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Vector fields and black holes

Tiago Vasques Fernandes

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Supervisor(s): Prof. Vítor Manuel dos Santos Cardoso

Examination Committee

Chairperson: Prof. Dr. José Pizarro de Sande e Lemos
Supervisor: Prof. Dr. Vítor Manuel dos Santos Cardoso
Member of the Committee: Prof. Dr. David Mathew Hilditch

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Resumo

A Relatividade Geral (GR) é uma descrição da interação gravitacional a diferentes escalas, com extremo sucesso. Uma das mais deslumbrantes e profundas consequências é que o colapso de estrelas massivas dão origem a buracos negros. Estes são objetos obíquos no nosso universo, não só são observados frequentemente e estudados na banda de ondas gravitationais (atualmente com a constelação LIGO/Virgo) mas também na banda eletromagnética (com o instrumento GRAVITY, o Event Horizon Telescope e telescópios X-ray). Para estudar buracos negros e testar a teoria da gravidade subjacente, é necessário um conhecimento preciso e completo das suas dinâmicas, além de um conhecimento de como a matéria se comporta em espaço-tempos curvos.

Nesta tese, nós focamo-nos em campos escalares e vetoriais à volta de buracos negros em rotação. Nós apresentamos novos resultados relativo a campos massivos vetoriais na proximidade de buracos negros Schwarzschild-anti-de Sitter. Em particular, nós fornecemos uma análise de primeiro princípio de campos vetoriais nestas geometrias, usando a decomposição em vetores harmónicos esféricos e uma ansatz recente que separa as equações relevantes em geometrias com rotação, a Frolov-Krtouš-Kubizňák-Santos (FKKS) ansatz. Nós mostramos que a FKKS ansatz consegue descrever duas polarizações: as polarizações longitudinal e transversal descritas pelos modos elétricos. Os modos quase-normais destes buracos negros e campos são calculados para ambas as abordagens no limite sem rotação, fornecendo suporte adicional aos nossos resultados.

Palavras-chave: Relatividade Geral, campos fundamentais, buracos negros, Schwarzschild-AdS, modos quase-normais

Abstract

General Relativity (GR) is an extremely successful description of the gravitational interaction at different scales. One of its most dazzling and profound consequences is that gravitational collapse of massive stars gives rise to black holes. These are ubiquitous objects in our universe, and are now routinely observed and studied in the gravitational-wave band (currently the LIGO/Virgo constellation) but also in the electromagnetic band (with the GRAVITY instrument, the Event Horizon telescope and X-ray telescopes). To study black holes and test the underlying theory of gravity, a precise and complete knowledge of their dynamics is required, in addition to a knowledge of how matter behaves in curved spacetime.

In the thesis, we focus on scalar and vector fields around spinning black holes. We present new results concerning massive vector fields in the vicinities of Schwarzschild-anti-de Sitter black holes. In particular, we provide a first principle analysis of vector fields in these geometries, using both a vector spherical harmonics decomposition and one recent ansatz to separate the relevant equations in spinning geometries, the Frolov-Krtouš-Kubizňák-Santos (FKKS) ansatz. We show that the FKKS ansatz is able to describe two polarizations: the longitudinal and the transversal polarization described by the electric modes. The quasinormal modes of such black holes and fields are calculated for both approaches in the non-rotating limit, providing further support to our results.

Keywords: General Relativity, fundamental fields, black holes, Schwarzschild-AdS, quasinormal modes

Notation

- The convention and definitions of the Riemann tensor and the lagrangian of the fields follow the book *Gravitation* (Misner, Thorne, Wheeler, 1973).
- The signature of the Lorentzian metric used is $(-,+,+,+)$.
- The indices for the components of a tensor, without defining an explicit chart, are Latin letters.
- In the document, there are sections where a frame is chosen. The identification of the basis vectors for the frame is done using Greek indices.
- The Einstein notation is assumed only for Latin indices, except in appendix B where Einstein notation is used also in Greek indices.
- Regarding tensors, the whole object is referred to when it is written in bold. For example, a (p,q) tensor is

$$\boldsymbol{T} = T^{a_1...a_p}{}_{b_1...b_q}\partial_{x^{a_1}}...\partial_{x^{a_p}}dx^{b_1}...dx^{b_q} \ . \tag{1}$$

- The partial and the covariant derivative can be represented by "," and ";", respectively, i.e

$$\nabla_a\nabla_b T_c = T_{c;ba} \quad , \quad \partial_a T = T_{,a} \tag{2}$$

- The dot operation between T and a second tensor Q is the contraction of the adjacent indices. For example, the dot of a (0,2) tensor T and a (0,2) tensor Q is given by

$$\boldsymbol{T}\cdot\boldsymbol{Q} = T_{ab}Q^b{}_c dx^a dx^c \ . \tag{3}$$

- For a p-form X (full anti-symmetric (0,n) tensor) and a q-form Y, the wedge is given by

$$(\boldsymbol{X}\wedge\boldsymbol{Y})_{a_1...a_pb_1...b_q} = \frac{(p+q)!}{p!q!}X_{[a_1...a_p}Y_{b_1...b_q]} \ , \tag{4}$$

  where the brackets $[\ ]$ correspond to the anti-symmetric part with respect to the indices inside them.
- The exterior derivative of a p-form is defined by

$$(\boldsymbol{dX})_{aa_1...a_p} = (p+1)\nabla_{[a}X_{a_1...a_p]} \ , \tag{5}$$

  where ∇_a is the covariant derivative using the Levi-Civita connection associated to the metric.
- Any spacetime considered is a orientable manifold. This means it is possible to define a volume form, also called Levi-Civita tensor

$$\boldsymbol{\epsilon} = \sqrt{-g}dx^1\wedge dx^2\wedge ...\wedge dx^D \ , \tag{6}$$

  where D is the number of dimensions.
- The Hodge dual of a p-form X is defined by

$$(\star\boldsymbol{X})_{b_1...b_{D-p}} = \frac{1}{p!}\epsilon_{a_1...a_pb_1...b_{D-p}}X^{a_1...a_p} \ . \tag{7}$$

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Chapter 1

Introduction

After the publication of Maxwell equations [1], it was found that electrodynamics was inconsistent with Galileo's relativity. To connect these two concepts, there was a theory that postulated the existence of a medium through which light could propagate (luminiferous aether). The Michelson-Morley experiment [2] disproved the existence of such a medium, leaving the problem unanswered. In 1905, Albert Einstein published the theory of special relativity [3], which was able to solve this inconsistency. To construct it, Einstein had to start from two postulates: the first admits that the form of the laws of physics is the same in all inertial frames (rest or constantly co-moving frames of reference), the second assumes that the speed of light is the same in each of these inertial frames. Therefore, Lorentz transformations become isometries (transformations in which the laws of physics assume the same form). Due to their nature, time and space cannot be separated as they must be fused into a 4-dimensional manifold called Minkowski spacetime [4]. The definitions of particle energy and momentum are reformulated to account for these isometries, being written as

$$E = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad , \quad \vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} \ . \tag{1.1}$$

From these two equations, Special Relativity reveals a strong prediction about the Universe: information cannot travel faster than the speed of light. In particular, massive objects can never reach this speed limit.

Nevertheless, there was still a conundrum to be solved for this theory to be complete. Newton's gravitational law was not covariant under Lorentz transformations. Therefore, the only description of gravity at the time was not compatible with special relativity. Following the work of mathematicians such as Marcel Grossmann and Tullio Levi-Civita, Einstein looked for answers in differential geometry to reformulate the law of gravity. This abstract mathematical formalism covers the differential calculus treatment of manifolds and it has embedded the description of general covariance. A manifold is defined as a topological space (a set of points that have certain properties considering a family of its subsets) that locally resembles the Euclidean (or Minkowski) space. The chart of a D-dimensional manifold translates the points into coordinates in a D-dimensional Euclidean space. On this manifold, there may be scalar functions defined that account for some physical properties, like temperature for example. Vector fields are defined as operators that can be applied to functions, giving their directional derivative, and they are embedded in the tangent space of a given point on a manifold. On the other side, covector fields are defined as operators that can be applied to vector fields, giving the scalar product, and they are embedded in the dual tangent space. By this logic, a rank-(p, q) tensor is an object such that when applied to p covectors and q vectors, gives a scalar.

In differential geometry, there are two types of derivatives: the covariant derivative and the Lie derivative. The covariant derivative is similar to the typical derivative in Euclidean space, but it requires an affine connection, an object that translates the notion of parallel transport. Since the manifold can be curved, the tangent spaces of two points near each other may not be parallel. The affine connection defines how these two can be compared, in a perspective of parallelism. The manifold can also be associated with a metric, which is a (0,2) tensor that gives the notion of length. The Levi-Civita connection is an affine connection that is torsion-free and that makes the covariant derivative of the metric vanish. This allows it to be written in terms of the metric and its derivatives. The Riemann tensor gives a notion of curvature and it can be expressed in terms of the Levi-Civita connection and its derivatives. Thus, it is possible to describe the geometry of the manifold only by knowing the metric.

Differential geometry was then used to formulate General Relativity (GR). This theory describes the whole spacetime as a manifold, which locally resembles Minkowski spacetime, with a metric satisfying the Einstein equation

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \ . \tag{1.2}$$

This equation generalizes Newton's law of gravitation since it relates the curvature of spacetime with its energy content. In this theory, gravity can be physically interpreted in the following way: the energy of each object deforms spacetime and, in turn, the deformed spacetime "tells" them how to move.

1.1 The testing of GR

A first test to the theory was provided 100 years ago by Eddington's experiment [5]. It consisted in observing the apparent position of stars near the Sun, to calculate the light deflection. Not only GR [6] could explain the "odd" precession of the perihelion of Mercury [7] (first reported in 1859), but also its predictions for light deflection near the Sun were in agreement with the observations.

To this day, much has been done to observe extensively the universe (driven by technological advances) and put GR under the test. Two examples of this tireless testing are the observation of gravitational lensing and the measurement of gravitational frequency shifts, both ending up corroborating the theory. Concerning this last example, the measurement of spectral lines from very distant stars revealed a general redshift, indicating that the Universe is expanding. From these experimental results, a new area of study using GR appeared: Cosmology.

From the solutions of the vacuum Einstein equations, it may arise regions of spacetime called black holes. These are mostly born from the collapse of very massive stars [8] in the Universe. Any object that falls into a black hole can never escape to infinity, not even light. This means they cannot be observed directly and must be inferred by the gravitational effect in the surrounding matter. The detection of the black hole's shadow, with the Event Horizon Telescope [9], and the detection of gravitational waves, from the Laser Interferometer Gravitational-wave Observatory and Virgo [10], have been corroborating the theory for now. These two instruments will provide more results to test GR soon.

Despite the success of GR, there are still observations of the Universe that arise some mystery. For example, gravitational effects in the galactic matter and the measured velocity dispersions of galaxies cannot be explained by taking into account the observed matter with GR. It was then postulated the existence of a new type of matter, called dark matter, that interacts very weakly with the observed one. It is required that approximately 25.8% of the Universe's content should be dark matter [11] so that GR remains consistent. Many candidates have been proposed to describe it. A famous one is the axion [12], that can additionally solve the strong CP problem [13], in Quantum Chromodynamics (QCD). The existence of such matter lead to the study of alternative theories of GR with additional fields that permeate the Universe and mediate gravity (for example, Scalar-Vector-Tensor Gravity [14]).

The best way to understand black holes is to analyze the behavior of matter near them. This is a way to test GR and all the arising modified theories. The motion of point particle matter is described by geodesics, which are curves that minimize the length defined by the metric. From Quantum Field Theory, it is known that fundamental particles are excitations of quantized fields [15], and the theory works well according to the experiments in CERN's particle accelerator, Large Hadron Collider. Since quantization in GR is still an open problem, classical field theory [16] is used to describe a fundamental particle. Some advances have been made recently with the AdS/CFT conjecture [17], which requires a 5-dimensional asymptotically anti-de Sitter spacetime. This is one of the motivations for the study of such spacetimes.

The study of GR and its corroborated predictions have been very important for physics and science in general. This is supported by the recent Physics Nobel prizes fully given to the detection of gravitational waves (Thorne, Bariss, and Weiss, 2017), partially given to Cosmology (Peebles, 2019) and to the prediction of black holes (Penrose, Ghez and Genzel, 2020).

For a deeper discussion on GR and its confrontation with the experiment, the reader should check Ref. [18].

1.2 State of the art

In the thesis, the main area of study is the behavior of classical fields in a spacetime containing a black hole. Generally, this can be done by using a given lagrangian for the field in curved spacetime and computing the corresponding Euler-Lagrange equations [19]. Then for a fixed background metric, the equations for the field perturbations can be obtained. Analytically, one tries to apply the method of separation of variables and then integrate each separated equation (which in most cases numerical integration must be done).

The most simple case treated and well studied in the literature is the scalar field. Bosonic vector fields are still being studied and their treatment presents more of a challenge in spacetimes with rotating black holes. There is a common feature of fields while interacting with rotating black holes: they exhibit superradiance [20]. This is characterized by the transfer of energy from a rotating object (in this case a black hole) to the field, if the ratio between its energy and angular momentum is lower than the rotation frequency of the object. Superradiance is a general effect also present in the interaction between particles and a medium in the form of Cherenkov radiation, as discussed in Ref. [21]. There is a recent work that showed the existence of superradiance in plasmas as well, check Ref. [22].

In Schwarzschild spacetime (static black hole), the electromagnetic perturbations were calculated in Ref. [23], using the vector spherical harmonics to separate the equations and the treatment was generalized afterward to a massive vector field [24]. The extension to Schwarzschild-(anti)-de Sitter spacetimes was also made in Refs. [25] and [26].

In Kerr spacetime (rotating black hole), the calculation of vector field perturbations is more complicated since there is no spherical symmetry. Nonetheless, the electromagnetic perturbations were computed by Teukolsky [27], using the Newman-Penrose formalism [28] to obtain separated equations. Another method of calculation regarding the electromagnetic field and the separation of its equations was done in Ref. [29] using an ansatz related to Teukolsky's solution, for Myers-Perry geometry [30] (a generalization of Kerr geometry in higher dimensions). Only on a recent work done by V .P. Frolov et al [31]–[33] the same was done for massive vector fields in Kerr-NUT-(A)dS spacetimes (another generalization of Kerr for higher dimensions, including Newmann-Unti-Tamburino parameters and the cosmological constant). They use an ansatz related to hidden symmetries that exist in Kerr-NUT-(A)dS, mostly referred to in the literature as Frolov-Krtouš-Kubizňák-Santos (FKKS) ansatz. It remains unclear if this ansatz describes all the degrees of freedom of the massive vector field. The purpose of the thesis is to investigate this issue in the Schwarzschild-AdS geometry.

Thus, a general review of the addressed developments is presented here. Additionally, the comparison between a generalization of the work in Ref. [24] for Schwarzschild-(A)dS and the corresponding limit of the FKKS ansatz is shown. For this purpose, analytical calculations are made, complemented with the calculation of the quasinormal modes.

Chapter 2

Scalar fields in curved spacetime

In this chapter, scalar fields in a curved spacetime will be analyzed. This case is well studied through the literature and it has a simpler treatment than the vector field case. For this reason, it is presented as a prelude.

The action for the scalar field Φ, including the Einstein-Hilbert Lagrangian (where R is the Ricci scalar), is given by

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[\frac{(\partial_a\Phi)(\partial^a\Phi)}{8\pi} + \frac{\mu^2}{8\pi}\Phi^2 + \frac{C(\Phi)}{4\pi}\,q(g_{ab}, R^a{}_{bcd}) - \frac{(R-2\Lambda)c^4}{16\pi G}\right], \tag{2.1}$$

where c is the speed of light in vacuum, G is the gravitational constant, μ is the mass of the scalar field and Λ is the cosmological constant. The functions $C(\phi)$ and $q(g_{ab}, R^a{}_{bcd})$ define the coupling of the curvature to the scalar field (where $R^a{}_{bcd}$ is the Riemann tensor). Through variational calculus, the minimum of the action can be found if each field satisfies the Euler-Lagrange equations [19]. The Klein Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_a(\sqrt{-g}g^{ab}\partial_b\Phi) = \left[\mu^2\Phi + C'(\Phi)q(g_{cd}, R^c{}_{def})\right], \tag{2.2}$$

and the Einstein equation (1.2) (obtained by varying g^{ab}) form the system that describes the scalar field in a curved spacetime. The tensors R_{ab} and T_{ab} are the Ricci tensor (a contraction of the Riemann tensor) and the stress-energy tensor, respectively. The stress-energy tensor of the scalar field can be expressed as

$$\begin{aligned} T_{ab} = \frac{1}{4\pi}&\left((\partial_a\Phi)(\partial_b\Phi) - g_{ab}\left(\frac{(\partial_c\Phi)(\partial^c\Phi)+\mu^2\Phi^2}{2}\right)\right) \\ &- \frac{C(\Phi)}{2\pi}\left(\frac{g_{ab}q(g,R)}{2} - \frac{\partial q(g,R)}{\partial g^{ab}} + \partial^c\frac{\partial q(g,R)}{\partial(\partial^c g^{ab})}\right). \end{aligned} \tag{2.3}$$

This system consists in nonlinear partial differential equations, thus they can only be solved numerically in most of the cases. For an analytical treatment, approximations must be done. Typically, only the first order perturbation of the field is considered and the metric is fixed. Therefore, the system reduces to equation (2.2) for a background metric that is a vacuum solution of equation (1.2) ($R_{ab} = 0$). Fortunately, in the astrophysical reality, this approximation is acceptable since the majority of the fields do not have energy density enough to have a considerable back-reaction in the metric. As an example, the existence of a laser in a spacetime can be considered. Typical intensities for very powerful lasers are in the order of $10^{17}\,W\,m^{-2}$, a frequency in the red colour band is of the order of 430×10^{12} Hz. Looking to the equation

(2.3), it can be estimated that $T_{00} \approx \frac{2I(r)}{c}$, using coordinates ($t$,$r$,$\theta$,$\phi$). In an extreme scenario, if lasers of this intensity are put all over the surface of the Earth ($R_0 = 6371$ Km), the right side of equation (1.2) would be of the order of $\frac{G}{c^4}2I(R_0)/c \approx 5\times10^{-36}m^{-2}$ at most (since $I(r) = \frac{I_0 R_0^2}{r^2}$). Compared to an ideal gas with the mass of Earth, which is $\frac{G}{c^4}T_{00} \approx \frac{G}{c^4}\rho c^2 \approx 4\times10^{-24}m^{-2}$, even an electromagnetic field this powerful won't cause any considerable curvature. If the lasers remain active until light reaches a certain radius r, the acceleration felt by a test particle can be calculated in the Newtonian limit and it would be

$$a = -\frac{GM_{Earth}}{R_0^2}\left(\frac{R_0}{r}\right)^2 - \frac{16\pi I_0 R_0}{c^3}\left(\frac{R_0}{r}\right) = -9.81\left(\frac{R_0}{r}\right)^2 - 8\times10^{-11}\left(\frac{R_0}{r}\right) , \tag{2.4}$$

thus the contribution due to presence of light is negligible. For scalars like the Higgs field, the stress-energy tensor would be even smaller. From now on, the gravitational constant G and the speed of light c are set to 1.

There is an important family of metrics that solve the 4-dimensional electrovacuum Einstein equations ($\Lambda = 0$), which is described by the Kerr-Newman metric [34]. The term "electrovacuum" is referred to a spacetime that contains only an electromagnetic field generated by the electric charge of the black hole. In 1968, Carter found a generalization of this family that solves the Einstein equation with a cosmological constant [35] described by the Kerr-(anti)-de Sitter metric. In this thesis, the most used metrics are the Kerr-AdS metric

$$ds^2 = \frac{\Sigma}{\Delta_\Lambda}dr^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 - \frac{\Delta_\Lambda}{\Sigma}\Big[dt - a\sin^2\theta d\phi\Big]^2 + \frac{\Delta_\theta \sin^2\theta}{\Sigma}\Big[adt - (a^2+r^2)d\phi\Big]^2 . \tag{2.5}$$

$$\Sigma = r^2 + a^2\cos^2\theta , \tag{2.6}$$

$$\Delta_\Lambda = r^2 - 2Mr + a^2 + \frac{r^2}{R_\Lambda^2}(r^2+a^2) = \Delta + \frac{r^2}{R_\Lambda^2}(r^2+a^2) , \tag{2.7}$$

$$\Delta_\theta = 1 - \frac{a^2}{R_\Lambda^2}\cos^2\theta , \tag{2.8}$$

and the Kerr metric, both in Boyer-Lindquist coordinates (t, r, θ, ϕ). The Kerr-AdS metric describes a rotating black hole with mass M, angular momentum $J = a\,M$ and negative cosmological constant ($R_\Lambda^2 = \frac{3}{|\Lambda|}$). The location of its event horizon r_+ corresponds to the greatest of the roots of $\Delta_\Lambda = 0$. The Kerr background metric can be obtained from (2.5) by setting $\Lambda = 0$. This metric is non-singular for a region $r \in]r_+, +\infty[$, where $r_+ = M + \sqrt{M^2 - a^2}$ is the location of the event horizon. Also, the determinant of the Kerr metric is given by $-\sin^2\theta\Sigma^2$. The Schwarzschild metric can be recovered by setting $\Lambda = a = 0$.

The inverse of (2.5) with $\Lambda = 0$ is

$$g^{ab} = \begin{pmatrix} 1 + \frac{2Mr(a^2+r^2)}{\Delta\Sigma} & 0 & 0 & \frac{2Mra}{\Delta\Sigma} \\ 0 & -\frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & -\Sigma^{-1} & 0 \\ \frac{2Mra}{\Delta\Sigma} & 0 & 0 & -\frac{\Delta - a^2 s_\theta^2}{\Delta\Sigma s_\theta^2} \end{pmatrix} . \tag{2.9}$$

Plugging (2.9) into (2.2), the equation for the field becomes

$$\left(\Sigma + \frac{2Mr(a^2+r^2)}{\Delta}\right)\partial_t^2\Phi + \frac{4Mra}{\Delta}\partial_{t\phi}^2\Phi - \partial_r(\Delta\partial_r\Phi) - \frac{1}{s_\theta}\partial_\theta(s_\theta\partial_\theta\Phi) - \left(\frac{1}{s_\theta^2} - \frac{a^2}{\Delta}\right)\partial_\phi^2\Phi$$
$$= -\Sigma(\mu^2\Phi + C'(\Phi)q(\boldsymbol{g}, \boldsymbol{R})) . \tag{2.10}$$

Considering that the field is minimally coupled to the curvature (i.e $C'(\Phi)q(\boldsymbol{g}, \boldsymbol{R}) = 0$), the method of separation of variables can be used. The ansatz for the field and separated equations are

$$\Phi(t,r,\theta,\phi) = R(r)Y(\theta,\phi)e^{-i\omega t} \tag{2.11}$$

$$\frac{1}{Y\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y) + \frac{1}{Y\sin^2\theta}\partial_\phi^2 Y + a^2\cos_\theta^2(\omega^2-\mu^2) = -\lambda\,, \tag{2.12}$$

$$\left(r^2 + \frac{2Mr(a^2+r^2)}{\Delta}\right)\omega^2 - m\omega\frac{4Mra}{\Delta} + \frac{1}{R}\partial_r(\Delta\partial_r R) - r^2\mu^2 + \frac{m^2a^2}{\Delta} = \lambda\,, \tag{2.13}$$

$$\partial_\phi Y = imY\,. \tag{2.14}$$

Quantities m and λ are separation constants. It must be noted that $Y(\theta,\phi)$ are the spheroidal harmonics. In the limit that a vanishes, the equations in a Schwarzschild background metric are recovered. In this case, the angular functions Y will turn into the well known spherical harmonics, Y_{lm}.

2.1 Superradiance and instabilities

Superradiance consists in the amplification of a field due to its interaction with a rotating object or with a dielectric medium. Thus, this effect should be present when considering a scalar field in Kerr. In this section, superradiance in scalar fields will be shown and the associated instabilities will be addressed.

First, one can start by stating

$$\frac{\mathrm{d}r}{\mathrm{d}r_*} = \frac{\Delta}{r^2+a^2} \quad , \quad R(r) = \frac{U(r)}{\sqrt{r^2+a^2}}\ . \tag{2.15}$$

Using the above definitions in (2.13), an equation for $U(r)$ can be obtained

$$\frac{\mathrm{d}^2U}{\mathrm{d}r_*^2} + \left[\frac{3r^2\Delta^2}{(r^2+a^2)^4} - \frac{\Delta^2+2r\Delta(r-M)}{(r^2+a^2)^3} + \frac{\mathcal{K}}{(r^2+a^2)^2} + \frac{2Mr\omega^2}{r^2+a^2}\right]U = 0\ , \tag{2.16}$$

where the function $\mathcal{K}$ is given by

$$\mathcal{K} = m^2a^2 - (\mu^2r^2+\lambda)\Delta - 4m\omega Mra + \omega^2r^2\Delta\ . \tag{2.17}$$

It must be noted that (2.16) assumes a form similar to the Schrödinger equation. Thus, it admits asymptotic plane wave solutions given by

$$U(r\to r_+) = \mathcal{T}e^{-ik_Hr_*} + \mathcal{O}e^{ik_Hr_*} \quad , \quad U(r\to\infty) = \mathcal{I}e^{-ik_\infty r} + \mathcal{R}e^{ik_\infty r}\ , \tag{2.18}$$

where

$$k_H = \omega - \frac{ma}{2Mr_+} \quad , \quad k_\infty = \sqrt{\omega^2-\mu^2}\ . \tag{2.19}$$

The parameter $\mathcal{O}$, that describes the amplitude of the outgoing wave at the event horizon, is set to 0 since supposedly nothing flows out from a black hole. The quantity $\Omega_H = \frac{a}{2Mr_+}$ is the angular velocity of an observer with zero angular momentum at the event horizon, measured at infinity. The dispersion relation at the event horizon allows negative values of k_H, which will shift the direction of propagation of the transmitted wave.

A conserved quantity can be evaluated for the equation (2.16), since it has a Schrodinger-like form.

The expression for this quantity (current) is[1]

$$J = \frac{1}{2i}\left(U^* \frac{\mathrm{d}U}{\mathrm{d}r_*} - U\frac{\mathrm{d}U^*}{\mathrm{d}r_*}\right) \ , \tag{2.20}$$

where U^* is the complex conjugate of U. The evaluation of the current in both asymptotic regions can be made. Since both values must be the same due to its conservation in space, the condition follows

$$|\mathcal{R}|^2 = |\mathcal{I}|^2 - \frac{\omega - m\Omega_H}{\sqrt{\omega^2 - \mu^2}}|\mathcal{T}|^2 \ , \tag{2.21}$$

It is evident now that a negative k_H will imply a reflectivity $\mathcal{R}$ larger than $\mathcal{I}$. This means the rotating black hole will transfer energy to the scalar field, amplifying it. In this process, the angular momentum and the mass of the black hole will decrease, by an argument of conservation of energy. This effect is the so called superradiance [36], [37], [20]. The condition that the field must obey for this effect to occur is

$$\omega < m\Omega_H \ . \tag{2.22}$$

The extraction of energy from the black hole is a source of instabilities in the equation (2.16). This can be seen in the following setting: consider a black hole with a spherical perfect mirror enclosing it at a radius $R_m > 2M$. An incoming wave inside the enclosing mirror would scatter into the black hole and the reflected wave would be amplified through superradiance, admiting that the condition (2.22) is satisfied. Due to the presence of the mirror, these amplified outgoing waves would be reflected towards the black hole again, where they would experience superradiance once more. This cycle gives rise to an exponential increase of the amplitude of the field inside the enclosing mirror. By solving equation (2.16) with the boundary conditions at the mirror and at the event horizon, these instabilities can be described by quasinormal modes with a positive imaginary part. Fixing the metric would make the amplitude to blow up. The Einstein equations with the stress-energy tensor of the scalar field should be treated in order to prevent this. The energy being transferred to the field comes from the black hole, that has a total mass and angular momentum. And so, during the process of superradiance, the black hole loses mass and slows the rotation up until the equality of (2.22) is reached. In the case of massive scalar fields, the potential in (2.16) could develop a potential barrier depending on the value of its mass. This barrier works as a non-perfect mirror, allowing the existence of instabilities. As an astrophysical example, accretion disks around black holes may also work as mirrors since they can reflect certain frequencies of light.

2.2 Gauss-Bonnet coupling associated to the scalar field

In the sections above, an analysis of a minimally coupled scalar field to curvature was done, around a rotating black hole. The action (2.1) with $q(\boldsymbol{g}, \boldsymbol{R}) = 0$ is not renormalizable [38], when one tries to quantize the theory. There has been a lot of interest in including higher order terms of curvature coupled to fields. One of the reasons is that it may help making the theory renormalizable. Here, the Gauss-Bonnet coupling to the scalar field is going to be addressed. It has been receiving a great interest since it can describe hairy black holes as well as the classic black hole solutions. The Gauss-Bonnet term [39]

[1] Like in the Schrodinger equation, this quantity can be obtained by evaluating the imaginary part of (2.16) multiplied by $\tilde{U}^*$. In Quantum Mechanics, this quantity is referred to the probability current.

couples the field to the curvature in the following fashion

$$q = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd} \ . \tag{2.23}$$

For the purpose of choosing a term that adds to the mass of the scalar and also to let the equation be linear, the following term is studied

$$C(\Phi) = -\frac{\Phi^2}{2} \ . \tag{2.24}$$

In the conditions of zeroth order perturbations, the metric is unperturbed and so R_{ab} vanishes since it satisfies the vacuum Einstein equation ($\Lambda = 0$). The only term that survives is $R_{abcd}R^{abcd}$ (Kretschmann scalar) and it is expressed, in the Kerr geometry, as

$$q = R_{abcd}R^{abcd} = \frac{48M^2(r^2 - a^2\cos^2\theta)\Big(\Sigma^2 - 16a^2r^2\cos^2\theta\Big)}{\Sigma^6} \ . \tag{2.25}$$

Replacing these two functions into (2.2), the equation becomes

$$\begin{aligned} &-\left(\Sigma + \frac{2Mr(a^2+r^2)}{\Delta}\right)\omega^2 - i\omega\frac{4Mar}{\Delta}\frac{\partial_\phi Y}{Y} - \frac{1}{F}\partial_r(\Delta\partial_r F) - \frac{1}{Y\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y) \\ &-\left(\frac{1}{Y\sin^2\theta} - \frac{a^2}{Y\Delta}\right)\partial_\phi^2 Y = -\Sigma\Big(\mu^2 - R_{abcd}R^{abcd}\Big) \ . \end{aligned} \tag{2.26}$$

The equation, excluding the Kretschmann scalar, has terms with only one coordinate dependence. Equation (2.14) holds due to the axisymmetric feature of Kerr metric, taking away any arising doubts on ϕ dependence. This means the equation can be expressed in the suggestive form

$$h(r) + f(\theta) = \Sigma\, R_{abcd}R^{abcd} \ . \tag{2.27}$$

where $h(r)$ and $f(\theta)$ are just the left hand side of equations (2.13) and (2.12). This sets the condition of separability of the equation. If (2.27) is not satisfied, then ansatz (2.11) fails to solve the equation of motion. First, one can consider putting $a = 0$ (Schwarzschild). In this limit, the Kretschmann scalar will only depend on the radius and so the condition (2.27) is satisfied. Both μ^2 and the Kretschmann scalar have the same multiplication factor, and in Schwarzschild, they only appear on the radial equation. So it can be seen that the only modifications to the equations would be equivalent to

$$\mu^2 \to \mu^2 - r^2 R_{abcd}R^{abcd} = \mu^2 - \frac{48M^2}{r^4} \tag{2.28}$$

This can be interpreted as a variable mass of the field, as it approaches the black hole. This can lead even to a negative mass term in the lagrangian for a certain region $2M < r \leq r_c$, where r_c is the radius at which the scalar becomes massless. Scalar fields that have these negative terms behave as tachyonic fields [40], i.e fields whose perturbations propagate faster than speed of light. In Kerr spacetime, this treatment is not valid. The Kretschmann scalar is not separable i.e. the condition (2.27) fails. Still, it might be that expanding in small a one could retain separating terms. The expansion of the Kretschmann scalar multiplied with Σ around $x = 0$ has the form

$$\Sigma R_{abcd}R^{abcd} = \frac{48M^2}{r^4}\sum_{j=0}^{\infty} b_j x^j \ , \tag{2.29}$$

where $x = \frac{a^2 \cos^2\theta}{r^2}$. The series coefficients are

$$b_j = \frac{(-1)^j}{3}(3 + 15j + 26j^2 + 18j^3 + 4j^4),\; n \geq 0\,. \tag{2.30}$$

The zeroth order of the Kretschmann scalar, as expected, corresponds to the value in Schwarzschild metric. Lets consider at least first order, one gets

$$\Sigma R_{abcd}R^{abcd} \approx \frac{48M^2}{r^4}\Big(1 - 22\frac{a^2\cos^2\theta}{r^2}\Big) \tag{2.31}$$

It seems that already at first order, there is a non-separate term. This happens because the power of r^{-1} is too large in the Kretschmann scalar. The multiplication factor on the equation has just a power of r^2 while the Kretschmann scalar will have a power of r^{-6}. This makes it impossible for the term ac_θ being isolated in the end, which would be the desired feature to plug it in the angular equation. Also, there is no other possible expansion to make due to the fact that only $0 \leq a \leq 1$ are physical solutions [34], whereas the radius can span from the event horizon (r_+) to infinity.

Considering this then, the only option is to retain only the zeroth order of the Kretschmann scalar, and so one obtains an untouched angular equation, (2.12) and (2.14), and a modified radial equation given by

$$\left(r^2 + \frac{2Mr(a^2+r^2)}{\Delta}\right)\omega^2 - m\omega\frac{4Mra}{\Delta} + \frac{1}{F}\partial_r(\Delta\partial_r F) - r^2\mu^2 + \frac{48M^2}{r^4} + \frac{m^2a^2}{\Delta} = \lambda\,. \tag{2.32}$$

In this case, since μ^2 appears in both the angular and radial equation, whilst the approximated Kretschmann scalar only appears in the radial one, the same interpretation as in the Schwarzschild case is not valid. The approximated Kretschmann scalar will only serve as a change in the potential felt by the field. There are recent works about this coupling. One example is Ref. [41] with a numerical analysis about the Gauss-Bonnet coupling to the scalar field in Schwarzschild geometry. Another is Ref. [42], where they analyze numerically spin-induced scalarization and tachyonic instabilities in Kerr.

Chapter 3

Vector fields in curved spacetime

In this chapter, vector fields in curved spacetimes will be addressed. Vector fields are important as they describe all the particles that mediate the forces of the Universe i.e. light, the bosons Z and $W^{\pm}$ (weak interaction), and the gluons (strong interaction). In curved spacetime, a vector field A^a can be described by the following action

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[\frac{F_{ab}F^{ab}}{16\pi} + \frac{m_A^2}{8\pi}A_aA^a - \frac{R-2\Lambda}{16\pi}\right] , \tag{3.1}$$

$$F_{ab} = \nabla_a A_b - \nabla_b A_a \, . \tag{3.2}$$

The stress-energy tensor of the vector field is given by

$$T_{ab} = \frac{1}{4\pi}\left[F_{ae}F_{bf}g^{ef} + m_A^2 A_aA_b - g_{ab}\frac{F_{ef}F^{ef} + 2m_A^2A_eA^e}{4}\right] . \tag{3.3}$$

The corresponding Euler-Lagrange equations for the field are

$$\nabla_b F^{ab} + m_A^2 A^a = 0 \; . \tag{3.4}$$

The strength field tensor, F_{ab}, holds internal equations

$$F_{[ab;c]} = 0 \; . \tag{3.5}$$

If the field is massless, then the lagrangian is invariant under the gauge transformation

$$A^a \to A^a + \chi^{,a} \; , \tag{3.6}$$

where χ is a scalar. It is possible to fix partially the gauge by imposing the Lorentz condition

$$\nabla_a A^a = 0 \, . \tag{3.7}$$

Of course, there will be a reminiscent gauge freedom even when this condition is imposed. A gauge transformation in which the additional scalar field obeys

$$g^{ab}\chi_{;ab} = 0 \; , \tag{3.8}$$

will satisfy (3.7). It is this reminescent gauge freedom that allows for the removal of the longitudinal polarization. For massive vector fields, this gauge freedom does not exist. Since the Ricci tensor is symmetric, the following is true

$$F^{ab}{}_{;ab} = (R_{ab} - R_{ba})\nabla^b A^a = 0 \ . \tag{3.9}$$

By applying ∇_c and contracting all indices in (3.4), the property (3.9) implies (3.7). And so in the case of massive vectors, (3.7) comes out from the field equations. This condition can then be used to simplify (3.4) into

$$g^{cd}\nabla_c\nabla_d A^a - m_A^2 A^a = g^{ca}R_{dc}A^d \ . \tag{3.10}$$

This expression is obtained by using the relation

$$(\nabla_b\nabla_c - \nabla_c\nabla_b)A_d = R^e{}_{bcd}A_e \ . \tag{3.11}$$

Since the metric will be fixed to the Kerr-AdS metric, then $R_{ab} = -\frac{3}{R_\Lambda^2}g_{ab}$ and so the Proca equations are equivalent to

$$g^{cd}\nabla_c\nabla_d A^a - \Big(m_A^2 - \frac{3}{R_\Lambda^2}\Big)A^a = 0 \ . \tag{3.12}$$

In general, this set of equations are quite hard to solve analytically. The fact is that all 4 Proca equations can be coupled due to the term $g^{cd}\nabla_c\nabla_d$ being a non-trivial operator. The analytical treatment for separation of variables of these equations can only be done by using the symmetries of the spacetime considered. In the case of Schwarzschild spacetime, the spherical symmetry allows this separation to be done. In the case of Kerr, Teukolsky [27] was the first to separate Maxwell's equations (massless vector field) and metric perturbations (gravitational waves). It turns out that Kerr metric holds a hidden symmetry that allows this separation to be done. In the work of V. Frolov et al [31], the authors used a symmetry, or rather a set of symmetries (Killing tower) of the Kerr-NUT-(A)dS metric (general Kerr metric for any dimensions, including NUT parameters and the cosmological constant) to separate the Proca field equations.

3.1 Proca fields in Schwarzschild geometry

In the Schwarzschild spacetime, a generalization of spherical harmonics for higher spin can be used to separate the Proca equations [24] since there is spherical symmetry. The treatment of massless vector fields was done in Ref. [23]. The set of equations (3.12) for the field in the Schwarzschild spacetime ($\Lambda = 0$ and $a = 0$) can be expanded into

$$\hat{\mathcal{J}}A_t + \frac{2M}{r^2}(\partial_t A_r - \partial_r A_t) = m_A^2 A_t \ , \tag{3.13}$$

$$\hat{\mathcal{J}}A_r + 2\frac{2M-r}{r^3}A_r - \frac{2\cot\theta}{r^3}A_\theta + \frac{2M\partial_t A_t}{(r-2M)^2} + \frac{2M}{r^2}\partial_r A_r - \frac{2\partial_\theta A_\theta}{r^3} - \frac{2\partial_\phi A_\phi}{r^3\sin^2\theta} = m_A^2 A_r \tag{3.14}$$

$$\hat{\mathcal{J}}A_\theta - \frac{1}{r^2\sin^2\theta}A_\theta + 2\frac{r-2M}{r^2}(\partial_\theta A_r - \partial_r A_\theta) - \frac{2\cot\theta}{r^2}\partial_\phi A_\phi = m_A^2 A_\theta \ , \tag{3.15}$$

$$\hat{\mathcal{J}}A_\phi + 2\frac{2M-r}{r^2}(\partial_r A_\phi - \partial_\phi A_r) - \frac{2\cot\theta}{r^2}(\partial_\theta A_\phi - \partial_\phi A_\theta) = m_A^2 A_\phi \ , \tag{3.16}$$

where the operator $\hat{\mathcal{J}}$ is given by

$$\hat{\mathcal{J}}A = \left(\frac{r}{2M-r}\partial_t^2 A + \Big(1-\frac{2M}{r}\Big)\partial_r^2 A + \frac{1}{\sin\theta r^2}\partial_\theta\big(\sin\theta\partial_\theta A\big) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 A\right) + 2\frac{r-M}{r^2}\partial_r A \ . \quad (3.17)$$

The system of differential equations can be solved by the method of separation of variables using vector spherical harmonics [43]

$$\boldsymbol{K}_{lm} = Y_{lm}\boldsymbol{e}_{(t)} \quad , \quad \boldsymbol{Y}_{lm} = Y_{lm}\boldsymbol{e}_{(r)} \quad , \quad \boldsymbol{\Psi}_{lm} = r\nabla Y_{lm} = \partial_\theta Y_{lm}\boldsymbol{e}_{(\theta)} + \frac{1}{\sin\theta}\partial_\phi Y_{lm}\boldsymbol{e}_{(\phi)} \ , \quad (3.18)$$

$$\boldsymbol{\Phi}_{lm} = \boldsymbol{r}\times\nabla Y_{lm} = -\frac{1}{\sin\theta}\partial_\phi Y_{lm}\boldsymbol{e}_{(\theta)} + \partial_\theta Y_{lm}\boldsymbol{e}_{(\phi)} \ , \quad (3.19)$$

where

$$\boldsymbol{e}_{(t)} = \partial_t \ , \ \boldsymbol{e}_{(r)} = \partial_r \ , \ \boldsymbol{e}_{(\theta)} = \frac{\partial_\theta}{r} \ , \ \boldsymbol{e}_{(\phi)} = \frac{1}{r\sin\theta}\partial_\phi \ . \quad (3.20)$$

Vector spherical harmonics are well known from quantum mechanics. They can be constructed by making the tensor product of the Hilbert space describing the angular momentum L with the three dimensional Euclidean space [44]. This last can be described as a spinor space of spin 1 with orthogonal spherical basis $\boldsymbol{e}_z$ and $\boldsymbol{e}_\pm = \frac{\mp 1}{\sqrt{2}}(\boldsymbol{e}_x \pm i\boldsymbol{e}_y)$. The resulting spaces will describe the total angular momentum $L-1$, L and $L+1$. The ansatz for the vector field can be made as

$$A_a = \frac{1}{r}\sum_{i=1}^{4}\sum_{\ell m} c_i u_{(i)}^{\ell m}(t,r) Z_a^{(i)\ell m}(\theta,\phi) \ , \quad (3.21)$$

where $c_1 = c_2 = 1$ and $c_3 = c_4 = (l(l+1))^{-\frac{1}{2}}$. The $Z_\mu^{(i)lm}$ are

$$Z_a^{(1)\ell m} = (1,0,0,0)Y^{\ell m} \ , \quad (3.22)$$

$$Z_a^{(2)\ell m} = (0,f^{-1},0,0)Y^{\ell m} \ , \quad (3.23)$$

$$Z_a^{(3)\ell m} = \frac{r}{\sqrt{\ell(\ell+1)}}(0,0,\partial_\theta,\partial_\phi)Y^{\ell m} \ , \quad (3.24)$$

$$Z_a^{(4)\ell m} = \frac{r}{\sqrt{\ell(\ell+1)}}\left(0,0,\frac{1}{\sin\theta}\partial_\phi, -\sin\theta\partial_\theta\right)Y^{\ell m} \ , \quad (3.25)$$

where $f = 1 - \frac{2M}{r}$. The separated equations become

$$\hat{\mathcal{D}}u_{(1)} + (\partial_r f)(\dot{u}_{(2)} - u'_{(1)}) = 0 \ , \quad (3.26)$$

$$\hat{\mathcal{D}}u_{(2)} + (\partial_r f)(\dot{u}_{(1)} - u'_{(2)}) + \frac{2f^2}{r^2}(u_{(3)} - u_{(2)}) = 0 \quad (3.27)$$

$$\hat{\mathcal{D}}u_{(3)} + \left[\frac{2f\ell(\ell+1)}{r^2}u_{(2)}\right] = 0 \ , \quad (3.28)$$

$$\hat{\mathcal{D}}u_{(4)} = 0 \ , \quad (3.29)$$

where $\dot{u}_{(i)} = \frac{\mathrm{d}u_{(i)}}{\mathrm{d}t}$, $u'_{(i)} = \frac{\mathrm{d}u_{(i)}}{\mathrm{d}r*}$ and $\frac{\mathrm{d}r*}{\mathrm{d}r} = f^{-1}$. The operator in these equations is given by

$$\hat{\mathcal{D}} = -\partial_t^2 + \partial_{r*}^2 - f\Big[\frac{l(l+1)}{r^2} + m_A^2\Big] \ . \quad (3.30)$$

The Lorenz condition (3.7) is given by

$$\nabla^a A_a = \frac{1}{rf(r)}\left[\partial_{r_*} u_{(2)} - \partial_t u_{(1)} + \frac{f(r)}{r}(u_{(2)} - u_{(3)})\right] = 0 \ . \tag{3.31}$$

This can be used to simplify (3.27) into

$$\hat{\mathcal{D}} u_{(2)} = \frac{2f}{r^2}\Big(1 - \frac{3M}{r}\Big)(u_{(2)} - u_{(3)}) \ . \tag{3.32}$$

It can be observed that the equation for $u_{(4)}$ is totally decoupled and that there are two coupled equations for $u_{(2)}$ and $u_{(3)}$. In the literature, solutions described by $u_{(4)}$ are called magnetic modes and the solutions described by the other components are called electric modes. The distinction comes from the change of sign under parity transformation. The components $u_{(2)}$ and $u_{(3)}$ can be decoupled (for $\ell > 0$) by making the substitution

$$u_2 = -\frac{r^2 \hat{\mathcal{D}} u_3}{2f\ell(\ell+1)} \ , \tag{3.33}$$

into equation (3.32), obtaining a fourth order equation for $u_{(3)}$

$$\hat{\mathcal{D}}\Big[\frac{r^2}{f}\hat{\mathcal{D}} u_3\Big] = \Big(1 - \frac{3M}{r}\Big)\Big[2\hat{\mathcal{D}} u_3 + \frac{4f}{r^2}\ell(\ell+1)u_3\Big] \ . \tag{3.34}$$

It is possible to simplify this equation even further by using an identity, shown in appendix A

$$\hat{\mathcal{D}}\Big[\frac{r^2}{f}\hat{\mathcal{D}} u_3\Big] = r\hat{\mathcal{D}}\Big[\frac{1}{f}\hat{\mathcal{D}}_1(ru_3)\Big] + \Big(1 - \frac{3M}{r}\Big)\Big[2\hat{\mathcal{D}} u_3 + \frac{4f}{r^2}\ell(\ell+1)u_3\Big] - \frac{4Mf}{r}m_A^2 u_3 \ , \tag{3.35}$$

where the new operator is defined by

$$\hat{\mathcal{D}}_1 = -\partial_t^2 + \partial_{r_*}^2 - f\Big[\frac{\ell(\ell+1)}{r^2} + m_A^2 + \frac{2M}{r^3}\Big] \ . \tag{3.36}$$

The operators $\hat{\mathcal{D}}$ and $\hat{\mathcal{D}}_1$ can also be called the Regge-Wheeler operators [45], each one corresponding to a different value of spin, in the massless case. Finally, the decoupled equation for $u_{(3)}$ becomes

$$\hat{\mathcal{D}}\Big[\frac{1}{f}\hat{\mathcal{D}}_1(ru_3)\Big] = \frac{4Mf}{r^2}m_A^2 u_3 \ . \tag{3.37}$$

In the massless limit, equation (3.37) factorizes completely. In this situation, it is possible to define

$$\Psi = \frac{1}{f}\hat{\mathcal{D}}_1(ru_3) \ , \tag{3.38}$$

and this scalar will obey the same equation as $u_{(4)}$

$$\hat{\mathcal{D}}\Psi = 0 \ . \tag{3.39}$$

The splitting of equation (3.37) makes clear the existence of two degrees of freedom given by Ψ and a solution for $u_{(3)}$ such that

$$\hat{\mathcal{D}}(ru_{(3)}) = 0 \ . \tag{3.40}$$

It turns out that this last degree of freedom gives a pure gauge solution for A^a, which can be totally removed by making a gauge transformation.

In sum, the three polarizations are present in $u_{(2)}$, $u_{(3)}$ and $u_{(4)}$. Even though $u_{(2)}$ can be totally described by $u_{(3)}$, when decoupling the equations, the decoupled equation for $u_{(3)}$ is of fourth order. Thus, the solution holds the extra degree of freedom for $u_{(2)}$. Of course, this is just one way to describe the set of equations. In the electromagnetic limit, one of these degrees of freedom is totally removed by a gauge transformation, which agrees with the fact that the electromagnetic quadrivector potential has only two polarizations.

In the case of the monopole ($\ell = 0$), the equations (3.26)-(3.29) are reduced to three: (3.26), (3.32) and (3.31). Since the spherical harmonic with $\ell = 0$ is constant, it means $u_{(3)}$ and $u_{(4)}$ vanish. One gets then the equation for $u_{(2)}$ from (3.32)

$$\partial^2_{r_*} u_{(2)} - \partial^2_t u_{(2)} - f\Big[m_A^2 + \frac{2}{r^2}\Big(1 - \frac{3M}{r}\Big)\Big]u_{(2)} = 0 \ . \tag{3.41}$$

The equation for $u_{(1)}$ can be split into a static and dynamic field. The dynamic field needs to obey the Lorentz condition

$$u_{(1)\mathsf{d}}(r,\omega) = \frac{i}{\omega}\Big[\partial_{r_*} u_{(2)} + \frac{f}{r}(u_2)\Big] \ . \tag{3.42}$$

The solution for a static field is obtained from (3.26) by using the linearity of the equations

$$u_{(1)} = u_{(1)\mathsf{d}}(r,t) + u_{(1)\mathsf{s}}(r) \quad , \quad \partial^2_r u_{(1)\mathsf{s}} = \frac{m_A^2}{f} u_{(1)\mathsf{s}} \ , \tag{3.43}$$

where ”d” and ”s” stand for ”dynamic” and ”static”, respectively. In the massless limit, the solution for the electromagnetic field will be

$$A_a dx^a = \frac{q}{r} dt + \chi_{,a} dx^a \ . \tag{3.44}$$

Thus the only physical solution will be the static field (corresponding to the field generated by a pointlike charge q), since one can remove the gradient by a gauge transformation. When the mass is non zero, the dynamic solution will correspond to the longitudinal polarization.

3.2 Proca field in Schwarzschild-AdS geometry

The Schwarzschild-AdS geometry describes a spherically symmetric spacetime with a negative cosmological constant. Its metric is given by

$$ds^2 = -f dt^2 + f^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \tag{3.45}$$

$$f = \frac{r^2}{R_\Lambda^2} + 1 - \frac{2M}{r} \ . \tag{3.46}$$

The Proca equations (3.12) can then be computed in the same way as in Schwarzschild case, using the vector spherical harmonics. The ansatz for the proca field can be given by (3.21) and the equations for the $u_{(i)}$ will be (3.26)-(3.29) with f having the expression (3.46). This also applies to the Lorenz condition (3.31). In this way, the second equation can be simplified into (3.32), using the Lorenz condition.

3.2.1 Asymptotic solutions

The Schwarzschild-AdS spacetime has one coordinate singularity corresponding to the positive root of f and it pinpoints to the event horizon. The boundaries of the spacetime are then at the event horizon and at infinity. For the purpose of comparison with the new ansatz in section 3.4, the asymptotic solutions of the Proca equation near infinity will be computed here.

The equations (3.26), (3.32), (3.28) and (3.29) near infinity can be found by replacing $f \approx \frac{r^2}{R_\Lambda^2}$ and taking the leading order in powers of r^{-1}. Also, it is assumed that functions $u_{(i)}(r,t) \propto e^{-i\omega t} u_{(i)}(r)$. The equations end up having the following expression

$$\partial_x^2 q_{(i)} + \Big[\omega^2 - \frac{j_i(j_i+1)}{R_\Lambda^2}\Big] q_{(i)} - \frac{R_\Lambda^2 m_A^2}{x^2} q_{(i)} = 0 \ , \tag{3.47}$$

where $x = \frac{R_\Lambda^2}{r}$ and

$$q_{(2)} = \frac{1}{2\ell+1}[u_{(3)} - (\ell+1)u_{(2)}] \ , \ q_{(3)} = \frac{1}{2\ell+1}(\ell u_{(3)} + u_{(2)}) \ , \ q_{(4)} = u_{(4)} \ . \tag{3.48}$$

$$j_2 = \ell + 1 \ , \ j_3 = \ell - 1 \ , \ j_4 = \ell \ . \tag{3.49}$$

The expression of $u_{(1)}$ can be found directly by using the Lorenz condition

$$u_{(1)} \approx \frac{i}{\omega}\Big[\frac{(u_{(2)} - u_{(3)})}{x} - \partial_x u_{(2)}\Big] \ . \tag{3.50}$$

The equation (3.47) can be put in the form of the Bessel's differential equation, by making the transformation $q_{(i)} = \sqrt{x}\ \chi_{(i)}$. The asymptotic solutions are then

$$q_{(i)} \approx \mathcal{A}_i \sqrt{\frac{R_\Lambda^2}{r}} J\Big[\alpha; \frac{k_i R_\Lambda^2}{r}\Big] + \mathcal{B}_i \sqrt{\frac{R_\Lambda^2}{r}} Y\Big[\alpha; \frac{k_i R_\Lambda^2}{r}\Big] \ , \tag{3.51}$$

$$\alpha = \frac{1}{2}\sqrt{1 + 4R_\Lambda^2 m_A^2} \ , \ \ k_i = \sqrt{\omega^2 - \frac{j_i(j_i+1)}{R_\Lambda^2}} \ , \tag{3.52}$$

where $J[\alpha, z]$ and $Y[\alpha, z]$ are the Bessel functions of First and Second Kind of order α, respectively.

By inspection of the asymptotic solution, it is possible to see that there are three polarizations corresponding to $q_{(2)}$, $q_{(3)}$ and $q_{(4)}$ with associated momentum k_2, k_3 and k_4. To connect these asymptotic solutions to the boundary conditions of quasinormal modes, a pure outgoing wave would have $\mathcal{B}_i = 0$. Since the Bessel function J is finite at $r \to \infty$, the condition of pure outgoing waves at infinity translates into $u_{(i)} \to 0$.

3.2.2 Series expansion and method for calculating quasinormal modes

With the separated equations found, the analysis of the system is concluded by integrating them. The quasinormal modes are defined by solutions that solve the equations of the field with a purely incoming wave at the event horizon and a purely outgoing wave at infinity, as boundary conditions. In this subsection, the calculation of the quasinormal modes of the Proca field is done, using the work of Horowitz and Hubeny [46]. The equations (3.26), (3.32), (3.28) and (3.29) may admit a series expansion as a solution given by

$$u_{(i)} = e^{-i(t+r_*)\omega} \sum_{n=0}^{\infty} a_n^{(i)} (x - x_+)^n \ , \tag{3.53}$$

where $x = \frac{1}{r}$ and $x_+ = \frac{1}{r_+}$ (the inverse of the radius of the event horizon). The operator $\hat{\mathcal{D}}$ becomes

$$\hat{\mathcal{D}}u_{(i)} = fe^{-i\omega(t+r_*)}\Big[(x-x_+)s(x)\partial_x^2 U_{(i)} + t(x)\partial_x U_{(i)} + \frac{u(x)}{x-x_+}U_{(i)}\Big] \ , \tag{3.54}$$

where $U_{(i)} = \sum_{n=0}^{\infty} a_n^{(i)}(x-x_+)^n$. The functions $s(x)$, $t(x)$ and $u(x)$ are polynomials

$$u(x) = -(x-x_+)\Big[x^2\ell(\ell+1) + m_A^2\Big] \ , \ t(x) = x^2\partial_x\Big(\frac{f}{r^2}\Big) + 2i\omega x^2 \ , \ s(x) = \frac{f}{r^2(x-x_+)}x^2 \ . \tag{3.55}$$

The function $\frac{f}{r^2}$ can be written in terms of x as

$$\frac{f}{r^2} = R_\Lambda^{-2} + x^2 - 2Mx^3 \ . \tag{3.56}$$

It is useful now to write the polynomials in function of $(x - x_+)$

$$s(x) = \sum_{j=0}^{N_t} s_j(x-x_+)^j \ , \ t(x) = \sum_{j=0}^{N_s} t_j(x-x_+)^j \ , \ u(x) = \sum_{j=0}^{N_d} u_j(x-x_+)^j \ . \tag{3.57}$$

According to the notation in Ref. [46], the recurrence relation coming from the equation for $u_{(4)}$ (3.29) can be described by

$$a_n = -\frac{1}{P_n}\sum_{j=0}^{n-1}\Big(j(j-1)s_{n-j} + jt_{n-j} + u_{n-j}\Big)a_j \ , \ P_n = n(n-1)s_0 + nt_0 \ . \tag{3.58}$$

For the electric modes, the equations are coupled. It is possible to use an extension of Horowitz-Hubeny's method by using matrices instead of scalars. The system can be described by

$$(x-x_+)s(x)\partial_x^2\boldsymbol{U} + t(x)\partial_x\boldsymbol{U} + \frac{u(x)}{x-x_+}\boldsymbol{U} + \frac{1}{x-x_+}\boldsymbol{K}.\boldsymbol{U} = 0 \ , \ \boldsymbol{U} = \begin{bmatrix} U_{(2)} \\ U_{(3)} \end{bmatrix}. \tag{3.59}$$

The matrix $\boldsymbol{K}$ contains the terms that are coupled in (3.32) and (3.28). The expression of $\boldsymbol{K}$ is

$$\boldsymbol{K} = (x-x_+)\begin{bmatrix} -2x^2(1-3Mx) & 2x^2(1-3Mx) \\ 2x^2\ell(\ell+1) & 0 \end{bmatrix} = \sum_{j=0}^{N_k}\boldsymbol{K}_j(x-x_+)^j \ , \tag{3.60}$$

where $\boldsymbol{K}_j$ can be obtained by making a Taylor expansion. It is then possible to obtain a formula similar to (3.58) for this case

$$\boldsymbol{M}_n = -\frac{1}{P_n}\sum_{j=0}^{n-1}\Big(j(j-1)s_{n-j}\boldsymbol{I} + jt_{n-j}\boldsymbol{I} + u_{n-j}\boldsymbol{I} + \boldsymbol{K}_{n-j}\Big).\boldsymbol{M}_j \ , \ \boldsymbol{a}_n = \boldsymbol{M}_n\boldsymbol{a}_0 \ , \tag{3.61}$$

$$\boldsymbol{a}_n = \begin{bmatrix} a_n^{(2)} \\ a_n^{(3)} \end{bmatrix} \ , \ \boldsymbol{M}_0 = \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.62}$$

The quasinormal modes can be obtained numerically by implementing the formulas (3.58) and (3.61) to calculate the coefficients $a_n^{(i)}$ up until a certain number N. Most of the calculations are done with $N = 40$. To compute ω, one needs to impose the boundary conditions at infinity (solution must vanish).

The computation reduces into finding a certain value ω for each following conditions

$$\sum_{j=0}^{N} a_j^{(4)}(-x_+)^j = 0 \ , \quad \left(\sum_{j=0}^{N} \boldsymbol{M}_j(-x_+)^j\right).\boldsymbol{a}_0 = \boldsymbol{M}.\boldsymbol{a}_0 = 0 \ . \tag{3.63}$$

It is straightforward to calculate the roots of the first condition numerically. The second condition requires some care. The value ω is obtained for $\det(\boldsymbol{M}) = 0$, then the eigenvector corresponding to the null eigenvalue can be computed. The numerical results can be found in section 3.5.

3.2.3 Normal modes in anti-de Sitter spacetime

By doing the limit of $M \to 0$ to the Schwarzschild-AdS metric, the anti-de Sitter spacetime is obtained. The equation for $u_{(4)}$ has then an analytic solution. First, the r_* coordinate is given by

$$\frac{\mathrm{d}r_*}{\mathrm{d}r} = \frac{1}{1+\frac{r^2}{R_\Lambda^2}} \to r_* = R_\Lambda \arctan\left(\frac{r}{R_\Lambda}\right) \ . \tag{3.64}$$

The interval $r \in [0, +\infty[$ corresponds to $r_* \in [0, R_\Lambda \frac{\pi}{2}[$. The equation (3.29) can be put in terms of $\frac{r_*}{R_\Lambda}$ in the following way

$$\partial_{r_*}^2 u_{(4)} + \left[\omega^2 - \frac{\ell(\ell+1)}{R_\Lambda^2 \sin\left(\frac{r_*}{R_\Lambda}\right)} - \frac{m_A^2}{\cos\left(\frac{r_*}{R_\Lambda}\right)}\right] u_{(4)} = 0 \ . \tag{3.65}$$

This equation has three singularities which are $r_* = \{0, R_\Lambda\frac{\pi}{2}, -R_\Lambda\frac{\pi}{2}\}$. This type of equations can be always reduced to the hypergeometric differential equation by a change of coordinates. The most convenient one is $z = \sin\left(\frac{r_*}{R_\Lambda}\right)$. Additionally, multiplying by $\frac{4}{R_\Lambda^2}$, the equation becomes

$$z(1-z)\partial_z^2 u_{(4)} + \frac{1}{2}\Big(1-2z\Big)\partial_z u_{(4)} + \Big[\frac{R^2\omega^2}{4} - \frac{\ell(\ell+1)}{4z} - \frac{m_A^2 R^2}{4(1-z)}\Big] u_{(4)} = 0 \ . \tag{3.66}$$

To remove the terms with z in the denominator, one can make the transformation $u_{(4)} = z^\alpha(1-z)^\beta\psi$. Thus

$$\begin{aligned} &z(1-z)\partial_z^2\psi + \Big[\Big(\frac{1}{2}+2\alpha\Big) - (1+2\alpha+2\beta)z\Big]\partial_z\psi \\ &+ \Big[\frac{\omega^2 R_\Lambda^2}{4} - (\alpha+\beta)^2 + \frac{4\alpha(\alpha+1) - 2\alpha - \ell(\ell+1)}{4z} + \frac{4\beta(\beta+1) - 2\beta - m_A^2 R_\Lambda^2}{4(1-z)}\Big]\psi = 0 \ . \end{aligned} \tag{3.67}$$

The expression for α and β can be found by setting the terms with z^{-1} and $(1-z)^{-1}$ to 0, obtaining

$$\alpha = \frac{1}{4}\Big[1 + \sqrt{1 + 4\ell(\ell+1)}\Big] \quad , \quad \beta = \frac{1}{4}\Big[1 + \sqrt{1 + 4m_A^2 R_\Lambda^2}\Big]. \tag{3.68}$$

There is also another expression for α and β that removes those terms but it implies that both are negative. This is not desired because the solution will behave badly. And so, the equation (3.67) can be identified as the hypergeometric differential equation [47]

$$z(1-z)\partial_z^2\psi + \Big[c - (a+b+1)z\Big]\partial_z\psi - ab\psi = 0 \ , \tag{3.69}$$

where

$$a=\alpha+\beta+\frac{\omega R_\Lambda}{2}\ ,\ b=\alpha+\beta-\frac{\omega R_\Lambda}{2}\ ,\ c=\frac{1}{2}+2\alpha\ . \tag{3.70}$$

The solution in the region of $z=0$ ($r=0$) can be written as

$$u_{(4)}=H_1 z^\alpha(1-z)^\beta\,{}_2F_1[a,b,c;z]+H_2 z^{\frac{1}{2}-\alpha}(1-z)^\beta\,{}_2F_1[1+a-c,1+b-c,2-c;z]\ , \tag{3.71}$$

where ${}_2F_1$ is a hypergeometric function. In the second term, one must note that the exponent $\frac{1}{2}-\alpha\le 0$. This means it will diverge at $z=0$. Since $u_{(4)}$ must be regular at this boundary then H_2 must vanish. The remaining solution can be prolonged to the singular point $z=1$ ($r\to+\infty$) by the following transformation

$$\begin{aligned}{}_2F_1[a,b,c;z]&=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1[a,b,1+a+b-c;1-z]\\&+(1-z)^{\frac{1}{2}-2\beta}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1[a,b,1+c-a-b;1-z]\ ,\end{aligned} \tag{3.72}$$

where Γ is the gamma function. The boundary conditions at $z=1$ require that $u_{(4)}$ must vanish. If all the gamma functions are finite, then $u_{(4)}\propto(1-z)^{\frac{1}{2}-\beta}$ which explodes. The only way that $u_{(4)}$ vanishes is if either a or b is a negative integer ($-n$), reducing the hypergeometric function into a nth order polynomial in z. Since ω must be positive, then the eigenvalues are obtained by

$$b=-n\ \to\omega R_\Lambda=2n+\ell+\frac{3}{2}+\frac{1}{2}\sqrt{1+4m_A^2R_\Lambda^2}\ . \tag{3.73}$$

The monopole case can be described by equation (3.41), which in the limit of $M\to 0$ is equivalent to (3.65) with $\ell=1$. It has been verified that in Ref. [26] there is a mistake in the formula of the normal modes and β. These results are backed by numerical calculation of the normal modes.

3.3 Massless vector fields in Kerr geometry

3.3.1 Newman-Penrose formalism

Unlike Schwarzschild, the Kerr spacetime only has two explicit symmetries: axisymmetry and time translation. For this reason, the Proca equations are harder to separate. There is a formalism built by Newman and Penrose [28] that makes the separation possible. This formalism consists in moving to a frame with four "null" basis vectors ($\boldsymbol{l}$,$\boldsymbol{n}$,$\boldsymbol{m}$,$\overline{\boldsymbol{m}}$) and they obey the following relations

$$\boldsymbol{l}\cdot\boldsymbol{n}=-1\ \ ,\ \ \boldsymbol{m}\cdot\overline{\boldsymbol{m}}=1\ \ ,\ \ \boldsymbol{m}^*=\overline{\boldsymbol{m}}\ , \tag{3.74}$$

where $\boldsymbol{m}^*$ is the complex conjugate of $\boldsymbol{m}$ and all the other scalar product combinations vanish. To connect with the tetrad formalism (see appendix B), each vector is labelled by

$$e^a_{(1)}=l^a\ ,\ e^a_{(2)}=n^a\ ,\ e^a_{(3)}=m^a\ ,\ e^a_{(4)}=\overline{m}^a\ . \tag{3.75}$$

The matrix η (equation (B.1)) is

$$\eta_{(\mu)(\nu)}=\begin{pmatrix}0&-1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\ . \tag{3.76}$$

The Kerr metric (equation (2.5) with $\Lambda = 0$) can be written as

$$g_{ab} = -l_a n_b - l_b n_a + m_a \overline{m}_b + m_b \overline{m}_a \quad , \tag{3.77}$$

$$l^a = \frac{1}{\Delta}(r^2 + a^2, \Delta, 0, a) \ , \ n^a = \frac{1}{2\Sigma}(r^2 + a^2, -\Delta, 0, a) \ , \tag{3.78}$$

$$m^a = \frac{1}{\sqrt{2}\ \overline{\rho}}(ia \sin\theta, 0, 1, i \csc\theta) \ , \ \overline{\rho} = r + ia\cos\theta \ . \tag{3.79}$$

In Kerr geometry, the vector fields $\boldsymbol{l}$ and $\boldsymbol{n}$ are the future directed and past directed null vectors in the equatorial plane. The vector $\boldsymbol{m}$ is a complex vector in which the real part and imaginary part are spatial-like vectors.

Teukolsky [27] used Newman-Penrose formalism to parameterize the strength field tensor in three complex scalar fields

$$\phi_0 = F_{(1)(3)} = F_{ab} l^a m^b \ , \tag{3.80}$$

$$\phi_1 = \frac{1}{2}(F_{(1)(2)} + F_{(4)(3)}) = \frac{F_{ab}}{2}(l^a n^b + \overline{m}^a m^b) \ , \tag{3.81}$$

$$\phi_2 = F_{(4)(2)} = F_{ab} \overline{m}^a n^b \ . \tag{3.82}$$

To clarify notation, the letter ϕ with no index is the azimuthal coordinate and the letter with index is a complex field. These three complex scalar fields cover all the non-zero components of the strength field tensor. It can be checked that applying the complex conjugate to a quantity with index 3 is the same as having the quantity with index 4, due to the relations in (3.74). Thus, the three complex scalars fields have 6 degrees of freedom that describes the 6 non-zero, independent elements of a 2-form tensor (anti-symmetric (0,2) tensor) in 4 dimensions. The equations for the Faraday tensor in this formalism are given by (B.10). After a series of manipulations, this set of equations can be reduced to

$$\phi_{1|(1)} - \phi_{0|(4)} = 0 \quad , \quad \phi_{1|(3)} - \phi_{0|(2)} = 0 \quad , \quad \phi_{2|(1)} - \phi_{1|(4)} = 0 \quad , \quad \phi_{2|(3)} - \phi_{1|(2)} = 0 \ . \tag{3.83}$$

Using (B.8) and (B.9), the following is obtained

$$\phi_{1,(1)} - \phi_{0,(4)} = 2\gamma_{(1)(3)(4)}\phi_1 + [\gamma_{(2)(1)(4)} + \gamma_{(3)(4)(4)} - \gamma_{(2)(4)(1)}]\phi_0 - \gamma_{(1)(3)(1)}\phi_2 \ , \tag{3.84}$$

$$\phi_{1,(3)} - \phi_{0,(2)} = [\gamma_{(2)(1)(2)} + \gamma_{(3)(4)(2)} - \gamma_{(2)(4)(3)}]\phi_0 + 2\gamma_{(1)(3)(2)}\phi_1 - \gamma_{(1)(3)(3)}\phi_2 \ , \tag{3.85}$$

$$\phi_{1,(4)} - \phi_{2,(1)} = 2\gamma_{(2)(4)(1)}\phi_1 - [\gamma_{(1)(3)(4)} + \gamma_{(2)(1)(1)} + \gamma_{(3)(4)(1)}]\phi_2 - \gamma_{(2)(4)(4)}\phi_0 \ , \tag{3.86}$$

$$\phi_{1,(2)} - \phi_{2,(3)} = 2\gamma_{(2)(4)(3)}\phi_1 - [\gamma_{(2)(1)(3)} + \gamma_{(3)(4)(3)} + \gamma_{(1)(3)(2)}]\phi_2 - \gamma_{(2)(4)(2)}\phi_0 \ . \tag{3.87}$$

The Ricci-rotation coefficients can be calculated using (B.8) and the definitions of the basis vectors (3.78) and (3.79), obtaining

$$\begin{aligned}
&\gamma_{(2)(3)(1)} = \frac{ia\sin\theta}{\sqrt{2}\overline{\rho}^2} \ , \ \gamma_{(1)(2)(2)} = \frac{r(Mr - a^2\sin^2\theta) - a^2 M\cos^2\theta}{\Sigma^2} \ , \ \gamma_{(1)(3)(2)} = -\frac{-ia\sin\theta}{\sqrt{2}\Sigma} \\
&\gamma_{(3)(4)(2)} = \frac{i\Delta a\cos\theta}{\Sigma^2} \ , \ \gamma_{(1)(2)(3)} = -\gamma_{(2)(3)(1)} \ , \ \gamma_{(1)(4)(3)} = -\frac{1}{\overline{\rho}} \ , \ \gamma_{(2)(4)(3)} = \frac{\Delta}{2\Sigma\overline{\rho}^*} \\
&\gamma_{(3)(4)(4)} = \frac{i(a + ir\cos\theta)}{\sqrt{2}\sin\theta(\overline{\rho}^*)^2} \ ,
\end{aligned} \tag{3.88}$$

where the other coefficients not displayed here vanish, except for the ones that are obtained by making the complex conjugate or by using the anti-symmetric property in the first two indices. Apparently, the equations for the complex scalars (3.84)-(3.87) are all coupled. It turns out that in Kerr, the Ricci-rotation coefficients $\gamma_{(1)(3)(1)}$, $\gamma_{(1)(3)(3)}$, $\gamma_{(2)(4)(4)}$ and $\gamma_{(2)(4)(2)}$ vanish. Therefore, the set of equations is only coupled in pairs. This is one of the characteristics of Kerr geometry that enables separation of variables.

Before going through the equations for the fields in Kerr geometry, it is useful to define the following operators

$$\mathcal{D}_n = \partial_r + i\frac{K}{\Delta} + 2n\frac{r-M}{\Delta} \quad , \quad \mathcal{D}_n^\dagger = \partial_r - i\frac{K}{\Delta} + 2n\frac{r-M}{\Delta} \ , \tag{3.89}$$

$$\mathcal{L}_n = \partial_\theta + Q + n\cot\theta \quad , \quad \mathcal{L}_n^\dagger = \partial_\theta - Q + n\cot\theta \ , \tag{3.90}$$

$$K = am - \omega(r^2+a^2) \quad , \quad Q = \frac{m}{\sin\theta} - a\sin\theta\omega \ , \tag{3.91}$$

where ω is a frequency and m is the azimutal quantum number. The directional derivatives can be expressed in terms of these operators taking into account that $\phi_i \propto e^{i(-\omega t + m\phi)}$

$$l^a\partial_a = \mathcal{D}_0 = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial\phi = \partial_r + i\frac{K}{\Delta} \ , \tag{3.92}$$

$$n^a\partial_a = -\frac{\Delta}{2\Sigma}\mathcal{D}_0^\dagger = \frac{\Delta}{2\Sigma}\left(\frac{(r^a+a^2)}{\Delta}\partial_t - \partial_r + \frac{a}{\Delta}\partial_\phi\right) = -\frac{\Delta}{2\Sigma}\Big(\partial_r - i\frac{K}{\Delta}\Big) \ , \tag{3.93}$$

$$m^a\partial_a = \frac{1}{\sqrt{2}\bar{\rho}}\mathcal{L}_0^\dagger = \frac{1}{\sqrt{2}\bar{\rho}}\left(ia\sin\theta\partial_t + \partial_\theta + \frac{i\partial_\phi}{\sin\theta}\right) = \frac{1}{\sqrt{2}\bar{\rho}}(\partial_\theta - Q) \ , \tag{3.94}$$

$$\bar{m}^a\partial_a = \frac{1}{\sqrt{2}\bar{\rho}^*}\mathcal{L}_0 = \frac{1}{\sqrt{2}\bar{\rho}^*}\left(-ia\sin\theta\partial_t + \partial_\theta - \frac{i\partial_\phi}{\sin\theta}\right) = \frac{1}{\sqrt{2}\bar{\rho}^*}(\partial_\theta + Q) \ . \tag{3.95}$$

It can be shown [48] that these operators have the following properties

$$\mathcal{D}_n^\dagger = (\mathcal{D}_n)^* \ , \ \mathcal{L}_n^\dagger(\pi-\theta) = -\mathcal{L}_n(\theta) \ , \tag{3.96}$$

$$\mathcal{D}_n\Delta = \Delta\mathcal{D}_{n+1} \ , \ \mathcal{L}_n\sin\theta = \sin\theta\mathcal{L}_{n+1} \ , \tag{3.97}$$

$$\Big(\mathcal{D}_n + \frac{\sigma}{\bar{\rho}}\Big)\Big(\mathcal{L}_n + i\frac{\sigma a\sin\theta}{\bar{\rho}^*}\Big) = \Big(\mathcal{L}_n + i\frac{\sigma a\sin\theta}{\bar{\rho}^*}\Big)\Big(\mathcal{D}_n + \frac{\sigma}{\bar{\rho}}\Big) \ , \tag{3.98}$$

$$\int_0^\pi g\mathcal{L}_n(f)\sin\theta d\theta = -\int_0^\pi f\mathcal{L}^\dagger_{-n+1}(g)\sin\theta d\theta \ , \tag{3.99}$$

where σ is an arbitrary number. The property (3.97) is also true for the operators with $\dagger$ and (3.98) is true as well if one replaces any operator by its $\dagger$. These properties will be relevant in the next section.

3.3.2 Teukolsky's Master Equations

The equations (3.84)-(3.87) can be expanded using (3.92)-(3.95) and the Ricci-rotation coefficients (3.88) to obtain

$$\Big(\mathcal{D}_0 + \frac{2}{\bar{\rho}^*}\Big)\phi_1 = \frac{1}{\sqrt{2}\bar{\rho}^*}\Big(\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\Big)\phi_0 \ , \tag{3.100}$$

$$\frac{1}{\sqrt{2}\bar{\rho}}\Big(\mathcal{L}_0^\dagger + \frac{2ia\sin\theta}{\bar{\rho}^*}\Big)\phi_1 = -\frac{\Delta}{2\Sigma}\Big(\mathcal{D}_1^\dagger - \frac{1}{\bar{\rho}^*}\Big)\phi_0 \ , \tag{3.101}$$

$$\frac{1}{\sqrt{2}\bar{\rho}^*}\Big(\mathcal{L}_0 + \frac{2ia\sin\theta}{\bar{\rho}^*}\Big)\phi_1 = \Big(\mathcal{D}_0 + \frac{1}{\bar{\rho}^*}\Big)\phi_2 \ , \tag{3.102}$$

$$-\frac{\Delta}{2\Sigma}\Big(\mathcal{D}_0^\dagger + \frac{2}{\bar{\rho}^*}\Big)\phi_1 = \frac{1}{\sqrt{2}\bar{\rho}}\Big(\mathcal{L}_1^\dagger + \frac{ia\sin\theta}{\bar{\rho}^*}\Big)\phi_2 \ . \tag{3.103}$$

These can be put in a more symmetric way by defining

$$\Phi_0 = \phi_0 \quad , \quad \Phi_1 = \sqrt{2}\bar{\rho}^* \phi_1 \quad , \quad \Phi_2 = 2(\bar{\rho}^*)^2 \phi_2 \ . \tag{3.104}$$

The previous definition follows Ref. [48], while in the original work [27] and [49], the complex scalars are defined without the factors of $\sqrt{2}$ and 2. The equations for the new complex scalar fields read

$$\Big[\mathcal{D}_0 + \frac{1}{\bar{\rho}^*}\Big]\Phi_1 = \Big[\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\Big]\Phi_0 \ , \tag{3.105}$$

$$\Big[\mathcal{L}_0^\dagger + \frac{ia\sin\theta}{\bar{\rho}^*}\Big]\Phi_1 = -\Delta\Big[\mathcal{D}_1^\dagger - \frac{1}{\bar{\rho}^*}\Big]\Phi_0 \ , \tag{3.106}$$

$$\Big[\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}^*}\Big]\Phi_1 = \Big[\mathcal{D}_0 - \frac{1}{\bar{\rho}^*}\Big]\Phi_2 \ , \tag{3.107}$$

$$-\Delta\Big[\mathcal{D}_0^\dagger + \frac{1}{\bar{\rho}^*}\Big]\Phi_1 = \Big[\mathcal{L}_1^\dagger - \frac{ia\sin\theta}{\bar{\rho}^*}\Big]\Phi_2 \ . \tag{3.108}$$

Applying $-\Delta\Big[\mathcal{D}_0^\dagger + \frac{1}{\bar{\rho}^*}\Big]$ into equation (3.107) together with the commutation rule in (3.98), equation (3.108) can be used to eliminate Φ_1 and so one obtains a second order differential equation only for Φ_2. The same can be done for Φ_0, but applying instead $\Big[\mathcal{L}_0^\dagger + \frac{ia\sin\theta}{\bar{\rho}^*}\Big]$ to (3.105) and using (3.106) to eliminate Φ_1. The decoupled equations become

$$\Big[\Delta\mathcal{D}_0^\dagger\mathcal{D}_0 + \mathcal{L}_0\mathcal{L}_1^\dagger - 2i\omega\bar{\rho}\Big]\Phi_2 = 0 \ , \tag{3.109}$$

$$\Big[\Delta\mathcal{D}_1\mathcal{D}_1^\dagger + \mathcal{L}_0^\dagger\mathcal{L}_1 + 2i\omega\bar{\rho}\Big]\Phi_0 = 0 \ , \tag{3.110}$$

Assuming an ansatz for each field

$$\Phi_2 = R_{-1}(r)S_{-1}(\theta)e^{-i\omega t + im\phi} \ , \ \Phi_0 = R_{+1}S_{+1}e^{-i\omega t + im\phi} \ , \tag{3.111}$$

the equations (3.109) and (3.110) can be separated into

$$\frac{1}{\Delta^s}\frac{d}{dr}\left[\Delta^{s+1}\frac{dR_s}{dr}\right] + \left[\frac{K^2 + 2isK(r-M)}{\Delta} + 4isR_\Lambda\omega - A_{slm} - a^2\omega^2 + 2ma\omega\right]R_s = 0 \ , \tag{3.112}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{dS_s}{d\theta}\right] + \left[(a\omega\cos\theta - s)^2 - s(s-1) - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + A_{slm}\right]S_s = 0 \ , \tag{3.113}$$

where A_{slm} is the separation constant and s takes the values of ± 1 (some factors of s are introduced by hand for suggestive purposes, maintaining the consistency of the original equations). This follows Ref. [20], with a definition of K that differs by a minus sign. It also follows Ref. [48], if one takes into account the different signature of the metric and that $A_{lm} = \lambda_C + a^2\omega^2 - 2ma\omega - s(s+1)$, where λ_C is the separation constant used in the referred book. Also, the solutions to equation (3.113) are referred in the literature as spin-weighted spheroidal harmonics [50], where in this case S_s is considered to be the θ dependent part of these functions. In posterior calculations, the factor $e^{i(-\omega t + m\phi)}$ is going to be dropped.

3.3.3 Teukolsky-Starobinski Identities

While it is true that Φ_0 and Φ_2 can be determined by solving equations (3.109) and (3.110), these two scalars are not independent. By applying $\Big[\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}^*}\Big]$ to equation (3.105) and using properties (3.98) and (3.107), one of the relations can be obtained

$$\left[\mathcal{D}_0 + \frac{1}{\bar{\rho}^*}\right]\left[\mathcal{D}_0 - \frac{1}{\bar{\rho}^*}\right]\Phi_2 = \left[\mathcal{L}_0 + \frac{ia\sin\theta}{\bar{\rho}^*}\right]\left[\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\right]\Phi_0 \ . \tag{3.114}$$

The other relation can be obtained by manipulating (3.106) and (3.108) in an analogous way

$$\left[\mathcal{L}_0^\dagger + \frac{ia\sin\theta}{\bar{\rho}^*}\right]\left[\mathcal{L}_1^\dagger - \frac{ia\sin\theta}{\bar{\rho}^*}\right]\Phi_2 = \Delta\left[\mathcal{D}_0^\dagger + \frac{1}{\bar{\rho}^*}\right]\left[\mathcal{D}_0^\dagger - \frac{1}{\bar{\rho}^*}\right](\Delta\Phi_0) \ . \tag{3.115}$$

Relations (3.114) and (3.115) can be simplified to

$$\frac{\mathcal{D}_0\mathcal{D}_0 R_{-1}}{R_{+1}} = \frac{\mathcal{L}_0\mathcal{L}_1 S_{+1}}{S_{-1}} \quad , \quad \frac{\Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger(\Delta R_{+1})}{R_{-1}} = \frac{\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger S_{-1}}{S_{+1}} \ . \tag{3.116}$$

The scalars R_{-1} and ΔR_{+1} are related by a constant as well as S_{+1} and S_{-1}. The Teukolsky-Starobinski identities are then

$$\Delta\mathcal{D}_0\mathcal{D}_0 R_{-1} = D_1\Delta R_{+1} \quad , \quad \Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger(\Delta R_{+1}) = D_2 R_{-1} \ , \tag{3.117}$$

$$\mathcal{L}_0\mathcal{L}_1 S_{+1} = D_1 S_{-1} \quad , \quad \mathcal{L}_0^\dagger\mathcal{L}_1^\dagger S_{-1} = D_2 S_{+1} \ . \tag{3.118}$$

It must be noted that the operators applying on the spin-weighted spheroidal harmonics are real and that S_s is also real. This means constants D_1 and D_2 must be real. The spin-weighted spheroidal harmonics can be submitted to the normalization condition

$$\int_0^\pi S_s^2 \sin\theta d\theta = 1 \ . \tag{3.119}$$

Using this condition, it is possible to get a restriction in the values of D_1 and D_2

$$\begin{aligned} D_1^2 &= D_1^2 \int_0^\pi S_{-1}^2 \sin\theta d\theta \\ &= \int_0^\pi (\mathcal{L}_0\mathcal{L}_1 S_{+1})(\mathcal{L}_0\mathcal{L}_1 S_{+1}) \sin\theta d\theta \\ &= \int_0^\pi (\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger\mathcal{L}_0\mathcal{L}_1 S_{+1}) S_{+1} d\theta \\ &= D_1 D_2 \ , \end{aligned} \tag{3.120}$$

where, in the passage from the second to the third line, the property (3.99) was used. Thus, the significance of imposing the normalization condition to the spin-weighted spheroidal harmonics is that the constants in equations (3.117)-(3.118) are the same (D_1 = D_2 = D). Finally to conclude this subsection, the value for this constant can be obtained by decoupling one of the equations (3.117)-(3.118). For example, applying $\Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger$ into the first equation in (3.117) and using the second equation in (3.117), it follows

$$\Delta\mathcal{D}_0^\dagger\mathcal{D}_0^\dagger\Delta\mathcal{D}_0\mathcal{D}_0 R_{-1} = D^2 R_{-1} \ . \tag{3.121}$$

It can be shown that the left-hand side of equation (3.121) can be simplified using Teukolsky equations (3.109) and (3.110) to obtain the expression for the constant

$$D^2 = \lambda_C^2 - 4\omega^2 a^2 + 4\omega a m \ . \tag{3.122}$$

The Teukolsky-Starobinski identities (3.117)-(3.118) can be put in a useful form, by using Teukolsky equations (3.109)-(3.110)

$$2iK\mathcal{D}_0 R_{-1} = D\Delta R_{+1} - (\lambda_C + 2i\omega r)R_{-1} \ , \tag{3.123}$$

$$2iK\mathcal{D}_0^\dagger(\Delta R_{+1}) = (\lambda_C - 2i\omega r)\Delta R_{+1} - DR_{-1} \ , \tag{3.124}$$

$$2Q\mathcal{L}_1 S_{+1} = DS_{-1} + (\lambda_C - 2a\omega\cos\theta)S_{+1} \ , \tag{3.125}$$

$$2Q\mathcal{L}_1^\dagger S_{-1} = -DS_{+1} - (\lambda_C + 2a\omega\cos\theta)S_{-1} \ . \tag{3.126}$$

The relations above can be used to express the derivatives of R_s and S_s in terms of themselves.

3.3.4 The solution for the remaining scalar

The solution for the scalars Φ_0 and Φ_2 are now fully characterized. For the solution of equations (3.105)-(3.108) to be complete, an expression for Φ_1 must be found. This can be done by looking to the equations (3.105) and (3.107). Multiplying them by $\bar{\rho}^*$, they can be simplified to

$$\Delta\mathcal{D}_0(\bar{\rho}^*\Phi_1) = (r - ia\cos\theta)\Delta R_{+1}\left[\mathcal{L}_1 - \frac{ia\sin\theta}{\bar{\rho}^*}\right]S_{+1} \ , \tag{3.127}$$

$$\mathcal{L}_0(\bar{\rho}^*\Phi_1) = (r - ia\cos\theta)S_{-1}\left[\mathcal{D}_0 - \frac{1}{\bar{\rho}^*}\right]R_{-1} \ . \tag{3.128}$$

The following functions can be defined

$$g_{+1} = \frac{1}{D}(r\mathcal{D}_0 R_{-1} - R_{-1}) \ \ , \ \ g_{-1} = \frac{1}{D}(r\mathcal{D}_0^\dagger(\Delta R_{+1}) - \Delta R_{+1}) \ , \tag{3.129}$$

$$f_{+1} = \frac{1}{D}(\cos\theta\mathcal{L}_1^\dagger S_{-1} + \sin\theta S_{-1}) \ \ , \ \ f_{-1} = \frac{1}{D}(\cos\theta\mathcal{L}_1 S_{+1} + \sin\theta S_{+1}) \ , \tag{3.130}$$

where g_s only depends in r and f_s only depends in θ. It can be shown that these functions have the following properties, using Teukolsky-Starobinski identities (3.117)-(3.118)

$$\Delta\mathcal{D}_0 \ g_{+1} = r\Delta R_{+1} \ , \ \Delta\mathcal{D}_0^\dagger \ g_{-1} = rR_{-1} \ , \tag{3.131}$$

$$\mathcal{L}_0^\dagger \ f_{+1} = \cos\theta S_{+1} \ , \ \mathcal{L}_0 \ f_{-1} = \cos\theta S_{-1} \ . \tag{3.132}$$

Thus, equations (3.127) and (3.128) can be reduced with the help of the newly defined functions and the Teukolsky-Starobinski identities

$$\Delta\mathcal{D}_0(\bar{\rho}^*\Phi_1) = \Delta\mathcal{D}_0\Big[g_{+1}\mathcal{L}_1 S_{+1} - iaf_{-1}\mathcal{D}_0 R_{-1}\Big] \ , \tag{3.133}$$

$$\mathcal{L}_0(\bar{\rho}^*\Phi_1) = \mathcal{L}_0\Big[g_{+1}\mathcal{L}_1 S_{+1} - iaf_{-1}\mathcal{D}_0 R_{-1}\Big] \ . \tag{3.134}$$

Since the homogeneous solutions of these equations will have singularities, then the expression for Φ_1 in terms of R_s and S_s is

$$\bar{\rho}^*\Phi_1 = g_{+1}\mathcal{L}_1 S_{+1} - iaf_{-1}\mathcal{D}_0 R_{-1} \ . \tag{3.135}$$

The same treatment can be done with equations (3.106) and (3.108) to obtain an alternative expression for Φ_1

$$\bar{\rho}^*\Phi_1 = iaf_{+1}\mathcal{D}_0^\dagger(\Delta R_{+1}) - g_{-1}\mathcal{L}_1^\dagger S_{-1} \ . \tag{3.136}$$

It can be shown that

$$ iaf_{+1}\mathcal{D}_0^\dagger(\Delta R_{+1}) - g_{-1}\mathcal{L}_1^\dagger S_{-1} = g_{+1}\mathcal{L}_1 S_{+1} - iaf_{-1}\mathcal{D}_0 R_{-1} \ , \tag{3.137} $$

And so everything is consistent. The expression (3.135) for Φ_1 reveals that, although this scalar is not separable, $\bar{\rho}^*\Phi_1$ is.

3.3.5 Expressions for the vector field

The vector field A_a can be obtained using (3.2) and (3.80)-(3.82), knowing the expressions of the complex scalars ϕ_0, ϕ_1 and ϕ_2. Starting from (3.80) and (3.82), the equations can be expanded into the following

$$ \phi_0 = \frac{1}{\bar{\rho}\sqrt{2}}\left[\mathcal{D}_0(\sqrt{2}\bar{\rho}m^aA_a) - \mathcal{L}_0^\dagger(l^aA_a)\right] \ , \tag{3.138} $$

$$ \phi_2 = \frac{1}{2\Sigma\bar{\rho}^*\sqrt{2}}\left[\mathcal{L}_0(2\Sigma n^aA_a) + \Delta\mathcal{D}_0^\dagger(\sqrt{2}\bar{\rho}^*\bar{m}^aA_a)\right] \ . \tag{3.139} $$

Using equations (3.104) and their ansatz (3.111), the above set turns into

$$ \Delta R_{+1}S_{+1}(r + ia\cos\theta) = \frac{\Delta}{\sqrt{2}}\Big(\mathcal{D}_0(\sqrt{2}\bar{\rho}m^aA_a) - \mathcal{L}_0^\dagger(l^aA_a)\Big) \ , \tag{3.140} $$

$$ R_{-1}S_{-1}(r + ia\cos\theta) = \frac{1}{\sqrt{2}}\Big(\mathcal{L}_0(2\Sigma n^aA_a) + \Delta\mathcal{D}_0^\dagger(\sqrt{2}\bar{\rho}^*\bar{m}^aA_a)\Big) \ . \tag{3.141} $$

Since there are terms proportional to r and $\cos\theta$, it is possible to use the properties (3.131) and (3.132) in order to get rid of the derivatives. The equations become

$$ \Delta\mathcal{D}_0\left(g_{+1}S_{+1} - \bar{\rho}m^aA_a\right) = -\mathcal{L}_0^\dagger\left(ia\Delta R_{+1}f_{+1} + \frac{\Delta}{\sqrt{2}}l^aA_a\right) \ , \tag{3.142} $$

$$ \Delta\mathcal{D}_0^\dagger\left(g_{-1}S_{-1} - \bar{\rho}^*\bar{m}^aA_a\right) = -\mathcal{L}_0\left(iaR_{-1}f_{-1} - \frac{2\Sigma n^aA_a}{\sqrt{2}}\right) \ . \tag{3.143} $$

Thus, the only solution for the internal product of A^a with the null frame vectors is

$$ l^aA_a = \sqrt{2}\Big(- iaR_{+1}f_{+1} - \mathcal{D}_0H_{+1}\Big) \ , \tag{3.144} $$

$$ n^aA_a = \frac{1}{\sqrt{2}\Sigma}\Big(iaR_{-1}f_{-1} + \Delta\mathcal{D}_0^\dagger H_{-1}\Big) \ , \tag{3.145} $$

$$ m^aA_a = \frac{1}{\bar{\rho}}\Big(g_{+1}S_{+1} - \mathcal{L}_0^\dagger H_{+1}\Big) \ , \tag{3.146} $$

$$ \bar{m}^aA_a = \frac{1}{\bar{\rho}^*}\Big(g_{-1}S_{-1} - \mathcal{L}_0H_{-1}\Big) \ , \tag{3.147} $$

where the functions H_{+1} and H_{-1} correspond to an additional degree of freedom, not present in the strength field tensor. Indeed, the terms that depend on these functions will vanish in equations (3.142) and (3.143). The above expressions can be put in a suggestive form, using (3.131),(3.132) and (3.92)-

(3.95)

$$l^a A_a = -\frac{ia\sqrt{2}}{r} l^a \partial_a (g_{+1} f_{+1}) - \sqrt{2} l^a \partial_a H_{+1} \ , \tag{3.148}$$

$$n^a A_a = -\frac{ia\sqrt{2}}{r} n^a \partial_a (g_{-1} f_{-1}) - \sqrt{2} n^a \partial_a H_{-1} \ , \tag{3.149}$$

$$m^a A_a = \frac{\sqrt{2}}{\cos\theta} m^a \partial_a (g_{+1} f_{+1}) - \sqrt{2} \bar{m}^a \partial_a H_{+1} \ , \tag{3.150}$$

$$\bar{m}^a A_a = \frac{\sqrt{2}}{\cos\theta} \bar{m}^a \partial_a (g_{-1} f_{-1}) - \sqrt{2} \bar{m}^a \partial_a H_{-1} \ . \tag{3.151}$$

Note that when comparing these expressions with the literature (Ref. [29] and Ref. [48]), there is an overall minus sign difference in A^a. This is due to a difference of a minus sign in the definition of (3.2). The components of A_a can be found to be

$$A_t = \Big(\frac{\Delta}{2\Sigma} l^a A_a + n^a A_a\Big) + \frac{ia\sin\theta}{\sqrt{2}} \Big(\frac{1}{\bar{\rho}^*} m^a A_a - \frac{1}{\bar{\rho}} \bar{m}^a A_a\Big) \ , \tag{3.152}$$

$$A_r = \frac{1}{2} l^a A_a - \frac{\Sigma}{\Delta} n^a A_a \ , \tag{3.153}$$

$$A_\theta = \frac{1}{\sqrt{2}} \Big(\bar{\rho} m^a A_a + \bar{\rho}^* \bar{m}^a A_a\Big) \ , \tag{3.154}$$

$$A_\phi = -\left[a\sin^2\theta \Big(\frac{\Delta}{2\Sigma} l^a A_a + n^a A_a\Big) + \frac{i}{\sqrt{2}} \sin\theta (r^2 + a^2) \Big(\frac{1}{\bar{\rho}^*} m^a A_a - \frac{1}{\bar{\rho}} \bar{m}^a A_a\Big)\right] \ . \tag{3.155}$$

It must be noted that these expressions must be multiplied by $e^{i(-\omega t + m\phi)}$ to obtain the complete solution and afterwards take the real part of A_a. Quantities A_a and F_{ab} are real by assumption. Still, there must be some care since complex basis vectors are being dealt with, which makes quantities like R_s or $\bar{m}^a A_a$ be complex.

To finalize, functions H_{+1} and H_{-1} are not independent. Looking to the definition of Φ_1 (3.104) and making analogous manipulations as done above, an equation that relates these two scalars arise

$$\mathcal{D}_0^\dagger \left(\frac{\Delta \mathcal{D}_0 H_{+1}}{(\bar{\rho}^*)^2}\right) + \mathcal{L}_1 \left(\frac{\mathcal{L}_0^\dagger H_{+1}}{(\bar{\rho}^*)^2}\right) - \mathcal{D}_0 \left(\frac{\Delta \mathcal{D}_0^\dagger H_{-1}}{(\bar{\rho}^*)^2}\right) - \mathcal{L}_1^\dagger \left(\frac{\mathcal{L}_0 H_{-1}}{(\bar{\rho}^*)^2}\right) = 0 \ . \tag{3.156}$$

In sum, Newman-Penrose formalism enables the separation of the Maxwell equations in Kerr metric. Furthermore, the problem is reduced to the Teukolsky equations, for one of the complex scalars Φ_0 or Φ_2. By just solving the equations for R_{-1} and S_{-1}, for example, it is possible to obtain R_{+1} and S_{+1}. Thus, the strength field tensor is fully described since ϕ_1 can be obtained in terms of these quantities. This only describes partially A_a though, as expected. In fact, the additional functions H_{+1} and H_{-1} that appear in the expressions are due to the gauge freedom of A_a. Talking about the three polarizations, R_{-1} holds the two irremovable ones, as well as R_{+1}. It must be reminded that these quantities are complex. Plus, the Teukolsky-Starobinski identities only restrict one of the R_s (it can be shown that the first equation in (3.117) can be obtained from the second equation in (3.117), which enables one to find the constant D in (3.121)). The third polarization, which would correspond to zero helicity, must be present in H_{-1} and in H_{+1}. Since these functions do not appear in the strength field tensor, the degree of freedom can be removed by a gauge transformation.

3.3.6 Superradiance

Superradiance has already been analyzed for the scalar field. Now, with this solution for a massless vector field in Kerr metric, it is important to check that this effect holds as well for this case. This analysis has been done originally in Ref. [49]. Analogously to the scalar field, one needs to look for the asymptotic solutions of R_s. The equation for these complex scalars is given by (3.112) and it can be turned into a Schrödinger-like equation. By making the following substitution in equation (3.112)

$$R_s = \frac{U_s}{\Delta^{\frac{s}{2}}\sqrt{r^2+a^2}} \ , \tag{3.157}$$

and multiplying everything by $\frac{\Delta^{\frac{s}{2}}}{\sqrt{r^2+a^2}}$, it is possible to obtain

$$\frac{\mathrm{d}^2U_s}{\mathrm{d}r_*^2} + \left[\frac{\Delta^2}{(r^2+a^2)^{\frac{3}{2}}}C(r) + \frac{\Delta}{(r^2+a^2)^2}X(r)\right]U_s = 0 \ , \tag{3.158}$$

$$C(r) = \frac{3r^2}{(r^2+a^2)^{\frac{5}{2}}} + \frac{2r(r-M)s-\Delta}{\Delta(r^2+a^2)^{\frac{3}{2}}} + \frac{(r-M)^2(s+2)s-\Delta s}{\Delta^2\sqrt{r^2+a^2}} \ , \tag{3.159}$$

$$X(r) = \frac{K^2+2i(r-M)K}{\Delta} + 4isR_\Lambda\omega - \lambda - 2(s+1)(r-M)\left(\frac{r}{r^2+a^2} + \frac{(r-M)s}{\Delta}\right) \ , \tag{3.160}$$

where r_* is given by the integration of (2.15). At $r\to\infty$, the type of behaviour that is expected for U_s is

$$U_s = \alpha_n r^n e^{ik_\infty r} \ , \ r\to\infty \ . \tag{3.161}$$

At the limit of large r, equation (3.158) takes the form

$$\omega^2 - k_\infty^2 + \frac{2i}{r}\Big(s\omega + nk_\infty\Big) + \mathcal{O}\Big(\frac{1}{r^2}\Big) = 0 \ , \tag{3.162}$$

where only the real and imaginary leading order terms were taken into account. The equation at leading order is satisfied if $k_\infty = \pm\omega$ and $n = \mp s$. So the asymptotic solution is

$$R_s \approx \frac{\mathcal{I}_s}{r}e^{-i\omega r} + \frac{\mathcal{R}_s}{r^{2s+1}}e^{i\omega r} \ , \ r\to\infty \tag{3.163}$$

Near the event horizon, by making use of $\Delta = 0$, the equation (3.158) becomes

$$\frac{\mathrm{d}^2U_s}{\mathrm{d}r_*^2} - \frac{s^2(r-M)^2}{(2Mr_+)^2}U_s + \frac{2is(r-M)}{2Mr_+}\Big(\frac{am}{2Mr_+} - \omega\Big)U_s + \Big(\omega - \frac{am}{2Mr_+}\Big)^2U_s = 0 \ . \tag{3.164}$$

This solution is trickier since the second derivative is done with respect to r_*. By making the ansatz

$$U_s = f(r)e^{ik_H r_*} \ , \tag{3.165}$$

the second derivative term has the following expression

$$e^{-ik_H r_*}\frac{\mathrm{d}^2U_s}{\mathrm{d}r_*^2} = -k_H^2 f + 2ik_H\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\Delta}{r^2+a^2} + \frac{d}{dr}\Big(\frac{\mathrm{d}f}{\mathrm{d}r}\frac{\Delta}{r^2+a^2}\Big)\frac{\Delta}{r^2+a^2} \ . \tag{3.166}$$

Inputing this into (3.164), it is found that the Schrödinger-like equation is satisfied if and only if

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \pm\frac{s\Delta'}{2\Delta}f \quad , \quad k_H = \pm(\omega - m\Omega_H) \ , \tag{3.167}$$

where again $\Omega_H = \frac{a}{2Mr_+}$. The differential equation for $f(r)$ can be solved and it is found that

$$f(r) = \sqrt{2Mr_+}\Delta^{\pm\frac{s}{2}} \ , \tag{3.168}$$

where the constant factor is chosen for convenience. Thus, the asymptotic solution near the event horizon is

$$R_s \approx \mathcal{O}_s e^{i(\omega - m\Omega_H)r_*} + \mathcal{T}_s \Delta^{-s} e^{-i(\omega - m\Omega_H)r_*} \ , \ r \to r_+ \ . \tag{3.169}$$

Since it is a black hole, any outgoing flux near the event horizon is not expected. This means it is considered $\mathcal{O}_s = 0$.

The coefficients in the asymptotic solutions are not all independent, because of the Teukolsky-Starobinski identities. The relations at infinity can be found by considering equation (3.123) for the coefficients $\mathcal{I}_s$ and equation (3.124) for the coefficients R_s, at leading order in powers of r, obtaining

$$\mathcal{I}_{+1} = -\frac{4\omega^2}{D}\mathcal{I}_{-1} \ \ , \ \ \mathcal{R}_{+1} = -\frac{D}{4\omega^2}\mathcal{R}_{-1} \ . \tag{3.170}$$

Near the event horizon, the relation can be found by using (3.124) for coefficients $\mathcal{T}_s$

$$D\mathcal{T}_{+1} = -16iM^2 r_+^2 k_H \Big(-ik_H + \frac{r_+ - M}{2Mr_+}\Big)\mathcal{T}_{-1} \ . \tag{3.171}$$

These three relations are present in Ref. [49], minding that there is a different factor of 2 between the definitions of Φ_2 used here and of Ref. [49]. The description of the asymptotic solutions for the complex scalars R_{-1} and R_{+1} is now complete. To proceed with the energy balance relative to these solutions, one should look for conserved quantities of Kerr spacetime. As showed in appendix C, the energy flux that enters the spacetime is given by (C.13)

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \lim_{r\to\infty}\int r^2 T_{tr} d\Omega = \lim_{r\to\infty}\int r^2 T_{ab}\Big(\frac{1}{2}l^a + n^a\Big)\Big(\frac{1}{2}l^a - n^a\Big) \\ &= \lim_{r\to\infty}\int \frac{r^2}{2\pi}\Big(-|\phi_2|^2 + \frac{|\phi_0|^2}{4}\Big)d\Omega \ ,\end{aligned} \tag{3.172}$$

where $\partial_t = (\frac{1}{2}l^a\partial_a + n^a\partial_a\Big)$ and $\partial_r = \Big(\frac{1}{2}l^a - n^a\Big)$ at the limit $r \to \infty$. This means that

$$\frac{d^2E_{out}}{dtd\Omega} = \frac{r^2}{2\pi}|\phi_2|^2 = \frac{|\mathcal{R}_{-1}|^2}{4}S_{-1}^2 = \frac{4\omega^4|\mathcal{R}_{+1}|^2}{D^2}S_{-1}^2 \ , \tag{3.173}$$

$$\frac{d^2E_{in}}{dtd\Omega} = \frac{r^2}{8\pi}|\phi_0|^2 = \frac{|\mathcal{I}_{+1}|^2}{4}S_{+1}^2 \ , \tag{3.174}$$

where (3.170) has been used, E_{in} corresponds to the ingoing energy and E_{out} to the outgoing energy. This result gives a physical meaning to ϕ_0 and ϕ_2. Indeed, ϕ_0 contains the information of ingoing waves and ϕ_2 contains the information of outgoing waves, asymptotically. Following the discussion in the previous subsection, the two polarizations must be contained in these scalars.

At the horizon, the variation of energy is given by (C.17)

$$\begin{aligned}\frac{\mathrm{d}E_{h_{in}}}{\mathrm{d}t} &= \frac{\omega}{2Mr_+k_H}\int T_{ab}\left(\frac{\Delta}{2}l^a + \Sigma n^a\Big)\Big(\frac{\Delta}{2}l^b + \Sigma n^b\Big)\right|_{\Delta=0} d\Omega \\ &= \frac{\omega}{16\pi Mr_+k_H}\int \Delta^2|\phi_0|^2\Big|_{\Delta=0} d\Omega \\ &= \frac{\omega}{8Mr_+k_H}|\mathcal{T}_{+1}|^2 \ ,\end{aligned} \tag{3.175}$$

where the normalization of S_s (3.119) has been applied, and the inward normal vector to the horizon is given by

$$N^a = -\left(\xi^a_{(t)} + \Omega_H \xi^a_{(\phi)}\right)\Big|_{\Delta=0} = -\frac{1}{2Mr_+}\left(\frac{\Delta}{2} l^a + \Sigma n^a\right) \ . \tag{3.176}$$

$E_{h_{out}}$ is the energy that flows outside the black hole and so it is considered to be 0. Integrating (3.173) and (3.174), the normalization of S_s can also be used to get rid of the angular functions. Now, for energy conservation to occur, the following statement needs to be satisfied

$$\frac{\mathrm{d}E_{out}}{\mathrm{d}t} = \frac{\mathrm{d}E_{in}}{\mathrm{d}t} - \frac{\mathrm{d}E_{h_{in}}}{\mathrm{d}t} \iff \frac{4\omega^4|\mathcal{R}_{+1}|^2}{D^2} = \frac{|\mathcal{I}_{+1}|^2}{4} - \frac{\omega}{8k_H M r_+}|\mathcal{T}_{+1}|^2 \ . \tag{3.177}$$

This expression tells that there is a condition leading to $\frac{\mathrm{d}E_{out}}{\mathrm{d}t} > \frac{\mathrm{d}E_{in}}{\mathrm{d}t}$, given by

$$k_H < 0 \iff \omega < m\Omega_H \ . \tag{3.178}$$

This is the superradiance for the case of electromagnetic waves.

3.4 Proca fields in Kerr-NUT-(A)dS geometry

The development of separating the Maxwell equations in Kerr spacetime took down in the 1970's [49]. The immediate question that arises is if there is a way to extend this formalism and apply the separation of variables to the Proca equations. It seems a possible answer has been given by Frolov et al [33], following the work of Lunin [29]. The machinery that allows the separation of variables are hidden symmetries present not only in Kerr but also a generalization of it for higher dimensions called Kerr-NUT-(A)dS (where NUT comes from the presence of NUT parameters [51], and (A)dS stands for (anti)-de Sitter, associated with a (negative) positive cosmological constant). These hidden symmetries are generated by a closed 2-form conformal Killing-Yano tensor h_{ab}. The following topics referred in this section are reviewed in Ref. [31].

3.4.1 Hidden Symmetries in Kerr-NUT-(A)dS spacetimes

All the symmetries of the geometry can be observed in the phase space. In the framework of General Relativity, the configuration space is defined as a D-dimensional manifold M (which corresponds to the spacetime). For a particle with momentum p_a, it is possible to define a $2D$-dimensional cotangent bundle (phase space). This bundle has associated a sympletic structure given by

$$\mathbf{\Omega} = dx^a \wedge dp_a \ . \tag{3.179}$$

Hamiltonian mechanics can be applied to a particle in General Relativity, using this structure. This is useful because symmetries can be depicted more easily in the Hamiltonian description. An important operator in this description is the Poisson bracket. Given two quantities F_1 and F_2, the Poisson bracket between these two quantities is given by

$$\{F_1, F_2\} = \frac{\partial F_1}{\partial x^a}\frac{\partial F_2}{\partial p_a} - \frac{\partial F_2}{\partial x^a}\frac{\partial F_1}{\partial p_a} \ . \tag{3.180}$$

Conserved quantities are directly related to symmetries of the spacetime. These can be mostly described by monomials in momenta

$$Q = k^{a_1 \dots a_m} p_{a_1} \dots p_{a_m} \ . \tag{3.181}$$

By choosing $m = 1$, one obtains the conserved quantity associated to a Killing vector field (appendix C). It seems that only symmetries associated to quantities with $m = 1$ can be explicitly seen in the configuration space (by an isometry). For higher m, the symmetries are called "hidden" for this reason. The tensor associated to such symmetries is called the Killing tensor and obeys the following equation

$$\nabla^{(a} k^{a_1) a_2 \dots a_m} = 0, \tag{3.182}$$

where the parentheses ( ) correspond to symmetrization in the covered indices. This condition allows Q to be conserved along a timelike or null geodesic. The case for $m = 1$ reduces to the condition for a Killing vector field. In the case of two conserved quantities Q_1 and Q_2 associated respectively to Killing tensors k_1 and k_2, the Poisson brackets of these quantities can be given by

$$\{Q_1, Q_2\} = ([k_1, k_2]_{NS})^{a_1 \dots a_{p-1} c b_1 \dots b_{q-1}} p_{a_1} \dots p_{a_{p-1}} p_c p_{b_1} \dots p_{b_{q-1}} \ , \tag{3.183}$$

$$([k_1, k_2]_{NS})^{a_1 \dots a_{p-1} c b_1 \dots b_{q-1}} = p k_1^{e(a_1 \dots a_{p-1}} \nabla_e k_2^{c b_1 \dots b_{q-1})} - q k_2^{e(b_1 \dots b_{q-1}} \nabla_e k_1^{c a_1 \dots a_{p-1})} \ , \tag{3.184}$$

where $[k_1, k_2]_{NS}$ is the Nijenhuis-Schouten bracket, p is the rank of k_1 and q is the rank of k_2.

There are also quantities

$$F = Q^{a_1 \dots a_m} l_{a_1} \dots l_{a_m} \ , \tag{3.185}$$

that are only conserved along null geodesics, where l_a is the momentum of light. In this case, $Q^{a_1 \cdots a_m}$ is a conformal Killing tensor and the condition for it can be found in [31].

In general, if and only if a spacetime contains r Killing vectors $\boldsymbol{l}_{(i)}$ ($i = 0, ..., r-1$) and $D - r$ Killing tensors $\boldsymbol{k}_{(\alpha)}$ ($\alpha = 1, ..., D - r$) such that

$$[\boldsymbol{l}_{(i)}, \boldsymbol{l}_{(j)}]_{NS} = 0 \ , \ [\boldsymbol{k}_{(\alpha)}, \boldsymbol{l}_{(i)}]_{NS} = 0 \ , \ [\boldsymbol{k}_{(\alpha)}, \boldsymbol{k}_{(\beta)}]_{NS} = 0 \ , \tag{3.186}$$

and that $D - r$ Killing tensors possess common eigenvectors $\boldsymbol{m}_{(\alpha)}$ that obey to

$$[\boldsymbol{m}_{(\alpha)}, \boldsymbol{m}_{(\beta)}]_{NS} = 0 \ , \ [\boldsymbol{m}_{(\alpha)}, \boldsymbol{l}_{(i)}]_{NS} = 0 \ , \ m^a_{(\alpha)} l^{(j)}_a = 0 \ , \tag{3.187}$$

then the spacetime admits the separability of the Hamilton-Jacobi equation. Also, the Klein-Gordon equation is separable if and only if the $\boldsymbol{m}_{(\alpha)}$ are eigenvectors of the Ricci tensor.

3.4.2 Killing-Yano family

It might happen that in a spacetime there are objects that are parallel transported along geodesics. Specifically, a form given by

$$A_{a_1 \dots a_m} = f_{c a_1 \dots a_m} p^c \ , \ f_{(c a_1 \dots a_m)} = 0 \ , \tag{3.188}$$

can be parallel transported along a timelike or null geodesic if and only if

$$\nabla_{(a} f_{b) a_1 \dots a_m} = 0 \ . \tag{3.189}$$

The form $f_{c a_1 \dots a_m}$ is called a Killing-Yano tensor and they are curiously related to 2-rank Killing tensors as

$$k^{ab} = f^a_{\ a_1 \dots a_m} f^{b a_1 \dots a_m} \ . \tag{3.190}$$

The Killing-Yano tensor can be interpreted as a ”square root” of a Killing tensor.

A broader definition of the Killing-Yano family can be given by looking into the covariant derivative of a form. It can be decomposed by the following

$$\nabla \boldsymbol{w} = \mathcal{A}\boldsymbol{w} + \mathcal{C}\boldsymbol{w} + \mathcal{T}\boldsymbol{w} \ , \tag{3.191}$$

$$(\mathcal{A}\boldsymbol{w})_{a b_1 \dots b_m} = \nabla_{[a} w_{b_1 \dots b_m]} \ , \tag{3.192}$$

$$(\mathcal{C}\boldsymbol{w})_{a b_1 \dots b_m} = \frac{m}{D - m + 1} g_{a[b_1} \nabla^c w_{|c| b_2 \dots b_m]} \ , \tag{3.193}$$

and $\mathcal{T}\boldsymbol{w}$ is given by rearranging (3.191). By assuming that $\mathcal{T}\boldsymbol{w}$ vanishes, one obtains the condition for the Killing-Yano family of forms. More precisely, every form that satisfies only $\mathcal{T}\boldsymbol{w} = 0$ is a conformal Killing-Yano form. These forms give rise to conformal Killing tensors in the same manner as (3.190). By making $\mathcal{C}\boldsymbol{w}$ also vanish, one obtains the condition for a Killing-Yano form. Indeed, this satisfies (3.189). There is another object in the Killing-Yano family that is defined by making $\mathcal{T}\boldsymbol{w}$ and $\mathcal{A}\boldsymbol{w}$ vanish. It is called a closed conformal Killing-Yano form, and this is the object of interest. Recapitulating, a closed conformal Killing-Yano form obeys

$$\nabla_a h_{b_1 \dots b_m} = m g_{a[b_1} \xi_{b_2 \dots b_m]} \quad , \quad \xi_{b_2 \dots b_m} = \frac{1}{D-m+1} \nabla^c h_{c b_2 \dots b_m} \ . \tag{3.194}$$

This set of equations, together with integrability conditions [31]

$$\nabla^a \nabla^b h_{c_1 \dots c_p} = -\frac{p}{D-p} \Big(R^a{}_e \delta^b_{[c_1} h^e{}_{c_2 \dots c_p]} + \frac{p-1}{2} R_{de}{}^a{}_{[c_1} \delta^b_{c_2} h^{de}{}_{\dots c_p]} \Big) \ , \tag{3.195}$$

$$2 R^{[a}_e \delta^{b]}_{[c_1} h^e{}_{c_2 \dots c_p]} - (D-p) R^{ab}{}_{e[c_1} h^e{}_{c_2 \dots c_p]} + (p-1) R_{de}{}^{[a}{}_{[c_1} \delta^{b]}_{c_2} h^{de}{}_{\dots c_p]} = 0 \ , \tag{3.196}$$

is an over-determined problem, meaning the closed conformal Killing-Yano forms exist for a special group of spacetimes. Some consequences of the integrability conditions are that

$$\mathscr{L}_{\boldsymbol{\xi}} \boldsymbol{h} = 0 \quad , \quad \mathscr{L}_{\boldsymbol{\xi}} \boldsymbol{g} = 0 \ . \tag{3.197}$$

So, ξ is a Killing vector field. These conditions are crucial in the determination of a spacetime metric in which a closed conformal Killing-Yano form exists.

A property of these objects is that their hodge dual gives a Killing-Yano form

$$f_{b_1 \dots b_{D-m}} = (\star h)_{b_1 \dots b_{D-m}} = \frac{1}{m!} \epsilon_{a_1 \dots a_m b_1 \dots b_{D-m}} h^{a_1 \dots a_m} \ , \tag{3.198}$$

where $\boldsymbol{\epsilon}$ is the volume form of the spacetime. This means it is possible to construct a Killing tensor with a closed conformal Killing-Yano tensor.

3.4.3 Principal tensor

Since there is overdeterminacy, one can start by assuming the existence of a non-degenerate closed conformal Killing-Yano 2-form h_{ab}, which it is called the principal tensor. Being non-degenerate means that $h^a{}_b$ as a matrix must possess non-vanishing eigenvalues in even dimensions and one vanishing eigenvalue in odd dimensions, in both cases being functionally independent. Since it is an anti-symmetric matrix, the eigenvalues need to be imaginary

$$h^a{}_b (m_\mu)^b = -i x_\mu m^a \quad , \quad h^a{}_b (\bar{m}_\mu)^b = i x_\mu \bar{m}^a \ , \tag{3.199}$$

$$h^a{}_b (\hat{e}_0)^b = 0 \text{ (for odd dimensions)}, \tag{3.200}$$

where $(m_\mu)^b$ are eigenvectors associated to the eigenvalues $-ix_\mu$, $(\hat{m}_\mu)^b$ are complex conjugate of these eigenvectors and $\mu = 1, 2, ..., n$. The number of dimensions is $D = 2n + \epsilon$, where $\epsilon = 0, 1$ if the spacetime has even or odd dimensions, respectively. As a reminder in notation, Greek indices in this section do not follow Einstein notation. As any non-degenerate matrix, it is possible to choose the orthogonality conditions. It can be defined then

$$(m_\mu)^a (m^\nu)_a = 0 \ , \ (m_\mu)^a (\bar{m}^\nu)_a = \delta^\nu_\mu \ , \ (\hat{e}_0)^a (\hat{e}^0)_a = 1 \tag{3.201}$$

where $(m^\nu)^a = g^{ab} (m_\nu)_b$. Thus, the metric can be expressed as

$$g^{ab} = \sum_{\mu=1}^{n} (m_\mu)^{(a} (\bar{m}_\mu)^{b)} + \epsilon (\hat{e}_0)^a (\hat{e}_0)^b \ , \tag{3.202}$$

Even though working with the eigenvectors gives the benefit that the principal tensor is diagonal, it is more useful to work with the following frame

$$\boldsymbol{e}_\mu = -\frac{i}{\sqrt{2}}(\boldsymbol{m}_\mu - \bar{\boldsymbol{m}}_\mu) \quad , \quad \hat{\boldsymbol{e}}_\mu = \frac{1}{\sqrt{2}}(\boldsymbol{m}_\mu + \hat{\boldsymbol{m}}_\mu) \ , \tag{3.203}$$

The expression of the principal tensor and the metric become in this frame

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu \quad , \quad \boldsymbol{g} = \sum_{\mu=1}^{n} (\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu \hat{\boldsymbol{e}}^\mu) + \epsilon\ \hat{\boldsymbol{e}}^0 \hat{\boldsymbol{e}}^0 \ . \tag{3.204}$$

This frame is called the special Darboux frame and it is closely related with the existence of the principal tensor. The construction presented above is done in the Euclidean signature $(+,+,+,+)$ because it is easier. Of course, it is possible to convert it to the Lorentzian signature by performing a Wick rotation, which will be clearer in the next sections.

It is possible to obtain more information about the spacetime using this construction. First, one applies the covariant derivative into (3.199), which can be simplified by using equations in (3.194). Projecting the expression into the eigenvectors $\boldsymbol{m}_\nu$, it is possible to obtain

$$\boldsymbol{e}^\mu = (Q_\mu)^{-\frac{1}{2}} \boldsymbol{d}x_\mu \ , \boldsymbol{\xi} = \sum_{\mu=1}^{n} \sqrt{Q_\mu} \hat{\boldsymbol{e}}_\mu + \epsilon \sqrt{Q_0} \hat{\boldsymbol{e}}_{\boldsymbol{0}} \ , \tag{3.205}$$

where Q_μ and Q_0 are functions related to the metric. These expressions reveal that functions x_μ can be chosen to be n of the $2n+\epsilon$ coordinates of the spacetime.

3.4.4 Kerr-NUT-(A)dS spacetime metrics

The Kerr-NUT-(A)dS metrics [52] can describe a series of spacetimes containing rotating black holes with cosmological constant in higher dimensions. Rotating black holes described by Myers-Perry metric [30] and Taub-NUT spacetimes [53] are subcases or limits of this family of metrics.

The expression of the metric, with Euclidean signature, is

$$\boldsymbol{g} = \sum_{\mu=1}^{n} \left[\frac{U_\mu}{X_\mu} dx_\mu^2 + \frac{X_\mu}{U_\mu} \Big(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \Big)^2 \right] + \epsilon \frac{c}{A^{(n)}} \Big(\sum_{j=0}^{n} A^{(j)} d\psi_j \Big)^2 \ , \tag{3.206}$$

where

$$A_\mu^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\nu_2<\dots<\nu_j \\ \nu\neq\mu}}^{n} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_j}^2 \quad , \quad A^{(j)} = \sum_{\substack{\nu_1,\dots,\nu_j=1 \\ \nu_1<\nu_2<\dots<\nu_j}}^{n} x_{\nu_1}^2 x_{\nu_2}^2 \dots x_{\nu_j}^2 \quad , \quad U_\mu = \prod_{\substack{\nu=1 \\ \nu\neq\mu}}^{n} (x_\nu^2 - x_\mu^2) \ . \tag{3.207}$$

The X_μ are pure functions on x_μ. These functions are arbitrary if the metric is off-shell. If the metric is on-shell, then the expression for X_μ is given as a solution of the Einstein equations. For the case of the vacuum Einstein equations, the X_μ have the following expression

$$X_\mu = \begin{cases} -2b_\mu x_\mu + \sum_{k=0}^{n} c_k x_\mu^{2k} & \text{for } D \text{ even,} \\ -\frac{c}{x_\mu^2} - 2b_\mu + \sum_{k=1}^{n} c_k x_\mu^{2k} & \text{for } D \text{ odd.} \end{cases} \tag{3.208}$$

The metric can be changed to Lorentzian signature by doing $x_1 \to ix_1$. The constant b_1 will then be related to the mass M of the black hole and c_n is related to the cosmological constant.

The inverse of the metric is given by

$$\boldsymbol{g}^{-1} = \sum_{\mu=1}^{n} \left[\frac{X_\mu}{U_\mu} \partial_{x_\mu}^2 + \frac{U_\mu}{X_\mu} \Big(\sum_{k=0}^{n-1+\epsilon} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} \partial_{\psi_k} \Big)^2 \right] + \epsilon \frac{1}{cA^{(n)}} \partial_{\psi_n}^2 \ . \tag{3.209}$$

The determinant of the metric is

$$det(g) = (cA^{(n)})^\epsilon U^2 \ , \ U = \prod_{\substack{\mu,\nu=1 \\ \mu<\nu}}^{n} (x_\mu^2 - x_\nu^2) \ . \tag{3.210}$$

The expression of these metrics is shown in a convenient way. In these spacetimes, the principal tensor exists and the metric can be described in the special Darboux frame. By making a quick check with the metric shown above and (3.204), the frame covectors are

$$\boldsymbol{e}^\mu = \sqrt{\frac{U_\mu}{X_\mu}} dx_\mu \ , \ \hat{\boldsymbol{e}}^\mu = \sqrt{\frac{X_\mu}{U_\mu}} \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \ , \ \hat{\boldsymbol{e}}^0 = \sqrt{\frac{c}{A^{(n)}}} \sum_{j=0}^{n} A^{(j)} d\psi_j \ , \tag{3.211}$$

and the corresponding vectors are

$$\boldsymbol{e}_\mu = \sqrt{\frac{X_\mu}{U_\mu}} \partial_{x_\mu} \ , \ \hat{\boldsymbol{e}}_\mu = \sqrt{\frac{U_\mu}{X_\mu}} \sum_{k=0}^{n-1+\epsilon} \frac{(-x_\mu^2)^{n-1-k}}{U_\mu} \partial_{\psi_k} \ , \ \hat{\boldsymbol{e}}_0 = \frac{1}{\sqrt{cA^{(n)}}} \partial_{\psi_n} \ . \tag{3.212}$$

Finally, the expression for the principal tensor is then

$$\boldsymbol{h} = \sum_{\mu=1}^{n} x_\mu \boldsymbol{e}^\mu \wedge \hat{\boldsymbol{e}}^\mu = \sum_{\mu=1}^{n} \sum_{k=0}^{n-1} x_\mu A_\mu^{(k)} dx_\mu \wedge d\psi_k \ . \tag{3.213}$$

The object $\boldsymbol{\xi}$ in (3.205) is given by

$$\boldsymbol{\xi} = \sum_{\mu=1}^{n} \sqrt{\frac{X_\mu}{U_\mu}} \hat{\boldsymbol{e}}_\mu + \epsilon \sqrt{\frac{c}{A^{(n)}}} \hat{\boldsymbol{e}}_0 = \partial_{\psi_0} \ . \tag{3.214}$$

It can be checked immediately that this is a Killing vector field, since the metric does not depend on ψ_0. The four dimensional Kerr metric can be obtained from (3.206) by changing to the following coordinates

$$(\psi_0, x_1, x_2, \psi_1) = (t - a\phi, ir, a\cos\theta, \phi/a) \ , \tag{3.215}$$

and verifying that

$$X_1 = 2Mr - r^2 - a^2 \ , \ X_2 = -a^2 \sin^2\theta \ . \tag{3.216}$$

3.4.5 Killing Tower

The existence of the principal tensor allows for a construction of conserved quantities associated with Killing vectors, Killing tensors, conformal Killing tensors and Killing-Yano tensors. It all starts from a crucial property of a closed conformal Killing-Yano form which states

$$\boldsymbol{h}^{(j)} = \boldsymbol{h}^{\wedge j} = \underbrace{\boldsymbol{h} \wedge \boldsymbol{h} \wedge ... \wedge \boldsymbol{h}}_{j} \tag{3.217}$$

is also a closed conformal Killing-Yano form.

With these objects, it is possible to construct the conformal Killing tensors

$$Q^{ab}_{(j)} = \frac{1}{(2j-1)!} h^{(j)a}{}_{c_1 \dots c_{2j-1}} h^{(j)bc_1 \dots c_{2j-1}} \ , \tag{3.218}$$

$$Q^{ab} \equiv Q^{ab}_{(1)} = h^a{}_c h^{bc} \ , \tag{3.219}$$

the Killing-Yano tensors

$$\boldsymbol{f}^{(j)} = \star \boldsymbol{h}^{(j)} \ , \tag{3.220}$$

the Killing tensors

$$k^{ab}_{(j)} = \frac{1}{(D-2j-1)!} f^{(j)a}{}_{c_1 \dots c_{D-2j+1}} f^{(j)bc_1 \dots c_{D-2j+1}} \ , \tag{3.221}$$

and Killing vectors

$$l^a_{(j)} = k^{ab}_{(j)} \xi_b \ . \tag{3.222}$$

It can be showed, by using the definition, that

$$k^{ab}_{(j)} + Q^{ab}_{(j)} = A^{(j)} g^{ab} \quad , \quad Q^{ab}_{(j)} = Q^a{}_c k^{cb}_{(j-1)} \ , \tag{3.223}$$

where $A^{(j)} = \frac{1}{(2j)!} h^{(j)a_1 \dots a_{2j}} h^{(j)}_{a_1 \dots a_{2j}}$ is equivalent to (3.207).

The expressions for these Killing objects, in the Kerr-NUT-(A)dS spacetime, is given by

$$\boldsymbol{Q}_{(j)} = \sum_{\mu=1}^{n} x_\mu^2 A_\mu^{(j-1)} (\boldsymbol{e}_\mu \boldsymbol{e}_\mu + \hat{\boldsymbol{e}}_\mu \hat{\boldsymbol{e}}_\mu) \tag{3.224}$$

$$\boldsymbol{k}_{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)} (\boldsymbol{e}_\mu \boldsymbol{e}_\mu + \hat{\boldsymbol{e}}_\mu \hat{\boldsymbol{e}}_\mu) + \epsilon \ \hat{\boldsymbol{e}}_0 \hat{\boldsymbol{e}}_0 \ , \tag{3.225}$$

$$\boldsymbol{l}_{(j)} = \sum_{\mu=1}^{n} A_\mu^{(j)} \sqrt{\frac{X_\mu}{U_\mu}} \hat{\boldsymbol{e}}_\mu + \epsilon A^{(j)} \sqrt{\frac{c}{A^{(n)}}} \hat{\boldsymbol{e}}_0 = \partial_{\psi_j} \ . \tag{3.226}$$

It is possible to obtain the explicit dependence on the coordinates by using (3.211) and (3.212). It must be noted that all the Killing tensors $k_{(j)}$ and Killing vectors $l_{(j)}$ commute with each other in the sense of the Nijenhuis-Schouten bracket. This means these objects constitute the separability structure of the Kerr-NUT-(A)dS metric. The important remark is that the principal tensor generates all of these Killing objects associated with the symmetries of the spacetime.

It is possible to define a Killing tensor that depends on a parameter β

$$\boldsymbol{k}(\beta) = \sum_{j=0}^{n} k_{(j)} \beta^{2j} \ . \tag{3.227}$$

Using the relations (3.223), it is possible to find that

$$\boldsymbol{k}(\beta) = A(\beta) \boldsymbol{q}^{-1} \quad , \quad q_{ab} = g_{ab} + \beta^2 Q_{ab} \ , \tag{3.228}$$

$$A(\beta) = \sum_{j=0}^{n} A^{(j)} \beta^{2j} = \prod_{\nu=1}^{n} (1 + \beta^2 x_\nu^2) = \sqrt{\frac{det(\boldsymbol{q})}{det(\boldsymbol{g})}} \ . \tag{3.229}$$

Aditionally, the β dependent Killing vector can be defined

$$\boldsymbol{l}(\beta) = \boldsymbol{k}(\beta) \cdot \boldsymbol{\xi} = \sum_{j=0}^{n} l_{(j)} \beta^{2j} \ . \tag{3.230}$$

From now on, the indication of the dependence of β is dropped. In the Kerr-NUT-(A)dS spacetime, the Killing objects referred above have the following form

$$\boldsymbol{q} = \sum_{\mu=1}^{n} (1 + \beta x_\mu^2)(\boldsymbol{e}^\mu \boldsymbol{e}^\mu + \hat{\boldsymbol{e}}^\mu \hat{\boldsymbol{e}}^\mu) + \epsilon \ \hat{\boldsymbol{e}}^0 \hat{\boldsymbol{e}}^0 \ , \tag{3.231}$$

$$\boldsymbol{k} = \sum_{\mu=1}^{n} A_\mu (\boldsymbol{e}_\mu \boldsymbol{e}_\mu + \hat{\boldsymbol{e}}_\mu \hat{\boldsymbol{e}}_\mu) + \epsilon A \hat{\boldsymbol{e}}_0 \hat{\boldsymbol{e}}_0 \ , \tag{3.232}$$

$$\boldsymbol{l} = \sum_{j=0}^{n-1+\epsilon} \beta^{2j} \partial_{\psi_j} \ , \tag{3.233}$$

$$A_\mu = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{n} (1 + \beta^2 x_\nu^2) = \sum_{k=0}^{n} A_\mu^{(k)} \beta^{2j} \ . \tag{3.234}$$

There are additional properties of the principal tensor. One is related to the β-dependent Killing tensor

$$k^a{}_b h^b{}_c = h^a{}_b k^b{}_c \ , \tag{3.235}$$

which means the principal tensor commutes with the said Killing tensor. Also, from the integrability conditions (appendix D), it can be shown that

$$R^a{}_b h^b{}_c = h^a{}_b R^b{}_c \ , \tag{3.236}$$

where R_{ab} is the Ricci tensor. The covariant derivatives of the Killing objects of interest are

$$\nabla_c q_{ab} = 2\beta^2 \Big(g_{c(a} h_{b)d} - g_{d(a} h_{b)c} \Big) \xi^d \ , \tag{3.237}$$

$$\nabla_a A = 2\beta^2 \xi_c k^{cd} h_{ad} \ , \tag{3.238}$$

$$\nabla^d k^{ab} = \frac{2\beta^2}{A} \Big[k^{ab} k^{dn} h_n{}^m + k^{d(a} k^{b)e} h^m_{\ e} + k^{m(a} k^{b)c} h_c{}^d \Big] \xi_m \ , \tag{3.239}$$

$$\nabla_n k^{na} = \frac{\beta^2 \nabla_n A}{2} \Big(k^{na} - g^{na} \frac{k^c{}_c}{2} \Big) \ . \tag{3.240}$$

3.4.6 Proca field ansatz and equations

The Kerr-NUT-(A)dS spacetime possesses a set of symmetries that allow for the separation of the Hamilton-Jacobi equations. All these symmetries are generated only by one object, a closed conformal Killing-Yano 2-form called the principal tensor h_{ab}. A breakthrough was made in regards to the separation of the Proca equations [33] using these symmetries and the computation of the quasinormal modes of the Proca field particularly in the 4 dimensional Kerr spacetime was done [32]. Following Lunin's work [29], Frolov et al. [33] made an ansatz for the Proca field given by

$$A^a = B^{ab}\nabla_b Z \quad , \quad B^{ab}(g_{bc} - \beta h_{bc}) = \delta^a_c \ . \tag{3.241}$$

The notation in Ref. [32] is being followed. In Ref. [33], Frolov et al. use μ instead of the parameter β. They are related by the following

$$\beta = -i\mu \ . \tag{3.242}$$

Also, A^a with a latin index is referred to the Proca field, while A_μ with a greek index is referred to the β-dependent polynomial (3.234). The tensor B^{ab} is referred to as the polarization tensor and an explicit expression for it can be found by multiplying the second equation in (3.241) with its transpose and contracting subjacent indices

$$\begin{aligned}
&B^{ab}g_{bc}(B^T)^{cd}(g_{de} + \beta h_{de})(\delta^e{}_k - \beta h^e{}_k) = \delta^a{}_k \iff \\
&B^{ab}g_{bc}B^{dc}(g_{dk} - \beta^2 h_{de}h^e{}_k) = \delta^a{}_k \iff \\
&B^{ab}g_{bc}B^{dc}q_{dk} = \delta^a{}_k \iff \\
&B^{ab}g_{bc}B^{dc} = (q^{-1})^{ad} \iff \\
&B^{ab}g_{bc}B^{dc} = \frac{k^{ad}}{A} \iff \\
&B^{ab} = \frac{g^{am}}{A}(g_{mn} + \beta h_{mn})k^{nb} \ ,
\end{aligned} \tag{3.243}$$

where the definitions (3.228) were used from the second to the third line and from the fourth to the fifth line. In the last step, the inverse of B^T was used. In terms of Darboux frame, the polarization tensor $\boldsymbol{B}$ has the following expression

$$\boldsymbol{B} = \sum_{\mu=1}^{n} \frac{1}{1+\beta^2 x_\mu^2}\Big[\boldsymbol{e}_\mu\boldsymbol{e}_\mu + \hat{\boldsymbol{e}}_\mu\hat{\boldsymbol{e}}_\mu + x_\mu\beta(\boldsymbol{e}_\mu\hat{\boldsymbol{e}}_\mu - \hat{\boldsymbol{e}}_\mu\boldsymbol{e}_\mu)\Big] \ . \tag{3.244}$$

The covariant derivative of the polarization tensor and its contractions can be written as

$$\nabla_c B^{ab} = \beta\Big(B^a{}_c\xi_n B^{nb} - B_c{}^b B^{am}\xi_m\Big) \ , \tag{3.245}$$

$$\nabla_n B^{na} = \frac{\beta}{A}\Big(k^n{}_n\xi_m B^{ma} - k^{am}\xi_m\Big) \ , \tag{3.246}$$

$$\nabla_n B^{an} = \frac{\beta}{A}\Big(k^{ac}\xi_c - k^n{}_n B^{am}\xi_m\Big) \ . \tag{3.247}$$

Starting with the testing of the ansatz, the Lorentz condition becomes

$$
\begin{aligned}
\nabla_a A^a &= \nabla_a (B^{ab}\nabla_b Z) = \nabla_a\Big(\frac{k^{ab}}{A}\nabla_b Z\Big) + \beta\nabla_a\Big(\frac{h^a{}_n k^{nb}}{A}\Big)\nabla_b Z \\
&= \nabla_a\Big(\frac{k^{ab}}{A}\nabla_b Z\Big) + \frac{\beta}{A}\Big(\frac{k^n{}_n}{A} - 1\Big)\xi_m k^{mb}\nabla_b Z \\
&= \frac{1}{A}\nabla_a\Big(k^{ab}\nabla_b Z\Big) + \frac{1}{A}\Big(-\frac{k^{ab}\nabla_a A}{A} + \Big(\frac{k^n{}_n}{A} - 1\Big)l^b\Big)\nabla_b Z = 0\ ,
\end{aligned}
\tag{3.248}
$$

Considering that the spacetime has an even number of dimensions, it can be shown that

$$
\nabla_a(k^{ab}\nabla_b Z) = \sum_{\nu=1}^{n}\frac{A_\nu}{U_\nu}\left[\partial_{x_\nu}\Big(X_\nu\partial_{x_\nu}Z\Big) + \frac{1}{X_\nu}\Big(\sum_{k=0}^{n-1}(-x_\nu^2)^{n-1-k}\partial_{\psi_k}\Big)^2 Z\right]\ . \tag{3.249}
$$

Also from expressions (3.229), (3.232) and (3.233), it is possible to obtain

$$
\frac{k^{ab}(\nabla_a A)\nabla_b Z}{A^2} = \sum_{\nu=1}^{n}\frac{2\beta^2 x_\nu}{(1+\beta^2 x_\nu^2)^2}\frac{X_\nu}{U_\nu}\partial_{x_\nu}Z\ , \tag{3.250}
$$

$$
\frac{1}{A}\Big(\frac{k^n{}_n}{A} - 1\Big)l^b\nabla_b Z = \beta\sum_{\nu=1}^{n}\sum_{k=0}^{n-1}\beta^{2-2n+2k}\frac{1-\beta^2 x_\nu^2}{U_\nu(1+\beta^2 x_\nu^2)^2}\partial_{\psi_k}Z\ . \tag{3.251}
$$

Putting all these expressions in (3.248), the Lorentz condition has the following coordinate dependent form

$$
\nabla_a A^a = \sum_{\nu=1}^{n}\frac{1}{(1+\beta^2 x_\nu^2)U_\nu}\tilde{\mathcal{C}}_\nu Z = 0\ , \tag{3.252}
$$

$$
\begin{aligned}
\tilde{\mathcal{C}}_\nu Z = &\ (1+\beta^2 x_\nu^2)\partial_{x_\nu}\Big(\frac{X_\nu}{1+\beta^2 x_\nu^2}\partial_{x_\nu}Z\Big) + \frac{1}{X_\nu}\left(\sum_{k=0}^{n-1}(-x_\nu^2)^{n-1-k}\partial_{\psi_k}\right)^2 Z \\
&+ \beta\sum_{k=0}^{n-1}\beta^{2-2n+2k}\frac{1-\beta^2 x_\nu^2}{1+\beta^2 x_\nu^2}\partial_{\psi_k}Z\ .
\end{aligned}
\tag{3.253}
$$

The Lorentz condition can then be separated. The Proca equation (3.10), using the ansatz of the field, can be given by

$$
\begin{aligned}
&-\nabla_c\Big(g^{cd}\nabla_d A^a\Big) + m_A^2 A^a + R^a{}_d A^d = 0 \iff \\
&-\nabla_c\Big(g^{cd}\nabla_d(B^{ab}\nabla_b Z)\Big) + m_A^2(B^{ab}\nabla_b Z) + R^a{}_d A^d = 0\ .
\end{aligned}
\tag{3.254}
$$

In appendix E, it has been shown that

$$
\begin{aligned}
&-\nabla_c\Big(g^{cd}\nabla_d(B^{ab}\nabla_b Z)\Big) + R^a{}_d A^d = -B^{am}\nabla_m\Big(\nabla_c\nabla^c Z + 2\beta\xi_n B^{nb}\nabla_b Z\Big) \\
&+ 2\beta(\nabla_m A^m)B^{an}\xi_n\ .
\end{aligned}
\tag{3.255}
$$

The term with the divergence of A^a vanishes due to the Lorentz condition. This ansatz allows to decompose the Proca equations into an equation for the complex scalar field as it becomes

$$
B^{am}\nabla_m\Big(\nabla_c\nabla^c Z + 2\beta\xi_n B^{nb}\nabla_b Z - m_A^2 Z\Big) = 0 \iff \nabla_c\nabla^c Z + 2\beta\xi_n B^{nb}\nabla_b Z - m_A^2 Z = 0\ . \tag{3.256}
$$

The box operator can be shown to be

$$\nabla_c(g^{cd}\nabla_d Z) = \sum_{\nu=1}^{n} \frac{1}{U_\nu}\Big[\partial_{x_\nu}(X_\nu \partial_{x_\mu} Z) + \frac{1}{X_\nu}\Big(\sum_{k=0}^{n-1}(-x_\nu^2)^{n-1-k}\partial_{\psi_k}\Big)^2 Z\Big] \ . \tag{3.257}$$

Also, the term with the first derivatives in Z can be written as

$$2\beta\xi_k B^{kn}\nabla_n Z = -\sum_{\nu=1}^{n} \frac{2\beta^2 x_\nu^2}{1+\beta^2 x_\nu^2}\frac{X_\nu}{U_\nu}\partial_{x_\nu} Z + \beta\sum_{\nu=1}^{n}\sum_{k=0}^{n-1}\frac{\beta^{2-2n+2k}(1-\beta^2 x_\nu^2)}{U_\nu(1+\beta^2 x_\nu^2)}\partial_{\psi_k} Z \ . \tag{3.258}$$

Once again, putting these terms in the equation for the complex scalar Z (3.256), it becomes

$$\Big(\sum_{\nu=1}^{n}\frac{1}{U_\nu}\tilde{\mathcal{C}}_\nu Z\Big) - m_A^2 Z = 0 \ . \tag{3.259}$$

The ansatz for the scalar Z can be given as

$$Z = \prod_{\nu=1}^{n} R_\nu(x_\nu) exp\left(i\sum_{k=0}^{n-1} L_k\psi_k\right) \ . \tag{3.260}$$

The Lorenz condition and the Proca equation become

$$\sum_{\nu=1}^{n}\frac{1}{(1+\beta^2 x_\nu^2)U_\nu R_\nu}\tilde{\mathcal{C}}_\nu R_\nu = 0 \ , \tag{3.261}$$

$$\Big(\sum_{\nu=1}^{n}\frac{1}{U_\nu R_\nu}\tilde{\mathcal{C}}_\nu R_\nu\Big) - m_A^2 = 0 \ , \tag{3.262}$$

where the operators $\partial_{\psi_k} \to iL_k$ trivially due to the ansatz given. These equations do not show explicitly the eigenvalue problem. The following transformation of the operator $\tilde{\mathcal{C}}_\nu$ must be done to make it clearer

$$\tilde{\mathcal{C}}_\nu Z = \sum_{k=0}^{n-1}(-x_\nu^2)^{n-1-k}\hat{\mathcal{C}}_k Z \ , \tag{3.263}$$

where $\hat{\mathcal{C}}_k$ are the new operators. The eigenvalue problem can then be described by

$$\hat{\mathcal{C}}_k Z = C_k Z \ , \tag{3.264}$$

$$i\partial_{\psi_k} Z = -L_k Z \ , \tag{3.265}$$

where C_k are the separation constants. Now, the Lorenz condition and the Proca equation become

$$\Big(\sum_{\nu=1}^{n}\frac{1}{U_\nu}\sum_{k=0}^{n-1}(-x_\nu^2)^{n-1-k}C_k Z\Big) - m_A^2 Z = (C_0 - m_A^2)Z = 0 \ , \tag{3.266}$$

$$\sum_{\nu=1}^{n}\frac{1}{(1+\beta^2 x_\nu^2)U_\nu}\sum_{k=0}^{n-1}(-x_\nu^2)^{n-1-k}C_k Z = \frac{1}{A}\sum_{k=0}^{n-1} C_k\beta^{2k} Z = 0 \ , \tag{3.267}$$

where it has been used that

$$\sum_{\nu=1}^{n} \frac{(-x_\nu^2)^{n-1-k}}{U_\nu} = \delta_0^k \ , \tag{3.268}$$

$$A_\mu = \sum_{k=0}^{n-1} A_\mu^{(k)} \beta^{2k} \ , \tag{3.269}$$

$$\sum_{\nu=1}^{n} \frac{A_\mu^{(j)}(-x_\nu^2)^{n-1-k}}{U_\nu} = \delta_k^j \ . \tag{3.270}$$

Equations (3.266) and (3.267) set the value of $C_0 = m_A^2$ and restrict the possible values of β to be the solutions of

$$\sum_{k=0}^{n-1} C_k \beta^{2k} = 0 \ , \tag{3.271}$$

thus concluding the analysis of the separation of the Proca equations.

3.4.7 Proca Field Equations in Kerr-AdS spacetime

The family of Kerr-NUT-(A)dS on-shell metrics in $D = 4$ has the following expression

$$ds^2 = \frac{x_2^2 - x_1^2}{X_1} dx_1^2 + \frac{x_1^2 - x_2^2}{X_2} dx_2^2 + \frac{X_1}{(x_2^2 - x_1^2)} \Big(d\psi_0 + x_1^2 d\psi_1\Big)^2 + \frac{X_2}{(x_1^2 - x_2^2)} \Big(d\psi_0 + x_2^2 d\psi_1\Big)^2 \ , \tag{3.272}$$

where

$$X_1 = -2b_1 x_1 + c_0 + c_1 x_1^2 + c_2 x_1^4 \ , \tag{3.273}$$

$$X_2 = -2b_2 x_2 + c_0 + c_1 x_2^2 + c_2 x_2^4 \tag{3.274}$$

This can be seen by looking at (3.206) and set $n = 2$ and $\epsilon = 0$. One of the spacetimes that can be described by this metric is the Kerr-AdS. The Kerr-AdS spacetime metric describes the geometry containing a rotating black hole and the presence of a negative cosmological constant Λ. For convenience, it is going to be used $R_\Lambda = \frac{3}{|\Lambda|}$ instead of $|\Lambda|$, like in the previous section. The correspondence between Kerr-NUT-(A)dS and Kerr-AdS is

$$b_1 = iM \ , \ b_2 = 0 \ \ c_0 = a^2 \ , \ c_1 = 1 + \frac{a^2}{R_\Lambda^2} \ , \ c_2 = -\frac{1}{R_\Lambda^2} \ , \tag{3.275}$$

and

$$(\psi_0, x_1, x_2, \psi_1) = \left(t - a\phi, ir, a\cos\theta, \frac{\phi}{a}\right) \ . \tag{3.276}$$

Since the metric presented in (3.272) has Riemannian signature, one needs to perform a Wick rotation to turn it into Lorentzian signature $(-, +, +, +)$. This is the reason for the presence of the imaginary unit in the expressions above. The metric with this settings is given by (2.5). The functions X_1 and X_2 are

$$X_1 = -\Delta_\Lambda \ , \tag{3.277}$$

$$X_2 = -a^2 \sin^2\theta \Delta_\theta \ . \tag{3.278}$$

The Proca field equations using the FKKS ansatz are

$$\tilde{\mathcal{C}}_1 Z = -C_0 x_1^2 Z + C_1 Z \quad , \quad \tilde{\mathcal{C}}_2 Z = -C_0 x_2^2 Z + C_1 Z \ , \tag{3.279}$$

$$C_0 = m_A^2 \quad , \quad C_1 = \beta^{-2} C_0 = \beta^{-2} m_A^2 \ , \tag{3.280}$$

$$\partial_{\psi_0} Z = i L_0 Z \quad , \quad \partial_{\psi_1} Z = i L_1 Z \ , \tag{3.281}$$

The ansatz for the complex scalar Z can be given by

$$Z = R(x_1) R(x_2) exp\Big(i L_0 \psi_0 + i L_1 \psi_1 \Big) = R(r) S(\theta) exp\Big(-i\omega t + i m_\phi \phi \Big) \ , \tag{3.282}$$

The correspondence between constants are

$$L_0 = -\omega \quad , \quad L_1 = a(m_\phi - \omega a) \ . \tag{3.283}$$

Using the coordinate transformation (3.276) in (3.253) and using the above correspondence for the constants, the equations for $R(r)$ and $S(\theta)$ are

$$\partial_r \left[\frac{\Delta_\Lambda}{q_r} \partial_r R(r) \right] + \left[\frac{K_r^2}{q_r \Delta_\Lambda} + i \frac{2 - q_r}{q_r^2 \beta} \sigma + \frac{m_A^2}{\beta^2} \right] R(r) = 0 \ , \tag{3.284}$$

$$\frac{1}{\sin\theta} \partial_\theta \left[\frac{q_\Lambda \sin\theta}{q_\theta} \partial_\theta S(\theta) \right] - \left[\frac{K_\theta^2}{q_\theta q_\Lambda \sin^2\theta} + i \frac{2 - q_\theta}{q_\theta^2 \beta} \sigma + \frac{m_A^2}{\beta^2} \right] S(\theta) = 0 \ , \tag{3.285}$$

where

$$q_\Lambda = 1 - \frac{a^2}{R_\Lambda^2} \cos^2\theta \quad , \quad q_r = 1 - \beta^2 r^2 \quad , \quad q_\theta = 1 + \beta^2 a^2 \cos^2\theta \ , \tag{3.286}$$

$$K_r = a m_\phi - (a^2 + r^2)\omega \quad , \quad K_\theta = m_\phi - a\omega \sin^2\theta \quad , \quad \sigma = a\beta^2 (m_\phi - \omega a) - \omega \quad . \tag{3.287}$$

With an expression for the scalar Z, it is possible to obtain the corresponding Proca field by (3.241), where the polarization tensor is given by

$$\boldsymbol{B} = \boldsymbol{B}_{\text{sym}} + \boldsymbol{B}_{\text{anti}} \ , \tag{3.288}$$

$$\boldsymbol{B}_{\text{sym}} = \frac{\Delta_\Lambda}{q_r \Sigma} \partial_r^2 + \frac{q_\Lambda}{q_\theta \Sigma} \partial_\theta^2 - \frac{1}{q_r \Delta_\Lambda \Sigma} \Big[(r^2 + a^2)\partial_t + a\partial_\phi \Big]^2 + \frac{1}{\Sigma q_\theta q_\Lambda \sin^2\theta} \Big[a \sin^2\theta \partial_t + \partial_\phi \Big]^2 \ , \tag{3.289}$$

$$\begin{aligned} \boldsymbol{B}_{\text{anti}} = & \frac{\beta r}{q_r \Sigma} \Big[(r^2 + a^2)(\partial_r \partial_t - \partial_t \partial_r) + a(\partial_r \partial_\phi - \partial_\phi \partial_r) \Big] \\ & + \frac{\beta a \cos\theta}{\Sigma q_\theta} \Big[a \sin\theta (\partial_t \partial_\theta - \partial_\theta \partial_t) + \frac{1}{\sin\theta} (\partial_\phi \partial_\theta - \partial_\theta \partial_\phi) \Big] \ , \end{aligned} \tag{3.290}$$

where $\boldsymbol{B}_{\text{sym}}$ and $\boldsymbol{B}_{\text{anti}}$ are the symmetric and the anti-symmetric part of $\boldsymbol{B}$, respectively.

3.4.7.1 Asymptotic solutions

The equation for the complex scalar that generates the Proca field (3.284) can be written in a Schrodinger-like equation given by

$$\partial^2_{r_*} U + \frac{1}{(r^2+a^2)^2}\left[K_r^2 + i\frac{2-q_r}{q_r\beta}\sigma\Delta_\Lambda + \frac{m_A^2}{\beta^2}\Delta_\Lambda q_r\right]U - (1+a^2\beta^2)\mathcal{V}U = 0 \ , \tag{3.291}$$

$$\mathcal{V} = \left[\frac{\Delta_\Lambda^2}{q_r(r^2+a^2)^3} + \frac{2\Delta_\Lambda r\Big(r - M + \frac{1}{R_\Lambda^2}(2r^3+a^2r)\Big)}{q_r(r^2+a^2)^3} + \frac{3\Delta_\Lambda^2 r^2\beta^2}{q_r^2(r^2+a^2)^3} - \frac{3\Delta_\Lambda^2 r^2}{q_r(r^2+a^2)^4}\right] \ , \tag{3.292}$$

where the scalar U has the following relation with R

$$R(r) = \sqrt{\frac{q_r}{r^2+a^2}}U(r) \ , \tag{3.293}$$

and the radial tortoise coordinate in Kerr-dS is given by the integration of

$$\frac{dr_*}{dr} = \frac{r^2+a^2}{\Delta_\Lambda} \ . \tag{3.294}$$

In Kerr-AdS spacetime, there are two boundaries that must be consider which is the event horizon and infinity. The event horizon is located at r_+, the greatest positive root of $\Delta_\Lambda = 0$.

The asymptotic solution of (3.284) at $r \to r_+$ can be easily found by looking at (3.291) and set $\Delta_\Lambda = 0$. The equation at this limit is given by

$$\partial^2_{r_*} U + \Big(\omega - m_\phi\Omega_\Lambda\Big)^2 U = 0 \ \ , \ \ \Omega_\Lambda = \frac{a}{r_+^2+a^2} \ . \tag{3.295}$$

The solution of this equation is given by

$$R(r) \to \mathcal{T}e^{-ik_H r_*} + \mathcal{O}e^{ik_H r_*} \ , \tag{3.296}$$

where $k_H = \omega - m_\phi\Omega_\Lambda$.

Now, for the asymptotic solution at $r \to +\infty$, it is better to start from (3.284). The strategy is to expand the derivative term and eliminate higher powers in r^{-1}. For the term with the first derivative of R, only the smallest power in r^{-1} is taken into account. For the other terms proportional to R, they are accounted for until r^{-4}. The equation with this conditions turns into

$$\partial_r^2 R(r) + \frac{2}{r}\partial_r R(r) + \Big[R_\Lambda^4\omega^2 - i\frac{R_\Lambda\sigma}{\beta} + \frac{R_\Lambda^2 m_A^2}{\beta^2}\Big]\frac{R(r)}{r^4} - \frac{R_\Lambda^2 m_A^2}{r^2}R(r) = 0 \ . \tag{3.297}$$

Multiplying by r^2, the equation has a more pleasing expression given by

$$r^2\partial_r^2 R(r) + 2r\partial_r R(r) + \Big[R_\Lambda^4\omega^2 - i\frac{R_\Lambda^2\sigma}{\beta} + \frac{R_\Lambda^2 m_A^2}{\beta^2}\Big]\frac{R(r)}{r^2} - R_\Lambda^2 m_A^2 R(r) = 0 \ . \tag{3.298}$$

The next step is to make the change of variables $x = \frac{R_\Lambda^2}{r}$. It follows

$$\partial_x^2 R(x) + \Big[R_\Lambda^2\omega^2 - i\frac{\sigma}{\beta} + \frac{m_A^2}{\beta^2}\Big]R - \frac{R_\Lambda^2 m_A^2}{x^2}R = 0 \ . \tag{3.299}$$

Finally, the last step is making the following transformation

$$R = \sqrt{x}\Phi \ , \tag{3.300}$$

and multiply by $x^{\frac{3}{2}}$. The equation turns into

$$x^2\partial_x^2\Phi + x\partial_x\Phi + \left[(qx)^2 - \left(\frac{1}{2}\sqrt{1+4R_\Lambda^2 m_A^2}\right)^2\right]\Phi = 0 \ , \tag{3.301}$$

$$q = \sqrt{\omega^2 + \frac{m_A^2}{R_\Lambda^2\beta^2} - i\frac{a}{R_\Lambda^2}\beta(m_\phi - \omega a) + i\frac{\omega}{R_\Lambda^2\beta}} \ . \tag{3.302}$$

The equation (3.301) can be identified as the Bessel's differential equation. This means automatically that the asymptotic solution of R at $r \to \infty$ is given by

$$R(r) \to \frac{\mathcal{A}R_\Lambda}{\sqrt{r}} J\left[\frac{1}{2}\sqrt{1+4R_\Lambda^2 m_A^2}; \frac{qR_\Lambda^2}{r}\right] + \frac{\mathcal{B}R_\Lambda}{\sqrt{r}} Y\left[\frac{1}{2}\sqrt{1+4R_\Lambda^2 m_A^2}; \frac{qR_\Lambda^2}{r}\right] \ , \tag{3.303}$$

where $J[\alpha; x]$ and $Y[\alpha; x]$ are the Bessel functions of First and Second Kind, respectively, of order α.

3.4.8 The Schwarzschild-AdS limit

The FKKS ansatz separates the Proca equations in Kerr-AdS geometry, as shown above. Since Schwarzschild-AdS can be obtained by doing the limit $a \to 0$ from Kerr-AdS, this subsection is dedicated to verify if the same statement can be said for Schwarzschild-AdS.

The equations for the complex scalar $R(r)$ and $S(\theta)$ in this limit are given by

$$\partial_r\Big[\frac{r^2 f}{q_r}\partial_r R\Big] + \Big[\frac{\omega^2 r^2}{f q_r} - i\omega\frac{2-q_r}{q_r^2\beta} + \frac{m_A^2}{\beta^2}\Big]R = 0 \ , \tag{3.304}$$

$$\frac{1}{\sin\theta}\partial_\theta\Big[\sin\theta\partial_\theta S\Big] - \frac{m_\phi^2}{\sin^2\theta}S + \Big[i\frac{\omega}{\beta} - \frac{m_A^2}{\beta^2}\Big]S = 0 \ , \tag{3.305}$$

where f is given by (3.46). The angular equation (3.305) can be identified as the spherical harmonics equation with

$$i\frac{\omega}{\beta} - \frac{m_A^2}{\beta^2} = \ell(\ell+1) \ . \tag{3.306}$$

Thus, there are two different values for β for each value of $\ell > 0$: β_+ and β_- given by

$$\beta_\pm = i\omega\frac{1 \pm \sqrt{1 + \frac{4m_A^2\ell(\ell+1)}{\omega^2}}}{2\ell(\ell+1)} \ . \tag{3.307}$$

For the monopole case ($\ell = 0$), the parameter β is given by

$$\beta_{\text{mono}} = -i\frac{m_A^2}{\omega} \ . \tag{3.308}$$

These two different β's correspond to a polarization. A further quick analysis on the expression suggests that β_- describes the longitudinal polarization, since setting $m_A = 0$ makes it vanish.

After making the limit $a \to 0$, the tensor B^{ab} given in (3.288)-(3.290) becomes

$$\boldsymbol{B}_{\text{sym}} = -\frac{1}{q_r f}\partial_t^2 + \frac{f}{q_r}\partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 \ , \tag{3.309}$$

$$\boldsymbol{B}_{\text{anti}} = \frac{\beta r}{q_r}(\partial_r\partial_t - \partial_t\partial_r) \ . \tag{3.310}$$

Thus, it is possible to obtain the expression for the covariant components of the massive vector field in function of the scalar $R(r)$ and the spherical harmonics $Y(\theta,\phi)$

$$A_a = \Big(-\frac{i\omega}{q_r} + \frac{\beta r f}{q_r}\partial_r, \frac{1}{q_r}\partial_r - i\frac{\omega\beta r}{q_r f}, \partial_\theta, \partial_\phi \Big) R(r)Y(\theta,\phi) \ . \tag{3.311}$$

Comparing (3.311) with (3.21), it can be seen that the FKKS ansatz in Schwarzschild-AdS limit ($a \to 0$) does not describe the magnetic polarization ($u_{(4)}$ in (3.21)). A possible cause for this might be found by looking at the definition of the principal tensor (3.213), where one of its eigenvalues is given by $x_2 = a\cos\theta$. By doing the Schwarzschild-AdS limit, the eigenvalue goes to zero and so the principal tensor is degenerate, which violates one of the necessary conditions for the separation of equations using this object.

3.4.8.1 An analytical verification of FKKS ansatz as a solution of the Proca equations

In principle, the FKKS ansatz must describe the electric modes, in Schwarzschild-AdS. The most important requirement is that the ansatz obeys the system of equations (3.26)-(3.29), as well as the Lorenz condition (3.31). Firstly, it is important to note the following correspondence from comparing (3.21) and (3.311)

$$u_{(1)} = -\frac{i\omega r}{q_r}R(r) + \frac{\beta r^2 f}{q_r}\partial_r R(r) \ , \tag{3.312}$$

$$u_{(2)} = \frac{rf}{q_r}\partial_r R(r) - i\frac{\omega\beta r^2}{q_r}R(r) \ , \tag{3.313}$$

$$u_{(3)} = \ell(\ell+1)R(r) \ . \tag{3.314}$$

Using the correspondence above and equation (3.304), the objective of this section is to get the equivalent equations of (3.28), (3.27) and (3.31) as these describe the dynamical evolution of the electric modes, that contain a longitudinal and a transversal polarization.

First, one can rearrange equation (3.304) in the following manner

$$\frac{r^2}{q_r}\partial_r(f\partial_r R) + \partial_r\Big(\frac{r^2}{q_r}\Big)f\partial_r R + \Big[\frac{\omega^2 r^2}{f q_r} - i\omega\frac{2-q_r}{q_r^2\beta} + \frac{m_A^2}{\beta^2}\Big]R = 0 \ . \tag{3.315}$$

Multiplying by $f\frac{q_r}{r^2}$ and using the definition of r_* coordinate ($f\partial_r = \partial_{r_*}$), one obtains

$$\partial_{r_*}^2 R + \frac{q_r f}{r^2}\partial_r\Big(\frac{r^2}{q_r}\Big)\partial_{r_*}R + \Big[\omega^2 - \frac{2-q_r}{q_r r^2\beta}i\omega f + \frac{q_r m_A^2 f}{\beta^2 r^2}\Big]R = 0 \ . \tag{3.316}$$

Expanding it further and taking into account that $2 - q_r = q_r + 2\beta^2 r^2$, the equation becomes

$$\omega^2 R + \partial_{r_*}^2 R - f\Big[m_A^2 + \frac{i\omega}{\beta r^2} - \frac{m_A^2}{\beta^2 r^2}\Big]R + \frac{2f}{rq_r}\partial_{r_*}R - \frac{2\beta f}{q_r}i\omega R = 0 \ . \tag{3.317}$$

Using the fact that β needs to obey (3.306) and multiplying by $\ell(\ell+1)$, it yields

$$\Big(\omega^2+\partial_{r_*}^2-f\Big[m_A^2+\frac{\ell(\ell+1)}{r^2}\Big]\Big)\ell(\ell+1)R+\frac{2f\ell(\ell+1)}{r^2}\Big[\frac{rf}{q_r}\partial_r R-\frac{i\omega\beta r^2}{q_r}R\Big]=0\ . \tag{3.318}$$

Finally, one can use the correspondence (3.313) and (3.314) to see that this is equivalent to (3.28).

The Lorenz condition (3.31) is also obeyed by the FKKS ansatz. This can be showed by substituting $u_{(1)}$, $u_{(2)}$ and $u_{(3)}$ by the correspondence (3.312)-(3.314) in the referred equation. It yields that

$$r^2\nabla_a A^a=r\partial_r\Big[\frac{fr}{q_r}\partial_r R-\frac{i\beta\omega r^2}{q_r}R\Big]+\frac{\omega^2 r^2}{fq_r}R+\frac{i\omega\beta r^3}{q_r}\partial_r R-\ell(\ell+1)R+\frac{fr}{q_r}\partial_r R-\frac{i\omega\beta r^2}{q_r}R\ , \tag{3.319}$$

where the replacement $\partial_t\to -i\omega$ has been made. One possible manipulation to simplify this equation is to notice that

$$r\partial_r\Big[\frac{fr}{q_r}\partial_r R-\frac{i\beta\omega r^2}{q_r}R\Big]=\partial_r\Big[\frac{fr^2}{q_r}\partial_r R\Big]-i\omega\beta r\partial_r\Big(\frac{r^2}{q_r}\Big)R-\frac{i\omega\beta r^3}{q_r}\partial_r R-\frac{fr}{q_r}\partial_r R\ . \tag{3.320}$$

Since these last two terms can be found in (3.319) with opposite sign, the referred equation simplifies into

$$r^2\nabla_a A^a=\partial_r\Big[\frac{fr^2}{q_r}\partial_r R\Big]-i\omega\beta r\partial_r\Big(\frac{r^2}{q_r}\Big)R+\frac{\omega^2 r^2}{fq_r}R-\ell(\ell+1)R-\frac{i\omega\beta r^2}{q_r}R\ . \tag{3.321}$$

The equation (3.304) can be used to get rid of the derivative term and the equation (3.306) can also be used to put ℓ in terms of β and ω. Equation (3.321) becomes

$$\begin{aligned}r^2\nabla_a A^a&=-\Big[\frac{\omega^2 r^2}{fq_r}-i\omega\frac{2-q_r}{q_r^2\beta}+\frac{m_A^2}{\beta^2}\Big]R-i\omega\beta r\partial_r\Big(\frac{r^2}{q_r}\Big)R+\frac{\omega^2 r^2}{fq_r}R-i\frac{\omega}{\beta}R+\frac{m_A^2}{\beta^2}R-\frac{i\omega\beta r^2}{q_r}R\\&=i\omega\frac{2-q_r}{q_r^2\beta}R-i\omega\beta r\partial_r\Big(\frac{r^2}{q_r}\Big)R-i\frac{\omega}{\beta}R-\frac{i\omega\beta r^2}{q_r}R\\&=i\omega\frac{2-q_r-2r^2\beta^2-q_r^2-\beta^2r^2q_r}{q_r^2\beta}R=i\omega\frac{1-2\beta^2r^2+\beta^4r^4-q_r^2}{q_r^2\beta}R=0\ .\end{aligned} \tag{3.322}$$

Thus, the Lorenz condition is satisfied.

Finally, the equation (3.27) can also be obtained from (3.311) and (3.304). One can look to (3.27) and notice that the terms of the derivative of u_2 can be simplified into

$$\partial_{r_*}^2u_{(2)}-(\partial_r f)\partial_{r_*}u_{(2)}=f^2\partial_r^2u_{(2)}\ . \tag{3.323}$$

Thus, calculating $f^2\partial_r^2u_{(2)}$ seems to be a good starting point. Using (3.311), it follows that

$$\begin{aligned}f^2\partial_r^2u_{(2)}&=f^2\partial_r\Big[\partial_r\Big(\frac{fr}{q_r}\partial_r R\Big)-i\omega\beta\partial_r\Big(\frac{r^2}{q_r}R\Big)\Big]\\&=f^2\partial_r\Big[\frac{1}{r}\partial_r\Big(\frac{fr^2}{q_r}\partial_r R\Big)-\frac{f}{q_r}\partial_r R\Big]-i\omega\beta f^2\Big[\partial_r^2\Big(\frac{r^2}{q_r}\Big)R+\partial_r\Big(\frac{r^2}{q_r}\Big)\partial_r R+\partial_r\Big(\frac{r^2}{q_r}\partial_r R\Big)\Big]\ ,\end{aligned} \tag{3.324}$$

The idea is to simplify every term that resembles $\partial_r\Big(\frac{fr^2}{q_r}\partial_r R\Big)$, so it can be substituted by (3.304). The second term inside the derivative and the last term can be put in this form. Thus, expanding the derivative in the first and second term, and doing the manipulation so that $\partial_r\Big(\frac{fr^2}{q_r}\partial_r R\Big)$ appears, one obtains

$$
\begin{aligned}
f^2\partial_r^2 u_{(2)} &= -\frac{2f^2}{r^2}\partial_r\Big(\frac{fr^2}{q_r}\partial_r R\Big) + \frac{f^2}{r}\partial_r^2\Big(\frac{fr^2}{q_r}\partial_r R\Big) + \frac{2f^3}{q_r r}\partial_r R \\
&- i\omega\beta\Big[f^2\partial_r^2\Big(\frac{r^2}{q_r}\Big)R + f^2\partial_r\Big(\frac{r^2}{q_r}\Big)\partial_r R + f\partial_r\Big(\frac{fr^2}{q_r}\partial_r R\Big) - f(\partial_r f)\frac{r^2}{q_r}\partial_r R\Big] \ . \qquad (3.325)
\end{aligned}
$$

In the above expression, the third and the last term of the right hand side (RHS) are already desired terms. Those can be put in the left hand side (LHS). The second and the sixth term can be developed using the equation (3.304) as

$$
\begin{aligned}
&\frac{f^2}{r}\partial_r^2\Big(\frac{fr^2}{q_r}\partial_r R\Big) - i\omega\beta f\partial_r\Big(\frac{fr^2}{q_r}\partial_r R\Big) = \Big(\frac{\omega^2 r^2}{fq_r} - i\omega\frac{2-q_r}{q_r^2\beta} + \frac{m_A^2}{\beta^2}\Big)\Big(-\frac{f^2}{r}\partial_r R + i\omega\beta fR\Big) \\
&- \frac{f^2}{r}R\partial_r\Big(\frac{\omega^2 r^2}{fq_r} - i\omega\frac{2-q_r}{q_r^2\beta}\Big) \ . \qquad (3.326)
\end{aligned}
$$

It is possible to simplify (3.326) using that

$$
\Big(\frac{\omega^2 r^2}{fq_r} - i\omega\frac{2-q_r}{q_r^2\beta} + \frac{m_A^2}{\beta^2}\Big) = \frac{r^2}{fq_r}\Big(\omega^2 - V(r)\Big) - i\omega\beta\frac{2r^2}{q_r^2} \ , \qquad (3.327)
$$

where $V(r) = f\Big[\frac{\ell(\ell+1)}{r^2} + m_A^2\Big]$, obtaining

$$
\begin{aligned}
&\frac{f^2}{r}\partial_r^2\Big(\frac{fr^2}{q_r}\partial_r R\Big) - i\omega\beta f\partial_r\Big(\frac{fr^2}{q_r}\partial_r R\Big) = -\Big[\omega^2 - V(r)\Big]u_{(2)} + 2i\omega\beta\frac{f^2 r}{q_r^2}\partial_r R + 2\omega^2\beta^2\frac{fr^2}{q_r^2}R \\
&- \frac{f^2}{r}R\partial_r\Big(\frac{\omega^2 r^2}{fq_r} - i\omega\frac{2-q_r}{q_r^2\beta}\Big) \ . \qquad (3.328)
\end{aligned}
$$

By making the substitution of the terms given by (3.328) into (3.325), the equation becomes

$$
\begin{aligned}
&\hat{\mathcal{D}}u_{(2)} - (\partial_r f)\Big(\partial_{r_*}u_{(2)} + i\omega\beta\frac{r^2}{q_r}\partial_{r_*}R\Big) - \frac{2f^3}{q_r r}\partial_r R \\
&= -\frac{2f^2}{r^2}\partial_r\Big(\frac{fr^2}{q_r}\partial_r R\Big) - i\omega\beta f^2\Big[\partial_r^2\Big(\frac{r^2}{q_r}\Big)R + \partial_r\Big(\frac{r^2}{q_r}\Big)\partial_r R\Big] + 2i\omega\beta\frac{f^2 r}{q_r^2}\partial_r R + 2\omega^2\beta^2\frac{fr^2}{q_r^2}R \\
&- \frac{f^2}{r}R\partial_r\Big(\frac{\omega^2 r^2}{fq_r} - i\omega\frac{2-q_r}{q_r^2\beta}\Big) \ . \qquad (3.329)
\end{aligned}
$$

Since $\partial_r\Big(\frac{r^2}{q_r}\Big) = \frac{2r}{q_r^2}$, the third term and the fourth term in the RHS cancel each other. Simplifying more the equation, one gets

$$
\begin{aligned}
&\hat{\mathcal{D}}u_{(2)} - (\partial_r f)\Big(\partial_{r_*}u_{(2)} + i\omega\beta\frac{r^2}{q_r}\partial_{r_*}R\Big) - \frac{2f^3}{q_r r}\partial_r R \\
&= \frac{2f}{q_r}(\omega^2 - V(r))R - 4i\omega\beta\frac{f^2}{q_r^2}R - i\omega\beta f^2\partial_r^2\Big(\frac{r^2}{q_r}\Big)R + 2\omega^2\beta^2\frac{fr^2}{q_r^2}R \\
&- \frac{f^2}{r}R\partial_r\Big(\frac{\omega^2 r^2}{fq_r} - i\omega\frac{2-q_r}{q_r^2\beta}\Big) \ . \qquad (3.330)
\end{aligned}
$$

Expanding the second to last term in the RHS, three terms arise. Two of them will cancel with the first term and fifth term in the RHS and the other can be put into the LHS. Hence,

$$\begin{aligned}&\hat{\mathcal{D}}u_{(2)} - (\partial_r f)\Big(\partial_{r_*}u_{(2)} + i\omega\beta\frac{r^2}{q_r}\partial_{r_*}R + \frac{\omega^2 r}{q_r}R\Big) - \frac{2f^3}{q_r r}\partial_r R\\ &= -\frac{2f}{q_r}V(r)R - 4i\omega\beta\frac{f^2}{q_r^2}R - i\omega\beta f^2\partial_r^2\Big(\frac{r^2}{q_r}\Big)R + i\omega\frac{f^2}{r\beta}R\partial_r\Big(\frac{2-q_r}{q_r^2}\Big)\ .\end{aligned} \tag{3.331}$$

Fortunately, it can be shown that the last three terms in the RHS cancel each other. Also, in the LHS, the two last terms proportional to $\partial_r f$ can be identified as $-\dot{u}_{(1)}$. The equation becomes

$$\hat{\mathcal{D}}u_{(2)} + (\partial_r f)\Big(\dot{u}_{(1)} - \partial_{r_*}u_{(2)}\Big) - \frac{2f^3}{q_r r}\partial_r R = -\frac{2f}{q_r}V(r)R\ . \tag{3.332}$$

The final term in the RHS can be manipulated in the convenient way

$$\begin{aligned}&-\frac{2f}{q_r}V(r)R = -\frac{2f^2}{r^2 q_r}\ell(\ell+1)R - \frac{2f^2}{q_r}m_A^2 R + i\omega\beta\frac{2f^2}{q_r}R - i\omega\beta\frac{2f^2}{q_r}R\\ &= -\frac{2f^2}{r^2}\ell(\ell+1)R - \frac{2f^2}{r^2}\Big(\frac{i\omega\beta r^2 R}{q_r}\Big)\ .\end{aligned} \tag{3.333}$$

Finally, substituting into (3.332) and putting every term into the LHS, one obtains

$$\hat{\mathcal{D}}u_{(2)} + (\partial_r f)\Big(\dot{u}_{(1)} - \partial_{r_*}u_{(2)}\Big) + \frac{2f^2}{r^2}\Big(\ell(\ell+1)R - \frac{rf}{q_r}\partial_r R - \frac{i\omega\beta r^2}{q_r}R\Big) = 0\ . \tag{3.334}$$

This equation is the same as (3.27), by identifying the last three terms inside the parenthesis as $u_{(3)} - u_{(2)}$.

With these three equations proven, it seems that the FKKS ansatz is a solution for the dynamic system of equations and describes the electric polarizations in Schwarzschild-anti-dS.

3.4.8.2 Asymptotic solution

The asymptotic solution of $R(r)$ of Schwarzschild-AdS can be obtained by solving (3.301) in the limit $a \to 0$. The solution in Schwarzschild-AdS limit is given by

$$R(r) \to \frac{\mathcal{A}}{\sqrt{r}}J\left[\frac{1}{2}\sqrt{1+4R_\Lambda^2 m_A^2};\frac{kR_\Lambda^2}{r}\right] + \frac{\mathcal{B}}{\sqrt{r}}Y\left[\frac{1}{2}\sqrt{1+4R_\Lambda^2 m_A^2};\frac{kR_\Lambda^2}{r}\right]\ , \tag{3.335}$$

$$k = \sqrt{\omega^2 + \frac{m_A^2}{\beta^2 R_\Lambda^2} + i\frac{\omega}{\beta R_\Lambda^2}}\ . \tag{3.336}$$

Using (3.307), one obtains the momentum for each polarization

$$k_\pm = \sqrt{\omega^2\left[1 + \frac{-1 \pm \sqrt{1 + \frac{4m_A^2\ell(\ell+1)}{\omega^2}}}{R_\Lambda^2 m_A^2}\right] - \frac{\ell(\ell+1)}{R_\Lambda^2}}\ . \tag{3.337}$$

It seems that $k_\pm$ corresponds to the k_2 and k_3 (in section 3.2.1) only in the case of $\omega = m_A$. Nevertheless, the leading order expansion in $\frac{kR_\Lambda^2}{r}$ around zero will have the same behaviour in both treatments, since k can be swallowed by the constant $\mathcal{A}$.

3.4.8.3 Method for calculating quasinormal modes

In this subsection, the quasinormal modes of the FKKS ansatz in Schwarzschild-AdS will be calculated to conclude its analysis.

The equation (3.318) can be solved by a series expansion given by

$$R = e^{-i\omega r_*} \sum_{n=0}^{\infty} a_n (x - x_+)^n \ . \tag{3.338}$$

where $x = \frac{1}{r}$ and $x_+ = \frac{1}{r_+}$. Substituting this solution into (3.318), the equation becomes

$$e^{-i\omega r_*} \frac{r^2}{q_r} \left[(x - x_+) s(x) \partial_x^2 U_R + t(x) \partial_x U_R + \frac{u(x)}{x - x_+} U_R \right] = 0 \ , \tag{3.339}$$

where $U_R = \sum_{n=0}^{\infty} a_n (x - x_+)^n$. The polynomials in this case are given by

$$s(x) = x^2 \frac{f}{r^2} \frac{(x^2 - \beta^2)}{x - x_+} \ , \tag{3.340}$$

$$t(x) = (x^2 - \beta^2)(2i\omega x^2 + 2x^3 - 6Mx^4) - 2x^3 \frac{f}{r^2}, \tag{3.341}$$

$$u(x) = (x - x_+) \Big[(x^2 - \beta^2)(m_A^2 + \ell(\ell+1)x^2) - 2i\omega(x^3 + \beta x^2) \Big] \ . \tag{3.342}$$

It must be noted that $\frac{f}{r^2}$ can be written in terms of x, as shown in (3.56). This applies to the term $\frac{\partial_r f}{r}$ as well

$$\frac{\partial_r f}{r} = 2 \Big(R_\Lambda^{-2} + \frac{M}{r^3} \Big) = 2 \Big(R_\Lambda^{-2} + Mx^3 \Big) \ . \tag{3.343}$$

To reach a recurrence relation, one needs to put the polynomials written in terms of $(x - x_+)$. The method used here is from Horowitz and Hubeny [46]. Since the introduction of the method has been done in 3.2.2, it will not be repeated here. The recurrence relation is described by the formula (3.58), using the expressions for the polynomials s, t and u shown above. The formula can be implemented numerically and the quasinormal modes can be obtained by solving (3.63). The numerical results can be found in section 3.5, for comparison purposes.

3.5 Proca's quasinormal modes in Schwarzschild-AdS: numerical results and comparison

In this section, the numerical results for the quasinormal modes of the Proca field in Schwarzschild-AdS are presented. A comparison is done between two ansatz: the ansatz using the vector spherical harmonics (section 3.2, from now on this ansatz will be referred as VSH) and the FKKS ansatz in the Schwarzschild-AdS limit (subsection 3.4.8).

The magnetic quasinormal modes ($u_{(4)}$) for $\ell = 1$ were calculated following section 3.2 and are shown in table 3.1.

$m_A R_\Lambda$	ωR_Λ ($r_+ = R_\Lambda$)	ωR_Λ ($r_+ = 100 R_\Lambda$)
0.01	$2.163 - 1.699\ i$	$(0) - 150.069\ i$
0.10	$2.171 - 1.710\ i$	$(0) - 152.187\ i$
0.20	$2.193 - 1.743\ i$	$(0) - 158.526\ i$
0.30	$2.228 - 1.795\ i$	$(0) - 168.922\ i$
0.40	$2.273 - 1.863\ i$	$(0) - 183.419\ i$
0.50	$2.327 - 1.944\ i$	$(0) - 202.860\ i$

(a) ωR_Λ for $r_+ = \{R_\Lambda, 100R_\Lambda\}$, with variable m_A (low).

r_+/R_Λ	ωR_Λ ($m_A R_\Lambda = 0.50$)
2	$2.453 - 4.312\ i$
3	$2.504 - 6.770\ i$
4	$2.426 - 9.269\ i$
7	$1.069 - 16.808\ i$
8	$(0) - 18.103\ i$
10	$(0) - 21.479\ i$

(b) ωR_Λ for $m_A R_\Lambda = 0.5$, with variable r_+.

$m_A R_\Lambda$	ωR_Λ ($r_+ = R$)
1.50	$3.072 - 3.016\ i$
4.50	$5.891 - 6.681\ i$
7.50	$8.885 - 10.424\ i$
10.50	$11.928 - 14.188\ i$

(c) ωR_Λ for $r_+ = R_\Lambda$, with variable m_A.

n	ωR_Λ ($r_+ = R_\Lambda$, $m_A R_\Lambda = 0.01$)
1	$3.844 - 4.152\ i$
2	$5.650 - 6.572\ i$
3	$7.818 - 8.693\ i$
4	$11.765 - 10.664\ i$

(d) Higher monotones for $r_+ = R_\Lambda$ and $m_A R_\Lambda = 0.01$.

Table 3.1: Fundamental magnetic quasinormal modes for the Proca field ($u_{(4)}$) in Schwarzschild-AdS, for $\ell = 1$.

The quasinormal modes for the monopole case in Schwarzschild-AdS were calculated by Konoplya [26]. To reproduce these results, the quasinormal modes were computed using the same method [46] and they are shown in table 3.2. It seems the modes are very close to the ones in Ref. [26]. To complete the analysis of the monopole, the quasinormal modes for higher values of the mass were calculated.

$m_A R_\Lambda$	ωR_Λ ($r_+ = R_\Lambda$)	ωR_Λ ($r_+ = 100 R_\Lambda$)
0.01	$2.798 - 2.671\ i$	$184.96 - 266.396\ i$
0.25	$2.852 - 2.737\ i$	$188.762 - 272.899\ i$
1.25	$3.562 - 3.672\ i$	$236.459 - 362.729\ i$
3.25	$5.316 - 5.942\ i$	$352.280 - 581.241\ i$
4.25	$6.227 - 7.105\ i$	$411.952 - 692.567\ i$
5.25	$7.156 - 8.278\ i$	$472.731 - 804.344\ i$
8.25	$10.031 - 11.856\ i$	$661.865 - 1142.58\ i$

Table 3.2: Fundamental quasinormal modes for the Proca field monopole in Schwarzschild-AdS, for $r_+ = \{R_\Lambda, 100R_\Lambda\}$ and changing the values of the mass.

Modes with a lower modulus of ω were not found. Still, there was an interesting mode found with $m_A R_\Lambda = 4.25$ corresponding to the frequency $4.624 - 100.603\ i$. This mode will converge into a pure imaginary number for $m_A R_\Lambda \notin [3.25, 5.25]$. The quasinormal modes shown in Table 3.3 were obtained for the monopole using the FKKS ansatz ($\beta = -i\frac{m_A^2}{\omega}$).

$m_A R$	ωR $(r_+ = R)$	$\omega R(r_+ = 100R)$
0.01	$2.798 - 2.671\ i$	$184.960 - 266.397\ i$
0.05	$2.800 - 2.674\ i$	$185.109 - 266.673\ i$
0.10	$2.807 - 2.683\ i$	$185.109 - 266.673\ i$
0.15	$2.818 - 2.698\ i$	$186.322 - 268.921\ i$
0.20	$2.833 - 2.717\ i$	$187.347 - 270.820\ i$
0.25	$2.851 - 2.742\ i$	$188.618 - 273.176\ i$

Table 3.3: Fundamental quasinormal modes of the monopole using FKKS ansatz, in Schwarzschild-AdS.

It seems these values are in agreement with Konoplya's work [26]. The quasinormal modes for the polarizations described by $u_{(2)}$ and $u_{(3)}$ were also calculated, using both ansatz. The results for the transversal and longitudinal polarizations are in table 3.4 and 3.5, respectively.

	$r_+ = R_\Lambda$		$r_+ = 100R_\Lambda$	
$m_A R_\Lambda$	ωR_Λ (VSH)	ωR_Λ (FKKS)	ωR_Λ (VSH)	ωR_Λ (FKKS)
0.01	$1.554 - 0.542\ i$	$1.554 - 0.542\ i$	$(0) - 149.984\ i$	$(0) - 149.984\ i$
0.10	$1.557 - 0.552\ i$	$1.557 - 0.552\ i$	$(0) - 152.099\ i$	$(0) - 152.099\ i$
0.20	$1.568 - 0.583\ i$	$1.568 - 0.584\ i$	$(0) - 158.432\ i$	$(0) - 158.432\ i$
0.30	$1.585 - 0.633\ i$	$1.584 - 0.633\ i$	$(0) - 168.817\ i$	$(0) - 168.817\ i$
0.40	$1.607 - 0.699\ i$	$1.606 - 0.699\ i$	$(0) - 183.291\ i$	$(0) - 183.291\ i$
0.50	$1.634 - 0.777\ i$	$1.632 - 0.777\ i$	$(0) - 202.684\ i$	$(0) - 202.684\ i$

(a) ωR_Λ for $r_+ = \{R_\Lambda, 100R_\Lambda\}$, with variable m_A (low).

	$m_A R_\Lambda = 0.5$	
r_+/R_Λ	ωR_Λ (VSH)	ωR_Λ (FKKS)
2	$1.049 - 1.836\ i$	$1.049 - 1.836\ i$
3	$(0) - 1.496\ i$	$(0) - 1.499\ i$
7	$(0) - 13.270\ i$	$(0) - 13.270\ i$
10	$(0) - 19.578\ i$	$(0) - 19.578\ i$

(b) ωR_Λ for $m_A R_\Lambda = 0.5$, with variable r_+.

	$r_+ = R_\Lambda$	
$m_A R_\Lambda$	ωR_Λ (VSH)	ωR_Λ (FKKS)
1.50	$2.019 - 1.946\ i$	$2.019 - 1.946\ i$
2.50	$2.570 - 3.474\ i$	$2.570 - 3.474\ i$
4.50	-	$5.936 - 8.318\ i$
7.50	-	$8.568 - 10.803\ i$

(c) ωR_Λ for $r_+ = R_\Lambda$, with variable m_A.

	$r_+ = R_\Lambda, m_A R_\Lambda = 0.01$	
n	ωR_Λ (VSH)	ωR_Λ (FKKS)
1	$2.972 - 2.928\ i$	$2.972 - 2.928\ i$
2	$4.748 - 5.372\ i$	$4.748 - 5.372\ i$
3	$6.579 - 7.675\ i$	$6.579 - 7.675\ i$
4	$9.605 - 9.653\ i$	$9.609 - 9.655\ i$

(d) Higher monotones for $r_+ = R_\Lambda$ and $m_A R_\Lambda = 0.01$.

Table 3.4: Fundamental electric transversal quasinormal modes for the Proca field (β_+) in Schwarzschild-AdS, for $\ell = 1$. Comparison between the two ansatz.

	$r_+ = R_\Lambda$		$r_+ = 100R_\Lambda$	
$m_A R_\Lambda$	ωR_Λ (VSH)	ωR_Λ (FKKS)	ωR_Λ (VSH)	ωR_Λ (FKKS)
0.01	$3.331 - 2.489\ i$	$3.330 - 2.489\ i$	$184.968 - 266.394\ i$	$184.968 - 266.395\ i$
0.10	$3.339 - 2.500\ i$	$3.339 - 2.501\ i$	$185.604 - 267.461\ i$	$185.578 - 267.524\ i$
0.20	$3.362 - 2.531\ i$	$3.362 - 2.534\ i$	$187.452 - 270.612\ i$	$187.355 - 270.817\ i$
0.30	$3.399 - 2.581\ i$	$3.398 - 2.586\ i$	$190.309 - 275.606\ i$	$190.117 - 275.939\ i$
0.40	$3.446 - 2.645\ i$	$3.444 - 2.652\ i$	$193.925 - 282.119\ i$	$193.650 - 282.498\ i$
0.50	$3.501 - 2.722\ i$	$3.498 - 2.729\ i$	$198.077 - 289.799\ i$	$197.761 - 290.138\ i$

(a) ωR_Λ for $r_+ = \{R_\Lambda, 100R_\Lambda\}$, with variable m_A (low).

	$m_A R_\Lambda = 0.5$	
r_+/R_Λ	ωR_Λ (VSH)	ωR_Λ (FKKS)
2	$4.866 - 5.701\ i$	$4.860 - 5.710\ i$
3	$6.575 - 8.629\ i$	$6.565 - 8.641\ i$
4	$8.406 - 11.543\ i$	$8.394 - 11.557\ i$
7	$14.146 - 20.257\ i$	$14.123 - 20.281\ i$
8	$16.091 - 23.159\ i$	$16.066 - 23.187\ i$
10	$20.004 - 28.960\ i$	$19.972 - 28.994\ i$

(b) ωR_Λ for $m_A R_\Lambda = 0.5$, with variable r_+.

	$r_+ = R_\Lambda$	
$m_A R_\Lambda$	ωR_Λ (VSH)	ωR_Λ (FKKS)
1.50	$4.215 - 3.720\ i$	$4.215 - 3.720\ i$
2.50	$5.036 - 4.815\ i$	$5.036 - 4.815\ i$
4.50	$6.793 - 7.100\ i$	$6.792 - 7.100\ i$
7.50	$9.580 - 10.669\ i$	$9.580 - 10.669\ i$
8.50	-	$10.536 - 11.884\ i$
10.50	-	$12.475 - 14.335\ i$

(c) ωR_Λ for $r_+ = R_\Lambda$, with variable m_A.

	$r_+ = R_\Lambda,\ m_A R_\Lambda = 0.01$	
n	ωR_Λ (VSH)	ωR_Λ (FKKS)
1	$5.172 - 4.888\ i$	$5.174 - 4.865\ i$
2	-	$8.379 - 7.069\ i$
3	-	$13.089 - 8.127\ i$
4	-	$18.509 - 10.070\ i$

(d) Higher monotones for $r_+ = R_\Lambda$ and $m_A R_\Lambda = 0.01$.

Table 3.5: Fundamental electric longitudinal quasinormal modes for the Proca field (β_-) in Schwarzschild-AdS, for $\ell = 1$. Comparison between the two ansatz.

It seems that the quasinormal modes obtained from the FKKS ansatz coincide or, at least, they are very near to the ones obtained from the VSH ansatz (maximum deviation of $\mathcal{O}(0.1)$). It must be pointed out that the empty entries in the tables mean that it was not possible to find the corresponding quasinormal modes. This might be due to the fact that these modes in the VSH ansatz are obtained from a system of equations, and so it must be needed an higher N to find them while using the method (subsection 3.2.2). Also in this method, only by inference one can distinguish each polarization. Fortunately, this is well characterized in the FKKS ansatz by the different values of β, allowing an easier distinction.

Chapter 4

Conclusion

4.1 Achievements

In this thesis, the topics of scalar fields and vector fields in spacetimes containing a black hole were reviewed. This subject is important to understand black holes better, to test GR and to possibly describe new physics such as dark matter. The equations for the scalar field in Kerr geometry were separated. The treatment of a minimally coupled scalar field is well studied in the literature. For a scalar field with Gauss-Bonnet coupling in Kerr geometry, it was demonstrated that the equations couldn't be separated due to the angular dependence of the Kretschmann scalar. The equations for the vector field in Kerr were also separated in the thesis. The treatment of massless vector fields was originally done by Teukolsky [27]. It is known that both fields can exhibit superradiance [49] in Kerr geometry, which was also shown. The generalization of Teukolsky's work for a massive vector field took more or less 30 years to appear with the construction of the FKKS ansatz [33], which uses hidden symmetries of Kerr-NUT-(A)dS to separate the Proca equations. A review about this ansatz, the principle tensor and the Kerr-NUT-(A)dS spacetime was presented.

A question still lingers about the FKKS ansatz: Does it describe all the degrees of freedom of the massive vector field? The objective of the thesis was to investigate this in Schwarzschild-AdS geometry, the non-rotating limit of Kerr-AdS. The objective was accomplished by making analytical and numerical comparisons between this ansatz and the typical treatment with vector spherical harmonics.

It was concluded that the FKKS ansatz is able to describe two of the three polarizations of the massive vector field in Schwarzschild-AdS: the longitudinal and the transversal polarizations corresponding to the electric modes [24] of the field. The absence of the magnetic modes in the ansatz may be due to the degeneracy of the principal tensor in the non-rotating limit. The analytical correspondence between the two treatments was obtained and revealed a transformation that decouples the two polarizations in the Proca equations. Indeed, an advantage of working with the FKKS ansatz is the natural decoupling of the polarizations, opposed to the typical treatment. The numerical comparison consisted in the calculation of the quasinormal modes for each ansatz and they seem to coincide with a maximum deviation of $\mathcal{O}(0.1)$, thus corroborating the drawn conclusion. Finally, it is worth mentioning that the calculation of the monopole's normal modes in anti-de Sitter done in Ref. [26] had a mistake and was corrected here.

4.2 Future work

To extend this work, the FKKS ansatz in Kerr-AdS spacetime should be studied. The Newman-Penrose formalism used by Teukolsky in Kerr is also valid in Kerr-AdS [54]. This means a possible comparison can be made between the FKKS ansatz in the massless limit and the treatment with this formalism. The comparison has important significance since a better identification of the polarizations can be made. Also, the principal tensor is non-degenerate in this geometry, in opposite to Schwarzschild-AdS. Thus, one should be hopefully able to find the magnetic modes.

Another work of interest would be to study extensions of the FKKS ansatz and the principal tensor even further. It is known that the principal tensor generates a Killing tower: a set of Killing objects that translate to symmetries. A legitimate question would be for example: Are these symmetries able to separate the equations of vector fields with higher order coupling terms to the curvature? If such ansatz exists, it may imply a different dependence in the principal tensor. Also, an interesting question would be: Is it possible to extend this for field with tensor nature, such as massive gravitons? An affirmative answer to these questions would be important for developments in the study of such fields.

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Appendix A

Identity proof of Regge-Wheeler operator

To prove the identity (3.35), a useful rule can be shown

$$\hat{\mathcal{D}}(ru_{(3)}) = r\hat{\mathcal{D}}(u_{(3)}) + fu_{(3)}\partial_r f + 2f^2\partial_r u_{(3)} \ . \tag{A.1}$$

Then it follows that

$$\begin{aligned}
\hat{\mathcal{D}}\Big[\frac{r^2}{f}\hat{\mathcal{D}}u_{(3)}\Big] &= \hat{\mathcal{D}}\Big[\frac{r}{f}\Big(\hat{\mathcal{D}}(ru_3) - fu_{(3)}\partial_r f - 2f^2\partial_r u_{(3)}\Big)\Big] \\
&= r\hat{\mathcal{D}}\Big[\frac{1}{f}\Big(\hat{\mathcal{D}}(ru_3) - fu_{(3)}\partial_r f - 2f^2\partial_r u_{(3)}\Big)\Big] + \frac{1}{f}\Big(\hat{\mathcal{D}}(ru_3) - fu_{(3)}\partial_r f - 2f^2\partial_r u_{(3)}\Big)f\partial_r f \\
&\quad + 2f^2\partial_r\Big[\frac{1}{f}\Big(\hat{\mathcal{D}}(ru_3) - fu_{(3)}\partial_r f - 2f^2\partial_r u_{(3)}\Big)\Big] \\
&= r\hat{\mathcal{D}}\Big[\frac{1}{f}\hat{\mathcal{D}}_1(ru_{(3)})\Big] - 2r\hat{\mathcal{D}}(\partial_{r_*}u_{(3)}) + f(\partial_r f)\frac{r}{f})\hat{\mathcal{D}}u_{(3)} + 2f^2\partial_r\Big[\frac{r}{f}\hat{\mathcal{D}}u_{(3)}\Big] \ ,
\end{aligned} \tag{A.2}$$

where $\hat{\mathcal{D}_1}$ is defined by (3.36). To simplify this further, another rule can be shown

$$\partial_{r_*}\hat{\mathcal{D}}u_{(3)} = \hat{\mathcal{D}}(\partial_{r_*}u_{(3)}) - \partial_{r_*}\Big(f\Big[\frac{\ell(\ell+1)}{r^2} + m_A^2\Big]\Big)u_{(3)} \ . \tag{A.3}$$

And so the third order derivative terms of (A.2) can be expanded into

$$2f^2\partial_r\Big[\frac{r}{f}\hat{\mathcal{D}}u_{(3)}\Big] - 2r\hat{\mathcal{D}}(\partial_{r_*}u_{(3)}) = 2f^2\partial_r\Big(\frac{r}{f}\Big)\hat{\mathcal{D}}u_{(3)} - 2r\partial_{r_*}\Big(f\Big[\frac{\ell(\ell+1)}{r^2} + m_A^2\Big]\Big) \ . \tag{A.4}$$

The equation (A.2) then becomes

$$\hat{\mathcal{D}}\Big[\frac{r^2}{f}\hat{\mathcal{D}}u_{(3)}\Big] = r\hat{\mathcal{D}}\Big[\frac{1}{f}\hat{\mathcal{D}}_1(ru_{(3)})\Big] + \Big[2f^2\partial_r\Big(\frac{r}{f}\Big) + r\partial_r f\Big] - 2r\partial_{r_*}\Big(f\Big[\frac{\ell(\ell+1)}{r^2} + m_A^2\Big]\Big)u_{(3)} \ . \tag{A.5}$$

By calculating the derivative of f and expanding the terms, the identity (3.35) is obtained.

Appendix B

Tetrad formalism

In General Relativity, tensor fields are displayed as objects idependent of the coordinate basis. It is common to manipulate them in this way throughout calculations and then move to a coordinate basis. In certain cases, this might be more difficult to obtain physical quantities (since these are invariant under change of coordinate basis) or even to carry out with the calculations. For this reason, it is useful to define a frame. Using a coordinate basis, each point of the manifold has a tangent space associated with it. Typically, the basis for this space consists of vector fields $\frac{\partial}{\partial x^a}$. A frame is defined by a set of vector fields $e^a_{(\mu)}$ (where the lower index in parenthesis represents the identifier of the vector field and the upper index represents its component) that make a basis for the tangent space, in a certain point. Moreover, it is useful to define such set of vector fields that obey

$$e^a_{(\mu)} e^b_{(\nu)} g_{ab} = \eta_{(\mu)(\nu)} \ , \tag{B.1}$$

where $\eta_{(\mu)(\nu)}$ are the components of a non-singular matrix. If η has the same matrix form as the Minkowski metric, then the frame is called a tetrad and it has the physical significance of an inertial frame. It must be noted that quantities that only possess indices in parentesis are scalar quantities. This is the advantageous characteristic of this formalism. The following relation can be defined

$$e^a_{(\mu)} \eta^{(\mu)(\nu)} = e^{(\nu)a} \ , \tag{B.2}$$

where

$$\eta_{(\mu)(\nu)} \eta^{(\nu)(\sigma)} = \delta^{(\sigma)}_{(\nu)} \ . \tag{B.3}$$

Then it is possible to parametrize the metric as

$$g_{ab} = \eta_{(\mu)(\nu)} e^{(\mu)}_a e^{(\nu)}_b \ . \tag{B.4}$$

In this formalism, tensor fields can be described by a set of scalars defined as

$$T_{(\mu_1)...(\mu_i)} = e^{a_1}_{(\mu_1)} .... e^{a_i}_{(\mu_i)} T_{a_1...a_i} \ , \tag{B.5}$$

where T is a $(0, i)$ rank tensor.

Since equations of interest involve covariant derivatives of tensors, it is useful to see its counterpart in this formalism. The quantities (B.5) are scalars, so it is possible to define their directional derivatives as

$$\mathbf{e}_{(\mu)}(T_{(\mu_1)...(\mu_i)}) \equiv e^a_{(\mu)} \nabla_a (T_{(\mu_1)...(\mu_i)}) = e^a_{(\mu)} \partial_a (T_{(\mu_1)...(\mu_i)}) \ . \tag{B.6}$$

These directional derivatives have a relation with the covariant derivative of the tensor. For the purpose of section 3.3.1, the following relation is presented for a $(0,2)$ anti-symmetric tensor F (a form called Faraday tensor)

$$F_{(\mu)(\nu),(\tau)} \equiv e^a_{(\tau)} \nabla_a (e^b_{(\mu)} e^c_{(\nu)} F_{bc})$$

$$= \eta^{(\sigma)(\delta)} (\gamma_{(\sigma)(\mu)(\tau)} F_{(\delta)(\nu)} + \gamma_{(\sigma)(\nu)(\tau)} F_{(\mu)(\delta)}) + F_{(\mu)(\nu)|(\tau)} \ , \tag{B.7}$$

$$\gamma_{(\sigma)(\mu)(\nu)} = e^a_{(\sigma)} e_{(\mu)a;b} e^b_{(\nu)} \ , \tag{B.8}$$

$$F_{(\mu)(\nu)|(\tau)} = e^a_{(\mu)} e^b_{(\nu)} e^c_{(\tau)} F_{ab;c} \ . \tag{B.9}$$

The scalars $\gamma_{(\sigma)(\mu)(\nu)}$ are called Ricci-rotation or spin coefficients. These are related to the Levi-Civita connection and they are anti-symmetric in their first two indices.
Thus, it is possible to write equations (3.4) (with a zero mass) and (3.5) as

$$F_{[(\mu)(\nu)|(\tau)]} = 0 \ , \ \eta^{(\mu)(\nu)} F_{(\tau)(\mu)|(\nu)} = 0 \ . \tag{B.10}$$

Appendix C

Explicit symmetries of axisymmetric stationary spacetimes

In Kerr spacetime, there are two explicit symmetries: time translation and rotation around a special axis. Symmetries correspond to transformations that are isometries, i.e. the metric as an object remains the same. Putting this in a mathematical form, the transformation

$$X^a \to X^a + s\xi^a \ , \tag{C.1}$$

is an isometry if the following is satisfied

$$\mathscr{L}_{\boldsymbol{\xi}}\mathbf{g} = 0 \iff \nabla_{(a}\xi_{b)} = 0 \ , \tag{C.2}$$

where $\mathscr{L}_{\boldsymbol{\xi}}$ is the Lie derivative along the vector field $\boldsymbol{\xi}$. The easiest isometries that can be found correspond to the fact that the metric is independent of a certain coordinate. The Kerr metric has this feature since does not depend on either t or ϕ. Indeed, by choosing

$$\boldsymbol{\xi}_{(t)} = \partial_t \ , \tag{C.3}$$

$$\boldsymbol{\xi}_{(\phi)} = \partial_\phi \ , \tag{C.4}$$

the Lie derivative of the metric corresponds to

$$\mathscr{L}_{\boldsymbol{\xi}_{(t)}} g_{ab} = \partial_t g_{ab} = 0 \ , \tag{C.5}$$

$$\mathscr{L}_{\boldsymbol{\xi}_{(\phi)}} g_{ab} = \partial_\phi g_{ab} = 0 \ . \tag{C.6}$$

The object $\boldsymbol{\xi}$ is called a Killing vector field and it is intimately related with symmetries and conserved quantities. In Kerr, for point-like particles, it is trivial to check that quantities defined as

$$E = -\xi^a_{(t)} p_a \ , \tag{C.7}$$

$$L = \xi^a_{(\phi)} p_a \ , \tag{C.8}$$

are conserved along timelike geodesics, where p_a is the 4-momentum. More generally, for fields, conserved currents can be defined by

$$P^a = -T^a_b \xi^b_{(t)} \ , \tag{C.9}$$

$$J^a = T^a_b \xi^b_{(\phi)} \ . \tag{C.10}$$

The physical meaning of P^a and J^a correspond to the flow of energy and angular momentum of the fields. It is possible to define the energy and angular momentum of the fields that flow out to infinity by

$$E = -\int T^a_b \xi^b_{(t)} d\Sigma_a \ , \tag{C.11}$$

$$L = \int T^a_b \xi^b_{(\phi)} d\Sigma_a \ . \tag{C.12}$$

where $d\Sigma_a = N_a r^2 dt d\Omega$ corresponds to the surface element of the hypersurface of constant r at $r \to \infty$, and $N_a = -(0,1,0,0)$ is its inward normal vector (at this limit). A stationary observer at infinity can measure the rate of these quantities as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{r\to\infty} \int r^2 T_{tr} d\Omega \ , \tag{C.13}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\lim_{r\to\infty} \int r^2 T_{r\phi} d\Omega \ , \tag{C.14}$$

where it was used that $g_{rr} \to 1$ at $r \to \infty$.
For an energy balance analysis, the event horizon must be also considered. In this case,

$$\frac{\mathrm{d}E_h}{\mathrm{d}t} - \Omega_H \frac{\mathrm{d}L_h}{\mathrm{d}t} = \int T_{ab} N^a N^b 2Mr_+ d\Omega \ , \tag{C.15}$$

where the 3-surface element is $d\Sigma_a = N_a 2Mr_+ d\Omega dt$ and $N^a = -\xi^a_{(t)} - \Omega_H \xi^a_{(\phi)}$ is its inward normal (also a Killing vector field, as the hypersurface is a Killing horizon). To simplify the expression, the relation between energy and angular momentum can be used [55]

$$\frac{\delta E}{\delta L} = \frac{\omega}{m} \ . \tag{C.16}$$

This holds for scalar fields, electromagnetic waves and gravitational waves. The proof of this relation for the electromagnetic case in a background Kerr spacetime can be seen in the next section C.1. Physically, it can be justified by the following argument: each quanta of the field has an energy $E = \hbar\omega$ and angular momentum $L = \hbar m$; if a quanta is absorbed by the black hole or leaves to infinity, the change on the energy and angular momentum of the field corresponds to the ones that the quanta was carrying, and so (C.16) follows. Inserting this into (C.15), then it is obtained that

$$\frac{\mathrm{d}E_h}{\mathrm{d}t} = \frac{\omega}{k_H} 2Mr_+ \int T_{ab} N^a N^b d\Omega \ , \tag{C.17}$$

$$\frac{\mathrm{d}L_h}{\mathrm{d}t} = \frac{m}{k_H} 2Mr_+ \int T_{ab} N^a N^b d\Omega \ . \tag{C.18}$$

C.1 Relation between Angular momentum and Energy

C.1.1 Electromagnetic waves in Schwarzschild spacetime

In this subsection, it is intended to prove relation (C.16) in the case of the eletromagnetic waves on Schwarzschild spacetime. To this purpose, an asymptotic solution near infinity will be derived and the fluxes of energy and angular momentum through a sphere surface will be calculated.
The asymptotic solution can be found by neglecting the $\frac{1}{r^2}$ terms in the equations (3.26)-(3.29). The equation that these solutions must obey is

$$\Big[-\partial_t^2+\partial_{r_*}^2\Big]u_{(s)}=0\ . \tag{C.19}$$

The solution for this is very trivial and can be found to be

$$u_{(s)}=\mathcal{I}_s e^{-i\omega(t+r_*)}+\mathcal{R}_s e^{-i\omega(t-r_*)}\ . \tag{C.20}$$

Looking into the Lorenz condition (3.31), it is possible to fix the gauge by setting

$$\partial_{r_*}u_{(2)}=\partial_t u_{(1)}\ ,\ u_{(2)}=u_{(3)}\ . \tag{C.21}$$

These conditions are translated to the following

$$\mathcal{I}_1=\mathcal{I}_2\ ,\ \mathcal{R}_1=-\mathcal{R}_2\ ,\ \mathcal{I}_2=\mathcal{I}_3\ ,\ \mathcal{R}_2=\mathcal{R}_3\ . \tag{C.22}$$

The complex asymptotic solution is then

$$\begin{aligned}\bar{A}_a dx^a = &\ e^{-i\omega t}\Big(\mathcal{I}_3e^{-i\omega r_*}-\mathcal{R}_3e^{i\omega r_*}\Big)Y_{lm}\frac{dt}{r}+e^{-i\omega t}\Big(\mathcal{I}_3e^{-i\omega r_*}+\mathcal{R}_3e^{i\omega r_*}\Big)Y_{lm}\frac{dr}{r}\\ &+e^{-i\omega t}\Big[\Big(\mathcal{I}_3e^{-i\omega r_*}+\mathcal{R}_3e^{i\omega r_*}\Big)\partial_\theta Y_{lm}+\Big(\mathcal{I}_4e^{-i\omega r_*}+\mathcal{R}_4e^{i\omega r_*}\Big)\frac{\partial_\phi Y_{lm}}{\sin\theta}\Big]\frac{d\theta}{\ell(\ell+1)}\\ &+e^{-i\omega t}\Big[\Big(\mathcal{I}_3e^{-i\omega r_*}+\mathcal{R}_3e^{i\omega r_*}\Big)\partial_\phi Y_{lm}-\Big(\mathcal{I}_4e^{-i\omega r_*}+\mathcal{R}_4e^{i\omega r_*}\Big)\sin\theta\partial_\theta Y_{lm}\Big]\frac{d\phi}{\ell(\ell+1)}\ .\end{aligned} \tag{C.23}$$

Without loss of generality, the coefficients can be taken as real values. Even though it is useful to work with the complex solution, the stress energy tensor calculated is only for a real vector field. So it is needed to take the real part of (C.23). Also it must be noted that Y_lm is complex as well, and so for the sake of notation

$$Y_{lm}=e^{im\phi}Y'_{lm}\ . \tag{C.24}$$

The real part of the vector field is

$$\begin{aligned}A_a dx^a =&\Big(\mathcal{I}_3\cos(\phi_-)-\mathcal{R}_3\cos(\phi_+)\Big)Y'_{lm}\frac{dt}{r}+\Big(\mathcal{I}_3\cos(\phi_-)+\mathcal{R}_3\cos(\phi_+)\Big)Y'_{lm}\frac{dr}{r}\\ &+\Big[\Big(\mathcal{I}_3\cos(\phi_-)+\mathcal{R}_3\cos(\phi_+)\Big)\partial_\theta Y'_{lm}-\Big(\mathcal{I}_4\sin(\phi_-)+\mathcal{R}_4\sin(\phi_+)\Big)\frac{mY'_{lm}}{\sin\theta}\Big]\frac{d\theta}{\ell(\ell+1)}\\ &-\Big[\Big(\mathcal{I}_3\sin(\phi_-)+\mathcal{R}_3\sin(\phi_+)\Big)mY'_{lm}+\Big(\mathcal{I}_4\cos(\phi_-)+\mathcal{R}_4\cos(\phi_-)\Big)\sin\theta\partial_\theta Y'_{lm}\Big]\frac{d\phi}{\ell(\ell+1)}\ ,\end{aligned} \tag{C.25}$$

where

$$\phi_+ = -\omega(t-r) + m\phi \ , \tag{C.26}$$

$$\phi_- = -\omega(t+r) + m\phi \ . \tag{C.27}$$

The r_* was dropped to r because asymptotically f goes to 1. In fact if the r_* was to be maintained, more terms with a power higher in $\frac{1}{r}$ would arise. It seems that these terms will only contribute for the terms in $\frac{1}{r^3}$, on the evaluation of the stress energy tensor and so it adequate in this kind of analysis to do this approximation.
Now for the calculation of the flux, as shown in appendix C, the formula of the flux of energy and angular momentum through the surface of a sphere near infinity are

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \lim_{r\to\infty} \int r^2 T_{tr} d\Omega \ , \tag{C.28}$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = -\lim_{r\to\infty} \int r^2 T_{r\phi} d\Omega \ , \tag{C.29}$$

and so it is needed to evaluate the stress energy tensor components. Asymptotically, the components are

$$4\pi r^2 T_{tr} = \partial_t A_\theta \partial_r A_\theta + \partial_t A_\phi \partial_r A_\phi \csc^2\theta \ , \tag{C.30}$$

$$4\pi r^2 T_{\phi r} = (\partial_\phi A_\theta - \partial_\theta A_\phi)\partial_r A_\theta - r^2 \partial_t A_\phi (\partial_r A_t - \partial_t A_r) \ . \tag{C.31}$$

The other terms that are not considered have a dependence in higher powers of r^{-1}. The following results can be shown

$$\begin{aligned}(\ell(\ell+1))^2 \partial_t A_\theta \partial_r A_\theta = \ & \omega^2 \Big[\mathcal{I}_3^2 \sin^2(\phi_-) - \mathcal{R}_3^2 \sin^2(\phi_+)\Big] (\partial_\theta Y'_{lm})^2 \\ & + \omega^2 \Big[\mathcal{I}_4^2 \cos^2(\phi_-) - \mathcal{R}_4^2 \cos^2(\phi_+)\Big] \frac{m^2 (Y'_{lm})^2}{\sin^2\theta} \\ & + 2m\omega^2 \Big[\mathcal{I}_3\mathcal{I}_4 \sin(\phi_-)\cos(\phi_-) - \mathcal{R}_3\mathcal{R}_4 \cos(\phi_+)\sin(\phi_+)\Big] \frac{Y'_{lm}\partial_\theta Y_{lm}}{\sin\theta} \ ,\end{aligned} \tag{C.32}$$

and

$$\begin{aligned}(\ell(\ell+1))^2 \partial_t A_\phi \partial_r A_\phi \csc^2\theta = \ & \omega^2 \Big[\mathcal{I}_3^2 \cos^2(\phi_-) - \mathcal{R}_3 \cos^2(\phi_+)\Big] \frac{m^2 (Y'_{lm})^2}{\sin^2\theta} \\ & + \omega^2 \Big[\mathcal{I}_4^2 \sin^2(\phi_-) - \mathcal{R}_4 \sin^2(\phi_+)\Big] (\partial_\theta Y'_{lm})^2 \\ & - 2m\omega^2 \Big[\mathcal{I}_3\mathcal{I}_4 \sin(\phi_-)\cos(\phi_-) - \mathcal{R}_3\mathcal{R}_4 \cos(\phi_+)\sin(\phi_+)\Big] \frac{Y'_{lm}\partial_\theta Y'_{lm}}{\sin\theta}\end{aligned} \tag{C.33}$$

The crossed terms in the flux of energy cancel and the terms that go with $\sin(2\phi)$ or $\cos(2\phi)$ vanish when integrating in the azimutal angle ϕ. By considering

$$\cos(\phi)^2 \approx \frac{1}{2} \ , \ \sin(\phi)^2 \approx \frac{1}{2} \ , \tag{C.34}$$

then

$$4\pi r^2 T_{tr} = \frac{\omega^2}{2(\ell(\ell+1))^2} \Big[\Big(\mathcal{I}_3^2 + \mathcal{I}_4^2\Big) - \Big(\mathcal{R}_3^2 + \mathcal{R}_4^2\Big)\Big] \Big((\partial_\theta Y'_{lm})^2 + \frac{m^2 (Y'_{lm})^2}{\sin^2\theta}\Big) \ . \tag{C.35}$$

Using the equation of the spherical harmonics

$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta Y_{lm}) + \frac{1}{\sin^2\theta}\partial_\phi^2 Y_{lm} = -\ell(\ell+1)Y_{lm} \ , \tag{C.36}$$

it is possible to simplify the angular part into

$$(\partial_\theta Y'_{lm})^2 + \frac{m^2(Y'_{lm})^2}{\sin^2\theta} = \frac{1}{\sin\theta}\partial_\theta(\sin\theta Y'_{lm}\partial_\theta Y'_{lm}) + \ell(\ell+1)Y'_{lm} \ . \tag{C.37}$$

The derivative when integrated in the solid angle will vanish since $\sin\theta = 0$ at 0 and π. Thus the expression that does not vanish is

$$4\pi r^2 T_{tr} = \frac{\omega^2}{2\ell(\ell+1)}\Big[\Big(\mathcal{I}_3^2 + \mathcal{I}_4^2\Big) - \Big(\mathcal{R}_3^2 + \mathcal{R}_4^2\Big)\Big](Y'_{lm})^2 \ . \tag{C.38}$$

For the flux of the angular momentum, the terms are

$$\begin{aligned}
&\ell(\ell+1)(\partial_\phi A_\theta - \partial_\theta A_\phi)\partial_r A_\theta = \ m\omega\Big[\mathcal{R}_4^2\cos^2(\phi_+) - \mathcal{I}_4^2\cos^2(\phi_-)\Big](Y'_{lm})^2 \\
&-\omega\sin\theta Y'_{lm}\partial_\theta Y'_{lm}\Big[\mathcal{I}_3\mathcal{I}_4\cos(\phi_-)\sin(\phi_-) - \mathcal{I}_4\mathcal{R}_3\cos(\phi_-)\sin(\phi_+) \\
&+\mathcal{I}_3\mathcal{R}_4\cos(\phi_+)\sin(\phi_-) - \mathcal{R}_3\mathcal{R}_4\cos(\phi_+)\sin(\phi_-)\Big] \ ,
\end{aligned} \tag{C.39}$$

and

$$\begin{aligned}
&\ell(\ell+1)\partial_t A_\phi(\partial_r A_t - \partial_t A_r) = \ \omega m\Big(\mathcal{R}_3^2\cos^2(\phi_+) - \mathcal{I}_3^2\cos^2(\phi_-)\Big)(Y'_{lm})^2 \\
&+\omega\sin\theta Y'_{lm}\partial_\theta Y'_{lm}\Big[\mathcal{I}_3\mathcal{I}_4\cos(\phi_-)\sin(\phi_-) + \mathcal{R}_4\mathcal{I}_3\cos(\phi_-)\sin(\phi_+) \\
&-\mathcal{R}_3\mathcal{I}_4\cos(\phi_+)\sin(\phi_-) - \mathcal{R}_3\mathcal{R}_4\cos(\phi_+)\sin(\phi_+)\Big] \ .
\end{aligned} \tag{C.40}$$

Combining these two expressions gives

$$\begin{aligned}
&4\pi r^2\ell(\ell+1)T_{\phi r} = \ \omega m\Big(\mathcal{R}_3^2\cos^2(\phi_+) - \mathcal{I}_3^2\cos^2(\phi_-) + \mathcal{R}_4^2\cos^2(\phi_+) - \mathcal{I}_4^2\cos^2(\phi_-)\Big)(Y'_{lm})^2 \\
&+\omega\sin\theta Y'_{lm}\partial_\theta Y'_{lm}\Big[\mathcal{I}_3\mathcal{R}_4 - \mathcal{I}_4\mathcal{R}_3\Big]\sin(\phi_- + \phi_+) \ .
\end{aligned} \tag{C.41}$$

Since

$$\sin(\phi_+ + \phi_-) = \sin(-2\omega t + 2m\phi) \ , \tag{C.42}$$

the integration over the azimutal angle will make this vanish. So the final expression for the component of the stress energy tensor is

$$4\pi r^2 T_{\phi r} = \frac{\omega m}{2\ell(\ell+1)}\Big[-\Big(\mathcal{I}_3^2 + \mathcal{I}_4^2\Big) + \Big(\mathcal{R}_3^2 + \mathcal{R}_4^2\Big)\Big](Y'_{lm})^2 \ . \tag{C.43}$$

And so the fraction of the fluxes through the surface of a sphere near infinity is given by

$$\frac{\delta E}{\delta L} = -\frac{T_{tr}}{T_{\phi r}} = \frac{\omega}{m} \ . \tag{C.44}$$

C.1.2 Electromagnetic waves in Kerr spacetime

The proof done in the previous subsection in Schwarzschild spacetime can be extended to the case of Kerr spacetime, since both spacetimes are asymptotically flat. Still, it might be more satisfatory to prove relation (C.16) using Newman-Penrose formalism and Teukolsky's complex scalars.
Following the calculations done in 3.3.6, the ingoing fluxes of the energy and angular momentum at the horizon are defined by

$$\frac{d^2E}{dtd\Omega} = -2Mr_+T_{ab}\xi^a_{(t)}N^b \ , \tag{C.45}$$

$$\frac{d^2L}{dtd\Omega} = 2Mr_+T_{ab}\xi^a_{(\phi)}N^b \ . \tag{C.46}$$

The Killing vetor fields can be evaluated in terms of the tetrad near the horizon by

$$\xi_{(t)} = \partial_t = \frac{1}{\Sigma}\Big(\frac{\Delta}{2}l^a\partial_a + \Sigma n^a\partial_a\Big) + \frac{ia\sin\theta}{\sqrt{2}}\Big(\frac{1}{\bar{\rho}^*}m^a\partial_a - \frac{1}{\bar{\rho}}\bar{m}^a\partial_a\Big) \ , \tag{C.47}$$

$$\xi_{(\phi)} = \partial_\phi = -\left[\frac{a\sin^2\theta}{\Sigma}\Big(\frac{\Delta}{2}l^a\partial_a + \Sigma n^a\partial_a\Big) + \frac{i}{\sqrt{2}}\sin\theta(2Mr_+)\Big(\frac{1}{\bar{\rho}^*}m^a\partial_a - \frac{1}{\bar{\rho}}\bar{m}^a\partial_a\Big)\right] \ , \tag{C.48}$$

$$N^a\partial_a = -\left(\xi^a_{(t)} + \Omega_H\xi^a_{(\phi)})\partial_a\right|_{\Delta=0} = -\frac{1}{2Mr_+}\Big(\frac{\Delta}{2}l^a + \Sigma n^a\Big)\partial_a \ . \tag{C.49}$$

Substituting this into the formula for the fluxes, it reads

$$\frac{d^2E}{dtd\Omega} = \frac{T_{ab}}{\Sigma}\Big(\frac{\Delta}{2}l^b + \Sigma n^b\Big)\Big(\frac{\Delta}{2}l^a + \Sigma n^a\Big) + \frac{ia\sin\theta T_{ab}}{\sqrt{2}}\Big(\frac{\Delta}{2}l^b + \Sigma n^b\Big)\Big(\frac{1}{\bar{\rho}^*}m^a - \frac{1}{\bar{\rho}}\bar{m}^a\Big) \ , \tag{C.50}$$

$$\begin{aligned}\frac{d^2L}{dtd\Omega} &= \frac{a\sin^2\theta T_{ab}}{\Sigma}\Big(\frac{\Delta}{2}l^b + \Sigma n^b\Big)\Big(\frac{\Delta}{2}l^a + \Sigma n^a\Big) \\ &+ \frac{iT_{ab}}{\sqrt{2}}\sin\theta(2Mr_+)\Big(\frac{\Delta}{2}l^b + \Sigma n^b\Big)\Big(\frac{1}{\bar{\rho}^*}m^a - \frac{1}{\bar{\rho}}\bar{m}^a\Big) \ .\end{aligned} \tag{C.51}$$

The necessary contractions of the stress-energy tensor (3.3) with the tetrad vectors give

$$T_{ab}\Big(\frac{\Delta}{2}l^a + \Sigma n^a\Big)\Big(\frac{\Delta}{2}l^b + \Sigma n^b\Big)\Big|_{\Delta=0} = \frac{\Delta^2|\phi_0|^2}{8\pi} \ , \tag{C.52}$$

$$T_{ab}\Big(\frac{\Delta}{2}l^a + \Sigma n^a\Big)\Big(\frac{1}{\bar{\rho}^*}m^b - \frac{1}{\bar{\rho}}\bar{m}^b\Big)\Big|_{\Delta=0} = \frac{\Delta}{4\pi}\Big(\frac{\phi_0\phi_1^*}{\bar{\rho}^*} - \frac{\phi_0^*\phi_1}{\bar{\rho}}\Big) + \frac{\Sigma}{2\pi}\Big(\frac{\phi_2^*\phi_1}{\bar{\rho}^*} - \frac{\phi_2\phi_1^*}{\bar{\rho}}\Big) \ . \tag{C.53}$$

Near the horizon, Φ_0 and Φ_2 were already calculated asymptotically. The asymptotic expression of Φ_1 must be calculated to move forward. Following the expression (3.135), it is possible to obtain

$$\begin{aligned}\bar{\rho}^*\Phi_1 &= \frac{1}{D}\Big(\bar{\rho}^*\mathcal{D}_0R_{-1} - R_{-1}\Big)\mathcal{L}_1S_{+1} - \frac{ia\sin\theta}{D}\mathcal{D}_0R_{-1}S_{+1} \ , &\text{(C.54)}\\ &= \frac{\mathcal{D}_0R_{-1}}{D}\Big(\bar{\rho}^*\mathcal{L}_1S_{+1} - ia\sin\theta S_{+1}\Big) \ , &\text{(C.55)}\\ &= \frac{i\mathcal{T}_{+1}e^{-ik_Hr_*}}{4Mr_+k_H}\Big(\bar{\rho}^*\mathcal{L}_1S_{+1} - ia\sin\theta S_{+1}\Big) \ , &\text{(C.56)}\end{aligned}$$

where it has been used that R_{-1} goes with Δ, and that $\mathcal{D}_0 R_{-1} = \frac{D\Delta R_{+1}}{2iK}$ (Starobinski-Teukolsky identity (3.123), near the horizon). Now, it is possible to proceed with the calculation of (C.50) and (C.51)

$$\frac{d^2E}{dtd\Omega} = \frac{\Delta^2|\phi_0|^2}{8\pi\Sigma} + \frac{ia\sin\theta}{\sqrt{2}}\left(\frac{\Delta}{4\pi}\Big(\frac{\phi_0\phi_1^*}{\bar{\rho}^*} - \frac{\phi_0^*\phi_1}{\bar{\rho}}\Big) + \frac{\Sigma}{2\pi}\Big(\frac{\phi_2^*\phi_1}{\bar{\rho}^*} - \frac{\phi_2\phi_1^*}{\bar{\rho}}\Big)\right) , \tag{C.57}$$

$$\frac{d^2L}{dtd\Omega} = \frac{a\Delta^2\sin^2\theta|\phi_0|^2}{8\pi\Sigma} + \frac{2iMr_+\sin\theta}{\sqrt{2}}\left(\frac{\Delta}{4\pi}\Big(\frac{\phi_0\phi_1^*}{\bar{\rho}^*} - \frac{\phi_0^*\phi_1}{\bar{\rho}}\Big) + \frac{\Sigma}{2\pi}\Big(\frac{\phi_2^*\phi_1}{\bar{\rho}^*} - \frac{\phi_2\phi_1^*}{\bar{\rho}}\Big)\right) . \tag{C.58}$$

This can be put in terms of the Φ's, using the definitions in (3.104)

$$\frac{d^2E}{dtd\Omega} = \frac{1}{8\pi\Sigma}\left[\Delta^2|\Phi_0|^2 + ia\sin\theta\left(\Delta\Big(\Phi_0\Phi_1^* - \Phi_0^*\Phi_1\Big) + \Big(\Phi_2^*\Phi_1 - \Phi_2\Phi_1^*\Big)\right)\right] , \tag{C.59}$$

$$\frac{d^2L}{dtd\Omega} = \frac{1}{8\pi\Sigma}\left[a\Delta^2\sin^2\theta|\Phi_0|^2 + 2iMr_+\sin\theta\left(\Delta\Big(\Phi_0\Phi_1^* - \Phi_0^*\Phi_1\Big) + \Big(\Phi_2^*\Phi_1 - \Phi_2\Phi_1^*\Big)\right)\right] . \tag{C.60}$$

The terms in Φ_2 should vanish since $\Phi_2^*\Phi_1$ is proportional to Δ. Using the asymptotic solutions (3.169) and the calculated asymptotic solution for Φ_1 (C.56), it can be shown that

$$|\Phi_0|^2 = \frac{|\mathcal{T}_{+1}|^2}{\Delta^2}S_{+1}^2 , \tag{C.61}$$

$$\Phi_0\Phi_1^* - \Phi_0^*\Phi_1 = -\frac{i|\mathcal{T}_{+1}|^2}{2Mr_+\Delta k_H}\Big(\mathcal{L}_1 S_{+1} + \frac{a^2\sin\theta\cos\theta}{\Sigma}S_{+1}\Big)S_{+1} . \tag{C.62}$$

The flux then can be put in terms of the amplitude coefficients and the angular functions as

$$\frac{d^2E}{dtd\Omega} = \frac{1}{8\pi\Sigma}\left[|\mathcal{T}_{+1}|^2 S_{+1}^2 + a\sin\theta\frac{|\mathcal{T}_{+1}|^2}{2Mr_+k_H}\Big(\mathcal{L}_1 S_{+1} + \frac{a^2\sin\theta\cos\theta}{\Sigma}S_{+1}\Big)S_{+1}\right] , \tag{C.63}$$

$$\frac{d^2L}{dtd\Omega} = \frac{1}{8\pi\Sigma}\left[a\sin^2\theta|\mathcal{T}_{+1}|^2 S_{+1}^2 + \sin\theta\frac{|\mathcal{T}_{+1}|^2}{k_H}\Big(\mathcal{L}_1 S_{+1} + \frac{a^2\sin\theta\cos\theta}{\Sigma}S_{+1}\Big)S_{+1}\right] . \tag{C.64}$$

It is possible to simplify this expressions even further by putting in evidence $(2Mr_+k_H)^{-1}$ and k_H^{-1}, respectively in each equation. These expressions can be massaged into

$$\frac{d^2E}{dtd\Omega} = \omega\frac{|\mathcal{T}_{+1}|^2 S_{+1}^2}{16\pi Mr_+k_H} + \frac{|\mathcal{T}_{+1}|^2 S_{+1}a\sin\theta}{16\pi Mr_+k_H\Sigma}\Big(\mathcal{L}_1 S_{+1} - QS_{+1} + \frac{a^2\sin\theta\cos\theta}{\Sigma}S_{+1}\Big) , \tag{C.65}$$

$$\frac{d^2L}{dtd\Omega} = m\frac{|\mathcal{T}_{+1}|^2 S_{+1}^2}{16\pi Mr_+k_H} + \frac{|\mathcal{T}_{+1}|^2 S_{+1}\sin\theta}{8\pi k_H\Sigma}\Big(\mathcal{L}_1 S_{+1} - QS_{+1} + \frac{a^2\sin\theta\cos\theta}{\Sigma}S_{+1}\Big) . \tag{C.66}$$

All the terms except the first, in each equation, have an angular dependence expressed by

$$\begin{aligned}
&\frac{S_{+1}\sin\theta}{\Sigma}(\mathcal{L}_1 S_{+1} - QS_{+1}) + \frac{a^2\cos\theta\sin^2\theta S_{+1}^2}{\Sigma^2} \\
&= \frac{S_{+1}}{\Sigma}\partial_\theta(\sin\theta S_{+1}) + \frac{a^2\cos\theta\sin^2\theta S_{+1}^2}{\Sigma^2} \\
&= \frac{1}{\sin\theta}\left(\frac{1}{2\Sigma}\partial_\theta(\sin^2\theta S_{+1}^2) + \frac{2a^2\cos\theta\sin^3\theta S_{+1}^2}{2\Sigma^2}\right) \\
&= \frac{1}{\sin\theta}\partial_\theta\left(\frac{\sin^2\theta S_{+1}^2}{2\Sigma}\right) ,
\end{aligned} \tag{C.67}$$

where in the first step has been used that

$$\sin\theta\mathcal{L}_1 S_{+1} = \mathcal{L}_0(\sin\theta S_{+1}) \ , \tag{C.68}$$

$$\mathcal{L}_0(\sin\theta S_{+1}) - Q\sin\theta S_{+1} = \partial_\theta(\sin\theta S_{+1}) \ . \tag{C.69}$$

Substituting this into the fluxes and integrating in the solid angle, it gives

$$\frac{dE}{dt} = \omega\frac{|\mathcal{T}_{+1}|^2}{8Mr_+k_H} + \frac{|\mathcal{T}_{+1}|^2 a}{8Mr_+k_H}\int_0^\pi \partial_\theta\left(\frac{\sin^2\theta S_{+1}^2}{2\Sigma}\right)d\theta \ , \tag{C.70}$$

$$\frac{dL}{dt} = m\frac{|\mathcal{T}_{+1}|^2}{8Mr_+k_H} + \frac{|\mathcal{T}_{+1}|^2}{4k_H}\int_0^\pi \partial_\theta\left(\frac{\sin^2\theta S_{+1}^2}{2\Sigma}\right)d\theta \ , \tag{C.71}$$

where it has been used the normalization of S_{+1} in the sphere

$$\int_0^\pi (S_{+1})^2 \sin\theta d\theta = 1 \ . \tag{C.72}$$

It is clear that

$$\int_0^\pi \partial_\theta\left(\frac{\sin^2\theta S_{+1}^2}{2\Sigma}\right)d\theta = \left.\left(\frac{\sin^2\theta S_{+1}^2}{2\Sigma}\right)\right|_0^\pi = 0 \ . \tag{C.73}$$

And so (C.16) is proved and the result in (3.175) is also obtained.

Appendix D

Integrability conditions of the closed conformal Killing-Yano form

It is important to prove the integrability conditions (3.195) and (3.196) of the closed conformal Killing-Yano tensor, since these conditions lead to ξ being a Killing vector field and to the commutativity of $\boldsymbol{h}$ with the Ricci tensor R_{ab}. This last feature is required for the proof of separability of the Proca equations. One needs to start from the Ricci identity for a $(0,p)$ tensor

$$\nabla_{[a}\nabla_{b]}T_{c_1\ldots c_p} = -\sum_{j=1}^{p} T_{c_1\ldots c_{j-1}dc_{j+1}\ldots c_p}R^{d}{}_{c_j ab} \ . \tag{D.1}$$

The identity simplifies for a p-form $\boldsymbol{h}$

$$\nabla_{[a}\nabla_{b]}h_{c_1\ldots c_p} = -pR_{ab}{}^{d}{}_{[c_1}h_{|d|c_2\ldots c_p]} \ . \tag{D.2}$$

This can be shown by using the properties of the anti-symmetrization. Considering now that $\boldsymbol{h}$ is a closed conformal Killing-Yano tensor, the first equation in (3.194) can be used to get

$$g_{b[c_1}\nabla_{|a|}\xi_{c_2\ldots c_p]} - g_{a[c_1}\nabla_{|b|}\xi_{c_2\ldots c_p]} + R_{ab}{}^{d}{}_{[c_1}h_{|d|c_2\ldots c_p]} = 0 \ . \tag{D.3}$$

From this condition, it is possible to obtain the covariant derivative of ξ in terms of the Riemann tensor and $\boldsymbol{h}$. The trick resides in rewrite the terms

$$pg_{a[c_1}\nabla_{|b|}\xi_{c_2\ldots c_p]} = g_{ac_1}\nabla_b\xi_{c_2\ldots c_p} - (p+1)g_{a[c_1}\nabla_b\xi_{c_2\ldots c_p]} \ , \tag{D.4}$$

$$pg_{b[c_1}\nabla_{|a|}\xi_{c_2\ldots c_p]} = g_{bc_1}\nabla_a\xi_{c_2\ldots c_p} - (p-1)g_{b[c_2}\nabla_{|a}\xi_{c_1|c_3\ldots c_p]} \ . \tag{D.5}$$

Again, this is possible due to the properties of the anti-symmetrization. Now, substituting this into (D.3), applying g^{bc_1} and using the cyclic property of the Riemann tensor, it follows that

$$\nabla_a\xi_{c_2\ldots c_p} = -\frac{1}{D-p}\Big[R^{d}{}_{a}h_{dc_2\ldots c_p} - \frac{(p-1)}{2}R_{dea[c_2}h^{de}{}_{c_3\ldots c_p]}\Big] \ . \tag{D.6}$$

By applying the covariant derivative to the first equation in (3.194), one obtains

$$\nabla^a\nabla^b h_{c_1\ldots c_p} = pg^{b}{}_{[c_1}\nabla^a\xi_{c_2\ldots c_p]} \ . \tag{D.7}$$

Plugging (D.6) in the above equation, the first integrability condition (3.195) is obtained

$$\nabla^a\nabla^b h_{c_1\ldots c_p} = -\frac{p}{D-p}\Big(R^a{}_e\delta^b_{[c_1}h^e{}_{c_2\ldots c_p]} + \frac{p-1}{2}R_{de}{}^a{}_{[c_1}\delta^b_{c_2}h^{de}{}_{\ldots c_p]}\Big)\ . \tag{D.8}$$

The second integrability condition (3.196) comes from plugging in (D.6) into (D.3)

$$2R^{[a}{}_e\delta^{b]}_{[c_1}h^e{}_{c_2\ldots c_p]} - (D-p)R^{ab}{}_{e[c_1}h^e{}_{c_2\ldots c_p]} + (p-1)R_{de}{}^{[a}{}_{[c_1}\delta^{b]}_{c_2}h^{de}{}_{\ldots c_p]} = 0\ . \tag{D.9}$$

For the case of the principal tensor, these integrability conditions read

$$\nabla^a\nabla^b h_{c_1c_2} = -\frac{2}{D-2}\Big(R^a{}_e\delta^b_{[c_1}h^e{}_{c_2]} + \frac{1}{2}R_{de}{}^a{}_{[c_1}\delta^b_{c_2]}h^{de}\Big)\ , \tag{D.10}$$

$$2R^{[a}{}_e\delta^{b]}_{[c_1}h^e{}_{c_2]} - (D-2)R^{ab}{}_{e[c_1}h^e{}_{c_2]} + R_{de}{}^{[a}{}_{[c_1}\delta^{b]}_{c_2]}h^{de} = 0\ . \tag{D.11}$$

Additionally, the integrability condition for ξ associated to the principal tensor is given by

$$\nabla_a\xi_{c_2} = -\frac{1}{D-2}\Big[R^d{}_a h_{dc_2} - \frac{1}{2}R_{deac_2}h^{de}\Big]\ . \tag{D.12}$$

An important property of the principal tensor can be obtained by looking at (D.11) and choosing $b = c_1$, lower the index a and symmetrize indices a and c_2. Note that making such choice is different from applying $\delta^b_{c_1}$ (contracting indices). The result is the following

$$R^{c_1}_e h^e{}_{(c_2}g_{a)c_1} = 0 \iff R_{ae}h^e{}_{c_2} = h_a{}^e R_{ec_2}\ . \tag{D.13}$$

Thus, the principal tensor and the Ricci tensor commute with each other. The implications of this can be seen when one symmetrizes indices a and c_2 in (D.12)

$$\nabla_{(a}\xi_{c_2)} = -\frac{1}{2(D-2)}\Big[R_a{}^d h_{dc_2} - h_{ad}R^d{}_{c_2}\Big] = 0\ , \tag{D.14}$$

where in the last step has been used (D.13). Raising the indices a and c_2, this corresponds to the Killing equation for a Killing vector field. Thus, $\xi^a\partial_a$ is a Killing vector field.

Appendix E

The ansatz and the separability of Proca equation

The Proca equation takes a simple form (3.255) using the ansatz (3.241). In this appendix, this expression will be derivated. Starting by substituting the ansatz into the Proca equation, it follows

$$\begin{aligned}
&\nabla_b \nabla^b A^a - R^a_{\ n} A^n = \nabla_m \nabla^m \Big(B^{ab} \nabla_b Z \Big) - R^a_{\ b} B^{be} \nabla_e Z \\
&= \nabla_m \Big[(B^{nb} B^{am} \xi_n - B^{mb} B^{an} \xi_n) \beta \nabla_b Z + B^{ab} \nabla^m \nabla_b Z \Big] - R^a_{\ b} B^{be} \nabla_e Z \ , \\
&= \nabla_m \Big[(B^{nb} B^{am} \xi_n - B^{mb} B^{an} \xi_n) \beta \nabla_b Z \Big] + \nabla_m (B^{ab}) \nabla^m \nabla_b Z + B^{ab} \nabla_m \nabla^m \nabla_b Z - R^a_{\ b} B^{be} \nabla_e Z \ ,
\end{aligned} \tag{E.1}$$

where, in the second step, it has been used (3.245). The fourth term contains the highest derivative of the scalar Z and it gives a light on how the identity (3.255) can be reached. First, one needs to commute the operator ∇_b with the box operator

$$B^{ab} \nabla_m \nabla^m \nabla_b Z = B^{ab} \nabla_b \nabla_m \nabla^m Z + B^{ab} R^e_{\ b} \nabla_e Z \ . \tag{E.2}$$

Since the Ricci tensor commutes with the principal tensor (D.13), it will also commute with the Killing tensor $\boldsymbol{k}$. Due to the dependence of $\boldsymbol{B}$ in the principal tensor and in the Killing tensor (also the Killing tensor and the principal tensor commute with each other), then $\boldsymbol{B}$ commutes with the Ricci tensor. This term will then cancel the Ricci dependence of the Proca equation

$$\begin{aligned}
&\nabla_b \nabla^b A^a - R^a_{\ n} A^n = \nabla_m \Big[(B^{nb} B^{am} \xi_n - B^{mb} B^{an} \xi_n) \beta \nabla_b Z \Big] + \nabla_m (B^{ab}) \nabla^m \nabla_b Z \\
&+ B^{ab} \nabla_b (\nabla^m \nabla_m Z) \ .
\end{aligned} \tag{E.3}$$

The next step is to isolate terms with B^{am} where a is the free index. So one has

$$\begin{aligned}
&\nabla_b \nabla^b A^a - R^a_{\ n} A^n \\
&= B^{am} \nabla_m \Big[\nabla^m \nabla_m Z + \beta \xi_n B^{nb} \nabla_b Z \Big] - \beta \nabla_m (B^{mb} \nabla_b Z) B^{an} \xi_n + \beta \nabla_m (B^{am}) B^{nb} \xi_n \nabla_b Z \\
&- \beta \nabla_m (B^{an} \xi_n) B^{mb} \nabla_b Z + \nabla_m (B^{ab}) \nabla^m \nabla_b Z \ ,
\end{aligned} \tag{E.4}$$

This last term must be expanded using (3.245)

$$\begin{aligned}
&\nabla_m(B^{ab})\nabla^m\nabla_b Z = \beta B^{am}\xi_n B^{nb}\nabla_m\nabla_b Z - \beta B^{mb}B^{an}\xi_n\nabla_m\nabla_b Z \\
&= B^{am}\nabla_m\Big[\beta\xi_n B^{nb}\nabla_b Z\Big] - \beta\nabla_m(B^{mb}\nabla_b Z)B^{an}\xi_n \\
&- \beta B^{am}\nabla_m(\xi_n B^{nb})\nabla_b Z + \beta\nabla_m(B^{mb})B^{an}\xi_n\nabla_b Z \ .
\end{aligned} \tag{E.5}$$

Inputing this into (E.4), then one has

$$\begin{aligned}
&\nabla_b\nabla^b A^a - R^a_{\ n}A^n \\
&= B^{am}\nabla_m\Big[\nabla^m\nabla_m Z + 2\beta\xi_n B^{nb}\nabla_b Z\Big] - 2\beta\nabla_m(B^{mb}\nabla_b Z)B^{an}\xi_n \\
&+ \beta\nabla_m(B^{am})B^{nb}\xi_n\nabla_b Z + \beta\nabla_m(B^{mb})B^{an}\xi_n\nabla_b Z \\
&- \beta B^{am}\nabla_m(\xi_n B^{nb})\nabla_b Z - \beta\nabla_m(B^{an}\xi_n)B^{mb}\nabla_b Z \ .
\end{aligned} \tag{E.6}$$

Now, focusing in the last four terms, it follows that

$$\begin{aligned}
&\nabla_m(B^{am})B^{nb}\xi_n\nabla_b Z + \nabla_m(B^{mb})B^{an}\xi_n\nabla_b Z - B^{am}\nabla_m(\xi_n B^{nb})\nabla_b Z - \nabla_m(B^{an}\xi_n)B^{mb}\nabla_b Z \\
&= \nabla_m(B^{am})B^{nb}\xi_n\nabla_b Z + \nabla_m(B^{mb})B^{an}\xi_n\nabla_b Z - B^{am}\xi_n\nabla_m(B^{nb})\nabla_b Z - B^{mb}\xi_n\nabla_m(B^{an})\nabla_b Z \\
&= \frac{1}{A}\Big[k^{ac}\xi_c B^{nb}\xi_n - k^{bm}\xi_m B^{an}\xi_n + k^{bc}\xi_c B^{an}\xi_n - k^{an}\xi_n B^{cb}\xi_b\Big] = 0 \ ,
\end{aligned} \tag{E.7}$$

where it has been used that ξ^a is a Killing vector field (D.14), that the derivatives of tensor $\boldsymbol{B}$ are given by (3.245)-(3.247) and that $B^{am}B^n_{\ m} = k^{an}$ (see (3.243)). Thus, it is concluded that

$$\begin{aligned}
&\nabla_b\nabla^b A^a - R^a_{\ n}A^n \\
&= B^{am}\nabla_m\Big[\nabla^m\nabla_m Z + 2\beta\xi_n B^{nb}\nabla_b Z\Big] - 2\beta\nabla_m(B^{mb}\nabla_b Z)B^{an}\xi_n \ .
\end{aligned} \tag{E.8}$$