such, the two-body pseudo-potential is obtained from Eqs. (6) and (8) by setting $N = 2$, and therefore defining $H_{\rm gas} \simeq H_{\rm pair} \equiv H(s,v)$, where

$$\begin{aligned} H(s,v) &\simeq H_{\rm kin}(s,v) + H_{\rm int}(s,v) \\ &- \tfrac{8}{3}|M_*|c^2\left(1-\tfrac{v^2}{c^2}\right)^{-3/2}. \end{aligned} \tag{18}$$

Here,

$$\begin{aligned} H_{\rm kin}(s,v) &= -\tfrac{1}{3}\tfrac{gn_0^2\xi}{\gamma^{3/2}}\left[2\beta^2 - \cosh(\zeta) - 1\right] \\ &\times \operatorname{csch}(\zeta)\left[\cosh(2\zeta)\right. \\ &- \left. 6\zeta\coth(\zeta) + 5\right], \end{aligned} \tag{19}$$

is the contribution from the kinetic term in Eq. (2) and

$$\begin{aligned} H_{\rm int}(s,v) &= \tfrac{1}{48}\tfrac{gn_0^2\xi}{\gamma^{3/2}}\operatorname{csch}^7(\zeta) \\ &\times \left[\left(16\beta^2-35\right)\sinh(5\zeta)\right. \\ &+ \sinh(7\zeta) + 12\zeta\cosh(5\zeta) \\ &+ \left(-432\beta^4+448\beta^2-171\right)\sinh(\zeta) \\ &+ \left(-176\beta^4+464\beta^2-207\right)\sinh(3\zeta) \\ &+ 24\left(36\beta^4-64\beta^2+29\right)\zeta\cosh(\zeta) \\ &+ \left. 12\left(8\beta^4-32\beta^2+21\right)\zeta\cosh(3\zeta)\right], \end{aligned} \tag{20}$$

results from the interaction (non-linear) term, where $\zeta = s/(\gamma\xi)$ and $v = (v_1+v_2)/2$ is the average velocity of the soliton pair. The third term in Eq. (18) corresponds to twice the energy of a single soliton, $s \to \infty$, and does not contribute to the force term in the Vlasov equation. As it can be seen in Fig. (2), the pairwise DS potential is attractive. However, since the DS mass is negative (and hence the reduced mass of the DS pair), the resulting interaction is repulsive. Moreover, it is observed that the potential has a range of order $\sim \xi$, and becomes weaker for more relativistic (less massive, in modulus) solitons.

3.2. Sound Modes of the Dark-Soliton Gas

In order to illustrate some of the features of the transport equation, a starting point is to consider small amplitude perturbations around a certain equilibrium configuration,

$$f \simeq f_0 + \tilde{f}_1, \quad \text{with} \quad f_1 \ll f_0. \tag{21}$$

Inserting this into Eq. (17), the linearised Vlasov equation is obtained,

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial s}\right) f_1 - \frac{\partial\langle V_1\rangle}{\partial s}\frac{\partial f_0}{\partial p} = 0, \tag{22}$$

where $\langle V_1(s,p)\rangle = \int H(s-s',p-p')f_1(s',p')ds'dp'$ with H being given by Eq. (18). Upon Fourier transforming the latter (i.e. by making $f_1(s,p,t) = \sum_{k,\omega}\tilde{f}_1(k,p,\omega)e^{i(ks-\omega t)}$, the kinetic dispersion relation of the DS gas is formally obtained as

$$1 = k\int \frac{\tilde{H}(k,v)}{(\omega - kv)}\frac{\partial f_0}{\partial v}\frac{\partial v}{\partial p}\, dv, \tag{23}$$

where $\tilde{H}(k,v)$ is the spatial Fourier transform of Eq. (18). The Jacobian $\partial v/\partial p$ allows one to eliminate p and corresponds to a generalised mass term that can be determined with the help of Eq. (5). The dispersion relation can be numerically solved for generic equilibrium configurations, accounting for i) the relativistic nature of dark solitons, ii) the velocity-dependence of their pairwise interaction, and iii) the negativity of their mass ("hole-like" nature). A particularly interesting and analytically tractable case is that of a non-relativistic DS gas, distributed such that $v \ll c$. In that case, $H(k,v) \simeq \tilde{H}(k,0)$ and $\partial p/\partial v \simeq -1/|M_*|$ can be set, which yields

$$1 \simeq -\frac{k\tilde{H}(k,0)}{|M_*|}\int \frac{1}{(\omega-kv)}\frac{\partial f_0}{\partial v}\, dv, \tag{24}$$

where

$$\tilde{H}(k,0) \simeq -\frac{1}{2}|M_*|c_s^2\xi\left(\frac{28}{9} + (7\pi^2-15)k^2\xi^2\right) + \mathcal{O}(k^4). \tag{25}$$

It can immediately be seen that the negative signs in Eqs. (24) and (25) cancel, thus confirming that the nature of the DS interaction is, indeed, repulsive as earlier stated.

The distribution function is formed by considering fluctuations on top of homogeneous soliton gases, $f_0(s,v) = N_0 g_0(v)$. An interesting situation corresponds to that of a cold dark-soliton gas, for which solitons are prepared at rest, distributed in velocity as $g_0(v) = \delta(v)$. Although this distribution is correct for classical particles, it may not be generally accurate to describe solitons. The reason stems from the fact that solitons are fermions, as one can immediately see from Eq. (6) for a two-DS wavefunction [17],

$$\psi_2(x_2,x_1;0,0) = -\psi_2(x_1,x_2;0,0), \tag{26}$$

in agreement with Lieb-Liniger theory [11, 12]. The fermionisation of the DS gas must therefore be included in the equilibrium at a semi-classical level. For that task, the distribution function

$$g_0(v) = \frac{1}{2v_F}\Theta(v_F - |v|), \tag{27}$$

may be used, where $v_F = \pi\hbar N_0/|M_*|$ is the 1D Fermi velocity of the DS gas. As a matter of fact, the later reduces to the Dirac-delta distributed gas in the limit $v_F \to 0$, i.e. for sufficiently diluted DS gases. The dispersion relation in (24) thus provides

$$\omega^2 = v_F^2 k^2 + \frac{N_0 k^2 \tilde{H}(k,0)}{M_*}. \qquad (28)$$

In the long-wavelength limit $k\xi \to 0$, this corresponds to a *solitonic first sound*, $\omega \simeq v_1 k$, where

$$v_1 = c\sqrt{\frac{\pi^2}{4}\Gamma^2 + \frac{14}{9\pi}\Gamma} \qquad (29)$$

is the first sound speed and $\Gamma = N_0\xi$ is the DS gas dilution parameter. In the validity range of Eq. (17), $\Gamma \ll 1$, $v_1/c \simeq 0.703\sqrt{\Gamma}$, meaning that the effects of the Fermi pressure are not relevant for very dilute gases. Indeed, the effect of the fermionisation becomes relevant for densities of the order $\Gamma \gtrsim (56/(9\pi^3)) \simeq 0.203$.

3.3. Numerical Method

In order to validate the kinetic approach developed in this work, a computer code was developed, using C++ language, which integrates Eq. (1) with the initial condition of Eq. (6).

There exist several well described and studied methods for discretising the GP equation in literature [31, 32, 33, 34, 35, 36, 37]. The one employed is a time-splitting finite difference (TSFD) method [38],

$$\begin{cases} \Psi_j^{(1)} = e^{-i[U_j + |\Psi_j^{(2)}|^2]\frac{\Delta t}{2}}\Psi_j^n \,;\; 0 \leq j \leq J \\ i\frac{\Psi_j^{(2)} - \Psi_j^{(1)}}{\Delta t} = -\frac{1}{4}\left(\delta_x^2 \mathbf{\Psi}_j^{(2)} + \delta_x^2 \mathbf{\Psi}_j^{(1)}\right) \,;\; 0 \leq j \leq J \\ \Psi_j^{n+1} = e^{-i[U_j + |\Psi_j^{(2)}|^2]\frac{\Delta t}{2}}\Psi_j^{(2)} \,;\; 0 \leq j \leq J \end{cases} \qquad (30)$$

Which can be adapted to a choice of boundary conditions.

This method consists in the separation of the GP equation into two splitting steps known as Strang-Splitting [39, 36], which are then solved sequentially in the code: whilst the nonlinear part is solved exactly, the linear part is discretised through a Crank-Nicolson finite-difference (CNFD) method [32, 36, 34].

The TSFD method is particularly favourable because it is unconditionally stable, time reversible, second-order accurate in both time and space, it conserves mass (or total particle number), and is time transverse invariant [36, 32]. In addition, as it is an implicit scheme where only a linear system needs be solved at each time step, it has a memory cost of only $\mathcal{O}(J)$ operations in the one-dimensional case.

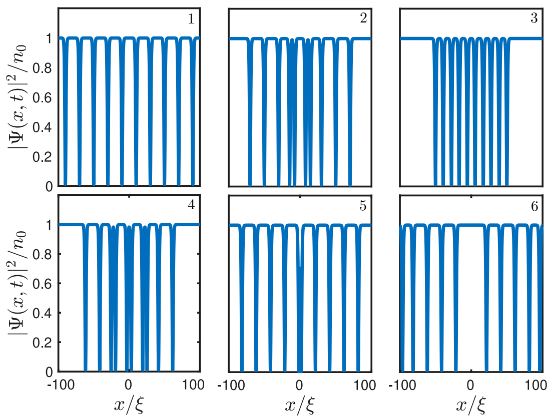

Figure 3: Evolution of a small $N = 20$ dark soliton gas ($\Gamma = 0.05$) in a homogeneous BEC through several snapshots (1 through 6) of the density profile during simulation. Between each snapshot $n_i = 2 \times 10^6$ iterations have elapsed.

The developed code integrates the GP equation statically an dynamically and can be adapted to virtually any specific set of initial conditions in one-dimension. The code allows one to choose the applied external potential, if any; allows for different sets of boundary conditions — homogeneous infinite systems or bounded trapped BECs are the most common; allows one to introduce the soliton gas with any initial velocity distribution, and in any density configuration; and it is also possible to introduce other excitations to the system such as phonons, or to include more than one soliton gas.

Fig. (3) depicts the time evolution of the density profile in the representative case of an initialised very diluted, $\Gamma = 0.05$, dark soliton gas on a homogeneous BEC. In this picture one may observe a soliton chain-like repulsion upon interaction.

4. Results

As can be observed in Fig. 4, the DS gas seems to sustain a collective mode which is acoustic (non-dispersive) in the long wavelength limit. In order to characterise this mode quantitatively and to compare with the theory, the dynamical structure factor, (5), may be used

$$S(\omega, k) = \iint e^{-i\omega t + ikx}|\Psi(x,t)|^2 dx dt, \qquad (31)$$

as depicted in Fig. 5. As can be seen, there is a good agreement between the first sound mode in Eq. (28) and the numerics is obtained in the region $k \lesssim N_0$, i.e. for wavelengths lying above the inter-particle separation $1/N_0$. Above this value, the coarse-graining assumption of the phase space breaks down. Moreover, for shorter wavelengths in

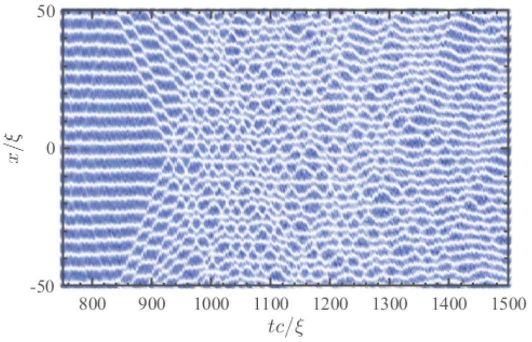

Figure 4: Numerical simulation of a cold dark-soliton gas, $g_0(v) = \delta(v)$, depicted for a concentration parameter $\Gamma = 0.2$. One observes the formation of an acoustic mode, which preludes the solitonic turbulence.

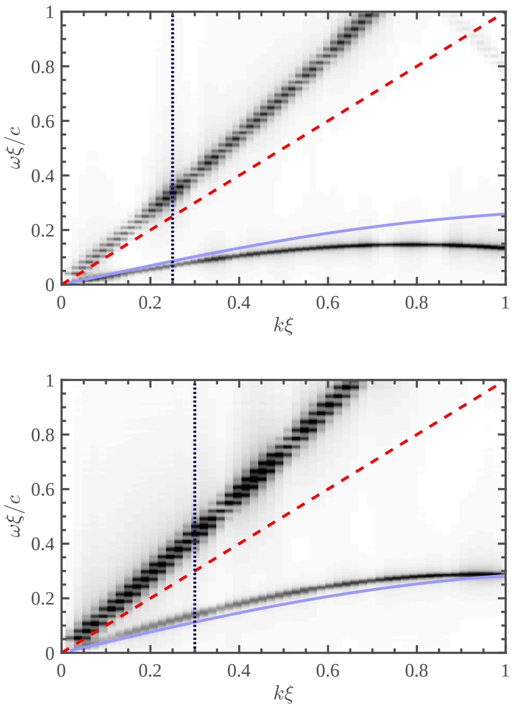

Figure 5: Dynamic structure factor $S(\omega, k)$ as obtained from the numerical integration of Eq. (1), for $\Gamma = 0.25$ (top) and $\Gamma = 0.3$ (bottom). In both panels, the solid line corresponds to the first-sound mode in Eq. (28) (no free parameters), while the top curve indicates an hybridisation mode between Bogoliubov excitations (dashed lines) and DS first sound with dispersion $\omega \simeq v_2 k$ (not shown, see text). The vertical dotted lines depicts the border of the expected validity of the Vlasov equation (17), $k = N_0$.

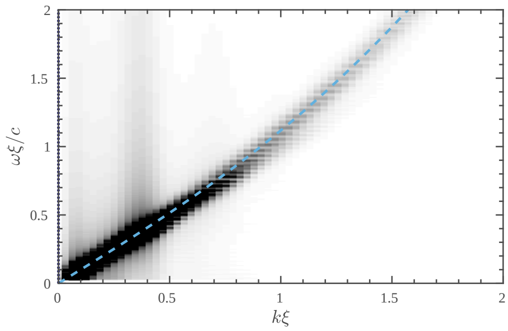

Figure 6: Dynamic structure factor $S(\omega, k)$ as obtained from the numerical integration of Eq. (1), for a density $N_b = 0.1\xi$ of small amplitude oscillations in the BEC. The dashed curve indicates the Bogoliubov excitation spectrum.

the range $k \sim \xi$, the description of DS as hole-like particles fails and the internal structure of the solitons becomes important, what hinders the validity of the kinetic equation. Finally, the effects of collision integral in Eq. (16) are expected to play a prominent role for sufficiently dense DS gases $N_0 \sim \xi$, as well as for sufficiently large velocities, $v \sim c$. In both situations, the short-range binary collisions between solitons needs to be taken into account.

An additional feature can be observed in Fig. (5): the emergence of an energetic mode $\omega \simeq v_2 k$, with $v_2 > c$. This mode does not correspond to the low-lying (Bogoliubov) excitations on top of the superfluid - a fact that was verified numerically - and is certainly not that of a soliton gas (for which the slope is $v_1 \ll c$, as discussed above). At this stage, this mode is understood to be a consequence of the hybridisation between the Bogoliubov (fast) and the first-sound (slow) modes, as a result of the interactions between solitons and phonons.

This fact is further cemented by the observation that, in a system initialised with small (fast) excitations and in the absence of solitons, Fig. (6), the only collective mode corresponds precisely to the Bogoliubov excitations on top of the Bose-Einstein condensate.

5. Conclusions

This work establishes the foundations of a kinetic theory of dark soliton gases in one-dimensional superfluids based on the Klimontovich approach. By considering that dark solitons behave as particles of negative mass, and assuming that they interact via an ensemble averaged pairwise potential, it was possible to define a kinetic equation governing the phase-space distribution function of an array of dark solitons, or rather a dark soliton gas. The approach described here allows one to describe solitons statistically in analytical grounds, a feature that is virtually impossible with studies based on the Gross-Pitaevskii equation. One important feature of the soliton gas is that it supports a first sound mode, which is much less energetic than the phonon modes on top of the condensate. This is a consequence of solitons behaving as weakly interacting particles of negative mass.

Another significant accomplishment in this thesis was the development of a computer code which integrates the GP equation for a versatile initial set of conditions. The numerical simulations provided the confirmation and visual materialisation of the studied systems.

Through this work, a more comprehensive, microscopic description of solitonic turbulence is possibly en route, an aspect of central importance when dealing with strong turbulence regimes in one-dimensional superfluids. In a near future, with the help of a hydrodynamic model that can be directly obtained from the developed kinetic equation, it may be possible to describe solitonic turbulence as a Kolmogorov theory of weak turbulence of dark soliton gases. If successful, this description will unveil important mechanisms underlying the spectral properties of strong quantum turbulence in low-dimensional superfluids, to be experimentally produced either in atomic or polaritonic Bose-Einstein condensates, or in quantum fluids of light [40].

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