Some theoretical aspects of Multi-Higgs-Doublet Models

Francisco S. Faro
Under supervision of *João P. Silva and Jorge C. Romão*
Centro de Física Teórica de Particulas, Departamento de Física,
Instituto Superior Técnico, Universidade de Lisboa, Lisboa, Portugal
(Dated: October 2019)

In this thesis, we study the decoupling limit of Multi-Higgs-Doublet Models (NHDMs). We give a new procedure to analytically scrutinize such limit and consider the situation for symmetric potentials. Depending on the model and on the vacuum structure, we show which parameters must be included and how these depend on the decoupling energy scales. We obtain the bounded from below (BFB) conditions for the $\mathbb{U}(1) \times \mathbb{U}(1)$ and for the $\mathbb{U}(1) \times \mathbb{Z}_2$ symmetric Three-Higgs-Doublet Model (3HDM). We give a pathological example of a potential which has a good-looking normal minimum but is unbounded from below in the charge-breaking directions. This remark is not limited to the particular model that we considered. It represents a rather general feature of elaborate NHDMs which must be carefully dealt with. The work produced for this thesis resulted in a paper published in *Physical Review D* [1], and in a second paper yet to be submitted.
Keywords: Multi-Higgs-Doublet Models, decoupling limit, Standard Model, bounded from below, charge-breaking directions.

I. INTRODUCTION

Experimental data do not force the Standard Model (SM) scalar sector to be as minimal as postulated. Building models with Multi-Higgs sectors is an attractive option [2], as it helps to resolve various open problems. When studying these, one has to deal with technical issues related to the properties of the scalar potential. First theoretical constraints must be applied for the model to be physically plausible, and then experimental constraints for the model to be compatible with our universe. In this thesis we focus on Multi-Higgs-Doublet Models (NHDMs) [3–6]. These have several Higgs bosons, both charged and neutral, can accommodate novel forms of $\mathcal{CP}$ violation, tree-level Flavour Changing Neutral Couplings (FCNCs), scalar Dark Matter candidates and modifications of the SM couplings.

With such a rich phenomenology, one must be careful not to contradict the experimental measurements [7]. From a practical perspective, our models should be close to the SM predictions at the electroweak scale. This can be accomplished through the decoupling limit [8], and new physics would be found at a higher energy scale. Not yet explored by our colliders, but with cosmological consequences in our universe. However, we usually do not work with the most general NHDM, but rather with one that is invariant under a certain symmetry. These limit the number of parameters and may yield restrictions that render the decoupling unattainable.

The decoupling limit conditions are known for the $\mathcal{CP}$ conserving Two-Higgs-Doublet Model (2HDM) [8, 9]. Nonetheless, such conditions are not known for models whose vevs have physical phases, neither for models with additional scalar doublets. In practice, one cannot make use of the literature methods to study the decoupling limit of a theory with more than two doublets. By extending the literature methods [8, 9] to investigate an NHDM, one would have to respectively obtain the analytical expression for the eigenvalues of an $N \times N$ or $(N-1) \times (N-1)$ matrix. Even for Three-Higgs-Doublet Models (3HDMs), these expressions are so complicated that they are hardly of any use. In this thesis, we introduce a procedure that can be applied to study the decoupling limit of Multi-Higgs-Doublet Models with an *arbitrary number of doublets*. Through this technique, we explicitly investigate the situation for models with abelian symmetries. We clarify in which situations it is, and in which situations it is not, necessary to include all quadratic parameters in order to have a decoupling limit. Additionally, our method allows to examine how the quadratic parameters depend on the decoupling energy scale.

When building extended Higgs sectors, one has to deal with some technical issues related to the properties of the scalar potential. In particular, these must have a global minimum, which requires the potential to be Bounded from Below (BFB). Even for sophisticated scalar sectors, it is often easy to give a set of *sufficient* BFB conditions. Models satisfying them are safe and can be used in phenomenological analyses. However, such conditions can be overly restrictive, leaving out parts of the parameter space with potentially intriguing phenomenological consequences. Thus, when exploring the parameter space in a class of Multi-Higgs Models, it is always desirable to establish the exact BFB conditions that are simultaneously necessary and sufficient. This technical issue is rather challenging and has only been solved for sufficiently simple cases. In this thesis, we obtain the set of necessary and sufficient BFB conditions for two other models, the $\mathbb{U}(1) \times \mathbb{U}(1)$ and the $\mathbb{U}(1) \times \mathbb{Z}_2$ symmetric 3HDM. When deriving such conditions, we learned yet another lesson: one must always check stability along charge-breaking directions in the Higgs space. Even if one has a normal looking neutral minimum. We will show an example in which such minimum exists, and the potential is stable in all *neutral* directions. Nonetheless, it is unbounded from below along some charge-breaking directions.

This extended abstract is organised as follows. In section II, we give an introduction to Multi-Higgs-Doublet Models, with emphasis on the basis freedom of the scalar potential. In section III, we investigate the decoupling limit conditions for Multi-Higgs-Doublet Models. In section IV we discuss the bounded from below conditions for the $\mathbb{U}(1)\times\mathbb{U}(1)$ and for the $\mathbb{U}(1)\times\mathbb{Z}_2$ symmetric 3HDM. Finally, we draw our main conclusions in section V.

II. MULTI-HIGGS-DOUBLET MODELS

There is no reason to believe that there is only one Higgs doublet in our universe. In general, we can have N $\mathbb{SU}(2)_L\times\mathbb{U}(1)_Y$ complex scalar doublets with hypercharge $Y=1/2$,

$$\Phi_i = \begin{pmatrix} \phi_i^+ \\ \phi_i^0 \end{pmatrix}. \tag{1}$$

These interact with each other through the renormalizable Higgs potential

$$V = Y_{ij}\left(\Phi_i^\dagger\Phi_j\right) + Z_{ij,kl}\left(\Phi_i^\dagger\Phi_j\right)\left(\Phi_k^\dagger\Phi_l\right), \tag{2}$$

whose hermiticity implies,

$$Y_{ij}=Y_{ji}^*, \quad Z_{ij,kl}=Z_{kl,ij}=Z_{ji,lk}^*. \tag{3}$$

As in the SM, we are only interested in certain regions of the parameter space. In particular, the potential must have a global minimum where the scalar fields acquire non-zero vacuum expectation value (vev). Such minimum can lead to the existence of a massive photon [10], so we assume that it preserves the charge symmetry, $\mathbb{U}(1)_Q$, generated by $Q=T_3+Y$. The expansion of the fields around its vevs, ν_i, is given by

$$\Phi_i = \nu_i + \varphi_i = \begin{pmatrix} 0 \\ v_i/\sqrt{2} \end{pmatrix} + \begin{pmatrix} \phi_i^+ \\ \left(\rho_i + i\chi_i\right)/\sqrt{2} \end{pmatrix}, \tag{4}$$

where each v_i is in general complex. The stationary conditions [10] are given by

$$\left(Y_{ij} + Z_{ij,kl}v_k^*v_l\right)v_j = 0. \tag{5}$$

A. Basis freedom

The potential in eq. (2) can always be rewritten in a different basis. This can done through a transformation in the family space, $\Phi_i \to U_{ij}\Phi_j$ given by a $\mathbb{SU}(N)$ matrix. Which leaves the potential invariant by definition. Or through a global transformation in the multiplet space, $\Phi_i \to U\Phi_i$ given by a $\mathbb{U}(1)$ and/or $\mathbb{SU}(2)$ matrix. Which leaves the potential invariant by construction since the model is a $\mathbb{SU}(2)\times\mathbb{U}(1)$ gauge theory.

A general basis change, $\Phi_i \to \Phi_i' = U_{ij}\Phi_j$ given by a $\mathbb{U}(N) = \mathbb{SU}(N)\times\mathbb{U}(1)$ matrix, modifies the potential parameters by

$$Y_{ij} \to Y'_{ij} = U_{ik}Y_{kl}U_{jl}^*, \tag{6}$$

$$Z_{ij,kl} \to Z'_{ij,kl} = U_{im}U_{ko}Z_{mn,op}U_{jn}^*U_{lp}^*. \tag{7}$$

While a symmetry, $\Phi_i \to \Phi_i' = S_{ij}\Phi_j$ also given by a $\mathbb{U}(N)$ matrix, imposes a certain relation among the potential parameters,

$$Y_{ij} = Y_{ij}^S = S_{ik}Y_{kl}S_{jl}^*, \tag{8}$$

$$Z_{ij,kl} = Z_{ij,kl}^S = S_{im}S_{ko}Z_{mn,op}S_{jn}^*S_{lp}^*. \tag{9}$$

Ferreira and Silva [11] showed that potentials which satisfy symmetries that belong to the same conjugacy class, $S' = USU^\dagger$ where U is a unitary matrix, are related through a basis change. Since all physical phenomena must be basis independent, we can choose a basis where the symmetry has a diagonal representation. This is the so-called Symmetry basis [10, 11], which defines the minimum amount of magnitudes required to describe the potential. Later we shall clarify the situation when there is not a well-defined symmetry basis. Namely, when several doublets have the same group charge.

If one wishes to compare these NHDMs with the Standard Model scalar sector, there exists a more appropriate basis to do so. In a Charged Higgs basis, the vev is real and entirely aligned with the direction of a neutral scalar [12, 13]. The doublet with the vev is conventionally chosen to be ϕ_1^{CH}, and the charged scalars are mass eigenstates

$$\Phi_1^{CH} = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} + \begin{pmatrix} G^+ \\ \left(h_{SM} + iG^0\right)/\sqrt{2} \end{pmatrix}, \tag{10}$$

$$\Phi_i^{CH} = \begin{pmatrix} H_i^+ \\ \left(R_i + iI_i\right)/\sqrt{2} \end{pmatrix}. \tag{11}$$

The stationary conditions are given by [10]

$$\left(Y_{i1}^H + Z_{i1,11}^H v^2\right) = 0, \tag{12}$$

and and the mass matrices by,

$$\left(M_0^{CH}\right)^2 = \begin{pmatrix} \left(M_{RR}^{CH}\right)^2 & \left(M_{RI}^{CH}\right)^2 \\ \left(M_{IR}^{CH}\right)^2 & \left(M_{II}^{CH}\right)^2 \end{pmatrix} \tag{13}$$

$$\left(M_{RR}^{CH}\right)^2_{ij} = m_{H_i^\pm}^2\delta_{ij} + v^2\mathrm{Re}\left(Z_{i1,1j}^{CH} + Z_{i1,j1}^{CH}\right), \tag{14}$$

$$\left(M_{II}^{CH}\right)^2_{ij} = m_{H_i^\pm}^2\delta_{ij} + v^2\mathrm{Re}\left(Z_{i1,1j}^{CH} - Z_{i1,j1}^{CH}\right), \tag{15}$$

$$-\left(M_{RI}^{CH}\right)^2_{ij} = v^2\mathrm{Im}\left(Z_{i1,1j}^{CH} - Z_{i1,j1}^{CH}\right), \tag{16}$$

$$\left(M_{IR}^{CH}\right)^2_{ij} = v^2\mathrm{Im}\left(Z_{i1,1j}^{CH} + Z_{i1,j1}^{CH}\right), \tag{17}$$

$$\left(M_{\pm}^{CH}\right)^2_{ij} = Y_{ij}^{CH} + Z_{ij,11}^{CH}v^2 = m_{H_i^\pm}^2\delta_{ij}, \tag{18}$$

with $m_{H_1^\pm}^2 = 0$ due to the stationary conditions. Here one can identify Y_{11}^{CH} as being of $\mathcal{O}(v^2)$

$$Y_{11}^{CH} = -Z_{11,11}^{CH}v^2, \tag{19}$$

and also identify the off diagonal quadratic terms as being of $\mathcal{O}(v^2)$,

$$\left(Y_{ij}^{CH} + Z_{ij,11}^{CH} v^2\right)(1-\delta_{ij}) = 0. \tag{20}$$

Due to the unitarity bounds requiring the quartic parameters to be at most of order 1 [14, 15], $Z_{ij,kl} \lesssim \mathcal{O}(1)$.

Since only Φ^{CH} acquires a vev, this construction allows the identification of a state, h_{SM}, which has couplings equal to those of the Standard Model Higgs boson. It also has the advantage that it disentangles the SM Nambu-Goldstone bosons from the physical fields. Thereafter, one can identify Φ_1^H as having SM couplings with all the fermions and gauge bosons since its states have SM couplings. However, we emphasise that h_{SM} is not the SM Higgs since it is not a mass eigenstate. In the next section, we shall explicitly study how one can have a Multi-Higgs-Doublet Model with a SM Higgs.

It is important to note that a Charged Higgs basis trades the simplicity of the potential parameters, as in the Symmetry basis, for the simplicity of the vacuum structure and the charged scalars mass matrix.

B. The Standard Model limits

To be able to compare our models with the SM and with the experimental measurements, it is useful to examine under which conditions h_{SM} is a mass eigenstate [8]. This can occur in two different situations: the so-called alignment without decoupling [8, 9, 16, 17]; and the decoupling limit [8]. Both can be easily studied in the Charged Higgs basis. In the alignment limit, the neutral scalars mass matrix must satisfy,

$$\left(M_0^{CH}\right)^2_{1j\neq 1} \ll v^2. \tag{21}$$

Such that h_{SM} is a mass eigenstate by suppressing its mixing with other neutral scalars. Conversely, the decoupling limit is attained by setting the masses of the charged scalars to be much higher than the electroweak scale,

$$m^2_{H_i^\pm} \gg v^2. \tag{22}$$

Also restricting the mixing between h_{SM} and other neutral scalars, see eqs. (14) to (17).

One can generically write the masses of the scalars as

$$M^2 \sim Y + v^2 Z. \tag{23}$$

Due to the unitary bounds the quartic terms are at most of order 1 [14, 15], $Z_{ij,kl} \lesssim \mathcal{O}(1)$. Thus, only the quadratic terms are allowed to drive the masses to larger energy scales than $\mathcal{O}(v)$ [8]. In a Charged Higgs basis the decoupling conditions can be written as

$$m^2_{H_i^\pm} \gg v^2 \Rightarrow Y_{ii}^{CH} \equiv \mathcal{M}_i^2 \gg v^2. \tag{24}$$

Moreover, the mass matrices for the neutral states, eqs. (14) to (17), have diagonal terms which depend on $\mathcal{M}_i^2 \gg v^2$ and off-diagonal terms of $\mathcal{O}\left(v^2\right)$. Hence, all states that belong to the same Φ_i^{CH} doublet will decouple when $\mathcal{M}_i \gg v$. Additionally, the mixing between the i^{th} $\mathcal{CP}$-even and $\mathcal{CP}$-odd states will be suppressed by a factor of $v^2/\mathcal{M}_i^2$.

If the NHDM being studied does not have a symmetry, its $Y_{ii\neq 1}^{CH}$ are independent and the decoupling limit conditions are straightforward to satisfy. A decoupling to an (N-1)HDM can be achieved by setting $\mathcal{M}_N \gg v$, while a decoupling to the SM is attained by setting all $\mathcal{M}_{i\neq 1} \gg v$. On the other hand, if the NHDM being considered has a certain symmetry, there may be not enough *independent magnitudes* to accommodate radically different energy scales. Moreover, we recall that we calculate the mass eigenstates in a particular minimum. The vacuum structure in the Symmetry basis is intimately connected to the possibility of attaining a decoupling. Therefore we emphasise that it is not sufficient to establish the decoupling limit conditions in the Charged Higgs basis since its parameters are in general *not independent*. Such conditions must be established in terms of the Symmetry basis parameters, as in the usual methods [8, 9]. Indeed, for a particular vacuum, the central questions are: Which Symmetry basis parameters *must* be included; How these parameters depend on the decoupling energy scale.

III. THE DECOUPLING LIMIT

Here we shall derive a set of necessary and sufficient conditions for symmetric NHDMs to have a decoupling limit. This can either occur to an effective field theory (EFT) with less scalar doublets or to the SM. Our procedure to study these situations is: Identify the massive charged scalars as not being part of the Standard Model; Notice that $Y_{ii\neq 1}^{CH} \equiv \mathcal{M}_i^2 \gg v^2$ defines the energy scale of the decoupling; Trade the remaining Y_{ij}^{CH} for the corresponding Z^{CH}, as in eqs. (19) and (20); Parameterise the unitary matrix, U^{CH}, that transforms the potential from a Symmetry basis[1] to the Charged Higgs basis; Write the Z^{CH} as a function of Z^S; Write the Symmetry basis quadratic parameters as a function of the Charged Higgs basis quadratic parameters. We emphasise that this procedure enables a clearer analysis of what happens when: some doublets decouple from the effective electroweak theory, but the theory as a whole does not decouple to the SM; the scalar potential does not include certain parameters in the Symmetry basis. Additionally, it has the advantage that it explicitly shows how the Symmetry basis quadratic parameters depend on

[1] In the following section we will clarify what happens when there is no well-defined Symmetry basis.

the decoupling energy scale and if particular conditions are required.

A. Well-defined Symmetry basis with non-vanishing vevs

We proceed by discussing the situation in models that have a well-defined Symmetry basis, and non-vanishing vevs, $v_i \neq 0$. In the formalism of the Charged Higgs basis we have,

$$\Phi^{CH} = U^{CH}\Phi^S, \quad U^{CH}_{1j} = \frac{v_j^*}{v}, \tag{25}$$

$$Y^S_{ij} = U^{CH*}_{ki} Y^{CH}_{kl} U^{CH}_{lj}, \tag{26}$$

$$Y^{CH}_{ij} = \mathcal{M}_i^2 (1-\delta_{i1}\delta_{j1})\delta_{ij} + \Omega^{CH}_{ij}, \quad \text{(no sum)} \tag{27}$$

$$-\Omega^{CH}_{ij} = Z^{CH}_{ij,11} v^2 (1-\delta_{ij}) + Z^{CH}_{11,11} v^2 \delta_{i1}\delta_{j1}, \text{ (n.s.)} \tag{28}$$

$$Z^{CH}_{ij,kl} = U^{CH}_{im} U^{CH}_{ko} Z^S_{mn,op} U^{CH*}_{jn} U^{CH*}_{lp}, \tag{29}$$

where the matrix Ω^{CH} ensures that the stationary conditions are satisfied and that $(M^{CH}_{\pm})^2$ is diagonal, as it should. In section II B we identified that $\mathcal{M}_i^2 \gg v^2$ drives the decoupling and defines the energy scale of the states within the Φ_i^{CH} doublet. We want to know what are the dependencies of Y^S_{ij} on $\mathcal{M}_i^2$, which we obtain by combining eqs. (26) and (27),

$$Y^S_{ij} = U^{CH*}_{ki} \mathcal{M}_k^2 U^{CH}_{kj} (1-\delta_{1k}) + U^{CH*}_{ki} \Omega^{CH}_{kl} U^{CH}_{lj}. \tag{30}$$

Here, one can explicitly see that in order to have a decoupling limit to the SM, it is certainly sufficient to include all quadratic parameters. However, what we are interested in is the set of necessary and sufficient conditions. If those are not satisfied, then one can be sure that there will not be a decoupling in any region of the parameter space. Henceforth, we investigate eq. (30) by making use of some properties of unitary matrices. Indeed, the square of each line/row of a $\mathbb{U}(N)$ matrix must sum to 1,

$$\sum_i |U_{ik}|^2 = 1, \quad \sum_k |U_{ik}|^2 = 1. \tag{31}$$

By imposing in eq. (25) that $U^{CH}_{1j} \neq 0$ and $U^{CH}_{ij} \neq 1$, the properties in (31) require at least two non-zero entries in each row and in each line of the matrix. Then, eq. (30) implies that for every doublet, Φ_k^{CH}, that decouples from the low energy theory, at least three quadratic parameter in the Symmetry basis, $Y^S_{ab\neq a}$ Y^S_{aa} and Y^S_{bb}, will depend on $\mathcal{M}_k^2$. These findings applied in a decoupling limit to the SM (all $\mathcal{M}_{k\neq 1} \gg v$), yield that every Y^S_{ii} will depend on at least one of the $\mathcal{M}_k^2$.

In eq. (30) one can see that there are no contributions to a certain Y^S_{ab}, due to $\mathcal{M}_k^2$, if $U^{CH}_{ka} = 0$ or if $U^{CH}_{kb} = 0$. Then we want to control the entries of this matrix, such that some of them are exactly zero. Any $\mathbb{U}(N)$ matrix can be parameterised through $N(N-1)/2$ orthogonal angles and $N(N+1)/2$ phases. From the original $N(N-1)/2$ angles, $(N-1)$ are needed for the definition of the vevs. These we shall call β_i and cannot be multiples of $\pi/2$ since the vevs must be non-zero. Conversely, the remaining $(N-1)(N-2)/2$ angles can take any value. These we shall call ω_i and can be chosen such that there are some entries in the U^{CH} matrix that are zero.

We emphasise that when Y^S_{aa} and Y^S_{bb} depend on $\mathcal{M}_k^2 \gg v^2$, it is not possible to make both of them small[2]. This is not troublesome because we have already made use of the stationary conditions in eq. (28). If these are satisfied in one basis, they will be satisfied in every basis. Nonetheless, we shall momentarily look at the problem from the opposite perspective. From the requirement that there will be at least one $Y^S_{aa} \gg v^2$, we shall guess which Symmetry basis parameters must be included. Then we write the Symmetry basis stationary conditions as

$$Y^S_{ii} = -\sum_{j=1, j\neq i}^{N} \left(Y^S_{ij}\frac{v_j}{v_i} + Z^S_{ij,kl}\frac{v_k^* v_l v_j}{v_i} \right), \tag{32}$$

Here we distinguish two types of decoupling limits[3], which are made possible through: off-diagonal quadratic terms (first type); small vevs and certain quartic parameters (second type). In the following subsections, we shall make use of our procedure to verify such conjectures[4].

The easiest and most general way of achieving a decoupling limit is through off-diagonal quadratic parameters. The so-called decoupling limit of the first type. Even if an exact symmetry forbids these terms, one can always make use of a potential which softly breaks the symmetry by including more quadratic parameters. In this type of decoupling limit,

$$Y^S_{ii} + \sum_{j=1, j\neq i}^{N} \left(Y^S_{ij}\frac{v_j}{v_i} \right) = \mathcal{O}(v^2). \tag{33}$$

Conversely, in a decoupling limit of the second type we have $|Y^S_{ii}| \gg v^2$ when there is a quartic parameter and vevs such that

$$\left| Z^S_{ij,kl}\frac{v_k^* v_l v_j}{v_i} \right| \gg v^2 \quad \text{(no sum)}. \tag{34}$$

Yet, it is not possible to recover the SM through this type of decoupling. This stems from the fact that is not possible to set $v_k^* v_l v_j / v_i \gg v^2$ the $N-1$ times required to decouple $N-1$ doublets. We shall clarify this issue in

---

[2] If $U^{CH}_{ki} \to 0$ for $i \neq a, b$, then $|U^{CH}_{ka}|^2 + |U^{CH}_{kb}|^2 = 1$. Such that $|U^{CH}_{ka}| \to 0$ implies $|U^{CH}_{kb}| \to 1$.

[3] These two situations that were already identified for the $\mathcal{CP}$ conserving 2HDM [8]. However, since the model does not have a well-defined Symmetry basis, they do not seem to have any particular meaning.

[4] We recall that the two types of decouplings only exist for potentials with a well-defined Symmetry basis and non-vanishing vevs.

the following sections. As an example, we will see that it is possible to decouple a $\mathbb{Z}_3$ symmetric 3HDM to a 2HDM without including any $Y^S_{ij\neq i}$. To further decouple the resultant EFT, one has to make use of a decoupling limit of the first type and include one $Y^S_{ij\neq i}$.

B. The Two-Higgs-Doublet Model

As a prelude for the more involving situations considered throughout the thesis, we studied the decoupling limit conditions for $\mathcal{CP}$ conserving 2HDM with a softly broken $\mathbb{Z}_2$ symmetry. This model was already examined in the literature through two distinct approaches [8, 9], and it is the simplest NHDM that one may consider.

By applying our procedure to this model, we re-derive two expressions that were already obtained through the literature methods. The key difference between these results is the procedure and interpretation. In the literature methods, one starts with a specific model and then it is verified if it is possible to have a decoupling. In our method, the symmetry only limits the number of quartic parameters, and we impose that there is a decoupling by constraints on the Y^{CH}_{ij}. Then we verify which Symmetry basis quadratic parameters must be included. All results follow from here. In no place, prior to the final result, do we have to consider the different possibilities for the normal vacuum or if the symmetry is softly broken.

In our method, the decoupling limit is defined by $Y^{CH}_{22} \equiv \mathcal{M}^2_2 \gg v^2$, as in the Higgs basis method for the 2HDM [9]. The difference is how we trace back the dependencies of the decoupling energy scale to the Y^S_{ij}. In the literature methods, the decoupling energy scale is written as a function of the Symmetry basis parameters. Then it is found that two parameters are required for the definition of the decoupling limit in the Symmetry basis, m^2_{12} and β. In our method, we write the Y^S_{ij} as a function of the decoupling energy scales, and no other condition is required. Except for the fact that there is a decoupling $Y^{CH}_{22} \equiv \mathcal{M}^2_2 \gg v^2$.

For completeness, we also considered the situation when the $\mathcal{CP}$ symmetry is broken. Then we concluded that it is not possible to have a decoupling limit and spontaneous $\mathcal{CP}$ violation in a 2HDM with a softly broken $\mathbb{Z}_2$ symmetry. For a decoupling limit to exist when the vevs have physical phases, Y^S_{12} must be complex in a basis where the quartic parameters are real, which explicitly breaks the $\mathcal{CP}$ symmetry.

C. Decoupling limit of the first type

We started by examining the situation for 3HDMs. If all vevs are of order v, for instance when $v_1 \approx v_2 \approx v_3$, then all quadratic parameters are of order $\mathcal{M}^2_i \gg v^2$. In such conditions, it is possible to have a decoupling to an effective 2HDM with only one $Y^S_{ij\neq i}$. On the other hand, a decoupling to the SM requires all Symmetry basis quadratic parameters. In other situations, some vevs may be so small that the contributions due to $\mathcal{M}^2_i \gg v^2$ are suppressed. Nonetheless, all Y^S_{ij} must be included in a decoupling of the first type with distinct decoupling energy scales $\mathcal{M}_2 \gg \mathcal{M}_3 \gg v$, even when some $v_i \to 0$. One can conclude that by not including one of the $Y^S_{ij\neq i}$, it is not possible to decouple the 3HDM to the SM through different decoupling energy scales.

We proceeded by showing that it is impossible for a Y^S_{ij} not to depend on at least one of the $\mathcal{M}^2_i$ when all $v_k \neq 0$. Therefore, all Y^S_{ij} must be included to attain a decoupling of the first type to the SM through $|\mathcal{M}_i - \mathcal{M}_{j\neq i}| \gg v$.

D. Decoupling of the second type

A decoupling limit of the second type is made possible through small vevs and specific quartic parameters. Indeed, the Symmetry basis stationary conditions, eq. (32), give $|Y^S_{ii}| \gg v^2$ when there is a term that yields

$$\left| Z^S_{ij,kl} \frac{v^*_k v_l v_j}{v_i} \right| \gg v^2 \quad \text{(no sum)}. \tag{35}$$

This requires a quartic parameter, $Z^S_{ij,kl}$, such that there is an index, i, which is different from all the other indexes, j, k, l. A feature that generally enables the existence of a region of the parameter space in which the denominator of eq. (35) tends to zero, $|v_i| \to 0$, but the numerator does not tend to zero $|v^*_k v_l v_j| \not\to 0$. Since the mass matrices must be positive definite, one also has to require that $Y^S_{ii} > 0$ which implies that

$$\mathrm{Re}\left(Z^S_{ij,kl} \frac{v^*_k v_l v_j}{v_i} \right) < 0 \quad \text{(no sum in } i\text{)}. \tag{36}$$

We notice that certain symmetries with a well-defined Symmetry basis forbid a decoupling limit of the second type. If it is possible to decompose a symmetry into the product of $\mathbb{U}(1)$ and/or $\mathbb{Z}_2$ groups, then there will be two equal indexes in every $Z^S_{ij,kl}$. Such symmetries cannot have a decoupling limit of the second type. The simplest potentials that may have this type of decoupling are the $\mathbb{Z}_3$ and the $\mathbb{Z}_4$ 3HDM.

If the Symmetry basis stationary conditions yield $Y^S_{aa} \gg v^2$ when $v_a \to 0$, it is possible not to include some $Y^S_{ab\neq a} = 0$ when a 3HDM decouples to the SM through $|\mathcal{M}_2 - \mathcal{M}_3| \gg v$. Through this procedure, it is possible to decouple a 3HDM with an exact $\mathbb{Z}_3$ symmetry, to an effective $\mathbb{U}(1)$ symmetric 2HDM. To further decouple the theory to SM, one has to make use of a decoupling of the first type by including one Y^S_{ij}. The easiest way of seeing that it is possible to attain the SM without including all $Y^S_{ij\neq i}$ is by writing the Symmetry basis stationary conditions and making $v_i \to 0$. Nonetheless, the stationary conditions do not tell us if it is possible to have distinct decoupling energy scales, $|\mathcal{M}_i - \mathcal{M}_{j\neq i}| \gg v$, when we do not include some $Y^S_{ij\neq i}$. We comment that the number of $Y^S_{ij\neq i}$ that must be included depends on the model, and if the decoupling energy scales are distinct[5]. For NHDMs with a decoupling limit of the second type, we expect this behaviour to hold. Yet, we emphasise that it is not possible to have a decoupling limit of the second type to the SM. This would require a vacuum in which $v_1 \to v$, $v_{i\neq 1} \to 0$ and a symmetry that includes all quartic parameters of the type $Z^S_{11,1i}$. For these to exist in a Symmetry basis, all $\Phi^S_{i\neq 1}$ would have to have the same symmetry group charge. This is not compatible with the fact that here we are only considering potentials with a well-defined Symmetry basis. Hence, a decoupling limit of the second type cannot decouple an NHDM to the SM. To recover the SM, one will always have to make use of a decoupling limit of the first type by including some $Y^S_{ij\neq i}$.

E. A special case

Here we shall consider that the decoupling energy scale is the same for all charged scalars, $\mathcal{M}_{i\neq 1} = \mathcal{M} \gg v$. Then we can write eq. (30) as,

$$\begin{aligned}\frac{Y^S_{ij}}{\mathcal{M}^2} &= \sum_{k=2}^{N} U^{CH*}_{ki} U^{CH}_{kj} + \frac{\mathcal{O}(v^2)}{\mathcal{M}^2} \\ &= \delta_{ij} - \frac{v_i v_j^*}{v^2} + \frac{\mathcal{O}(v^2)}{\mathcal{M}^2}.\end{aligned} \tag{37}$$

For situations when there is no vev with a small value, it is not possible to obtain a decoupling limit without all Symmetry basis quadratic parameters. Indeed, all Y^S_{ij} will be of $\mathcal{O}(\mathcal{M}^2) \gg \mathcal{O}(v^2)$. However the situation is particularly interesting when there are small vevs. For the 3HDM potential, we concluded that it is possible not to include $Y^S_{ab\neq a}$ when there are two small vevs, $v_a \to 0$ and $v_b \to 0$. For the general NHDM, we do not know how many $Y^S_{ij\neq i}$ need to be included in order to attain $\mathcal{M}_i = \mathcal{M} \gg v$. Nevertheless, we suspect that it may be possible not to include $Y^S_{ab\neq a}$ when both $v_a \to 0$ and $v_b \to 0$.

IV. NECESSARY AND SUFFICIENT BFB CONDITIONS FOR THE $\mathbb{U}(1) \times \mathbb{U}(1)$ THREE-HIGGS-DOUBLET MODEL

We begin by reminding the reader of the Higgs space structure in a 3HDM and of the *copositivity conditions* [18]. A simple but powerful approach to establish the BFB conditions when the scalar potential can be written as a quadratic form of positive definite variables. The Higgs space of a 3HDM is spanned by three $\mathbb{SU}(2)_L \times \mathbb{U}(1)_Y$ scalar doublets with hypercharge $Y = 1/2$. When studying the structure of the potential, we replace the scalar field operators $\Phi_i(x)$ by c-numbers that can be conveniently parameterised as

$$\Phi_i = \sqrt{r_i} e^{i\gamma_i} \begin{pmatrix} \sin(\alpha_i) \\ \cos(\alpha_i) e^{i\beta_i} \end{pmatrix}, \quad i = 1, 2, 3. \tag{38}$$

Here the norms of the doublets are represented by $(\Phi_i^\dagger \Phi_i) = r_i \geq 0$, which we shall refer to as the radial variables. Conversely, the angles α_i and the phases β_i, γ_i are called angular variables. By allowing for any values of the phases, we can restrict α_i to lie within the first quadrant. We make use of the standard definition for the electric charge, where the upper components of the doublets are charged. *Neutral directions* in the Higgs space correspond to situations when all Φ_i are proportional to each other. One can define the non-negative charge sensitive quantities

$$z_{ij} = (\Phi_i^\dagger \Phi_i)(\Phi_j^\dagger \Phi_j) - (\Phi_i^\dagger \Phi_j)(\Phi_j^\dagger \Phi_i) \geq 0, \tag{39}$$

and notice that neutral directions in the Higgs space correspond to taking all $z_{ij} = 0$. Other directions, along which the strict proportionality of all three doublets does not hold, are called *charge-breaking* (CB) directions and have at least one $z_{ij} \neq 0$. This can be clearly seen by applying the $\mathbb{SU}(2) \times \mathbb{U}(1)$ gauge transformations to all three doublets[6], rewriting them as

$$\begin{aligned}&\Phi_1 = \sqrt{r_1} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \Phi_2 = \sqrt{r_2} \begin{pmatrix} \sin(\alpha_2) \\ \cos(\alpha_2) e^{i\beta_2} \end{pmatrix}, \\ &\Phi_3 = \sqrt{r_3} e^{i\gamma} \begin{pmatrix} \sin(\alpha_3) \\ \cos(\alpha_3) e^{i\beta_3} \end{pmatrix},\end{aligned} \tag{40}$$

so that $\alpha_{2,3}$ and can be identified as the charge breaking angles.

The $\mathbb{U}(1) \times \mathbb{U}(1)$ symmetry group can be represented, in a suitable basis, by arbitrary re-phasing transformations of individual doublets. The 3HDM potential invariant under this symmetry can be written as $V = V_2 + V_N + V_{CB}$, where

$$V_2 = m_{11}^2 (\Phi_1^\dagger \Phi_1) + m_{22}^2 (\Phi_2^\dagger \Phi_2) + m_{33}^2 (\Phi_3^\dagger \Phi_3), \tag{41}$$

$$\begin{aligned}V_N = &\frac{\lambda_{11}}{2} (\Phi_1^\dagger \Phi_1)^2 + \frac{\lambda_{22}}{2} (\Phi_2^\dagger \Phi_2)^2 + \frac{\lambda_{33}}{2} (\Phi_3^\dagger \Phi_3)^2 \\ &+ \lambda_{12} (\Phi_1^\dagger \Phi_1)(\Phi_2^\dagger \Phi_2) + \lambda_{13} (\Phi_1^\dagger \Phi_1)(\Phi_3^\dagger \Phi_3) \\ &+ \lambda_{23} (\Phi_2^\dagger \Phi_2)(\Phi_3^\dagger \Phi_3),\end{aligned} \tag{42}$$

$$V_{CB} = \lambda'_{12} z_{12} + \lambda'_{13} z_{13} + \lambda'_{23} z_{23}. \tag{43}$$

[5] The situation for the $\mathbb{Z}_4$ symmetric 3HDM is quite bizarre, and we shall not discuss it. Nonetheless, the conclusions stated here hold for both the $\mathbb{Z}_3$ and for the $\mathbb{Z}_4$

[6] To remove the upper phase in Φ_2, apply an $\mathbb{SU}(2)$ gauge transformation generated by $\tau_3 + Y$.

Along neutral directions all $z_{ij} = 0$, so that the scalar potential is given by $V = V_2 + V_N$. Along charge breaking directions, at least one $z_{ij} \neq 0$, meaning that V_{CB} switches on and contributes to the potential.

Let us now focus on the potential along neutral directions and establish conditions for its boundedness from below. As usual, this is equivalent to requiring that the quartic part of the potential is non-negative along all neutral directions. Using the parameterisation in (40), we express the potential as a quadratic form of the $r_i \geq 0$ variables:

$$\begin{aligned} V_N = &\frac{\lambda_{11}}{2} r_1^2 + \frac{\lambda_{22}}{2} r_2^2 + \frac{\lambda_{33}}{2} r_3^2 + \lambda_{12} r_1 r_2 \\ &+ \lambda_{13} r_1 r_3 + \lambda_{23} r_2 r_3 \end{aligned} \tag{44}$$

$$\equiv \frac{1}{2} A_{ij} r_i r_j \,. \tag{45}$$

The 3×3 real symmetric matrix A must be positive definite (or, at least, non-negative) in the first octant of the r_i space. Then, its entries must satisfy the following list of inequalities, known as the *copositivity conditions* [18]:

$$\begin{aligned} &A_{11} \geq 0, \quad A_{22} \geq 0, \quad A_{33} \geq 0, \\ &\bar{A}_{12} \equiv \sqrt{A_{11} A_{22}} + A_{12} \geq 0, \\ &\bar{A}_{13} \equiv \sqrt{A_{11} A_{33}} + A_{13} \geq 0, \\ &\bar{A}_{23} \equiv \sqrt{A_{22} A_{33}} + A_{23} \geq 0, \end{aligned} \tag{46}$$

and

$$\begin{aligned} &\sqrt{A_{11} A_{22} A_{33}} + A_{12}\sqrt{A_{33}} + A_{13}\sqrt{A_{22}} \\ &+ A_{23}\sqrt{A_{11}} + \sqrt{2 \bar{A}_{12} \bar{A}_{13} \bar{A}_{23}} \geq 0 \,. \end{aligned} \tag{47}$$

These are the necessary and sufficient conditions for V_N to be bounded from below.

A. Including charge breaking directions

The charge-breaking part of the potential, V_{CB}, depends not only on the radial variables, r_i, but also on the angular variables, α_i, β_i and γ_i. Thus, the quartic part of the potential, V_4, cannot yet be written as a quadratic form of independent and non-negative variables.[7] However, for each point in the r space, we can find the angular direction along which V_{CB} reaches a minimum. If V_{CB} evaluated along these special directions can be written as a quadratic form of r_i, then the quartic potential $V_N + V_{CB}$ can still be written in the same general form as eq. (45). The matrix A will receive, in addition to (44), contributions from V_{CB} and one just applies the same copositivity conditions in (46) and (47) to the total matrix A. The resulting conditions can be stronger than $V_N \geq 0$, but only in the case when V_{CB} can become negative along some angular directions. Since the $z_{ij} \geq 0$, this occurs whenever there is at least one negative λ'_{ij}, and those are the situations we shall focus on.

By setting the derivatives of eq. (43) with respect to the angular variables to zero, we obtain two equations that resemble the law of sines in a flat triangle[8]:

$$\frac{\sin(\theta_1)}{L_1} = \frac{\sin(\theta_2)}{L_2} = \frac{\sin(\theta_3)}{L_3} \,. \tag{48}$$

These two equations can have trivial and non-trivial solutions. Trivial solutions arise when α_2 and α_3 are equal to multiples of $\pi/2$. Substituting these values in the potential of eq. (43), yield three angular extrema of V_{CB}:

$$\begin{aligned} \lambda'_{12} r_1 r_2 + \lambda'_{23} r_2 r_3 \,, \quad \lambda'_{13} r_1 r_3 + \lambda'_{23} r_2 r_3 \,, \\ \lambda'_{12} r_1 r_2 + \lambda'_{13} r_1 r_3 \,. \end{aligned} \tag{49}$$

Clearly, it only makes sense to include those expressions for situations in which at least one of $\lambda'_{ij} < 0$.

Non-trivial solutions arise when the triangle inequalities are satisfied,

$$|L_1 - L_2| \leq L_3 \leq L_1 + L_2 \,, \tag{50}$$

where the lengths of the sides are given by,

$$L_1 = \frac{r_1}{|\lambda'_{23}|}, \quad L_2 = \frac{r_2}{|\lambda'_{13}|}, \quad L_3 = \frac{r_3}{|\lambda'_{12}|}. \tag{51}$$

On the other hand, the inner angles of the triangle must lie in the upper half of the trigonometric circle $\theta_i \in]0, \pi[$, and sum up to $\theta_1 + \theta_2 + \theta_3 = \pi$. The exact relations between θ's and the angular variables depends on the sign factors $\lambda'_{ij} = \sigma_{ij} |\lambda'_{ij}|$, where $\sigma_{ij} = \pm 1$. When these conditions are satisfied, one can make use of the law of cosines to write the angular variables as a function of the L's, such that V_{CB} can be expressed in a compact form

$$V_{CB}^{\text{non-triv.}} = \frac{\lambda'_{12} \lambda'_{23} \lambda'_{31}}{4} \left(\frac{r_1}{\lambda'_{23}} + \frac{r_2}{\lambda'_{13}} + \frac{r_3}{\lambda'_{12}} \right)^2 . \tag{52}$$

This expression is negative only in two cases: if exactly one among the three λ'_{ij} is negative, and if all three λ'_{ij} are negative. Therefore, only in these two cases one needs to consider non-trivial solutions for the angular minima of V_{CB} to establish the BFB conditions.

B. The set of necessary and sufficient BFB conditions

We are ready to formulate the set of necessary and sufficient BFB conditions for the $\mathbb{U}(1) \times \mathbb{U}(1)$ symmetric 3HDM, which we present as an algorithm.

[7] Although $z_{ij} \geq 0$, they are *not* independent from the radial variables.

[8] This triangle technique was already used by Branco [6, 19] to find $\mathcal{CP}$ breaking minima for the real $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric 3HDM, the so-called Branco's model.

Step 1. Apply the copositivity conditions in (46) and (47) to the matrix $A = A_N$ extracted from V_N, eq. (44).

Step 2. If at least one $\lambda'_{ij} < 0$, construct three new matrices $A_{1,2,3} = A_N + \Delta_{1,2,3}$, where $\Delta_{1,2,3}$ are extracted from (49). These three expressions for V_{CB} correspond to the trivial solutions of (48). Apply the copositivity conditions in (46) and (47) to each of $A_{1,2,3}$.

Step 3. If $\lambda'_{12}\lambda'_{23}\lambda'_{31} < 0$, consider $V_N + V_{CB}^{\text{non-triv.}}$ and extract from it the new matrix $A_4 = A_N + \Delta_4$. This matrix must be non-negative within the *open tetrahedron* in the r_i space, illustrated by fig. 1. Its apex is at the origin, lies inside the first octant, and is constrained by the triangle inequalities in (50).

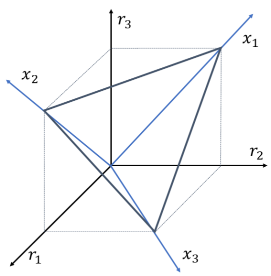

Figure 1. Location of the open tetrahedron defined by $x_i \geq 0$ inside the first octant of the r_i space. Each axis x_i lies in the plane orthogonal to the corresponding r_i axis.

Non-negativity inside this tetrahedron can be achieved through the same copositivity technique. Let us define new variables x_i, which are linearly related to r_i:

$$r_i = R_{ij}x_j, \; R = \begin{pmatrix} |\lambda'_{23}| & 0 & 0 \\ 0 & |\lambda'_{31}| & 0 \\ 0 & 0 & |\lambda'_{12}| \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \quad (53)$$

Here, the first matrix links r_i and L_i, while the second matrix aligns the axes x_i with the directions of $L_i = 0$ (and therefore two other L's being equal). The open tetrahedron defined by the inequalities in (50) corresponds to the first octant in terms of the new variables, $x_i \geq 0$. Since the relation $r_i = R_{ij}x_j$ is linear, the quartic potential can be written as a quadratic form in variables x_i with the matrix $R^T A_4 R$. Therefore, to complete step 3 we need to check that the entries of this matrix satisfy the set of copositivity conditions in (46) and (47).

These three steps represent the necessary and sufficient conditions for the potential of the $\mathbb{U}(1)\times\mathbb{U}(1)$ symmetric 3HDM to be bounded from below.

C. A pathological example

It is important to recall that when at least one $\lambda'_{ij} < 0$, the BFB conditions along the charge-breaking directions are more constraining than along the neutral ones. Nonetheless, this fact is not related to the existence of a neutral minimum. Let us illustrate this point with a pathological example whose pathology would be easily missed if one focused only on the neutral directions.

Consider a $\mathbb{U}(1)\times\mathbb{U}(1)$ symmetric 3HDM where the quadratic potential in eq. (41) contains only one negative coefficient: $m_{11}^2 < 0$. Then the minimum is at $v_1 = v$ with $v^2 = 2|m_{11}^2|/\lambda_{11}$, while $v_2 = v_3 = 0$. In fact, this is the only phenomenologically viable choice. It avoids spontaneous breaking of the $\mathbb{U}(1)\times\mathbb{U}(1)$ symmetry, and does not lead to Goldstone bosons. This minimum also leads to scalar dark matter candidates, which is one of the main attractive aspects of symmetry protected Multi-Higgs Models. By expanding the potential around this point, one obtains the masses of all physical scalars

$$\begin{aligned} M_{H_2}^2 &= M_{A_2}^2 = m_{22}^2 + \frac{v^2}{2}\lambda_{12}, \quad M_h^2 = v^2\lambda_{11}, \\ M_{H_3}^2 &= M_{A_3}^2 = m_{33}^2 + \frac{v^2}{2}\lambda_{13}, \\ M_{H_2^\pm}^2 &= m_{22}^2 + \frac{v^2}{2}(\lambda_{12}+\lambda'_{12}), \\ M_{H_3^\pm}^2 &= m_{33}^2 + \frac{v^2}{2}(\lambda_{13}+\lambda'_{13}). \end{aligned} \quad (54)$$

One can see that if $\lambda_{11} > 0$ and if the quadratic parameters m_{22}^2, m_{33}^2 are sufficiently large, then the masses squared are positive. Moreover, by choosing $\lambda_{11}, \lambda_{22}, \lambda_{33} > 0$ and

$$\lambda_{12} > 0, \quad \lambda_{13} > 0, \quad \lambda_{23} > 0, \quad (55)$$

we can immediately guarantee that the potential is BFB in all neutral directions of the Higgs space.

However, these conditions are *not* sufficient to guarantee that the potential is bounded from below along all angular directions. As an example, consider the following point in the parameter space:

$$\begin{aligned} \lambda_{11} = \lambda_{22} = \lambda_{33} = 0.1, \quad \lambda_{12} = \lambda_{13} = \lambda_{23} = 0.1, \\ \lambda'_{12} = \lambda'_{13} = \lambda'_{23} = -0.6, \end{aligned} \quad (56)$$

and explore how the quartic potential behaves along the ray[9] $r_1 = r_2 = r_3 \equiv r$, $\alpha_2 = \alpha_3 = \pi/2$. We find by direct calculation that $V_4 = -0.75r^2$. This clearly indicates that the potential is unbounded from below, despite the existence of a normal-looking neutral minimum. In order to avoid such pathological models, one must always look for BFB conditions valid in the entire Higgs space. Not only along neutral directions. This is a general remark and is not limited to the $\mathbb{U}(1)\times\mathbb{U}(1)$ symmetric 3HDM.

D. The necessary and sufficient BFB conditions for the $\mathbb{U}(1)\times\mathbb{Z}_2$ 3HDM

There is a symmetry which contains one additional phase-sensitive term when compared to the $\mathbb{U}(1)\times\mathbb{U}(1)$

[9] We do not claim that the potential is minimal along this direction. We simply show that there exists a direction along which the potential is unbounded from below.

3HDM. The $\mathbb{U}(1) \times \mathbb{Z}_2$ symmetry can be parameterised in a suitable diagonal basis by

$$S_{\mathbb{U}(1)\times\mathbb{Z}_2} = \text{diag}\left(1,\, -1,\, e^{i\theta}\right) \tag{57}$$

The invariant potential under this symmetry is given by $V_2 + V_N + V_{CB}$ from eqs. (41) to (43), together with,

$$V_{\mathbb{U}(1)\times\mathbb{Z}_2} = \frac{1}{2}\left[\bar{\lambda}_{12}\left(\Phi_1^\dagger \Phi_2\right)^2 + \text{h.c.}\right]. \tag{58}$$

For this model, the quartic part of the potential is given by $V_4 = V_N + V_{CB} + V_{\mathbb{U}(1)\times\mathbb{Z}_2}$. By applying the three steps of the algorithm in section IV B, together with the transformations $\lambda_{12} \to \lambda_{12} - |\bar{\lambda}_{12}|$ and $\lambda'_{12} \to \lambda'_{12} + |\bar{\lambda}_{12}|$, we arrive at the set of necessary and sufficient conditions for the $\mathbb{U}(1) \times \mathbb{Z}_2$ symmetric 3HDM to be BFB.

V. CONCLUDING REMARKS

In this work, we have seen a popular extension of the Standard Model scalar sector — the so-called Multi-Higgs-Doublet Models. We explored different basis in which its scalar sector can be studied and made several remarks regarding potentials that do not have a well-defined symmetry basis. These situations can occur when the potential has no symmetry, or when some doublets have the same symmetry group charge. In both, one should make use of the basis freedom to remove spurious parameters, or to simplify the vacuum structure.

We clarified some questions regarding the decoupling limit of Multi-Higgs-Doublet models. Its properties are straightforward to establish in a Charged Higgs basis since its charged scalars are mass eigenstates. Indeed, we start by identifying the charged scalars as not being part of the SM and require that $m^2_{H_i^\pm} \gg v^2$. Then, the Charged Higgs basis quadratic parameters define the decoupling energy scale $Y_i^{CH} \equiv \mathcal{M}_i^2 \gg v^2$ and all states that belong to the same Φ_i^{CH} doublet decouple from the electroweak theory. Even if there is $\mathcal{CP}$ violation, either explicit or spontaneous, the mixing between the i^{th} $\mathcal{CP}$-even and $\mathcal{CP}$-odd scalars is suppressed by a factor of $v^2/\mathcal{M}_i^2$.

We emphasised that when the potential has a certain symmetry, it is not sufficient to establish the decoupling limit conditions in the Charged Higgs basis. Its parameters are in general not independent, and there may be not enough independent magnitudes to accommodate radically different energy scales. Such conditions must be established in the Symmetry basis, where it is clear the minimum amount of magnitudes required to characterise the potential. As a prelude for the more involving situations considered throughout the thesis, we studied the decoupling limit conditions for $\mathcal{CP}$ conserving 2HDM with a softly broken $\mathbb{Z}_2$ symmetry. This model was already examined in the literature through two distinct procedures [8, 9], and it is the simplest NHDM that one may consider. There, we introduced a new method to derive the decoupling limit conditions in the Symmetry basis. The key difference between our procedure and the literature methods is how we trace back the dependencies of the decoupling energy scales. For completeness, we also considered the situation when there are complex vevs with physical phases. We found that there is no decoupling limit with spontaneous $\mathcal{CP}$ violation. When the phases of the vevs are physical, and there is a decoupling limit, Y_{12}^S must be complex in a basis where λ_5 is real. Therefore a $\mathbb{Z}_2$ symmetric 2HDM can only have a decoupling limit with complex vevs when we include a soft breaking quadratic parameter that explicitly breaks $\mathcal{CP}$.

In practice, one cannot extend the literature methods in [8, 9] to study the decoupling of a model with more than two doublets. Nonetheless, our method can be used to study models with an arbitrary number of doublets. It is implemented by placing constraints on the Charged Higgs basis parameters and writing the Symmetry basis quadratic parameters as a function of the Charged Higgs basis quadratic parameters. Our procedure enables a more precise analysis of what happens when: some doublets decouple from the effective electroweak theory, but the theory as a whole does not decouple to the SM; the scalar potential does not include specific parameters in the Symmetry basis. Then we clarified in which situations it is necessary to include all $Y^S_{ij\neq i}$, and in which situations it is sufficient to include some $Y^S_{ij\neq i}$. We also clarified how the Y^S_{ij} depend on the decoupling energy scale.

Models with an ill-defined Symmetry basis can always decouple to an EFT with a well-defined Symmetry basis. Indeed, not all quadratic parameters in an ill-defined Symmetry basis are necessary for the decoupling. It is possible not to include those related to the directions in which the vevs do vanish. In turn, one can explicitly obtain theses directions through a carefully chosen basis that we introduce, the Symmetric Higgs basis Φ^{SH}. Similarly, potentials with a well-defined symmetry basis and vanishing vevs can always be decoupled to an EFT with a well-defined symmetry basis and non-vanishing vevs. This does not require the $Y^S_{ij\neq i}$ related to the directions in which the vevs do vanish, $v_j = 0$.

To decouple an NHDM with a well-defined Symmetry basis and non-vanishing vevs, we identify two possibilities. The first is made possible through off-diagonal quadratic parameters in the Symmetry basis and is viable for any symmetry. In order to recover the SM through different decoupling energy scales, $|\mathcal{M}_i - \mathcal{M}_{j\neq i}| \gg v$, all Y^S_{ij} must be included. A decoupling limit of the second type is made possible through small vevs and specific quartic parameters. To decouple a $\mathbb{Z}_3$ or $\mathbb{Z}_4$ 3HDM to the SM, it is not necessary to include all Y^S_{ij}. If the Symmetry basis stationary conditions yield that a specific quadratic parameter is large when there is a small vev, $Y^S_{aa} \gg v^2$ when $v_a \to 0$, it is possible not to include some off-diagonal quadratic parameters related to the direction of the small vev, $Y^S_{ab\neq a} = 0$. We comment that the number of $Y^S_{ij\neq i}$ that must be included depends on the model and if the decoupling energy scales are distinct. For NHDMs with a decoupling limit of the second type, we expect this behaviour to hold. Yet, we emphasised that it is not possible to have a decoupling limit of the second type to the SM.

For the special case of a 3HDM with one decoupling energy scale, it is possible not to include $Y^S_{ab\neq a}$ when $v_a \to 0$ and $v_b \to 0$. Even for symmetries that cannot have a decoupling of the second type. Additionally, we comment that it is possible to have a decoupling limit of the second type with one decoupling energy scale $\mathcal{M}_2 = \mathcal{M}_3 \gg v$. For the $\mathbb{Z}_3$ symmetric 3HDM one still needs to include one $Y^S_{ij\neq i}$. Nevertheless, the situation for the $\mathbb{Z}_4$ symmetric 3HDM is quite bizarre and we shall not discuss it. For the general NHDM, we do not know how many $Y^S_{ij\neq i}$ need to be included in order to attain $\mathcal{M}_{i\neq 1} = \mathcal{M} \gg v$. Nevertheless, we suspect that it may be possible not to include $Y^S_{ab\neq a}$ when both $v_a \to 0$ and $v_b \to 0$.

Exploration of viable parameter space regions in models with extended Higgs sectors can be challenging due to sophisticated scalar potentials. In particular, requiring the potential to be Bounded from Below (BFB) places constraints on its parameters, but establishing the exact necessary and sufficient BFB conditions can be notoriously difficult. In this thesis, we found these conditions for the $\mathbb{U}(1) \times \mathbb{U}(1)$ Three-Higgs-Doublet Model, which can be used to construct viable models with degenerate scalar dark matter candidates. We also found the BFB conditions for the $\mathbb{U}(1) \times \mathbb{Z}_2$ 3HDM, which can be used as a model where one of its doublets contains dark matter candidates.

When deriving these conditions, we found that it is crucial to check not only neutral but also charge-breaking directions in the Higgs space. To highlight this point, we showed an example of a model which possesses a good-looking minimum. The masses are positive for all Higgses, and the potential is bounded from below in all neutral directions of the Higgs space. Nevertheless, the example is pathological because the potential is not bounded from below in charge-breaking directions. In short, having a neutral minimum is not an excuse to sidestep stability checks in the entire Higgs space.

It would be even more interesting to find the exact BFB conditions for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric 3HDM, which includes, in addition to our potential, additional terms. We have not yet solved this problem, but we notice that the approach taken in ref. [20] does not lead to the necessary and sufficient BFB conditions for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric 3HDM.

[1] Francisco S. Faro and Igor P. Ivanov, "Boundedness from below in the $U(1) \times U(1)$ Three-Higgs-doublet model," Physical Review D **100**, 035038 (2019).

[2] Igor P. Ivanov, "Building and testing models with extended Higgs sectors," Progress in Particle and Nuclear Physics **95**, 160–208 (2017).

[3] T. D. Lee, "A Theory of Spontaneous T Violation," Physical Review D **8**, 1226–1239 (1973).

[4] Steven Weinberg, "Gauge Theory of CP Nonconservation," Physical Review Letters **37**, 657–661 (1976).

[5] Gustavo C. Branco, "Spontaneous CP Nonconservation in Theories with More Than Four Quarks," Physical Review Letters **44**, 504–507 (1980).

[6] Gustavo C. Branco, "Spontaneous CP Nonconservation and Natural Flavor Conservation: A Minimal Model," Physical Review D **22**, 2901–2905 (1980).

[7] M. Tanabashi *et al.* (Particle Data Group), "Review of Particle Physics," Physical Review D **98** (2018), https://doi.org/10.1103/PhysRevD.98.030001.

[8] John F. Gunion and Howard E. Haber, "CP-conserving two-Higgs-doublet model: The approach to the decoupling limit," Physical Review D **67**, 075019 (2003).

[9] Jérémy Bernon, John F. Gunion, Howard E. Haber, Yun Jiang, and Sabine Kraml, "Scrutinizing the alignment limit in two-Higgs-doublet models: $m_h = 125$ GeV," Physical Review D **92**, 075004 (2015).

[10] A. Barroso, P. M. Ferreira, R. Santos, and João P. Silva, "Stability of the normal vacuum in multi-Higgs-doublet models," Physical Review D **74**, 085016 (2006).

[11] P. M. Ferreira and João P. Silva, "Discrete and continuous symmetries in multi-Higgs-doublet models," Physical Review D **78**, 116007 (2008).

[12] C. C. Nishi, "Structure of potentials with N Higgs doublets," Physical Review D **76**, 055013 (2007).

[13] Miguel P. Bento, Howard E. Haber, J. C. Romão, and João P. Silva, "Multi-Higgs doublet models: physical parametrization, sum rules and unitarity bounds," Journal of High Energy Physics **11**, 95 (2017).

[14] Benjamin W. Lee, C. Quigg, and H. B. Thacker, "Strength of Weak Interactions at Very High Energies and the Higgs Boson Mass," Physical Review Letters **38**, 883–885 (1977).

[15] Benjamin W. Lee, C. Quigg, and H. B. Thacker, "Weak interactions at very high energies: The role of the Higgs-boson mass," Physical Review D **16**, 1519–1531 (1977).

[16] Howard E. Haber, "The Higgs data and the Decoupling Limit," in *Proceedings, 1st Toyama International Workshop on Higgs as a Probe of New Physics 2013 (HPNP2013): Toyama, Japan, February 13-16, 2013* (2013) arXiv:1401.0152 [hep-ph].

[17] Jérémy Bernon, John F. Gunion, Howard E. Haber, Yun Jiang, and Sabine Kraml, "Scrutinizing the alignment limit in two-Higgs-doublet models. II. $m_H = 125$ GeV," Physical Review D **93**, 035027 (2016).

[18] Kristjan Kannike, "Vacuum Stability Conditions From Copositivity Criteria," The European Physical Journal C **72**, 2093 (2012).

[19] Gustavo C. Branco, Luis Lavoura, and Joao Paulo Silva, *CP violation*, Vol. 103 (Oxford University Press, 1999).

[20] B. Grzadkowski, OM. Ogreid, and P. Osland, "Natural multi-Higgs model with dark matter and CP violation," Physical Review D **80**, 055013 (2009).