# Vector fields and black holes

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General Relativity (GR) is an extremely successful description of the gravitational interaction at different scales. One of its most dazzling and profound consequences is that gravitational collapse of massive stars gives rise to black holes. These are ubiquitous objects in our universe, and are now routinely observed and studied in the gravitational-wave band (currently the LIGO/Virgo constellation) but also in the electromagnetic band (with the GRAVITY instrument, the Event Horizon telescope and X-ray telescopes). To study black holes and test the underlying theory of gravity, a precise and complete knowledge of their dynamics is required, in addition to a knowledge of how matter behaves in curved spacetime. In the thesis, we focus on scalar and vector fields around spinning black holes. We present new results concerning massive vector fields in the vicinities of Schwarzschild-anti-de Sitter black holes. In particular, we provide a first principle analysis of vector fields in these geometries, using both a vector spherical harmonics decomposition and one recent ansatz to separate the relevant equations in spinning geometries, the Frolov-Krtouš-Kubizňák-Santos (FKKS) ansatz. We show that the FKKS ansatz is able to describe two polarizations: the longitudinal and the transversal polarization described by the electric modes. The quasinormal modes of such black holes and fields are calculated for both approaches in the non-rotating limit, providing further support to our results.

Keywords: General Relativity, fundamental fields, black holes, Schwarzschild-AdS, quasinormal modes

## I. INTRODUCTION

Special Relativity [1] was formulated by Einstein in 1905 to solve the inconsistency between mechanics and electrodynamics, known to exist since the publication of Maxwell equations [2]. The main point of this theory states that the laws of physics are the same under Lorentz transformations. Thus, the dimensions of time and space must be fused into a 4-dimensional manifold called spacetime. Nevertheless, it was noticed that Newton's law of gravity couldn't be added to special relativity because it is not covariant under Lorentz transformations. Following the work of mathematicians such as Marcel Grossmann and Levi Civita, Einstein used differential geometry as a tool to formulate the theory of General Relativity (GR). Generally, spacetime is described by a manifold with an associated metric, which gives the notion of length. There is also the Riemann tensor, which gives a notion of curvature and can be written in terms of the metric and its derivatives. In Special Relativity, spacetime has the Minkowski metric [3], which is flat meaning the Riemann tensor vanishes everywhere. General Relativity extends this to include gravity by stating that the curvature of spacetime may be nonzero and is related to its energy content through the Einstein equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \ , \tag{1}$$

where $R_{ab}$ is the Ricci tensor, $R$ is the Ricci scalar, $T_{ab}$ is the stress-energy tensor, $G$ is the gravitational constant, $c$ is the speed of light and $\Lambda$ is the cosmological constant. In this theory, gravity gains a new interpretation: the energy of each object deforms spacetime and, in turn, the deformed spacetime "tells" them how to move.

Since the series of publications on GR in 1915, many observations have been done throughout the century to test GR. Back at the time, it was known that Mercury's orbit had an anomalous precession [4] that could be explained by GR [5] and not by Newton's gravity. The first observation that corroborated the theory was done by Eddington et al. [6], consisting of the measurement of light deflection near the sun. Nowadays, GR remains consistent with most of the experiments and observations such as the observation of gravitational lensing, the detection of gravitational redshifts, and most recently with the detection of gravitational waves by the Laser Interferometer Gravitational-wave Observatory (LIGO)/Virgo [7] and the observation of the M87 central supermassive black hole's shadow by the Event Horizon Telescope [8].

Still, some observations cannot be explained solely by GR and the observed matter. The theory is only consistent if an additional matter that interacts very weakly, called dark matter, is considered. A famous candidate to

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describe this matter is the axion [9] (scalar field), that can additionally solve the strong CP problem [10], in Quantum Chromodynamics (QCD). There is another unresolved question which is quantum gravity, an attempt to join GR with Quantum Field Theory. Recent advances have been made with the AdS/CFT conjecture [11], which requires a 5-dimensional asymptotically anti-de Sitter spacetime. This is one of the motivations to study these spacetimes.

From the solutions of the vacuum Einstein equation, it may arise regions of spacetime called black holes. These are mostly born from the collapse of very massive stars [12]. Any object that falls into a black hole can never escape to infinity, not even light, and so black holes can't be observed directly. So, the best way to study black holes is to analyze the behavior of matter, such as scalar or vector fields, in their vicinity.

In the thesis, the main area of study is the behavior of classical fields in a spacetime containing a black hole. Generally, this can be done by using a given lagrangian for the field in curved spacetime and computing the corresponding Euler-Lagrange equations [13]. Then for a fixed background metric, the equations for the field perturbations can be obtained. Analytically, one tries to apply the method of separation of variables and then integrate each separated equation (which in most cases numerical integration must be done).

The most simple case treated and well studied in the literature is the scalar field. Bosonic vector fields are still being studied and their treatment presents more of a challenge in spacetimes with rotating black holes. There is a common feature of fields while interacting with rotating black holes: they exhibit superradiance [14]. This is characterized by the transfer of energy from a rotating object (in this case a black hole) to the field, if the ratio between its energy and angular momentum is lower than the rotation frequency of the object. Superradiance is a general effect also present in the interaction between particles and a medium in the form of Cherenkov radiation, as discussed in Ref. [15]. There is a recent work that showed the existence of superradiance in plasmas as well, check Ref. [16].

In Schwarzschild spacetime (static black hole), the electromagnetic perturbations were calculated in Ref. [17], using the vector spherical harmonics to separate the equations and the treatment was generalized afterward to a massive vector field [18]. The extension to Schwarzschild-(anti)-de Sitter spacetimes was also made in Refs. [19] and [20].

In Kerr spacetime (rotating black hole), the calculation of vector field perturbations is more complicated since there is no spherical symmetry. Nonetheless, the electromagnetic perturbations were computed by Teukolsky [21], using the Newman-Penrose formalism [22] to obtain separated equations. Another method of calculation regarding the electromagnetic field and the separation of its equations was done in Ref. [23] using an ansatz related to Teukolsky's solution, for Myers-Perry geometry [24] (a generalization of Kerr geometry in higher dimensions). Only on a recent work done by V .P. Frolov et al [25–27] the same was done for massive vector fields in Kerr-NUT-(A)dS spacetimes (another generalization of Kerr for higher dimensions, including Newmann-Unti-Tamburino parameters and the cosmological constant). They use an ansatz related to hidden symmetries that exist in Kerr-NUT-(A)dS, mostly referred to in the literature as Frolov-Krtouš-Kubizňák-Santos (FKKS) ansatz. It remains unclear if this ansatz describes all the degrees of freedom of the massive vector field. The purpose of the thesis is to investigate this issue in the Schwarzschild-AdS geometry.

Thus, a general review of the addressed developments is presented in the thesis. Additionally, the comparison between a generalization of the work in Ref. [18] for Schwarzschild-(A)dS and the corresponding limit of the FKKS ansatz is shown. For this purpose, analytical calculations are made, complemented with the calculation of the quasinormal modes.

## II. PROCA FIELD PERTURBATIONS IN SCHWARZSCHILD-ADS

The action for a vector field $A^a$ in a curved spacetime is given by

$$S = -\int \mathrm{d}^4x\sqrt{-g}(L_A - L_E) \ , \tag{2}$$

$$L_E = \frac{R-2\Lambda}{16\pi} \ , \tag{3}$$

$$L_A = \frac{F_{ab}F^{ab}}{16\pi} + \frac{m_A^2}{8\pi}A_aA^a \ , \tag{4}$$

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \ , \tag{5}$$

following the notation in Ref. [28] with $G = c = 1$, except for a general minus sign in $F_{ab}$. From the Euler-Lagrange equations, it follows that the field must obey

$$\nabla_b F^{ab} + m_A^2 A^a = 0 \ , \tag{6}$$

with the internal equations

$$F_{[ab;c]} = 0 \ , \tag{7}$$

and the spacetime metric must obey the Einstein equation (1) where

$$T_{ab} = \frac{1}{4\pi}\left[F_{ae}F_{bf}g^{ef} + m_A^2 A_aA_b - g_{ab}L_A\right] . \tag{8}$$

The above system consists of nonlinear coupled partial differential equations. Due to its complexity, an approximation of weak fields can be made in order to proceed with the analysis. Only the perturbations of the field are considered and the metric is fixed as a background. Generally, known physical fields generate a negligible curvature compared with astrophysical objects. Thus, this approximation is valid in most scenarios. The perturbations of the field are then described by equation (6). The metric considered here is the Schwarzschild-AdS, given by

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad (9)$$

$$f = \frac{r^2}{R_\Lambda^2} + 1 - \frac{2M}{r} \ , \quad (10)$$

where $M$ is the mass of the black hole (in this case), and $R_\Lambda^2 = \frac{3}{|\Lambda|}$. The Schwarzschild-AdS metric describes an empty spherically symmetric spacetime with negative cosmological constant. At the center region, there is a static black hole with the event horizon located at $r_+$ (the positive root of $f$).

The lagrangean of the massless vector field is invariant under the gauge transformation $A^a \to A^a + \chi^{,a}$, where $\chi$ is a scalar. It is possible to partially fix the gauge by imposing the Lorenz condition

$$\nabla_a A^a = 0 \, . \quad (11)$$

For the massive case, the Lorenz condition needs to be obeyed since it naturally arises from the field equations. Thus, it can be used to simplify (6), obtaining the Proca equations

$$g^{cd}\nabla_c\nabla_d A^a - \Big(m_A^2 - \frac{3}{R_\Lambda^2}\Big) A^a = 0 \ . \quad (12)$$

### A. Separation of Variables

In the Schwarzschild-AdS spacetime, the Proca equations (12) can be separated using vector spherical harmonics [29]

$$\boldsymbol{K}_{lm} = Y_{lm}\boldsymbol{e}_{(t)} \ , \quad (13)$$

$$\boldsymbol{Y}_{lm} = Y_{lm}\boldsymbol{e}_{(r)} \ , \quad (14)$$

$$\boldsymbol{\Psi}_{lm} = \partial_\theta Y_{lm}\boldsymbol{e}_{(\theta)} + \frac{1}{\sin\theta}\partial_\phi Y_{lm}\boldsymbol{e}_{(\phi)} \ , \quad (15)$$

$$\boldsymbol{\Phi}_{lm} = -\frac{1}{\sin\theta}\partial_\phi Y_{lm}\boldsymbol{e}_{(\theta)} + \partial_\theta Y_{lm}\boldsymbol{e}_{(\phi)} \ , \quad (16)$$

where $Y^{lm}$ are the spherical harmonics and

$$\boldsymbol{e}_{(t)} = \partial_t \ , \ \boldsymbol{e}_{(r)} = \partial_r \ , \quad (17)$$

$$\boldsymbol{e}_{(\theta)} = \frac{\partial_\theta}{r} \ , \ \boldsymbol{e}_{(\phi)} = \frac{1}{r\sin\theta}\partial_\phi \ . \quad (18)$$

Inherent to the existence of spherical symmetry, these objects are well known from quantum mechanics [30] as they can be obtained by the sum of a spin $S = 1$ with the angular momentum $L$. Therefore, the field may assume the following ansatz

$$A_a = \frac{1}{r}\sum_{i=1}^{4}\sum_{lm} c_i u_{(i)}^{\ell m}(t,r) Z_a^{(i)\ell m}(\theta,\phi) \ , \quad (19)$$

where $c_1 = c_2 = 1$, $c_3 = c_4 = (\ell(\ell+1))^{-\frac{1}{2}}$ and

$$Z_a^{(1)\ell m} = (1,0,0,0)Y^{\ell m} \ , \quad (20)$$

$$Z_a^{(2)\ell m} = (0,f^{-1},0,0)Y^{\ell m} \ , \quad (21)$$

$$Z_a^{(3)\ell m} = \frac{r}{\sqrt{\ell(\ell+1)}}(0,0,\partial_\theta,\partial_\phi)Y^{\ell m} \ , \quad (22)$$

$$Z_a^{(4)\ell m} = \frac{r}{\sqrt{\ell(\ell+1)}}\left(0,0,\frac{\partial_\phi}{\sin\theta},-\sin\theta\partial_\theta\right)Y^{\ell m} \ . \quad (23)$$

Inserting this ansatz into the Proca equations (12), the system for the functions $u_{(i)}$ is obtained as follows

$$\hat{\mathcal{D}}u_{(1)} + (\partial_r f)(\dot{u}_{(2)} - u'_{(1)}) = 0 \ , \quad (24)$$

$$\hat{\mathcal{D}}u_{(2)} + (\partial_r f)(\dot{u}_{(1)} - u'_{(2)}) + \frac{2f^2}{r^2}(u_{(3)} - u_{(2)}) = 0 \quad (25)$$

$$\hat{\mathcal{D}}u_{(3)} + \left[\frac{2f\ell(\ell+1)}{r^2}u_{(2)}\right] = 0 \ , \quad (26)$$

$$\hat{\mathcal{D}}u_{(4)} = 0 \ , \quad (27)$$

where $\dot{u}_{(i)} = \frac{\mathrm{d}u_{(i)}}{\mathrm{d}t}$, $u'_{(i)} = \frac{\mathrm{d}u_{(i)}}{\mathrm{d}r*}$, $\frac{\mathrm{d}r*}{\mathrm{d}r} = f^{-1}$ and

$$\hat{\mathcal{D}} = -\partial_t^2 + \partial_{r*}^2 - f\Big[\frac{\ell(\ell+1)}{r^2} + m_A^2\Big] \ . \quad (28)$$

It must be noted that $u_{(4)}$ describes the magnetic modes and the other functions describe the electric modes. These definitions are associated to the different way the $Z_a^{(i)\ell m}$ transform under parity.

Inserting the ansatz in the Lorenz condition, one obtains

$$\nabla^a A_a = \frac{1}{rf}\left[u'_{(2)} - \dot{u}_{(1)} + \frac{f}{r}(u_{(2)} - u_{(3)})\right] = 0 \ . \quad (29)$$

This condition can be used to simplify equation (25) into

$$\hat{\mathcal{D}}u_{(2)} = \frac{2f}{r^2}\Big(1 - \frac{3M}{r}\Big)(u_{(2)} - u_{(3)}) \ . \quad (30)$$

Therefore, the system consists of equations (24), (30), (26) and (27), together with the Lorenz condition (29). The equations for $u_{(2)}$ and $u_{(3)}$ are coupled together, unlike the equation for $u_{(4)}$ which is totally decoupled. The dynamic part of $u_{(1)}$ can be described by the Lorenz condition, where as the static part must be obtained from equation (24).

This analysis is very similar to Ref. [18], where the equations in Schwarzschild spacetime were presented.

#### 1. Monopole case

The monopole case $\ell = 0$ for the massive vector field simplifies the system considerably. The functions $u_{(3)}$

and $u_{(4)}$ vanish since $Y^{00}$ is a constant. The equation for $u_{(2)}$ can then be written as

$$u''_{(2)} - \ddot{u}_{(2)} - f\Big[m_A^2 + \frac{2}{r^2}\Big(1 - \frac{3M}{r}\Big)\Big]u_{(2)} = 0\ . \tag{31}$$

The dynamic part of $u_{(1)}$ can be obtained directly from the Lorenz condition

$$\dot{u}_{(1)} = u'_{(2)} + \frac{f}{r}u_{(2)}\ , \tag{32}$$

whereas the static part ($u_{(1)\mathrm{s}}$) can be obtained from equation (24)

$$\partial_r^2 u_{(1)\mathrm{s}} = \frac{m_A^2}{f}u_{(1)\mathrm{s}}\ . \tag{33}$$

## III. HIDDEN SYMMETRIES AND THE FKKS ANSATZ

Symmetries are the most useful tools in physics since they provide conserved quantities. Explicit symmetries are the easiest to find because they correspond to isometries of the spacetime. Examples of these type of symmetries are the spherical symmetry and translation of time, corresponding to the conservation of angular momentum and energy of a particle, respectively. It is defined that such symmetries have conserved quantities of the form

$$Q = \xi^a p_a\ , \tag{34}$$

where $\xi^a$ is the Killing vector field associated to the symmetry and $p_a$ is the momentum of the particle. There is, though, another type of symmetries that correspond to conserved quantities of the form

$$Q = k^{a_1 \dots a_n} p_{a_1} \dots p_{a_n}\ , \tag{35}$$

where $\boldsymbol{k}$ is a Killing tensor. These are an example of the so called hidden symmetries. Unlike the explicit ones, they can only be observed in the phase space.

The object of interest here related to these hidden symmetries is the principal tensor $h_{ab}$, a closed conformal Killing-Yano 2-form (anti-symmetric tensor). This object is able to generated a set of symmetries, allowing the integration of the equations of motion. A thorough review is done in Ref. [25] and it is presented in the thesis as well. The principal tensor is described by

$$\nabla_a h_{bc} = 2g_{a[b}\xi_{c]} \quad , \quad \xi_b = \frac{1}{D-1}\nabla^c h_{cb}\ , \tag{36}$$

together with the integrability conditions

$$\nabla^a\nabla^b h_{cd} = \frac{2}{2-D}\Big(R^a{}_e \delta^b_{[c} h^e{}_{d]} + \frac{1}{2}R_{fe}{}^a{}_{[c}\delta^b_{d]}h^{fe}\Big)\ , \tag{37}$$

$$\frac{2R^{[a}{}_e\delta^{b]}_{[c}h^e{}_{d]}}{D-2} - R^{ab}{}_{e[c}h^e{}_{d]} + R_{fe}{}^{[a}{}_{[c}\delta^{b]}_{d]}h^{fe} = 0\ . \tag{38}$$

These conditions give the following properties about the principal tensor: it commutes with the Ricci tensor and $\xi_c$ is a Killing vector field. Additionally, the system composed by equations (36)-(38) is overdetermined, meaning the principal tensor only exists in special kinds of spacetimes. Fortunately, it was found that the Kerr-NUT-(A)dS family belongs to this group. This family describes a generalization of Kerr black holes for higher dimensions, with NUT parameters and cosmological constant. The Kerr-NUT-(A)dS metric for $D = 2n + \epsilon$ dimensions can be written as

$$\begin{aligned}\boldsymbol{g} = &\sum_{\mu=1}^{n}\Bigg[\frac{U_\mu}{X_\mu}dx_\mu^2 + \frac{X_\mu}{U_\mu}\Big(\sum_{k=0}^{n-1}A_\mu^{(k)}d\psi_k\Big)^2\Bigg] \\ &+ \epsilon\frac{c}{A^{(n)}}\Big(\sum_{j=0}^{n}A^{(j)}d\psi_j\Big)^2\ ,\end{aligned} \tag{39}$$

where

$$A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\nu_2<\dots<\nu_j \\ \nu\neq\mu}}^{n} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_j}^2\ , \tag{40}$$

$$A^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\nu_2<\dots<\nu_j}}^{n} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_j}^2\ , \tag{41}$$

$$U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n}(x_\nu^2 - x_\mu^2)\ , \tag{42}$$

and the functions $X_\mu$ are chosen such that the vacuum Einstein equation is satisfied, having the form

$$X_\mu = \begin{cases} -2b_\mu x_\mu + \sum_{k=0}^{n} c_k x_\mu^{2k} & \text{for } D \text{ even,} \\ -\frac{c}{x_\mu^2} - 2b_\mu + \sum_{k=1}^{n} c_k x_\mu^{2k} & \text{for } D \text{ odd.}\end{cases}\ . \tag{43}$$

The constants $c_k$, $b_\mu$ and $c$ are associated with the parameters of the spacetime, for example $b_1$ is related to the mass $M$ and $c_n$ is related to the cosmological constant. The principal tensor has then an expression given by

$$\boldsymbol{h} = \sum_{\mu=1}^{n}\sum_{k=0}^{n-1} x_\mu A_\mu^{(k)} dx_\mu \wedge d\psi_k\ . \tag{44}$$

### A. Kerr-AdS spacetime

The Kerr-AdS spacetime describes a spacetime containing a rotating black hole in 4 dimensions with negative cosmological constant and its metric is

$$ds^2 = \frac{\Sigma}{\Delta_\Lambda}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 - \frac{\Delta_\Lambda}{\Sigma}\Big[dt - a\sin^2\theta d\phi\Big]^2 + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\Big[adt - (a^2+r^2)d\phi\Big]^2 \ . \quad (45)$$

$$\Sigma = \ r^2 + a^2\cos^2\theta \ , \quad (46)$$

$$\Delta_\Lambda = \ r^2 - 2Mr + a^2 + \frac{r^2}{R_\Lambda^2}(r^2+a^2) \ , \quad (47)$$

$$\Delta_\theta = 1 - \frac{a^2}{R_\Lambda^2}\cos^2\theta \ , \quad (48)$$

where $a$ is related to the angular momentum of the black hole. This metric can be obtained from (39) by making the transformation

$$(\psi_0, x_1, x_2, \psi_1) = \left(t - a\phi, ir, a\cos\theta, \frac{\phi}{a}\right) \ . \quad (49)$$

The functions $X_\mu$ are

$$X_1 = -\Delta_\Lambda \ , \quad (50)$$

$$X_2 = -a^2\sin^2\theta\Delta_\theta \ . \quad (51)$$

### B. Proca field and FKKS ansatz

The equations of the massive vector field are separated naturally in Schwarzschild-(A)dS due to spherical symmetry. In spacetimes such as Kerr or more generically Kerr-AdS, there is only axisymmetry which by itself is not enough to separate the equations. Still, the equations for the massless vector field in Kerr were separated by Teukolsky [21] using the Newman-Penrose formalism [22], that consisted in moving to a frame of 4 null basis vectors. Extending this to the massive case was a very hard task. Only recently an ansatz given in Ref. [26, 27], called the FKKS ansatz, was able to achieve the separation in the Kerr-NUT-(A)dS spacetime, using the machinery of the principal tensor, and it can be written as

$$A^a = B^{ab}\nabla_b Z \ , \ B^{ab}(g_{bc} - \beta h_{bc}) = \delta^a_c \ , \quad (52)$$

where $\beta$ is a complex constant and $Z$ is a complex function given by

$$Z = \prod_{\nu=1}^{n} R_\nu(x_\nu) exp\left(i\sum_{k=0}^{n-1} L_k\psi_k\right) \ . \quad (53)$$

The polarization tensor $B_{ab}$ can be put in terms of a $\beta$ dependent Killing tensor generated by the principal tensor. Inserting (52) into the Proca equation (6) and the Lorenz condition (11), in summary, one obtains

$$\left(\sum_{\nu=1}^{n} \frac{1}{U_\nu R_\nu}\tilde{\mathcal{C}}_\nu R_\nu\right) - m_A^2 = 0 \ , \quad (54)$$

$$\sum_{\nu=1}^{n} \frac{1}{(1+\beta^2 x_\nu^2)U_\nu R_\nu}\tilde{\mathcal{C}}_\nu R_\nu = 0 \ , \quad (55)$$

where

$$\tilde{\mathcal{C}}_\nu Z = \ (1+\beta^2x_\nu^2)\partial_{x_\nu}\left(\frac{X_\nu}{1+\beta^2x_\nu^2}\partial_{x_\nu}Z\right) + \frac{1}{X_\nu}\left(\sum_{k=0}^{n-1}(-x_\nu^2)^{n-1-k}\partial_{\psi_k}\right)^2 Z + \beta\sum_{k=0}^{n-1}\beta^{2-2n+2k}\frac{1-\beta^2x_\nu^2}{1+\beta^2x_\nu^2}\partial_{\psi_k}Z \ . \quad (56)$$

The eigenvalue problem can be resumed into

$$\tilde{\mathcal{C}}_\nu Z = \sum_{k=0}^{n-1}(-x_\nu^2)^{n-1-k}C_k Z \ , \quad (57)$$

where $C_k$ are constants of separation. Again, making the substitution into the Proca equation and the Lorenz condition, one obtains

$$C_0 = m_A^2 \ , \ \sum_{k=0}^{n-1} C_k\beta^{2k} = 0 \ . \quad (58)$$

The last equation restricts the possible values of $\beta$. It can be interpreted that a different value for this parameter indicates a different polarization. Still, it is unclear if this ansatz describes all the polarizations.

In the Kerr-AdS geometry, the ansatz for the complex field $Z$ simplifies to

$$Z = R(r)S(\theta)exp\Big(-i\omega t + im_\phi\phi\Big) \ , \quad (59)$$

where the correspondence

$$L_0 = -\omega \quad , \quad L_1 = a(m_\phi - \omega a) \ , \quad (60)$$

has been used. The equations (57) turn into

$$\partial_r\left[\frac{\Delta_\Lambda}{q_r}\partial_r R(r)\right]+\left[\frac{K_r^2}{q_r\Delta_\Lambda}+i\frac{2-q_r}{q_r^2\beta}\sigma+\frac{m_A^2}{\beta^2}\right]R(r)=0\ , \tag{61}$$

$$\frac{1}{\sin\theta}\partial_\theta\left[\frac{q_\Lambda\sin\theta}{q_\theta}\partial_\theta S(\theta)\right]-\left[\frac{K_\theta^2}{q_\theta q_\Lambda\sin^2\theta}+i\frac{2-q_\theta}{q_\theta^2\beta}\sigma+\frac{m_A^2}{\beta^2}\right]S(\theta)=0\ , \tag{62}$$

$$q_\Lambda=1-\frac{a^2}{R_\Lambda^2}\cos^2\theta\ \ ,\ \ q_r=1-\beta^2r^2\ ,\ q_\theta=1+\beta^2a^2\cos^2\theta\ , \tag{63}$$

$$K_r=am_\phi-(a^2+r^2)\omega\ ,\ K_\theta=m_\phi-a\omega\sin^2\theta\ \ ,\ \ \sigma=a\beta^2(m_\phi-\omega a)-\omega\ \ . \tag{64}$$

The condition (58) can be satisfied by setting $C_1=\frac{m_A^2}{\beta^2}$. This allows one to write the eigenvalue equations solely in terms of $\beta$, as shown above. With an expression for the scalar $Z$, it is possible to obtain the corresponding Proca field by (52), where the polarization tensor is given by

$$\boldsymbol{B}=\boldsymbol{B}_{\rm s}+\boldsymbol{B}_{\rm a}\ , \tag{65}$$

$$\boldsymbol{B}_{\rm s}=\frac{\Delta_\Lambda}{q_r\Sigma}\partial_r^2+\frac{q_\Lambda}{q_\theta\Sigma}\partial_\theta^2-\frac{1}{q_r\Delta_\Lambda\Sigma}\Big[(r^2+a^2)\partial_t+a\partial_\phi\Big]^2+\frac{1}{\Sigma q_\theta q_\Lambda\sin^2\theta}\Big[a\sin^2\theta\partial_t+\partial_\phi\Big]^2\ , \tag{66}$$

$$\boldsymbol{B}_{\rm a}=\frac{\beta r}{q_r\Sigma}\Big[(r^2+a^2)(\partial_r\partial_t-\partial_t\partial_r)+a(\partial_r\partial_\phi-\partial_\phi\partial_r)\Big]+\frac{\beta a\sin2\theta}{2\Sigma q_\theta}\Big[a(\partial_t\partial_\theta-\partial_\theta\partial_t)+\frac{1}{\sin^2\theta}(\partial_\phi\partial_\theta-\partial_\theta\partial_\phi)\Big]\ , \tag{67}$$

where $\boldsymbol{B}_{\rm s}$ and $\boldsymbol{B}_{\rm a}$ are the symmetric and the anti-symmetric part of $\boldsymbol{B}$, respectively.

### 1. Schwarzschild-AdS limit

The case for the Schwarzschild-AdS can be obtained by doing the limit $a\to0$. Thus, the angular equation (62) turns into

$$\frac{1}{\sin\theta}\partial_\theta\Big[\sin\theta\partial_\theta S\Big]-\frac{m_\phi^2}{\sin^2\theta}S+\Big[i\frac{\omega}{\beta}-\frac{m_A^2}{\beta^2}\Big]S=0\ . \tag{68}$$

The solutions for this equation are the spherical harmonics $Y^{lm}$. The values for the parameter $\beta$ can then be found by setting

$$i\frac{\omega}{\beta}-\frac{m_A^2}{\beta^2}=\ell(\ell+1)\ . \tag{69}$$

Thus, there are two different values for $\beta$ for each $\ell>0$: $\beta_+$ and $\beta_-$ given by

$$\beta_\pm=i\omega\frac{1\pm\sqrt{1+\frac{4m_A^2\ell(\ell+1)}{\omega^2}}}{2\ell(\ell+1)}\ . \tag{70}$$

For the monopole case ($\ell=0$), the parameter $\beta$ is given by

$$\beta_{\rm mono}=-i\frac{m_A^2}{\omega}\ . \tag{71}$$

These two different $\beta$'s correspond to a polarization. A further quick analysis on the expression suggests that $\beta_-$ describes the longitudinal polarization, since setting $m_A=0$ makes it vanish.

The equation for $R(r)$ is given by

$$\partial_r\Big[\frac{r^2f}{q_r}\partial_rR\Big]+\Big[\frac{\omega^2r^2}{fq_r}-i\omega\frac{2-q_r}{q_r^2\beta}+\frac{m_A^2}{\beta^2}\Big]R=0\ , \tag{72}$$

and from (65)-(67) the tensor $B^{ab}$ becomes

$$\boldsymbol{B}_{\rm sym}=-\frac{1}{q_rf}\partial_t^2+\frac{f}{q_r}\partial_r^2+\frac{1}{r^2}\partial_\theta^2+\frac{1}{r^2\sin^2\theta}\partial_\phi^2\ , \tag{73}$$

$$\boldsymbol{B}_{\rm anti}=\frac{\beta r}{q_r}(\partial_r\partial_t-\partial_t\partial_r)\ . \tag{74}$$

Thus, it is possible to obtain the expression for the covariant components of the massive vector field in function of the scalar $R(r)$ and the spherical harmonics $Y(\theta,\phi)$

$$A_a=\Big(-\frac{i\omega}{q_r}+\frac{\beta rf}{q_r}\partial_r,\frac{1}{q_r}\partial_r-i\frac{\omega\beta r}{q_rf},\partial_\theta,\partial_\phi\Big)RY\ . \tag{75}$$

Comparing (75) with (19), it can be seen that the FKKS ansatz in Schwarzschild-AdS limit ($a \to 0$) does not describe the magnetic polarization ($u_{(4)}$ in (19)). By inspecting the definition of the principal tensor (44), one of the eigenvalues of $\boldsymbol{h}$ is given by $x_2 = a\cos\theta$. By doing the Schwarzschild-AdS limit, the eigenvalue goes to zero and so the principal tensor is degenerate. This fact might be the cause for the absence of the magnetic modes.

The important aspect of ansatz (75) is that there is a natural decoupling of the two polarizations contained in the electric modes, in opposite to the treatment of section II. In the thesis, it is proven analytically that this expression continues to obey the Proca equations (25), (26) and the Lorenz condition (29), by considering the correspondence

$$u_{(1)} = -\frac{i\omega r}{q_r}R(r) + \frac{\beta r^2 f}{q_r}\partial_r R(r) \ , \tag{76}$$

$$u_{(2)} = \frac{rf}{q_r}\partial_r R(r) - i\frac{\omega\beta r^2}{q_r}R(r) \ , \tag{77}$$

$$u_{(3)} = \ell(\ell+1)R(r) \ . \tag{78}$$

## IV. QUASINORMAL MODES IN SCHWARZSCHILD-ADS

With the separated equations found, the analysis of the system is concluded by integrating them. The quasinormal modes are defined by solutions that solve the equations of the field with a purely incoming wave at the event horizon and a purely outgoing wave at infinity, as boundary conditions. In the case of AdS spacetimes, the boundary conditions are equivalent to $u_{(i)} \to 0$ at infinity.

### A. Normal modes in AdS limit

In the limit of pure AdS ($M \to 0$), the equation (27) is simplified into

$$\frac{\partial^2_{r_*}u_{(4)}}{u_{(4)}} + \left[\omega^2 - \frac{\ell(\ell+1)}{R_\Lambda^2\sin\left(\frac{r_*}{R_\Lambda}\right)} - \frac{m_A^2}{\cos\left(\frac{r_*}{R_\Lambda}\right)}\right] = 0 \ , \tag{79}$$

where $r_* = R_\Lambda \arctan\left(\frac{r}{R_\Lambda}\right)$. Fortunately, this equation can be solved analytically. By performing the transformation $z = \sin\left(\frac{r_*}{R_\Lambda}\right)$ into (79) and assume an ansatz of the type

$$u_{(4)} = z^\alpha(1-z)^\beta\psi \ , \tag{80}$$

$$\alpha = \frac{1}{4}\left[1 + \sqrt{1+4\ell(\ell+1)}\right] \ , \tag{81}$$

$$\beta = \frac{1}{4}\left[1 + \sqrt{1+4m_A^2R_\Lambda^2}\right] \ , \tag{82}$$

The equation assumes the form of the hypergeometric differential equation

$$z(1-z)\partial_z^2\psi + \Big[c - (a+b+1)z\Big]\partial_z\psi - ab\psi = 0 \ , \tag{83}$$

where

$$a = \alpha + \beta + \frac{\omega R_\Lambda}{2} \ , \ b = \alpha + \beta - \frac{\omega R_\Lambda}{2} \ , \tag{84}$$

$$c = \frac{1}{2} + 2\alpha \ . \tag{85}$$

The solutions of this equation are described by the hypergeometric function ${}_2F_1$. Thus,

$$u_{(4)} = H_1 z^\alpha(1-z)^\beta \,{}_2F_1[a,b,c;z] + H_2 z^{\frac{1}{2}-\alpha}(1-z)^\beta \,{}_2F_1[d-c,e-c,2-c;z] \ , \tag{86}$$

where $H_1$, $H_2$ are constants, $d = 1 + a$ and $e = 1 + b$. Since there is no event horizon, the boundary condition that must be satisfied is $u_{(4)} \to 0$ at $r \to 0$ ($z \to 0$). The hypergeometric function assumes the value of unity when $z = 0$. Since $\alpha, \beta \geq \frac{1}{2}$, the second solution explodes, thus $H_2$ must be set to 0.

The solution must be transformed in order to be analyzed near $z = 1$ ($r \to +\infty$). Following Ref. [31], one has

$${}_2F_1[a,b,c;z] = \frac{\Gamma(c)\Gamma(w)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1[a,b,1+v;1-z] + (1-z)^{\frac{1}{2}-2\beta}\frac{\Gamma(c)\Gamma(v)}{\Gamma(a)\Gamma(b)}\,{}_2F_1[a,b,1+w;1-z] \ , \tag{87}$$

where $w = c - a - b$, $v = a + b - c$ and $\Gamma$ is the gamma function. If all the gamma functions are finite, the solution explodes at $z = 1$ since $\beta \geq \frac{1}{2}$. The only way for $u_{(4)}$ to vanish is if either $a$ or $b$ is a negative integer ($-n$). Without loss of generality, one can choose

$$b = -n \ \to \omega R_\Lambda = 2n + \ell + \frac{3}{2} + \frac{1}{2}\sqrt{1+4m_A^2R_\Lambda^2} \ , \tag{88}$$

thus obtaining the normal modes for $u_{(4)}$. Surprisingly, the equation for the monopole (31) in the AdS limit can be put also in the form of (79) with $\ell = 1$. By fixing this value of $\ell$, equation (88) describes the normal modes of the monopole as well. This treatment was done in Ref. [20] but it has a mistake in the expression for $b$. This statement is supported by numerical calculations.

### B. Treatment in Schwarzschild-AdS

In Schwarzschild-AdS, it is possible to write every component of the vector field as an expansion series

$$U_{(i)} = u_{(i)}e^{i(t+r_*)\omega} = \sum_{n=0}^{\infty} a_n^{(i)}(x - x_+)^n \ , \tag{89}$$

where $x = \frac{1}{r}$ and $x_+ = \frac{1}{r_+}$. The equation (27) for $u_{(4)}$ can be written as

$$\left[(x - x_+)s(x)\partial_x^2 + t(x)\partial_x + \frac{u(x)}{x - x_+}\right]U_{(4)} = 0 \ . \quad (90)$$

where

$$u(x) = -(x - x_+)\Big[x^2\ell(\ell+1) + m_A^2\Big] \ , \quad (91)$$

$$t(x) = x^2\partial_x\Big(\frac{f}{r^2}\Big) + 2i\omega x^2 \ , \quad (92)$$

$$s(x) = \frac{f}{r^2(x - x_+)}x^2 \ , \quad (93)$$

$$\frac{f}{r^2} = R_\Lambda^{-2} + x^2 - 2Mx^3 \ . \quad (94)$$

These polynomials can be expanded around $x_+$ so that it is possible to write for example $u(x) = \sum_{j=0}^{N_d} u_j(x - x_+)^j$. Equation (90) can then be reduced to a recurrence relation

$$a_n = -\frac{1}{P_n}\sum_{j=0}^{n-1}\Big(j(j-1)s_{n-j} + jt_{n-j} + u_{n-j}\Big)a_j \ , \quad (95)$$

$$P_n = n(n-1)s_0 + nt_0 \ . \quad (96)$$

The quasinormal modes can be found by imposing that the series vanishes at $x \to 0$ $(r \to +\infty)$

$$\sum_{j=0}^{N} a_j^{(4)}(-x_+)^j = 0 \ . \quad (97)$$

This is the Horowitz-Hubeny method [32] for the calculation of quasinormal modes in Schwarzschild-AdS.

The Proca equations (30) and (26) are coupled, so they required a more careful treatment. An extension of the method described above can be made, substituting the coefficients by matrices. Using expansion (89), this two equations turn into

$$(x - x_+)s(x)\partial_x^2\boldsymbol{U} + t(x)\partial_x\boldsymbol{U} + \frac{u(x)}{x - x_+}\boldsymbol{U} + \frac{1}{x - x_+}\boldsymbol{K}.\boldsymbol{U} = 0 \ , \quad (98)$$

$$\boldsymbol{U} = \begin{bmatrix} U_{(2)} \\ U_{(3)} \end{bmatrix}, \quad (99)$$

where all the polynomials are the same as in the case of $u_{(4)}$ and

$$\frac{\boldsymbol{K}}{(x - x_+)} = \begin{bmatrix} -2x^2(1 - 3Mx) & 2x^2(1 - 3Mx) \\ 2x^2\ell(\ell+1) & 0 \end{bmatrix} \ . \quad (100)$$

By defining

$$\boldsymbol{a}_n = \boldsymbol{M}_n\boldsymbol{a}_0 \ , \ \boldsymbol{a}_n = \begin{bmatrix} a_n^{(2)} \\ a_n^{(3)} \end{bmatrix} \ , \ \boldsymbol{M}_0 = \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (101)$$

The Proca equations for $u_{(2)}$ and $u_{(3)}$ are reduced to the recurrence relation

$$\boldsymbol{M}_n = -\frac{1}{P_n}\sum_{j=0}^{n-1}\boldsymbol{V}_{nj}.\boldsymbol{M}_j \ , \quad (102)$$

$$\boldsymbol{V}_{nj} = \big[j(j-1)s_{n-j} + jt_{n-j} + u_{n-j}\big]\boldsymbol{I} + \boldsymbol{K}_{n-j} \ . \quad (103)$$

The quasinormal modes can then be obtained by imposing that both series vanish simultaneously at $x \to 0$, meaning

$$\det\left(\sum_{j=0}^{N}\boldsymbol{M}_j(-x_+)^j\right) = 0 \ . \quad (104)$$

The treatment for the FKKS ansatz requires more complex manipulations in equation (72), as shown in the thesis. It is possible to apply the same method done for $u_{(4)}$ and a similar equation to (90) is obtained, with different polynomials

$$s(x) = x^2\frac{f}{r^2}\frac{(x^2 - \beta^2)}{x - x_+} \ , \quad (105)$$

$$t(x) = (x^2 - \beta^2)(2i\omega x^2 + 2x^3 - 6Mx^4) - 2x^3\frac{f}{r^2}, \quad (106)$$

$$u(x) = (x - x_+)\Big[(x^2 - \beta^2)(m_A^2 + \ell(\ell+1)x^2) - 2i\omega(x^3 + \beta x^2)\Big] \ . \quad (107)$$

### C. Results

The computation of the quasinormal modes was done numerically using *Mathematica*, with $N = 40$. The quasinormal modes for $u_{(4)}$ with $\ell = 1$ are presented in the thesis for different ranges in the parameters. In the massless case, the results are consistent with Ref. [19]. The quasinormal modes for the monopole using both the treatment in section II and the FKKS ansatz are also presented in the thesis and they seem consistent with Ref. [20]. The most important results are the comparison of the quasinormal modes between the treatment with vector spherical harmonics (VSH) in section II and the FKKS ansatz, for $\ell = 1$. A sample of these results are presented in table I. They seem to coincide up to a deviation of $\mathcal{O}(0.1)$.

In the computation of the quasinormal modes regarding the system of equations for $u_{(2)}$ and $u_{(3)}$, there were higher monotones that could not be found and that were present in the FKKS ansatz. It might be possible that the fact of having two polarizations in the system requires an higher value of $N$ so that these monotones can be found. Also regarding the system, only by inference one can distinguish each polarization. Fortunately, this is well characterized in the FKKS ansatz by the different values of $\beta$, allowing an easier distinction.

| | $r_+ = R_\Lambda$ | | $r_+ = 100R_\Lambda$ | |
|---|---|---|---|---|
| $m_A R_\Lambda$ | $\omega R_\Lambda$ (VSH) | $\omega R_\Lambda$ (FKKS) | $\omega R_\Lambda$ (VSH) | $\omega R_\Lambda$ (FKKS) |
| 0.01 | $1.554 - 0.542\ i$ | $1.554 - 0.542\ i$ | $(0) - 149.984\ i$ | $(0) - 149.984\ i$ |
| 0.10 | $1.557 - 0.552\ i$ | $1.557 - 0.552\ i$ | $(0) - 152.099\ i$ | $(0) - 152.099\ i$ |
| 0.20 | $1.568 - 0.583\ i$ | $1.568 - 0.584\ i$ | $(0) - 158.432\ i$ | $(0) - 158.432\ i$ |
| 0.30 | $1.585 - 0.633\ i$ | $1.584 - 0.633\ i$ | $(0) - 168.817\ i$ | $(0) - 168.817\ i$ |
| 0.40 | $1.607 - 0.699\ i$ | $1.606 - 0.699\ i$ | $(0) - 183.291\ i$ | $(0) - 183.291\ i$ |
| 0.50 | $1.634 - 0.777\ i$ | $1.632 - 0.777\ i$ | $(0) - 202.684\ i$ | $(0) - 202.684\ i$ |

(a) Transversal polarization ($\beta_+$).

| | $r_+ = R_\Lambda$ | | $r_+ = 100R_\Lambda$ | |
|---|---|---|---|---|
| $m_A R_\Lambda$ | $\omega R_\Lambda$ (VSH) | $\omega R_\Lambda$ (FKKS) | $\omega R_\Lambda$ (VSH) | $\omega R_\Lambda$ (FKKS) |
| 0.01 | $3.331 - 2.489\ i$ | $3.330 - 2.489\ i$ | $184.968 - 266.394\ i$ | $184.968 - 266.395\ i$ |
| 0.10 | $3.339 - 2.500\ i$ | $3.339 - 2.501\ i$ | $185.604 - 267.461\ i$ | $185.578 - 267.524\ i$ |
| 0.20 | $3.362 - 2.531\ i$ | $3.362 - 2.534\ i$ | $187.452 - 270.612\ i$ | $187.355 - 270.817\ i$ |
| 0.40 | $3.446 - 2.645\ i$ | $3.444 - 2.652\ i$ | $193.925 - 282.119\ i$ | $193.650 - 282.498\ i$ |
| 0.50 | $3.501 - 2.722\ i$ | $3.498 - 2.729\ i$ | $198.077 - 289.799\ i$ | $197.761 - 290.138\ i$ |

(b) Longitudinal polarization ($\beta_-$).

TABLE I: Quasinormal modes ($\omega R_\Lambda$) of the Proca's electric modes in Schwarzschild-AdS with $\ell = 1$, for $r_+ = \{R_\Lambda, 100R_\Lambda\}$, with variable $m_A$. A comparison between the treatment with vector spherical harmonics (VSH) and the FKKS ansatz is presented.

## V. CONCLUSIONS

In this thesis, the topics of scalar fields and vector fields in spacetimes containing a black hole were reviewed. This subject is important to understand black holes better, to test GR and to possibly describe new physics such as dark matter. The equations for the scalar field in Kerr geometry were separated. The treatment of a minimally coupled scalar field is well studied in the literature. For a scalar field with Gauss-Bonnet coupling in Kerr geometry, it was demonstrated that the equations couldn't be separated due to the angular dependence of the Kretschmann scalar. The equations for the vector field in Kerr were also separated in the thesis. The treatment of massless vector fields was originally done by Teukolsky [21]. It is known that both fields can exhibit superradiance [33] in Kerr geometry, which was also shown. The generalization of Teukolsky's work for a massive vector field took more or less 30 years to appear with the construction of the FKKS ansatz [27], which uses hidden symmetries of Kerr-NUT-(A)dS to separate the Proca equations. A review about this ansatz, the principle tensor and the Kerr-NUT-(A)dS spacetime was presented.

A question still lingers about the FKKS ansatz: Does it describe all the degrees of freedom of the massive vector field? The objective of the thesis was to investigate this in Schwarzschild-AdS geometry, the non-rotating limit of Kerr-AdS. The objective was accomplished by making analytical and numerical comparisons between this ansatz and the typical treatment with vector spherical harmonics.

It was concluded that the FKKS ansatz is able to describe two of the three polarizations of the massive vector field in Schwarzschild-AdS: the longitudinal and the transversal polarizations corresponding to the electric modes [18] of the field. The absence of the magnetic modes in the ansatz may be due to the degeneracy of the principal tensor in the non-rotating limit. The analytical correspondence between the two treatments was obtained and revealed a transformation that decouples the two polarizations in the Proca equations. Indeed, an advantage of working with the FKKS ansatz is the natural decoupling of the polarizations, opposed to the typical treatment. The numerical comparison consisted in the calculation of the quasinormal modes for each ansatz and they seem to coincide with a maximum deviation of $\mathcal{O}(0.1)$, thus corroborating the drawn conclusion. Finally, it is worth mentioning that the calculation of the monopole's normal modes in anti-de Sitter done in Ref. [20] had a mistake and was corrected here.

To extend this work, the FKKS ansatz in Kerr-AdS spacetime should be studied. The Newman-Penrose formalism used by Teukolsky in Kerr is also valid in Kerr-AdS [34]. This means a possible comparison can be made between the FKKS ansatz in the massless limit and the treatment with this formalism. The comparison has important significance since a better identification of the polarizations can be made. Also, the principal tensor is non-degenerate in this geometry, in opposite to Schwarzschild-AdS. Thus, one should be hopefully able to find the magnetic modes.

Another work of interest would be to study extensions of the FKKS ansatz and the principal tensor even further. It is known that the principal tensor generates a Killing tower: a set of Killing objects that translate to symmetries. A legitimate question would be for example: Are these symmetries able to separate the equations of vector fields with higher order coupling terms to the curvature? If such ansatz exists, it may imply a different dependence in the principal tensor. Also, an interesting question would be: Is it possible to extend this for field with tensor nature, such as massive gravitons? An affirmative answer to these questions would be important for developments in the study of such fields.

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[1] A. Einstein, Annalen der Physik **14**, 194 (2005).
[2] J. C. Maxwell, Philosophical Transactions of the Royal Society of London **155**, 459 (1865).
[3] H. Minkowski, Math. Annalen **68**, 472 (1910).
[4] U. L. Verrier, Comptes rendus hebdomadaires des séances de l'Académie des sciences **49**, 379 (1859).
[5] A. Einstein, Die grundlage der allgemeinen relativitätstheorie, Annalen der Physik **354**, 769 (1916).
[6] F. W. Dyson, A. S. Eddington, and C. Davidson, Philosophical Transactions of the Royal Society of London A, **220**, 291 (1920).
[7] B. P. Abbott *et al.* (The LIGO Scientific Collaboration and the Virgo Collaboration), Phys. Rev. D **100**, 104036 (2019).
[8] K. Akiyama *et al.* (The EHT collaboration), Astrophys. J. Lett. **875**, 10.3847/2041-8213/ab0e85 (2019).
[9] L. D. Duffy and K. van Bibber, New J. Phys. **11**, 105008 (2009), arXiv:0904.3346 [hep-ph].
[10] R. D. Peccei, Lect. Notes Phys. **741**, 3 (2008), arXiv:hep-ph/0607268 [hep-ph].
[11] J. M. Maldacena, The Large N limit of superconformal field theories and supergravity, Int. J. Theor. Phys. **38**, 1113 (1999), arXiv:hep-th/9711200.
[12] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).
[13] J. Lagrange, *Mécanique Analytique*, Vol. 1 (Cambridge University Press, 2009).
[14] R. Brito, V. Cardoso, and P. Pani, Lect. Notes Phys. **906**, pp.1 (2015), arXiv:1501.06570 [gr-qc].
[15] J. D. Bekenstein and M. Schiffer, Phys. Rev. D **58**, 064014 (1998), arXiv:gr-qc/9803033.
[16] J. Vieira, M. Pardal, J. T. Mendonça, and R. A. Fonseca, Nature Physics 10.1038/s41567-020-0995-5 (2020).
[17] R. Ruffini, Black holes: les astres occlus (Gordon and BreachScience Publishers, 1973) pp. 451–547.
[18] J. G. Rosa and S. R. Dolan, Phys. Rev. **D85**, 044043 (2012), arXiv:1110.4494 [hep-th].
[19] V. Cardoso and J. P. S. Lemos, Phys. Rev. **D64**, 084017 (2001), arXiv:gr-qc/0105103 [gr-qc].
[20] R. A. Konoplya, Phys. Rev. D **73**, 024009 (2006), arXiv:gr-qc/0509026.
[21] S. A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972).
[22] E. T. Newman and R. Penrose, Journal of Mathematical Physics **3**, 566 (1962).
[23] O. Lunin, JHEP **12**, 138, arXiv:1708.06766 [hep-th].
[24] R. C. Myers (2012) pp. 101–133, arXiv:1111.1903 [gr-qc].
[25] V. P. Frolov, P. Krtouš, and D. Kubizňák, Living Rev. Rel. **20**, 6 (2017), arXiv:1705.05482 [gr-qc].
[26] P. Krtouš, V. P. Frolov, and D. Kubizňák, Nucl. Phys. **B934**, 7 (2018), arXiv:1803.02485 [hep-th].
[27] V. P. Frolov, P. Krtouš, D. Kubizňák, and J. E. Santos, Phys. Rev. Lett. **120**, 231103 (2018), arXiv:1804.00030 [hep-th].
[28] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W. H. Freeman, San Francisco, 1973).
[29] R. G. Barrera, G. A. Estevez, and J. Giraldo, European Journal of Physics 10.1088/0143-0807/6/4/014 (1985).
[30] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics*, 2nd ed. (Cambridge University Press, 2017).
[31] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 9th ed. (Dover, New York, 1964).
[32] G. T. Horowitz and V. E. Hubeny, Phys. Rev. D **62**, 024027 (2000), arXiv:hep-th/9909056.
[33] S. A. Teukolsky and W. H. Press, Astrophys. J. **193**, 443 (1974).
[34] O. J. Dias, J. E. Santos, and M. Stein, JHEP **10**, 182, arXiv:1208.3322 [hep-th].