Using medical data to adjust flow boundary conditions in computational hemodynamics

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Abstract

The aorta coarctation (CoA) is one of the most common congenital heart diseases (CHDs). It is defined by a narrowing in the proximal thoracic aorta and is treated through surgery or stent placement.
Advances in medical imaging and computational techniques provided the research community with a unique opportunity to investigate CHDs, with computational fluid dynamics (CFD) models opening new ways to understand cardiovascular pathologies. Due to the complexity of the cardiovascular system, it is common to model merely the region of interest and represent the remaining circulation through boundary conditions (BCs), whose choice is critical for CFD since different BCs could lead to quantitative differences in the solution.
This work presents a framework to adjust Murray's law BC, using an optimisation approach. Taking Murray's law parameter as the control, the approach uses a discretise-then-optimise methodology to numerically solve the control problem. The framework was tested using generated in-silico data sets and then applied to a realistic 3D geometry representing a CoA, using patient-specific data, for validation.
The results show small errors over the domain, with absolute errors < 1% and relative errors <10%. The largest relative errors were found in the supra-aortic outlets when different types of BCs were attributed in the in-silico data and in the controlled problem, whilst the absolute errors kept minor (<1.5%). For the normal aorta and CoA, the parameters obtained are different from those found in the literature, with higher parameter values found for the CoA, which can be explained by the flow behaviour.

Keywords: cardiovascular system, aortic coarctation, computational fluid dynamics, boundary conditions, Murray's law, optimisation

1. Introduction

According to the World Health Organization (WHO), cardiovascular diseases (CVDs) are the leading cause of death worldwide, corresponding to 31% of all world deaths. Therefore, it is important to have a comprehensive knowledge of the cardiovascular system to understand the pathophysiology and the mechanism of the diseases, assist in the preventative and therapeutic measures, and understand the outcome of the CVDs treatments.
Computational fluid dynamics (CFD) field, which concerns the numerical methods for solving fluid dynamics, has become a widely used tool by researchers for the cardiovascular system because it explains the physical laws that govern blood flow (hemodynamics). Numerical description helps us quantifying parameters, such as pressure, flow velocity and orientation or forces applied in the vessels, that characterize the physiological and pathological states.
When studying the hemodynamics of blood flow in the circulatory system, two significant aspects affect the numerical simulation – the geometry of the domain of interest and the boundary conditions (BCs). To reduce the complexity of the blood flow simulation is essential to truncate the domain of interest by creating artificial sections, which separate the region of interest for the simulation from the remaining

part of the circulatory system. The choice of BCs on artificial boundaries is an essential issue for fluid dynamic computations since different BCs could lead to quantitative differences in the solution. [1]–[5] In an ideal numerical simulation, the BC should consist of patient-specific data (e.g., velocity or pressure), which are very complex to obtain. The most appropriate boundary conditions are still a topic of debate and continuous development, mainly due to the lack of patient-specific measurements, especially for the outflow. While at the inlet, the missing data can be replaced by adapting literature data to the case of interest, at the outlets is more challenging due to the morphological variation.

The inlet BC should truncate the upstream vasculature of the vessel of interest. The simplest BC include idealised velocity profiles such as the flat-velocity, the fully-developed or the Womersley profiles. This velocity profiles can also be translated into a volumetric flow rate (Q) when the section's diameter is known since the flow rate is equal to the measured average velocity times the area of the section where the velocity profile was measured.

Despite the importance of the inlet BC, the solutions to the NS equations in large arteries are also highly dependent on the imposed outlet BCs to represent the downstream vascular systems. It has been demonstrated that velocity and pressure fields of the same computational domain can change significantly depending on the choice of outflow BC.[6] The Murray's law boundary condition is a common strategies, stating that the flow in a vessel is proportional to the diameter of the vessel lumen cubed. Which in a generic outlet, translates to:

$$Q_{out} = Q_{in} \times \left(\frac{{D_{out}}^3}{{D_{in}}^3} \right) \quad (1)$$

Another example of prescribed BCs, still common amongst the cardiovascular research community, is the traction-free (TF), one of the simplest BC to apply in a simulation, consists in assigning a zero pressure to the boundary, similar to assuming the vessel is cut and exposed to atmospheric conditions, thus neglects the resistive effects of downstream vessels.

More advanced BCs, such as the three-element Windkessel has such, have been developed to better model the interaction between the computational domain and the downstream vasculature, by coupling the domain with simple models such as resistance, impedance, lumped parameter, to capture the resistance and compliance effects of the proximal and distal vessels in the arterial tree from corresponding outlets. [7]

Another approach observed in the literature is using the variational approach to obtain more accurate boundary conditions and other flow metrics. Variational approaches, which consists of varying a defined number of parameters to minimise a cost function, have been used successfully as a strategy to take advantage of real data measurements. The variational approach allows for increasing the accuracy of numerical simulations. [8]

The variational approach is a type of optimisation problem that consists of finding the parameters that optimise the objective - a quantitative measure of the system's performance being studied, which depends on certain system characteristics (parameters). These parameters might be subject to constraints, or bounds, depending on the problem and system being studied. The optimisation consists of the minimisation or maximisation of a function subjected to constraints on its parameters. [9], [10]

2. Methods

2.1. Mathematical model and problem definition

An aorta developed and studied by [11], was used to create a framework where the exponents of Murray's law (ML) were automatically adjusted in the boundary conditions (BCs), using an optimisation study in COMSOL Multhiphysics®. Adaptations were made to the model of [11], and an independent mesh convergence study. The refined study concluded that a mesh with average element size $h = 4.32 \times 10^{-3}\,\mathrm{m}$ was the most appropriated, consisting of 193,611 degrees of freedom.

This framework consists of two problems – the forward problem and the controlled problem. The forward problem was used to generate different in-silico datasets, since no patient-specific data was available, in the same aorta model where the optimisation tool was applied. [6], [12] The controlled problem

consisted of using a variational approach to adjust ML exponent – here designated as a parameter – that fitted better the in-silico data generated in the forward problem.
The three-dimension computational domain consists of the upper part of a patient-specific thoracic aorta, which was truncated by three planar surfaces: one at the entrance of the ascending aorta, to define the inlet; one in the intersection of the arteries brachiocephalic artery (BA), left common carotid artery (LCA) and left subclavian artery (LSA) with the aortic arch, to define the supra-aortic outlets; and one in the middle of the thorax, to define the descendent aorta (DA) outlet. The domain is divided into two by a virtual outlet, in the xy-plane, used as the optimisation baseline.
Blood was modelled as a stationary, incompressible, laminar, homogeneous and Newtonian fluid with a constant density (ρ_{blood}) of 1060 kg/m^3 [2], [13]–[15], and constant viscosity (μ_{blood}) of 0.004 Pa.s [2], [16]–[18]. Also, the interaction with the vessel wall was not considered since it is a stationary model [19] Under these assumptions, blood flow was described using the stationary Navier-Stokes (NS) equations for an incompressible fluid [20]:

$$\begin{cases} -\upsilon\Delta u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega \\ \nabla\cdot u = 0 & \text{in } \Omega \end{cases} \tag{2}$$

Where u is the fluid velocity, p is the pressure divided by density, f is the body forces, ρ is the density of the fluid, μ is the dynamic viscosity, and ν is the kinematic viscosity ($\nu = \mu/\rho$).
The wall was considered a rigid surface with a no-slip condition, which means the velocity of the elements of the wall was set to zero, which translates by:

$$u = 0 \tag{3}$$

At the inlet, patient-specific flow rate (Q_{in}) BC was imposed, where the maximum velocity (V_{max}) was obtain from [11]. It is described by:

$$Q_{in} = V_{mean} \times A_{in} = \frac{V_{max}}{2} \times A_{in} \tag{4}$$

At the outlet in the descendent aorta (Γ_{DA}), the BC imposed was a traction-free boundary condition (TFBC); this condition assumes that the normal stresses at the boundary, which are approximately equal to the pressure, are zero. The tangential stress component is also set to 0 Pa; therefore, the expression used to apply this BC was:

$$\left[-pI + \mu\left(\nabla u + (\nabla u)^T\right)\right]n = -P_0 n \tag{5}$$

where $P_0 = 0$.
The inlet and outlet DA BCs were set to be the same at the forward and the controlled problem. On the other hand, the BCs at the supra-aortic outlets were changed accordingly to the type of in-silico data being generated and the controlled test being performed.
In sum, two types of BCs were attributed in the supra-aortic outlets: TFBC, where the condition imposed was (5), and MLBC, where the flow rate in each branch (Qi) was given by:

$$Q_i = \left(\frac{D_i}{D_{in}}\right)^p \times Q_{in} \tag{6}$$

Where Q_{in} is the inflow given by (4). D_{in} is the diameter of the inlet, D_i is the diameter of the branch i, with $i \in (BA, LCA, LSA)$, and p is the exponent of the MLBC.
The parameter p is what will distinguish the forward problem from the controlled problem. When attributing MLBC, in the forward problem, the parameter was considered $\alpha = 3$, as stated in the literature [21], in the controlled problem, the parameter α was used as the value to be optimized.

- **Forward problem**

Three different datasets (Data A, Data B and Data C) were generated using different combinations of BCs in the supra-aortic outlets, while the outlet DA and the inlet BCs were given by (5) and (4), respectively. The BCs attributed to the supra-aortic outlets are summarized in Table 1.

Table 1: Boundary conditions attributed to the supra-aortic outlets (BA, LCA and LSA) in the forward problem.

Test	Inlet	Outlet DA	Outlet BA	Outlet LCA	Outlet LSA
Data A	▲	●	▲	▲	▲
Data B	▲	●	▲	●	▲
Data C	▲	●	●	●	●

● TFBC ▲ $Q_i = (V_{mean} * A_{in}) * \left(\frac{D_i}{D_{in}}\right)^3, i \in (in, BA, LCA, LSA)$

The forward problem consists of solving the equation (2), using the BCs above described.

- **Controlled problem**

For each in-silico data set, either one control optimising a single parameter (α_1) common to all outlets (A_1, B_1 and C_1) or three controls optimising a different parameter$(\alpha_1, \alpha_2, \alpha_3)$ in each outlet (A_3, B_3, C_3), was attributed to the boundary conditions. In Table 2, the summary of each BC applied to each test is stated.

The controlled problem consisted of looking at the parameter α_x such that the following cost function:

$$f_{\cos t}(U) = \int_{S_{obs}} \left(\|u\| - \|u_d\|\right)^2 dx \quad (7)$$

Will be minimised. Here u corresponds to the solution of the NS for the controlled problem, and u_d represents the solution of the NS for the forward problem, both at S_{obs}. u is subjected to the parameters obtained and their respective bounds.

An optimisation problem always needs an initial value, and sometimes might need bounds, for example, due to physical constraints. To reduce computational time, as the mesh was refined, the bounded interval was adapted and can be consulted in [22].

Table 2: Boundary conditions attributed to the supra-aortic outlets (BA, LCA and LSA) in the controlled problem.

Test	Forward Problem	Inlet	Outlet DA	Outlet BA	Outlet LCA	Outlet LSA
A_1	Data A	▲	●	α_1	α_1	α_1
A_3	Data A	▲	●	α_1	α_2	α_3
B_1^{LCA}	Data B	▲	●	▲	α_1	▲
B_1	Data B	▲	●	α_1	α_1	α_1
B_3	Data B	▲	●	α_1	α_2	α_3
C_1	Data C	▲	●	α_1	α_1	α_1
C_3	Data C	▲	●	α_1	α_2	α_3

α_x $Q_i = (V_{mean} * A_{in}) * \left(\frac{D_i}{D_{in}}\right)^{\alpha_x}, i \in (BA, LCA, LSA)$

● TFBC

▲ $Q_i = (V_{mean} * A_{in}) * \left(\frac{D_i}{D_{in}}\right)^3, i \in (in, BA, LCA, LSA)$

2.2. The solution to the controlled problem

When coupling the stationary NS equations for incompressible fluids, stated in (2), with the appropriate boundary conditions, parameters α_x to be adjusted and the bounds they are subjected. The resulting system of equations is:

$$\begin{cases} -\upsilon\Delta u + u\cdot\nabla u + \nabla p = 0 & \text{in } \Omega \\ \nabla\cdot u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_{wall} \\ \int_{\Gamma_{in}} \overrightarrow{u_{avg}^{in}}\cdot(-\vec{n})\, ds = Q_{in} & \text{on } \Gamma_{in} \\ \nu\partial_n u - pn = 0 & \text{on } \Gamma_{DA} \\ \int_{\Gamma_{BA}^{ext}} \overrightarrow{u_{avg}^{BA}}\cdot(\vec{n})\, ds = Q_{BA}(\alpha_1) & \text{on } \Gamma_{BA},\ \text{low}_1 < \alpha_1 < up_1 \\ \int_{\Gamma_{LCA}^{ext}} \overrightarrow{u_{avg}^{LCA}}\cdot(\vec{n})\, ds = Q_{LCA}(\alpha_2) & \text{on } \Gamma_{LCA},\ \text{low}_2 < \alpha_2 < up_2 \\ \int_{\Gamma_{LSA}^{ext}} \overrightarrow{u_{avg}^{LSA}}\cdot(\vec{n})\, ds = Q_{LSA}(\alpha_3) & \text{on } \Gamma_{LSA},\ \text{low}_3 < \alpha_3 < up_3 \end{cases} \tag{8}$$

Where α_x are the parameters to be obtained with the minimised cost function, and the low_x and the up_x with $x \in [1,3]$ are the lower and upper bound, respectively, for each parameter.
A discretise-then-optimize (DO) approach will be used to solve both (7) and (8), which consists of first discretizing the optimal control problem and then solving the optimisation problem, resulting from the discretization. Beginning by discretizing the cost functional $f_0(U)$ given by (7).

$$f_{\cos t}(u) = \int_{S_{obs}} \left(\|\vec{u}\| - \|\overrightarrow{u_d}\|\right)^2 dx = \int_{S_{obs}} \left(\|\vec{u}\|^2 - 2\langle \vec{u}, \vec{u}_d \rangle + \|\vec{u}_d\|\right) dx \tag{9}$$

Since the solution of the forward problem $(\vec{u}_d)$ and the solution of the controlled problem $(\vec{u})$ can be approximated as:

$$\vec{u} \approx \vec{u}_h = \sum_i^{N_u} u_i \vec{\phi}_i \quad \vec{u}_d \approx \vec{u}_{d_h} = \sum_i^{N_o} u_{d_i} \vec{\phi}_i \tag{10}$$

Equation (9) becomes:

$$\begin{aligned} f_0(U) &= \int_{S_{obs}} \left\langle \sum_i u_i\phi_i, \sum_j u_j\phi_j \right\rangle dx - 2\int_{S_{obs}} \left\langle \sum_i u_i\phi_i, \sum_j u_{d_j}\phi_j \right\rangle dx + \int_{S_{obs}} \left\langle \sum_i u_{d_i}\phi_i, \sum_j u_{d_j}\phi_j \right\rangle dx \\ &= \sum_i\sum_j u_i u_j \int_{S_{obs}} \langle \phi_i, \phi_j \rangle dx - 2\sum_i\sum_j u_i u_{d_j} \int_{S_{obs}} \langle \phi_i, \phi_j \rangle dx + \sum_i\sum_j u_{d_i} u_{d_j} \int_{S_{obs}} \langle \phi_i, \phi_j \rangle dx \\ f_0(U) &= U^T M U - 2U^T M U_d + U_D^T M U_D \end{aligned} \tag{11}$$

Where ‖·‖ is the norm induced by the inner product $(\cdot,\cdot)_M$ and M is symmetric $N_u \times N_o$ matrix where each element is given by:

$$mij = \int_{S_{obs}} \phi_i\phi_j dx, i = 1,2,\ldots,N_u, j = 1,2,\ldots,N_o \tag{12}$$

The finite element method (FEM) was used to discretize the NS equations, resulting in the system of algebraic equations is formed in the form:

$$\begin{cases} \mathrm{AU} + \mathrm{B^T P} + \mathrm{N(U)} = 0 \\ \mathrm{BU} = 0 \end{cases} \tag{13}$$

Where U is the matrix $(N_u \times 1)$ of the unknows coefficients u_h, and P is the matrix $(N_p \times 1)$ of the unknows coefficients p_h, AU is the discretised form of diffusion term, where $A_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j\ N(u)$ is the discretised form of nonlinear convection, $B^T P$ is the discretised pressure term, and BU is the discretised form of the divergence of U, where $B_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\psi_j$ [23]–[26]. The full discretization can be consulted in the thesis.

The discrete problem can be stated as:

$$\begin{aligned} &\text{minimise} \quad f_0(U) \\ &\text{subject to} \quad F(U,\alpha_1,\alpha_2,\alpha_3)=0 \quad \text{where} \quad \begin{array}{l} \text{low}_1 < \alpha_1 < up_1 \\ \text{low}_2 < \alpha_2 < up_2 \\ \text{low}_3 < \alpha_3 < up_3 \end{array} \end{aligned} \tag{14}$$

Where F is the discretization of the NS equations, seen in (13) taking into consideration the parameters α_x and its respective bounds.

2.3. Application to aortic coarctation

The framework developed applied to a model of an aorta with coarctation, for validation. The assumptions made regarding the flow were the same as those for developing the framework. The model consists of the upper part of a patient-specific aorta with a coarctation, with the same boundaries as the model in [11]. A virtual boundary (S_{obs}) was added to the aorta for the optimisation process.
The first step consisted of solving system the NS equations, using the BCs in Table 3. The flow rate at the inlet was obtained from the literature ($Q_{in} = 316\ mL/s$) [27], based on the geometry of the patient and the morphology of the aorta.

Table 3: Boundary conditions attributed to the supra-aortic outlets (BA, LCA and LSA) in the forward problem for the aortic coarctation.

Test	Inlet	Outlet DA	Outlet BA	Outlet LCA	Outlet LSA
Data P	▲	●	▲	▲	▲
Data Q	▲	●	●	●	●

● TFBC ▲ $Q_i = (V_{mean} * A_{in}) * \left(\frac{D_i}{D_{in}}\right)^3, i \in (in, BA, LCA, LSA)$

The second step, applying the optimisation framework, consisted of solving the system (8) using the BCs in Table 4, with the appropriate bounds (found in [22]). An additional simulation was performed for each dataset, where the laminar flow hypothesis was not considered (Test P_v and Test Q_v), and the fully-developed profile is not expected.

Table 4: Boundary conditions attributed to the supra-aortic outlets (BA, LCA and LSA) in the controlled problem for the aortic coarctation.

Test	Forward Problem	Inlet	Outlet DA	Outlet BA	Outlet LCA	Outlet LSA
P_1	Data P	▲	●	α_1	α_1	α_1
P_3	Data P	▲	●	α_1	α_2	α_3
P_v	Data P	▲	●	β_1	β_1	β_1
Q_1	Data Q	▲	●	α_1	α_1	α_1
Q_3	Data Q	▲	●	α_1	α_2	α_3
Q_v	Data Q	▲	●	β_1	β_1	β_1

▲ $Q_i = (V_{mean} * A_{in}) * \left(\frac{D_i}{D_{in}}\right)^3, i \in (BA, LCA, LSA)$ α_x $Q_i = (V_{mean} * A_{in}) * \left(\frac{D_i}{D_{in}}\right)^{\alpha_x}, i \in (BA, LCA, LSA)$

● TFBC β_x $V_i = V_{mean} * \left(\frac{D_i}{D_{in}}\right)^3, i \in (BA, LCA, LSA)$

In this case, the objective function was built using patient-specific data. The controlled problem consisted of looking at the parameter α_x such that the following cost function:

$$f_{cost}(U) = \int_{S_{obs}} \left(\|u - v\|\right)^2 dx \tag{15}$$

Where u is the controlled problem solution in the section S_{obs} and v is the mean velocity obtained from a patient-specific hemodynamic gradient. A hemodynamic gradient, in the coarctation area, was obtained from the hospital ($P = 30\ mmHG$) and using a simplification of Bernoulli's equation:

$$\Delta P = 4V^2 \Leftrightarrow V = \sqrt{\frac{\Delta P}{4}} = 2.7 \text{ m/s} \quad (16)$$

3. Results

The solutions of recovering Data A, Data B and Data C were compared and found in [22], together with the results of applying the framework to an aorta with coarctation. In this section, the recovery of the flow from Data B and the flow from the aorta with coarctation are analysed.

Simulations with controlled parameter

Table 5 contains the obtained optimized ML exponent parameters for each test, recovering Data B. Values when applying a control in only one outlet (Test $\mathrm{B}_1^{\mathrm{LCA}}$) are around 60% lower than when applying controls in all three supra-aortic outlets (Test B_1 and Test B_3). Therefore, the velocities obtained in the outlet LCA will be higher for Test $\mathrm{B}_1^{\mathrm{LCA}}$. On the other hand, since the remaining boundary conditions for Test $\mathrm{B}_1^{\mathrm{LCA}}$ have $\alpha = 3$ and $\alpha_1 < 3$, in Test B_1 and Test B_3, outlets LSA and BA will have higher values for both tests.
For Test B_3, as the diameter decreases, $\mathrm{D}_{\mathrm{BA}} > \mathrm{D}_{\mathrm{LSA}} > \mathrm{D}_{\mathrm{LCA}}$, the value of the parameter increases, $\alpha_1 < \alpha_3 < \alpha_2$. Considering the equation (6), as the diameter of a daughter's vessel increases the flow entering that vessel also increases. Additionally, as the parameter of ML increases, the flow declines, since it is powered to a number < 1. Therefore, as the diameter increases, the parameter adapts to the largest daughter's flow is proportional high. For Tests B_1/B_3, where the flow in all outlets was subjected to optimisation, the parameters obtained for Test B_3 are analogous between them - $\alpha_1 = 2.3107$, $\alpha_2 = 2.3122$ and $\alpha_3 = 2.3118$ – and between the parameter obtained for Test B_1 ($\alpha_1 = 2.3114$). Since the parameters are very close to each other, the velocity, pressure, and WSS solutions will be the same. Therefore, the errors derived will also be similar.

Table 5: Murray's Law parameters obtained by solving Test B_1^{LCA}, with and without noise, Test B_1, with and without noise, and Test B_3.

Test	α_1	α_2	α_3
$\mathrm{B}_1^{\mathrm{LCA}}$	1.4437	-	-
$\mathrm{B}_1^{\mathrm{LCA}}$ + noise	1.4441	-	-
B_1	2.3114	-	-
B_1 + noise	2.3039	-	-
B_3	2.3107	2.3122	2.3118

Table 6 contains the errors of the domain, of the section used for the optimisation problem (S_{obs}), and of the descendent aorta. The errors in S_{obs} can be neglected since it is used as the optimisation stop and therefore need to be as small as defined in the objective function (7).

Table 6: Absolute errors, relative errors and magnitude in the domain, section S_{obs} and outlet DA of the controlled solutions obtained by solving Test B_1^{LCA}, with and without noise, Test B_1, with and without noise, and Test B_3.

Test	Domain		S_{obs} Section			Outlet DA		
	Abs. Error	Rel. Error	Abs. Error	Rel. Error	Mag.	Abs. Error	Rel. Error	Mag.
$\mathrm{B}_1^{\mathrm{LCA}}$	2.4E-05	6.8E-03	6.8E-07	7.4E-05	-8.2E-12	3.3E-07	3.2E-05	3.8E-06
$\mathrm{B}_1^{\mathrm{LCA}}$ + noise	2.4E-05	6.8E-03	9.6E-07	1.0E-04	-6.3E-05	6.7E-07	6.4E-05	-5.9E-05
B_1	3.4E-04	9.7E-02	6.6E-05	7.2E-03	-1.2E-10	2.7E-05	2.6E-03	1.5E-04
B_1 + noise	3.4E-04	9.7E-02	7.1E-05	7.7E-03	1.6E-03	3.3E-05	3.2E-03	1.8E-03
B_3	3.4E-04	9.7E-02	6.6E-05	7.2E-03	-9.0E-11	2.7E-05	2.6E-03	1.4E-04

The three tests have a low absolute error in the domain, with Test $\mathrm{B}_1^{\mathrm{LCA}}$ presenting 0.002% and Tests B_1/B_3 0.034%. Tests B_1/B_3, nonetheless, present a considerable relative error (9.7%) while Test $\mathrm{B}_1^{\mathrm{LCA}}$ a small one (0.678%). Since the same BC was attributed for outlets BA and LSA, in Test $\mathrm{B}_1^{\mathrm{LCA}}$, this relative error is only driven by the solution in outlet LCA while for Tests B_1/B_3 all outlets contribute to the error. The attribution of the same boundary conditions from Data B explains the neglectable error found for $\mathrm{B}_1^{\mathrm{LCA}}$ in Table *7*, for those two outlets, in both absolute error and relative error. On the other hand, for LCA – the outlet where the parameter was imposed – the absolute error is around 0.12% and the relative error 32.58%. This value of error means that the velocity in these outlets, with an optimised parameter, is 1/3 higher or lower than the velocity obtained in the controlled solution. Relative errors take into consideration the differences in the magnitude of the velocity vectors and their direction. Since the MLBC was imposed in the optimised solution, this solution must have a parabolic shape while in the controlled solution, the flow does not necessarily achieve a parabolic flow.

Table 7: Absolute errors, relative errors and magnitude in the supra-aortic outlets (Outlet BA, Outlet LCA and Outlet LSA) of the controlled solutions obtained by solving Test B_1^{LCA}, with and without noise, Test B_1, with and without noise, and Test B_3.

Test	Outlet BA			Outlet LSA			Outlet LSA		
	Abs. Error	**Rel. Error**	**Mag.**	**Abs. Error**	**Rel. Error**	**Mag.**	**Abs. Error**	**Rel. Error**	**Mag.**
$\mathrm{B}_1^{\mathrm{LCA}}$	5.6E-09	5.8E-06	-9.5E-08	1.2E-03	3.3E-01	2.2E-02	1.1E-09	1.9E-06	-3.5E-08
$\mathrm{B}_1^{\mathrm{LCA}}$ + noise	7.2E-09	7.4E-06	-3.3E-08	1.2E-03	3.3E-01	2.3E-02	2.5E-09	4.5E-06	-3.0E-08
B_1	1.1E-03	1.1E+00	-1.1E+00	2.8E-03	7.6E-01	7.7E-01	8.4E-04	1.5E+00	-1.5E+00
B_1 + noise	1.1E-03	1.1E+00	-1.1E+00	2.8E-03	7.6E-01	7.6E-01	8.6E-04	1.6E+00	-1.6E+00
B_3	1.1E-03	1.1E+00	-1.1E+00	2.9E-03	7.6E-01	7.7E-01	8.4E-04	1.5E+00	-1.5E+00

Relatively to the errors obtained using a noise component in the objective function, Table 7 shows that they are in the same order of the errors obtained for the same tests without noise.

Application to aortic coarctation

When comparing all the CoA tests, the first observation is that the controlled parameter obtained (Table 8) is similar in the same type of tests, regardless of the data used (Data P or Data Q). For example, α_1 in Test P_3 is similar to α_1 in Test Q_3, while β_1 in Test P_v is similar to β_1 in Test Q_v. The fact that the same values were obtained regardless of the data used confirms the method's validity, even if the flux in the outlets is not well adjusted to the real data. Moreover, parameters show different values as seen in the literature [21], with the tests using a non-laminar profile approach with the closest values to $\alpha = 3$. In this case, the values obtained as parameters for the CoA were higher than the ones for the normal aorta.

Table 8: Murray's Law parameters obtained by solving the CoA tests

Test	$\boldsymbol{\alpha_1/\beta_1}$	$\boldsymbol{\alpha_2}$	$\boldsymbol{\alpha_3}$
P_1	5.5296	-	-
P_3	5.9972	4.2723	4.3380
P_v	3.3576	-	-
Q_1	5.4765	-	-
Q_3	5.9972	4.2723	4.3380
Q_v	3.3576	-	-

Additionally, unlike what was obtained for the tests in the previous section, when using a different control per each outlet (Test P_3 and Test Q_3) the values obtained are considerably difference, mainly between the parameter for the outlet BA (α_1) and the remaining ones. These higher values in α_1 reduce the flow rate going into this outlet which, in the end, approximates the proportion of flow in each supra-aortic outlet to what would be expected. However, this outlet BA does not have a circular shape, but an elliptical one, which might influence the results since Murray's Law was developed in pipes.

Errors - both absolute and relative – are similar for all tests, with the absolute error in the domain < 1%. Nevertheless, for tests with Data P, the relative error in the domain is <10% while for the test with Data Q is ~145%, driven by the errors in the outlet DA and the coarctation section. In the supra-aortic outlets, relative errors are in the order of 10^1 and absolute errors around 0.2% for tests with Data P and 1.4% for tests with Data Q.

Table 9: Absolute errors, relative errors and magnitude in the domain, section S_{obs} and outlet DA of the controlled solutions obtained by solving the CoA tests.

Test	Domain		S_{obs} Section			Outlet DA		
	Abs. Error	Rel. Error	Abs. Error	Rel. Error	Mag.	Abs. Error	Rel. Error	Mag.
P_1	1.1E-03	9.8E-02	2.6E-03	9.4E-02	-9.4E-02	1.6E-03	9.3E-02	-9.3E-02
P_3	1.1E-03	9.7E-02	2.6E-03	9.4E-02	-9.3E-02	1.6E-03	9.3E-02	-9.3E-02
P_v	1.1E-03	9.7E-02	2.6E-03	9.4E-02	-9.3E-02	1.6E-03	9.3E-02	-9.3E-02
Q_1	8.4E-03	1.5E+00	2.0E-02	2.0E+00	-2.0E+00	1.2E-02	2.0E+00	-2.0E+00
Q_3	8.4E-03	1.5E+00	2.0E-02	2.0E+00	-2.0E+00	1.2E-02	2.0E+00	-2.0E+00
Q_v	8.4E-03	1.5E+00	2.0E-02	2.0E+00	-2.0E+00	1.2E-02	2.0E+00	-2.0E+00

4. Conclusions

To conclude, using a variational approach is an acceptable method to attribute boundary conditions when patient-specific is not available to provide individual characteristics to the boundary conditions. The absolute errors over the domain obtained for both models are small (< 1%), with the normal aorta (where the framework was developed) showing smaller values (<0.03%). Relative errors in the domain are considerably higher but no more than 10%, for both models except for some tests in the aorta with coarctation. In general, relative errors in the supra-aortic outlets were higher driven by the laminar profile forced in the boundary conditions.

The ML parameters obtained in both development of the framework and validation using a CoA, are different from those found in the literature, with higher parameter values found for the CoA, which can be explained by the flow behaviour.

This work has some limitations primarily related to the model assumptions, for example, considering the flow stationary or not considering the interaction with the vessel wall. Additionally, the flow was considered laminar. A turbulence or transition model should be considered when studying the aortic arch's hemodynamics since, in pathologies like the aorta's coarctation, the flow is proved to be turbulent.

Other limitations include the mesh sizes, which were not the ideal one, since they have associated a relatively high error; the accuracy of the geometry used, since the acquired images have some noise associated; and the use of inflow data taken from the literature instead of patient-specific data. Nevertheless, this work was only the first stage of developing a method to help make better patient treatment decisions. Surpassing the limitations of the work and improving some of the assumptions, as a next step, could lead to better and more useful results.

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