Decentralized Navigation System Based on Bearing Measurements for Autonomous Vehicles

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Abstract

The applications of autonomous vehicle formations are expected to grow in the future. Part of this applications will be aimed at subaquatic environments and since the Global Positioning System is not available in such environments, the development of efficient and low-cost navigation systems is crucial. The existing underwater navigation solutions are based on acoustic signals to obtain range and bearing measurements. The solution proposed in this thesis uses these acoustic signals to measure bearings from each vehicle to its neighbours, which are then used to develop a decentralized navigation system. A system considering bearing measurements as outputs is nonlinear, and, in this approach, these systems are transformed into equivalent linear systems by trading the bearing measurement for a conveniently defined artificial linear output. For a linear system, it is possible to obtain theoretical guarantees of stability, which is the main advantage of the proposed solution. The final solution is composed by several local Kalman filters that rely only on local measurements and limited communication between vehicles. Monte Carlo simulations are carried out in order to compare the performance with the extended Kalman filter and the unscented Kalman filter. This comparison leads to the conclusion that the performance of the three solutions is similar. As so, since the proposed solution is the only with theoretical guarantees of convergence and does not present any disadvantage, it is concluded that this is the best option.

Keywords: underwater navigation, decentralized control, bearings, autonomous vehicles, AUV

1. Introduction

1.1. Motivation

A great research effort has been put into decentralized systems in recent years. Isolated systems are becoming more connected everyday, which leads to the need for scalable and robust solutions. Centralized solutions suffer from lack of scalability and are highly exposed to the failure of the central node. Scalability becomes a problem for centralized solutions since, as the systems’ size grows, the number of communications and computing power of the central node grows accordingly. Some politic and economic theories take a viewpoint of centralized versus decentralized to explain the differences between democracies and dictatorships or between free-market and planned economies, which takes the debate between centralized and decentralized far beyond the field of engineering. In the same way a failure of a powerful state has highly impacting consequences, a failure in, for example, a centralized navigation system will compromise the entire system.

An example of a large-scale system are robotic swarms. There are many applications where the use of robotic swarms can be an advantage, and it is probable that, in the future, they will take a role of increasing importance in our societies. Fields such as military, transportation, mobility and warehouse management are just some examples where robotic swarms may revolutionize the industry in the near future. For all this, there is a need to decentralize systems.

Decentralized solutions become even more important when leading with underwater robotic swarms. In such scenarios, communications are specially limited, which undermines most centralized control and navigation systems. One more problem associated with the underwater framework is the unavailability of the Global Positioning System (GPS) due to signal attenuation, which puts the development of feasible navigation systems as a top priority.

The development of decentralized navigation solutions for underwater vehicle formations is, therefore, of uttermost importance.
1.2. State of the art
There are many applications where the use of vehicle formations or robotic swarms is advantageous, and several works have approached this theme, such as [1], where a solution for naval minesweeping is proposed, or [2], which proposes a control method for formations of vehicles based on the formulation of an optimization problem.

In the field of marine robotics, several decentralized solutions can be found with the particularity of taking into account the limitation on communications, such as [3] or [4].

The unavailability of GPS in underwater applications motivated the research and development of underwater navigation systems. Unlike the electromagnetic field, which suffers high attenuation underwater, acoustic waves propagate well and with known velocity underwater. Because of this, acoustic positioning systems were developed for underwater applications. There are three broad classes of underwater acoustic positioning systems: Long Baseline (LBL), Short Baseline (SBL) and Ultra-Short Baseline (USBL) systems. [5] proposes a positioning system based on a LBL system with three acoustic transponders and an extended Kalman filter (EKF). Experimental results of a SBL system are presented in [6]. One example of the design and experimental validation of a USBL positioning system can be seen in [7]. The USBL allows to obtain the bearings between landmarks or vehicles, which can also be used to design bearing-based navigation systems.

The design of bearing-based navigation systems is related to the problem of bearings-only tracking, which has been widely studied, with [8] and [9] as examples of works addressing this problem.

All the previous solutions depend on either range or bearing measurements. These measurements are nonlinear with respect to the position of the agent. The most common approach to the non-linearities is to use the EKF or the unscented Kalman filter (UKF). An alternative is to pose the nonlinear observations as equivalent linear ones, by using state augmentation and artificial outputs. This is the case of in [10] and [11], following the introduction of the technique in [12].

1.3. Contributions
This thesis focuses on developing a decentralized navigation system with globally exponentially stable (GES) error dynamics for tiered formations based on bearing measurements. Local measurements, such as attitude, relative speed and, in some cases, depth are available. The vehicles in the formation also measure bearing to one or more neighbouring vehicles.

The design of the decentralized navigation system can be broken into several parts. At first, local observers are designed by using artificial outputs to cope with the bearings’ non-linearities. In the case when multiple bearings are available, results obtained in [13] are used. The cases when only one bearing is available, with or without depth measurement, are studied and observability conditions are obtained. A linear Kalman filter (LKF) is then used to obtain an observer for the resulting linear time-varying (LTV) system.

Next, the local observers are put together and the stability of the system as a whole is studied, following a similar approach to the one used in [14]. Monte Carlo simulations were performed to study the performance of the proposed solution and compare it with a decentralised EKF and a decentralised UKF.

1.4. Thesis outline
In Chapter 2 the notation used is introduced and the problem addressed in this thesis is formally posed. In Chapter 3 the proposed solution is designed in two phases: first, local observers are designed for the different possible systems; then the system as a whole is built by putting together the local observers. The EKF and UKF are introduced and adapted into a decentralized solution where each node has limited information in Chapters 4 and 5, respectively.

In Chapter 6, results are presented. First, the simulation set-up is introduced, followed by the parameters for each estimation solution. Finally, results of Monte Carlo analysis are presented to compare the performance of the three solutions, as well as the Bayesian Cramér-Rao Bound (BCRB).

Finally, in Chapter 7 some conclusions are drawn.

2. Problem statement and notation

2.1. Notation
Throughout the thesis, the symbol \( \mathbf{0} \) denotes a matrix of 0s of proper dimensions and \( \mathbf{I}_n \) denotes the \( n \times n \) identity matrix. A block diagonal matrix is represented by \( \text{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n) \). The special orthogonal group is denoted by \( SO(3) := \{ \mathbf{X} \in \mathbb{R}^{3 \times 3} : \mathbf{X}^T \mathbf{X} = \mathbf{I}, \det(\mathbf{X}) = 1 \} \), and the set of unit vectors is defined as \( S(2) := \{ \mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\| = 1 \} \). For \( \mathbf{x} \in \mathbb{R}^3 \), \( \mathbf{x}^\star, \mathbf{x}^\circ \), and \( \mathbf{x}^2 \) denote the first, second, and third component of \( \mathbf{x} \), respectively. The transpose operator is defined as \( (\cdot)^T \).

2.2. Problem statement
Consider a formation of \( N \) vehicles, indexed from 1 to \( N \). All the vehicles are evolving in a fluid whose velocity is assumed to have a time-invariant spatial distribution. Moreover, it is assumed that the velocity of the vehicles is small enough such that the
The fluid’s velocity can be considered constant for each vehicle. Since the vehicles may be operating in different regions of the space, it is assumed that the velocity of the fluid may differ from vehicle to vehicle. As so, \( \mathbf{v}_{fi}(t) \in \mathbb{R}^3 \) denotes the fluid velocity around vehicle \( i \), expressed in a local inertial frame, \( \{I\} \). The position of the vehicle \( i \), expressed in \( \{I\} \), is denoted by \( \mathbf{p}_i(t) \in \mathbb{R}^3 \).

Each vehicle is moving with a velocity relative to the fluid, measured by a relative velocity sensor, \( \mathbf{v}_i(t) \in \mathbb{R}^3 \), expressed in the body frame, \( \{B_i\} \). Each vehicle is also equipped with an attitude and heading reference system (AHRS) which provides a rotation matrix, \( \mathbf{R}_i(t) \in SO(3) \), from \( \{B_i\} \) to \( \{I\} \).

The kinematics of vehicle \( i \) are given by
\[
\begin{align*}
\dot{\mathbf{p}}_i(t) &= \mathbf{v}_{fi}(t) + \mathbf{R}_i(t)\mathbf{v}_i(t) \\
\mathbf{v}_{fi}(t) &= 0
\end{align*}
\]

The formation is assumed to be organized in a tiered topology, and each vehicle has access to either:

- An absolute position measurement, provided by, for example, GPS or a LBL system, if they are in the first tier; or
- Bearing measurements and position estimates of one or more vehicles in the tier above and, in some cases, depth measurements.

The focus of this paper is on the second case, since the position is available in the first one. In the second case, the outputs are available at discrete-time and are given by
\[
\begin{align*}
\mathbf{d}_{ij}(k) &= \mathbf{R}_i^T(t_k)\mathbf{p}_j(t_k) - \mathbf{p}_i(t_k), \quad j \in D_i \\
h_i(k) &= \mathbf{p}^T_i(t_k), \quad \text{if depth available},
\end{align*}
\]

where \( D_i \) is the set of vehicles to which vehicle \( i \) has bearing measurements.

From now on, and unless specified otherwise, it is considered
\[
\mathbf{d}_{ij}(k) = \frac{\mathbf{p}_j(t_k) - \mathbf{p}_i(t_k)}{\|\mathbf{p}_j(t_k) - \mathbf{p}_i(t_k)\|}, \quad j \in D_i,
\]

since this simplifies the computations. This is done without loss of generality since the matrix \( \mathbf{R}_i(t_k) \) is available and invertible. For simulation purposes, the original bearing measurement is used.

Because the communication and the bearing measurements between vehicles are only available at low frequency, the system must be discretized, which leads to
\[
\begin{align*}
\mathbf{p}(t_{k+1}) &= \mathbf{p}(t_k) + T\mathbf{v}_{fi}(t_k) + \mathbf{u}(k) \\
\mathbf{v}_{fi}(t_{k+1}) &= \mathbf{v}_{fi}(t_k) \\
\mathbf{d}_{ij}(k) &= \frac{\mathbf{p}_j(t_k) - \mathbf{p}_i(t_k)}{\|\mathbf{p}_j(t_k) - \mathbf{p}_i(t_k)\|}, \quad j \in D_i \\
h_i(k) &= \mathbf{p}^T_i(t_k), \quad \text{if depth available}
\end{align*}
\]

where \( T \) is the sampling period and \( \mathbf{u}(k) \) is given by
\[
\mathbf{u}(k) = \int_{t_k}^{t_{k+1}} \mathbf{R}_i(t)\mathbf{v}_i(t)dt.
\]

The problem addressed in this paper is that of designing a decentralized observer, with globally exponentially stable dynamics, for the position and local fluid velocity of each vehicle, \( \mathbf{p}_i(t_k) \) and \( \mathbf{v}_{fi}(t_k) \) respectively. The decentralized observer is composed by local observers, each one with access to the local measurements described before.

3. Linear Kalman filter

3.1. Local observer design

Depending on the available measurements, the design of a local observer for (2) differs. Three cases are discussed: i) when one bearing and depth are available; ii) when two or more bearings are available; and iii) when one bearing, without depth, is available. To simplify the notation, the index \( i \) will be omitted from this point onward, resulting in the system
\[
\begin{align*}
\mathbf{p}(t_{k+1}) &= \mathbf{p}(t_k) + T\mathbf{v}_{fi}(t_k) + \mathbf{u}(k) \\
\mathbf{v}_{fi}(t_{k+1}) &= \mathbf{v}_{fi}(t_k) \\
\mathbf{d}_{ij}(k) &= \frac{\mathbf{p}_j(t_k) - \mathbf{p}(t_k)}{\|\mathbf{p}_j(t_k) - \mathbf{p}(t_k)\|}, \quad j \in D \\
h(k) &= \mathbf{p}^T(t_k), \quad \text{if depth available}
\end{align*}
\]

3.2. Artificial output

The dynamic system (4) is nonlinear due to the bearing outputs. To address this issue, and obtain an LTV system, the bearing outputs will be replaced by the artificial output \( \mathbf{z}_j(k) \in \mathbb{R}^3 \), which is given by
\[
\mathbf{z}_j(k) := (\mathbf{I} - \mathbf{d}_j(k)d_j^T(k))\mathbf{p}_j(t_k)
\]

This quantity is known since \( (\mathbf{I} - \mathbf{d}_j(k)d_j^T(k))\mathbf{p}_j(t_k) \) can be computed using known measurements. Also, because \( \mathbf{d}_j(k) \) is a known measurement, \( \mathbf{z}_j(k) \) is linear on the state \( \mathbf{p}(t_k) \). Replacing \( \mathbf{d}_j(k) \) by \( \mathbf{z}_j(k) \) in (4) yields
\[
\begin{align*}
\mathbf{p}(t_{k+1}) &= \mathbf{p}(t_k) + T\mathbf{v}_{fi}(t_k) + \mathbf{u}(k) \\
\mathbf{v}_{fi}(t_{k+1}) &= \mathbf{v}_{fi}(t_k) \\
\mathbf{z}_j(k) &= (\mathbf{I} - \mathbf{d}_j(k)d_j^T(k))\mathbf{p}(t_k), \quad j \in D \\
h(k) &= \mathbf{p}^T(t_k), \quad \text{if depth available}
\end{align*}
\]
This is an LTV system and can be written in the form

\[
\begin{align*}
\mathbf{x}(k+1) &= \mathbf{A}_k \mathbf{x}(k) + \mathbf{B}_k \mathbf{u}(k) \\
\mathbf{y}(k) &= \mathbf{C}_k \mathbf{x}(k)
\end{align*}
\]

where

\[
\mathbf{x}(k) = [\mathbf{p}^T(t_k) \mathbf{v}^T(t_k)]^T \in \mathbb{R}^6,
\]

and

\[
\mathbf{y}(k) = [\mathbf{z}_1^T(k) \ldots \mathbf{z}_L^T(k) \ h(k)]^T \in \mathbb{R}^{3L+1},
\]

or

\[
\mathbf{y}(k) = [\mathbf{z}_1^T(k) \ldots \mathbf{z}_L^T(k)]^T \in \mathbb{R}^{3L},
\]

with \(L\) representing the number of elements in \(D\).

### 3.3. Observability

The state matrices when depth and only one bearing measurement are available are given by

\[
\mathbf{A}_k = \begin{bmatrix} \mathbf{I}_3 & \mathbf{T} \mathbf{I}_3 \\ 0 & \mathbf{I}_3 \end{bmatrix} \in \mathbb{R}^{6 \times 6}, \quad \mathbf{B}_k = \begin{bmatrix} \mathbf{I}_3 \\ 0 \end{bmatrix} \in \mathbb{R}^{6 \times 3},
\]

\[
\mathbf{C}_k = \begin{bmatrix} \mathbf{I}_3 - \mathbf{d}_j(k) \mathbf{d}_j^T(k) & 0 \\ \mathbf{e}_3^T & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 6},
\]

with \(\mathbf{e}_3 := [0 \ 0 \ 1]^T\). The following theorem addresses the observability of this system.

**Theorem 1.** The system (5) with depth and only one bearing available is observable on the interval \([k_a, k_a + 2]\) if and only if \(d_j(k_a) \neq 0\) and \(d_j(k_a + 1) \neq 0\).

**Proof.** The system is observable on \([k_a, k_a + 2]\) if and only if the observability matrix, \(\mathcal{O}(k_a, k_a + 2)\) has rank equal to the number of states. The proof follows by showing that the case. The observability matrix is given by

\[
\mathcal{O}(k_a, k_a + 2) = \begin{bmatrix} \mathbf{C}_{k_a} \\ \mathbf{C}_{k_a+1} \mathbf{A}_{k_a} \end{bmatrix} \in \mathbb{R}^{8 \times 6}.
\]

The rank condition on the observability matrix is equivalent to state that the only solution of

\[
\mathcal{O}(k_a, k_a + 2) \mathbf{c} = \mathbf{0}
\]

is \(\mathbf{c} = \mathbf{0}\). Considering \(\mathbf{c} = [\mathbf{c}_1^T \ \mathbf{c}_2^T]^T\), with \(\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^3\), this can be rewritten as

\[
\begin{align*}
(\mathbf{I} - \mathbf{d}_j(k_a) \mathbf{d}_j^T(k_a)) \mathbf{c}_1 &= \mathbf{0} \\
\mathbf{c}_1^j &= \mathbf{0} \\
(\mathbf{I} - \mathbf{d}_j(k_a + 1) \mathbf{d}_j^T(k_a + 1)) (\mathbf{c}_1 + \mathbf{T} \mathbf{c}_2) &= \mathbf{0} \quad (6)
\end{align*}
\]

The sufficiency of the conditions of the theorem is shown by direct proof. Suppose \(\mathbf{d}_j(k_a) \neq 0 \land \mathbf{d}_j(k_a + 1) \neq 0\). Next, it will be shown that the only solution for (6) is \(\mathbf{c} = \mathbf{0}\). The first equation of (6) allows to conclude that \(\mathbf{c}_1 = \alpha \mathbf{d}_j(k_a), \ \alpha \in \mathbb{R}\). Since \(\mathbf{d}_j(k_a) \neq 0\) and \(\mathbf{c}_1^j = \mathbf{0}\), it must be \(\alpha = 0\) and thus \(\mathbf{c}_1 = \mathbf{0}\). Then, the last two equations of (6) become

\[
\begin{align*}
(\mathbf{I} - \mathbf{d}_j(k_a + 1) \mathbf{d}_j^T(k_a + 1)) \mathbf{c}_2 &= \mathbf{0} \\
\mathbf{c}_2^j &= \mathbf{0}
\end{align*}
\]

Following the same steps, it can be concluded that \(\mathbf{c}_2 = \mathbf{0}\). This concludes the proof of sufficiency. The proof of necessity follows by contraposition. Suppose that the conditions of the theorem are not met. This may happen because \(\mathbf{d}_j(k_a) = 0\) or \(\mathbf{d}_j(k_a + 1) = 0\). In the first case, take

\[
\mathbf{c} = \begin{bmatrix} \mathbf{d}_j(k_a) \\ -\frac{1}{\alpha} \mathbf{d}_j(k_a) \end{bmatrix}.
\]

This non zero \(\mathbf{c}\) fulfils (6), which makes the system not observable. Suppose now that \(\mathbf{d}_j(k_a + 1) = 0\) and take

\[
\mathbf{c} = \begin{bmatrix} 0 \\ \mathbf{d}_j(k_a + 1) \end{bmatrix}.
\]

This non zero \(\mathbf{c}\) also fulfils (6), which implies that the system is not observable, thus concluding the proof of necessity. \(\square\)

When more than one bearing is available but there is no depth measurement, the state matrices are given by

\[
\mathbf{A}_k = \begin{bmatrix} \mathbf{I}_3 & \mathbf{T} \mathbf{I}_3 \\ 0 & \mathbf{I}_3 \end{bmatrix} \in \mathbb{R}^{6 \times 6}, \quad \mathbf{B}_k = \begin{bmatrix} \mathbf{I}_3 \\ 0 \end{bmatrix} \in \mathbb{R}^{6 \times 3},
\]

\[
\mathbf{C}_k = \begin{bmatrix} \mathbf{I}_3 - \mathbf{d}_j(k) \mathbf{d}_j^T(k) & 0 \\ \mathbf{e}_3^T & 0 \end{bmatrix} \in \mathbb{R}^{4 \times 6},
\]

where \(L\) is the number of vehicles in \(D\). This system has been studied in [13], from where the following theorem can be used.

**Theorem 2.** The system (5) with more than one bearing and no depth measurement is observable on the interval \([k_a, k_a + 2]\) if and only if there exists \(m, n, l, p \in \{1, \ldots, L\}\) such that

\[
\mathbf{d}_m(k_a) \neq \alpha_1 \mathbf{d}_n(k_a)
\]

and

\[
\mathbf{d}_l(k_a + 1) \neq \alpha_2 \mathbf{d}_p(k_a + 1)
\]

for all \(\alpha_1, \alpha_2 \in \mathbb{R}\).

When only one bearing is available but there is no depth measurement, the state matrices are given by

\[
\mathbf{A}_k = \begin{bmatrix} \mathbf{I}_3 & \mathbf{T} \mathbf{I}_3 \\ 0 & \mathbf{I}_3 \end{bmatrix}, \quad \mathbf{B}_k = \begin{bmatrix} \mathbf{I}_3 \\ 0 \end{bmatrix},
\]
The following theorem addresses the observability of this system.

**Theorem 3.** The system (5) with only one bearing available is observable on the interval [\(k_a, k_a + 3\)] if and only if \(d_1(k_a), d_j(k_a + 1)\) and \(d_j(k_a + 2)\) are linearly independent.

**Proof.** The proof is done by showing that the observability matrix has rank equal to the number of states if and only if the conditions of the theorem are met, in a similar way to Theorem 1. \(\square\)

With the observability studied, conditions for uniform complete observability (UCO) should be derived so that the design of a Kalman filter leads to an observer with guarantees of GES error dynamics. However, such proof implies long and tedious calculations, which follows similar steps considering uniform bounds in time, which were left outside of the scope of this work.

### 3.4. Decentralized System

The conditions for stability of the local observers have already been established. However, these local observers have access to the position of the vehicles to which bearings are measured. When the observers are put together into a decentralized system, they will only have access to position estimates, which can be written as

\[
\hat{p}_j(t_k) = P_j(t_k) + e_j(k),
\]

where \(\hat{p}_j(t_k)\) is an estimate of the position of vehicle \(j\) and \(e_j(k)\) is a term with GES dynamics representing the estimation error of \(p_j(t_k)\). This will alter the value of the artificial output, which will be given by

\[
z_j(k) = (I - d_j(k)d_j^T(k))p(t_k) + \bar{e}_j(k),
\]

where \(\bar{e}_j(k)\) is defined as

\[
\bar{e}_j(k) = (I - d_j(k)d_j^T(k))e_j(k).
\]

Since \(e_j(k)\) decays exponentially and \((I - d_j(k)d_j^T(k))\) is bounded, \(\bar{e}_j(k)\) will also decay exponentially.

As so, the effect of not having the true position of the other vehicles can be regarded as an exponentially decaying perturbation on the outputs of system (5). This will not alter the dynamics of the Kalman filter covariance matrix

\[
\begin{cases}
P_{k|k-1} = A_kP_{k-1|k-1}A_k^T + Q \\
K_k = P_{k|k-1}C_k^T(\text{R} + C_kP_{k-1|k-1}C_k^T)^{-1} \\
P_{k|k} = (I - K_kC_k)P_{k|k-1}(I - K_kC_k)^T + K_kR_kK_k^T\end{cases}
\]

where \(Q\) and \(R\) are the process and output noise covariance matrices, respectively. \(P\) is the estimation error covariance and \(K\) is the observer gain. Since these equations are not affected by the perturbation, they will remain bounded. The estimates will be given by

\[
\dot{x}(k + 1) = A_k\hat{x}(k) + B_ku(k) + K_k(z(k) - C_k\hat{x}(k)).
\]

Considering (7), the exponentially decaying perturbation will be multiplied by a bounded matrix, \(K\), which will cause an exponentially decaying error on the estimate of the state \(x\).

All the local observers of the vehicles of the second tier receive true information of the position of the vehicles of the tier above, as it is assumed the first tier has access to its own position. Therefore, they will produce estimates of their own position with GES error dynamics. As shown before, all the vehicles receiving position estimates with GES error dynamics will also produce positions estimates with GES error dynamics of their own. As so, the observers of all the tiers will converge, since the errors that are propagated will always have GES error dynamics.

### 4. Extended Kalman filter

To compare the performance of the proposed solution, a decentralized EKF was implemented. It is important to note that, unlike the LKF, the EKF has no stability guarantee, which represents an advantage of the proposed solution.

Consider a system in the form

\[
\begin{cases}
x(k + 1) = f(x(k), u(k)) + w(k) \\
y(k) = h(x(k)) + v(k)
\end{cases}
\]

where \(w(k) \in \mathbb{R}^n\) and \(v(k) \in \mathbb{R}^m\) are the process and output noise, respectively. \(w(k) \in \mathbb{R}^n\) and \(v(k) \in \mathbb{R}^m\) are assumed to be zero-mean white Gaussian noise with covariance matrices \(Q \in \mathbb{R}^{n \times n}\) and \(R \in \mathbb{R}^{m \times m}\), respectively.

To adapt the LKF to a nonlinear model, the nonlinear model is linearised. Let \(F_k\) and \(H_k\) be the Jacobian matrices of \(f\) and \(h\) evaluated at \(\hat{x}\). This approximations will result in the predict step

\[
\begin{cases}
\dot{x}(k + 1) = f(\hat{x}(k), u(k)) \\
\dot{P}(k + 1) = F_kP(k|k)F_k^T + Q
\end{cases}
\]

and the update step

\[
\begin{cases}
K_k = P(k|k - 1)H_k^T(H_kP(k|k - 1)H_k^T + R)^{-1} \\
\dot{x}(k|k) = \hat{x}(k|k - 1) + K_k(y(k) - h(\hat{x}(k|k - 1))) \\
\dot{P}(k|k) = (I - K_kH_k)P(k|k - 1)
\end{cases}
\]

More details on the deduction of the EKF can be found in [15].
5. Unscented Kalman filter

Consider a nonlinear function $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$. Also, consider a random variable $\mathbf{x} \in \mathbb{R}^n$ with expected value $\bar{\mathbf{x}} \in \mathbb{R}^n$ and covariance $\mathbf{P}_x \in \mathbb{R}^{n \times n}$.

The Unscented Transform (UT) is a method of approximating the statistical characterization of the random variable $\mathbf{y} = g(\mathbf{x})$. The UT starts by using the statistical characterization of $\mathbf{x}$ to compute a set of sigma points and corresponding weights that capture its mean and covariance. Then, these sigma points are used as argument to the nonlinear function, and the resulting values are used to compute the approximated statistics of $\mathbf{y}$.

Instead of using first-order linearisation to deal with the nonlinearities of the model, as the EKF does, the UKF uses the UT. This allows to take into the account non-Gaussian noise, as well as better approximate the non-linearities. More details on the implementation used, as well as the advantages of the UKF over the EKF can be found in [16]. However, it is important to note that the UKF does not present any guarantee of convergence either.

6. Simulations results

Monte Carlo simulations were performed to access the performance of the proposed solution when the measurements are subject to noise. A decentralized implementation used, as well as the advantages of the UKF over the EKF can be found in [16].

Besides, it is assumed that $\mathbf{p}_j(t_k)$ is received at low frequency in case of a vehicle in tier 1, provided by GPS or a LBL system available in tier 0. In all the other tiers, an estimate of $\mathbf{p}_j(t_k)$ communicated by vehicle $j$ is received instead.

The attitude angles are used to compute the rotation matrix from the body frame to the inertial frame;

- Inclination and azimuth angles, $\theta_{ij}(t)$ and $\phi_{ij}(t)$ respectively, to vehicle $j$, expressed in the body frame and available at low frequency;
- Depth sensor providing $p_i^z(t)$ at high frequency.

The computation of the bearing is done using the azimuth and inclination angles

$$
\mathbf{d}_{ij}^{B_i}(t_k) = \begin{bmatrix} 
\sin(\theta_{ij}(t_k))\cos(\phi_{ij}(t_k)) \\
\sin(\theta_{ij}(t_k))\sin(\phi_{ij}(t_k)) \\
\cos(\theta_{ij}(t_k)) 
\end{bmatrix},
$$

where $\mathbf{d}_{ij}^{B_i}(t_k)$ is the bearing measurement presented in the body frame $\{B_i\}$. From this, it can be obtained

$$
\mathbf{d}_{ij}(t_k) = \mathbf{R}(t_k)\mathbf{d}_{ij}^{B_i}(t_k).
$$

The sampling periods were set to $T = 1$ s and $T_h = 0.01$ s, for the low and high frequency measurements, respectively. Additive white zero-mean Gaussian noise was introduced in every measurement available. For the position of vehicles in tier 0, additive white zero-mean Gaussian noise with covariance matrix

$$
0.01 \times \begin{bmatrix} 
1 & 0.1 & 0.1 \\
0.1 & 1 & 0.1 \\
0.1 & 0.1 & 1 
\end{bmatrix},
$$

was

$$
\mathbf{u}_i(t) = \int_{t_k}^{t_{k+1}} \mathbf{R}_i(t)\mathbf{v}_i(t)dt.
$$

To implement the solutions proposed, $\mathbf{u}_i(t)$, $\mathbf{d}_{ij}(t_k)$, $h_i(t_k)$ and $\mathbf{p}_j(t_k)$ are needed. However, to make the simulation model as realistic as possible, it was not considered that this quantities were directly available. Instead, the set of sensors available are

- Relative velocity sensor, such as a DVL, providing $\mathbf{v}_i(t) = \mathbf{p}_i(t) - \mathbf{v}_{fi}(t)$ at high frequency and expressed in the body frame;
- AHRS providing the attitude angles at high frequency, which allow the computation of the rotation matrix from the body frame to the inertial frame;
- Depth sensor providing $p_i^z(t)$ at high frequency;
which was not set to a diagonal matrix to introduce some correlation between the components, as usually happens in such measurements. White zero-mean Gaussian noise was added to the pitch and roll angles with a standard deviation of 0.01° and 0.03°, respectively. For the azimuth and inclination angles, used to compute the bearing, additive zero-mean white Gaussian noise with a standard deviation of 1° was considered. The relative velocity measurements were corrupted by additive, uncorrelated, zero-mean white Gaussian noise with standard deviation of 0.01 m/s. Finally, the depth measurements were corrupted by additive zero-mean white Gaussian noise with a standard deviation of 0.1 m.

The formation was organized according to the graph in Fig. 1. Vehicles 3, 4 and 5 have access to depth measurements while the vehicles 6 and 7 have not. The two vehicles in tier 0 have access to measurements of their own position and, for that reason, observers for this tier are not simulated. The formation includes, at least, one of each of the three cases studied: one bearing without depth; one bearing and depth; and multiple bearings without depth.

The attitudes of all vehicles was set to be constant and equal to the identity rotation matrix.

For the LKF, the state and output noise covariance matrices were set to, respectively, $\mathbf{Q} = diag(0.01^2 \mathbf{I}_3, 0.001^2 \mathbf{I}_3)$ and $\mathbf{R} = diag(0.1^2, 10 \mathbf{I})$ or $\mathbf{R} = 10 \mathbf{I}$, depending on whether depth is available or not. The matrix $\mathbf{P}$ was set to $diag(10^2 \mathbf{I}_3, \mathbf{I}_3)$.

For the EKF, the state and output noise covariance matrices were set to, respectively, $\mathbf{Q} = diag(0.01^2 \mathbf{I}_3, 0.001^2 \mathbf{I}_3)$ and $\mathbf{R} = diag(0.1^2, 0.001 \mathbf{I})$ or $\mathbf{R} = 0.001 \mathbf{I}$, depending on whether depth is available or not. The matrix $\mathbf{P}$ was set to $diag(10^2 \mathbf{I}_3, \mathbf{I}_3)$ and the initial state estimates were generated as in

<table>
<thead>
<tr>
<th>Vehicle</th>
<th>Initial Position (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[0 0 0]</td>
</tr>
<tr>
<td>2</td>
<td>[100 100 0]</td>
</tr>
<tr>
<td>3</td>
<td>[1 1 -50]</td>
</tr>
<tr>
<td>4</td>
<td>[0 10 -60]</td>
</tr>
<tr>
<td>5</td>
<td>[110 100 -40]</td>
</tr>
<tr>
<td>6</td>
<td>[90 90 -30]</td>
</tr>
<tr>
<td>7</td>
<td>[50 50 -100]</td>
</tr>
</tbody>
</table>

Table 1: Initial positions used in the simulations.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Position (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>[0 0 0]</td>
</tr>
<tr>
<td>100</td>
<td>[50 0 0]</td>
</tr>
<tr>
<td>200</td>
<td>[50 20 0]</td>
</tr>
<tr>
<td>300</td>
<td>[20 20 0]</td>
</tr>
<tr>
<td>400</td>
<td>[20 40 0]</td>
</tr>
<tr>
<td>500</td>
<td>[50 40 0]</td>
</tr>
<tr>
<td>600</td>
<td>[50 60 0]</td>
</tr>
<tr>
<td>800</td>
<td>[5 30 -30]</td>
</tr>
<tr>
<td>1000</td>
<td>[5 0 -30]</td>
</tr>
</tbody>
</table>

Table 2: Trajectory waypoints for vehicle 1

The trajectories of the vehicles were generated using waypoints. All the vehicles, with exception for vehicle 2, followed the same trajectory, but with different starting points. The starting points of each vehicle can be seen in Table 1 and the waypoints of vehicle 1 can be seen in Table 2.

The acceleration of each vehicle was limited to 0.01 m/s², which resulted in the curve represented in Fig. 2.

For vehicle 2, the trajectory followed the same curve of the other vehicles, to which it was added $[10\sin(0.1t) 50\cos(0.13) 0]^T$. This is done to enrich the bearing values of vehicle 6 relative to vehicle 2, so that the system becomes observable.
the previous section.

For the UKF, the matrices were chosen the same as in the EKF, with the parameters responsible for the generation of sigma points being set to $\alpha = 1$, $k = 0$ and $\beta = 2$.

6.2. Bayesian Cramér-Rao Bound

Consider a system in the form of

\[
\begin{aligned}
    x(k+1) &= A(k)x(k) + B(k)u(k) + n_x(k)
    \\
    y(k) &= h(x(k)) + n_y(k)
\end{aligned}
\]

where $x(k)$ is the state, $u(k)$ is a deterministic input, and $y(k)$ is the output, which depends on the state through a nonlinear function $h(x(k))$. Both $n_x(k)$ and $n_y(k)$ follow a zero-mean Gaussian distribution with covariance matrices $Q_x(k)$ and $Q_y(k)$ respectively. For a system in this form, the BCRB is provided in [17], which is a lower bound on the performance achievable by any unbiased estimator.

The recursion used to compute the BCRB is the same as the one in the EKF, with the difference that the Jacobian of $h$ is evaluated at the true state instead of the state estimate. As so, any unbiased observer for (2) should perform worst than the BCRB. However, due to non-linearities, it is possible that the observers designed are slightly biased.

6.3. Monte Carlo simulations

To compare the performance of the solutions, Monte Carlo simulations were performed. 1000 simulations were carried out and, for each, different randomly generated noise signals were considered. Monte Carlo analysis is used to analyse the variance of the solutions, by computing the mean estimation error, and the filter performance in terms of error covariance, using the root-mean squared error. Also, both allow the study of the convergence speed.

The formation consists of 7 vehicles, with only 5 of them having estimation results. Due to space limitations only the results for vehicle 3 will be presented. The results concerning the others are similar. Also, because conclusions drawn from the position and velocity estimation are the same, only the position results are presented.

Table 3: Vehicle 3: Steady-state RMSE

<table>
<thead>
<tr>
<th></th>
<th>Position (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$x$</td>
</tr>
<tr>
<td>LKF</td>
<td>0.0863</td>
</tr>
<tr>
<td>EKF</td>
<td>0.1024</td>
</tr>
<tr>
<td>UKF</td>
<td>0.1024</td>
</tr>
</tbody>
</table>

In Figs. 3, 4, and 5, the average position estimation errors of vehicle 3 for the LKF, EKF and UKF, respectively, are presented.

While the EKF and UKF present the same results, the LKF shows a smaller RMSE in steady-state.

In sum, it was possible to see that the proposed solution presented a slower behaviour but with smaller error covariance. All are comparable, without significant differences and are highly dependent on the tuning of the noise covariance matrices. As so, it is possible to conclude that the three solutions are similar in terms of performance, while it is impossible to determine if one outperforms the others. However, the proposed solution has the advantage of having theoretical guarantees of convergence.
7. Conclusions
In underwater scenarios the communication bandwidth is very limited, rendering centralized navigation solutions impossible to implement. This thesis presents a cooperative, decentralized navigation solution for formations of underwater vehicles based on bearing measurements. Three different cases were studied to design a flexible solution for heterogeneous formations. In order to cope with the nonlinear nature of the outputs, artificial outputs are employed that render the dynamics linear, thus allowing for the design of local Kalman filters with GES errors dynamics. Then, the error dynamics of the formation as a whole are also shown to be GES. Finally, Monte Carlo simulations are presented, including the comparison with the EKF, the UKF and the BCRB.

References


Figure 5: Vehicle 3: UKF Average Position Estimation Error

Figure 6: Vehicle 3: Position RMSE - x

Figure 7: Vehicle 3: Position RMSE - y

Figure 8: Vehicle 3: Position RMSE - z


