On the Derivation of the Navier–Stokes Equations from a Nondeterministic Variational Principle

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Abstract

The mathematical description of fluid dynamics, which models the flow of matter in all the possible states with the exception of the solid state, is of fundamental importance to many branches of physics, due to the wide range of phenomena it predicts and explains. The Navier–Stokes equations, among the most important equations of mathematical physics, provide a very accurate description of the motion of viscous, incompressible fluids with uniform density. Discovering a general theory of these equations remains one of the great unsolved problems of mathematics. In this article we start by reviewing the derivation of the Navier–Stokes equations from the principles of conservation of matter and conservation of momentum. We proceed by defining the Wiener process, mentioning one of its constructions, and we provide a few comments on the problematic of functional integration and of the construction of integrals with respect to martingales of various types. Next we define stochastic differential equations, with and without reflecting boundaries, relating the former to the Skorokhod problem and to the concept of local time. Finally we describe how the Navier–Stokes equations with periodic boundary conditions can be derived from a nondeterministic variational principle, and adapt the results obtained to the case when we have Neumann boundary conditions in a different domain.

Keywords: Navier–Stokes equations, Wiener process, stochastic integration, reflecting stochastic differential equations, local time, nondeterministic variational principle.

1 The Laws of Continuum Mechanics and Classical Hydrodynamics

Continuum mechanics is the branch of classical mechanics that describes the dynamics of bodies with the assumption that their substance completely fills the space it occupies. This is an idealization (as is any physical theory), since, according to quantum theory, matter is composed of discrete particles, and therefore not continuous. We devote our attention to one of the branches of continuum mechanics, namely fluid dynamics. We are only interested in describing a fluid, which is any material that is not in a solid state, regarded as a continuum, that is, as a macroscopic phenomenon (at the microscopic level, the flow of a fluid is governed by models in the field of statistical mechanics).

Any physical theory rests on a few essential, universal principles, namely (and among others) the conservation of mass, which states that mass is neither created nor destroyed, and the conservation of momentum, which states that the interaction of bodies composing an isolated system leads only to an exchange in momentum between them that does not affect the motion of the system as a whole. Basing our arguments on these two laws successively, we are able to derive a (partial) system of conservation laws of classical fluid dynamics, known as the Navier–Stokes equations (the full system of conservation laws takes into account the principle of conservation of energy, which is irrelevant to our purposes since we assume that the forces that act on a fluid do not depend on the tem-

perature).

There are two classical representations for the mathematical description of the motion of a fluid: the Lagrange representation and the Euler representation. In the Lagrange representation, the state of a particle of the fluid at any given time is described with reference to its initial position, that is, the trajectory of the particle is given. In the Euler representation, the values of attributes of the fluid are given, such as the velocity, the pressure, or the density, at arbitrary times and positions.

It is important to bear in mind that when we speak of a particle of the fluid, that we are referring to a volume element always supposed to be so large that it contains a very great number of molecules, and at the same time is very small compared with the total volume of the body under consideration (consider, for example, that a fluid such as water contains around 3×10^{22} water molecules per cubic centimetre).

When deriving the Navier–Stokes equations in the next section, we operate in the Euler representation, the representation followed on all the monographs we consulted. Nevertheless, we use concepts from the Lagrange representation, such as the notion of material derivative. Note that the two representations are not equivalent in general (according to [1, p. 71], the Lagrange representation is more disadvantageous to use due to the fact that it does not give the spatial gradients of velocity of the fluid directly).

2 Derivation of the Navier–Stokes Equations from Physical Principles

To derive the Navier–Stokes equations from the physical principles of conservation of mass and conservation of momentum, we need the concept of material derivative, which corresponds to the time rate at which a physical quantity $f(t,x)$, such as the velocity u or the density ρ, varies while moving with a fluid particle.

Definition 2.1. The *material derivative* $\mathrm{D}/\mathrm{D}t$ of f is defined as follows:

$$\frac{\mathrm{D}f}{\mathrm{D}t} := \frac{\partial f}{\partial t} + u \cdot \nabla f. \tag{1}$$

The term $\partial f/\partial t$ corresponds to the local rate of change due to the temporal changes at position x, and the term $u \cdot \nabla f$ corresponds to the rate of change of f due to the transport of a material element to a different position.

Another result we will need is the transport theorem, which describes the rate of change not of a physical quantity but that of any volume integral of a physical quantity.

Theorem 2.1 (Transport)**.** [1] *Let $\Omega(t)$ denote an arbitrary volume that is moving with the fluid. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega(t)} f(t,x)\,\mathrm{d}x = \int\limits_{\Omega(t)} \left(\frac{\mathrm{D}f}{\mathrm{D}t}(t,x) + f(t,x)\nabla \cdot u(t,x) \right) \mathrm{d}x. \tag{2}$$

By the principle of conservation of mass, the mass of a fluid in a volume $\Omega(t)$ does not change as $\Omega(t)$ moves with the fluid, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Omega(t)} \rho\,\mathrm{d}x = 0, \tag{3}$$

where the integral of the density corresponds to the mass. Using this fact, taking $f(t,x) = \rho(t,x)$ in equation (2), and applying the definition of material derivative, we obtain an equation that is valid for an integration over an arbitrary material volume $\Omega(t)$, which allows us to conclude that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0. \tag{4}$$

We call this equation the equation of continuity.

Definition 2.2. We call a fluid *incompressible* if for any domain $\Omega(0)$ and any t, the density is constant, that is, if $\mathrm{vol}(\Omega(0)) = \mathrm{vol}(\Omega(t))$.

Taking $f(t,x) = 1$ on the left-hand side of equation (2), using the fact that the element of volume $\mathrm{d}x$ changes according to the formula $\mathrm{d}x = J\mathrm{d}X$ (where J is the Jacobian determinant), and using the equation (whose proof can be found in [4, pp. 10–11])

$$\frac{\mathrm{d}}{\mathrm{d}t} J(t,X) = \nabla \cdot u(t,x) J(t,X), \tag{5}$$

[1] In this article all proofs are omitted. They, or the references to where they can be seen, can be found in [4].

we obtain successively

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{vol}(\Omega(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega(t)} \mathrm{d}x = \int\limits_{\Omega(0)} \frac{\mathrm{d}}{\mathrm{d}t} J(t,X)\,\mathrm{d}X$$
$$= \int\limits_{\Omega(t)} \nabla\cdot u(t,x)\,\mathrm{d}x = 0. \tag{6}$$

Since the choice of the domain $\Omega(t)$ is arbitrary, we deduce that a fluid is incompressible if and only if

$$\nabla\cdot u(t,x) = 0. \tag{7}$$

Definition 2.3. The *Cauchy stress tensor* $\boldsymbol{T}_{ij}$ is defined as follows:

$$\boldsymbol{T}_{ij} := \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) - p\delta_{ij}, \tag{8}$$

where the first term on the right-hand side is obtained from the rate-of-strain tensor (which describes the rate of change of the deformation of a material in the neighbourhood of a certain point), and the second term corresponds to the hydrostatic pressure (a pressure that acts perpendicularly to the local surface).

The forces acting on an element of a fluid can be external, such as gravitational or electromagnetic forces, or internal (also called contact forces), which act on an element of volume $\Omega(t)$ through its bounding surface.

Definition 2.4. The *normal stress* t_n is defined as follows: $t_n(t,x,n) := \boldsymbol{n}(t,x)\boldsymbol{T}(t,x)$, where $\boldsymbol{n}$ is the unit outward normal at a point of the boundary $\partial\Omega(t)$, and $\boldsymbol{T} = \boldsymbol{T}_{ij}$ is the Cauchy stress tensor. The normal stress corresponds to the force per unit area exerted at a point of the surface $\partial\Omega(t)$.

By the Cauchy principle we know that, at any given time, t_n depends only on the position and orientation of a surface element dS. We know also by the principle of conservation of linear momentum that the rate of change of the linear momentum of a volume equals the resultant force on the volume. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}(mu) = \int\limits_{\Omega(t)} \rho f\,\mathrm{d}x + \int\limits_{\partial\Omega(t)} t_n\,\mathrm{d}S, \tag{9}$$

where the first term on the right-hand side of the equality corresponds to how an external force f (per unit mass) acts on $\Omega(t)$, and the second term corresponds to the surface force exerted on the volume.

Lemma 2.1.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega(t)} \rho f\,\mathrm{d}x = \int\limits_{\Omega(t)} \rho\frac{\mathrm{D}f}{\mathrm{D}t}\,\mathrm{d}x. \tag{10}$$

Using the definition of mass (the quantity being differentiated in equation (3)), lemma 2.1, and applying the divergence theorem to the second term on the right-hand side of equation (9), we transform equation (9) into an equation whose terms are integrated over an arbitrary domain. Therefore we obtain the conservation of momentum equation

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = \nabla\cdot\boldsymbol{T} + \rho f. \tag{11}$$

Expanding the term $\nabla\cdot(\rho u)$ in equation (4), and using the definition of material derivative, we obtain the following, alternative form of the equation of continuity:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot u = 0. \tag{12}$$

Equations (11) and (12) constitute the (partial) system of conservation laws of classical hydrodynamics.

If we substitute definition (8) in equation (11); if we assume that the fluid is incompressible (i.e. that equation (7) is valid); if we use the definition of material derivative on the equations we obtained; and if we assume that ρ is uniform, then we obtain the system

$$\begin{cases} \dfrac{\partial u}{\partial t} + u\cdot\nabla u = \nu\Delta u - \dfrac{1}{\rho}\nabla p + f, & (13)\\ \nabla\cdot u = 0, & (14)\end{cases}$$

where $\nu := \mu/\rho$ is kinematic viscosity coefficient (μ is the shear viscosity coefficient). These are the Navier–Stokes equations of motion of viscous, incompressible fluids with uniform density, where u and p, the velocity and the pressure, respectively, are the dependent variables and t and x_i, $i = 1, 2, 3$, are the independent variables. These equations are second order, non-linear partial differential equations on account of the presence of the

Laplace operator and of the product between u_j and the partial derivatives of u_i with respect to x_j, respectively.

If the fluid is inviscid, that is, if $\nu = 0$, then we call equation (13) the Euler equation. Note that the equations of hydrodynamics have as their domain $\mathbb{R}_+ \times \mathbb{R}^n$, for n equal to 2 or n equal to 3 only, since it does not make sense to study these equations in higher dimensions.

All physical problems described by partial differential equations have a domain of application, which includes boundaries of some kind. To adequately describe a problem it is essential to define both initial conditions and boundary conditions. The initial condition for the equations of hydrodynamics describes the initial distribution of velocities and is simply written as $u(0, x) = u_0(x)$ for $x \in \Omega$, where u_0 is a given function and Ω is the bounded domain. Boundary conditions can be of various types, but for our purposes, we need only to define the so-called Neumann boundary conditions. These conditions consist of the specification on the boundary of the directional derivative of u in the outward normal direction. This derivative is represented as $\partial u/\partial \boldsymbol{n} = \nabla u \cdot \boldsymbol{n}$, and is proportional to the inward flux. When there is no flux across the boundary, the most common case in the context of the Navier–Stokes equations, the condition is written as

$$\nabla u \cdot \boldsymbol{n} = 0 \qquad \text{on } [0, T] \times \partial\Omega. \tag{15}$$

3 The Wiener Process and Stochastic Integration

The notion of Wiener process, which underpins the whole field of stochastic analysis, is the mathematical idealization of the Brownian motion phenomenon: the irregular motion of small particles of pollen suspended in water, caused by the random impact of the water molecules with the particles.

Definition 3.1. A continuous stochastic process W defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a *Wiener process* if it satisfies the following three properties:

1) $W(0) = 0$ a.s.

2) $W(t) - W(s) \sim N(0, t - s)$ for all $0 \leq s \leq t$.

3) The increments are independent of $\mathcal{F}_s$, that is, for all times $0 \leq s < t$, the random variables $W(t) - W(s)$ are independent of $\mathcal{F}_s$.

The construction of the Wiener process was originally obtained in 1923 by Norbert Wiener (the proof can be found, along with construction of the Wiener measure, in [6]). We mention here only a generalization of the original construction (for more details on other constructions see [4, pp. 28–30]), which consists of proving that the Wiener process can be represented in the form

$$W(t) = \sum_{k=0}^{\infty} Z_k \int_0^t \phi_k(s)\,\mathrm{d}s, \tag{16}$$

where $\{\phi_k\}_{k=0}^{\infty}$ is an arbitrary complete, orthonormal basis of the Hilbert space $L^2([0, 1])$, and Z_k, $k \in \mathbb{N}$, are independent Gaussian random variables with zero mean. The Wiener process has several important properties, such as the a.s.-continuity and the nowhere-differentiability of its sample paths.

Unlike Lebesgue integration, where the domain of the integral is a region of n-dimensional space, functional integration consists of defining an integral which has its domain in a space of functions, a significantly more complicated mathematical object. There are at least two major approaches to functional integration, one that arises in the vast fields of probability theory and stochastic analysis, and the other in the path integral approach to quantum mechanics and in quantum field theory. We provide here a few remarks on the first approach. Adapting and extending the concept of the Daniell integral, an integral which is defined axiomatically (in contrast with the Lebesgue integral, where it is the concept of a measure that is axiomatic), Norbert Wiener defined an integral with respect to the Wiener measure, or more specifically to a class of Brownian motion paths (the Wiener measure corresponds to the probability of finding a particle in a certain path, and its definition is based on the fundamental solution of the diffusion equation; see [4, pp. 23–28] for more details). This concept of integral was later extended by Paley, Wiener and Zygmund to an integral with respect to the Wiener process, where the integrand is a deterministic func-

tion satisfying certain conditions (this integral is in fact a random variable). Further extensions were later obtained by defining integrals with respect to martingales (of which the Wiener process is an example) of various types, and where the integrand can belong to $\mathcal{L}_2$, the space of real-valued, $\mathcal{F}_t$-adapted, measurable stochastic processes X that have a finite expectation of the integration of X^2 with respect to time for every $t \in \mathbb{R}_+$. The particular definition we make extensive use of is the definition of the Itô stochastic integral of a process belonging to $\mathcal{L}_2$ with respect to the Wiener process.

4 Semimartingales and Stochastic Differential Equations

Definition 4.1. We call a Markovian-type process X, i.e. a process independent of its past history, a *diffusion process* if the sample function $X_t(\omega)$ is continuous in t with probability one until a certain random time, called the terminal time, after which $X_t(\omega)$ stays at the terminal point.

The Wiener process is an example of a diffusion process.

Definition 4.2. A *continuous semimartingale* X is an adapted stochastic process which has the unique decomposition

$$X_t = X_0 + M_t + V_t \qquad \text{a.s.,} \tag{17}$$

where X_0 is a random variable, $M \in \mathcal{M}^{\text{c,loc}}$ (this is the space of continuous, local martingales, and its definition can be seen in [4, p. 35]) and V_t is a right-continuous, $\mathcal{F}_t$-adapted, integrable process (verifying $V_0 = 0$) such that $t \mapsto V_t$ is of bounded variation a.s. We call M the martingale part of the decomposition and V the drift part.

A particular case of a continuous semimartingale is an Itô process, which has the stochastic differential $\mathrm{d}X = F\mathrm{d}t + G\mathrm{d}W$ (i.e. M_t in definition (17) is the Itô integral), where F and G are adapted processes belonging to certain martingale spaces, provided the corresponding integrals are well-defined (see the definition of several martingale spaces, and in some cases of the corresponding stochastic integrals, in [4, pp. 33–36]).

Of fundamental importance is the change-of-variable, or Itô, formula, sometimes also called the chain rule for stochastic calculus.

Theorem 4.1 (Multidimensional Itô Formula)**.** *Let $f\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ be a function of class $C^{1,2}$, and let X be a multidimensional continuous semimartingale, where in that case we write the martingale part of its decomposition as the vector $(M_t^1, \ldots, M_t^d)$, where $M_t^i \in \mathcal{M}^{\text{c,loc}}$, and the drift part as the vector $(V_t^1, \ldots, V_t^d)$, V_t^i being an adapted process of bounded variation verifying $V_0^i = 0$, $i = 1, \ldots, d$. We call*

$$\mathrm{d}f(t, X_t) = \frac{\partial}{\partial t} f(t, X_t)\,\mathrm{d}t + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f(t, X_t)\,\mathrm{d}X_t^i + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} f(t, X_t)\,\mathrm{d}\langle M^i, M^j\rangle_t \tag{18}$$

the multidimensional Itô formula.

Observe that $f(t, X_t)$ is a continuous semimartingale belonging to $\mathcal{M}^{\text{c,loc}}$. The last integral on the right-hand side of equation (18) is an integral with respect to the cross-variation of M and N (the definition of the cross-variation, also known as the quadratic variation, can be seen in [4, pp. 35–36]). We will deal frequently with the particular case where $\mathrm{d}M_t = \mathrm{d}W_t$ and $\mathrm{d}V_t = \mathrm{d}t$, and in that case we know that $\langle M^i, M^j\rangle_t = \delta_{ij} t$.

Definition 4.3. Let $b_i(t, x), B_{ij}(t, x)\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$, for $1 \le i \le d$ and $1 \le j \le r$, be Borel-measurable functions. We call $b(t, x) = (b_1(t, x), \ldots, b_d(t, x))$ the drift vector and $B(t, x) = [B_{ij}]$ the dispersion matrix. Let also X be a progressively measurable stochastic process. With these assumptions we call

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + B(t, X_t)\,\mathrm{d}W_t \tag{19}$$

a *stochastic differential equation* (SDE) and X_0 its *initial value*.

We can interpret a SDE as being an ordinary differential equation with an extra term, which accounts for the random perturbations of the trajectory of its solution. These random perturbations are accounted for in the term with the differential $\mathrm{d}W$, which is therefore called the random term, or diffusion term.

Theorem 4.2 (Existence and uniqueness of solution of a SDE)**.** *Let b and B be measurable func-*

tions satisfying, for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^d$, the following Lipschitz conditions, for some constant K:

1) $\sup_t \sup_x |b(t,x)| \leq K$.

2) $\sup_t \sup_x \|B(t,x)\| \leq K$.

3) $|b(t,x)| + \|B(t,x)\| \leq K(1+|x|)$.

Let also X_0 be a random variable verifying $\mathrm{E}[|X_0|^2] < \infty$. Then there exists a unique solution $X \in \mathcal{L}_2$ of the stochastic differential equation (19).

Conditions 1) and 2) tell us that the functions b and B are Lipschitz continuous. Condition 3), which ensures that there is linear growth in x, is actually a consequence of the previous two conditions.

5 The Skorokhod Problem, Stochastic Differential Equations with Reflection, and Local Time

Let us assume that there are particles located inside an open set $\mathbb{R}^d_+$, the boundary of which is impermeable, and that $t \in \mathbb{R}_+$. The Skorokhod problem is a mathematical formalization of the problem of finding the position of the particles after they have hit the boundary and been reflected. The definition presented next is not the most general one (which can be found in [5, pp. 24–25]), but simply the one that we will need.

Definition 5.1. Let $f = (f_1, \ldots, f_d)\colon C(\mathbb{R}_+) \to \mathbb{R}^d$ be a continuous function that verifies $f(0) \in \mathbb{R}^d_+$, and let $g = (g_1, \ldots, g_d)\colon \mathbb{R}_+ \to \mathbb{R}_+ \times \mathbb{R}^d$ and $l\colon C(\mathbb{R}_+) \to \mathbb{R}$ be two other continuous functions. We call the pair of functions (g, l) a solution of the *Skorokhod problem* in $\mathbb{R}^d_+$ with normal reflections at the boundary $\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1}_+ \times \{0\}$ if the following conditions are met:

1) $g(t) \in \mathbb{R}^d_+$, where $g(t) = f(t) + nl(t)$ and $n = (0, \ldots, 0, 1)$.

2) l is a non-decreasing function verifying $l(0) = 0$ and $l(t) = \int_0^t \mathbb{1}_{\partial\mathbb{R}^d_+}(g(s))\,\mathrm{d}l(s)$.

We interpret $g(t)$ as indicating the position of a particle at the instant t, and $f(t)$ as being a trajectory of a Wiener process. The moment t when a particle hits the boundary corresponds to the position $g(t) = 0$. The conditions in item 2) of definition 5.1 signify that l does not increase when $g(s) \notin \partial\mathbb{R}^d_+$, that is, that l may only increase at those instants s when the particle is hitting the boundary.

It is clear from the definition that $g_1(t) = f_1(t)$, $\ldots$, $g_{d-1}(t) = f_{d-1}(t)$. We make the identification $g(t) = \Gamma f(t) = (f_1(t), \ldots, f_{d-1}(t), \Gamma f_d(t))$, calling the Lipschitz-continuous map $\Gamma\colon C(\mathbb{R}_+, \mathbb{R}^d) \to C(\mathbb{R}_+, \mathbb{R}^d)$ the multidimensional Skorokhod map.

One of our goals is to study stochastic differential equations, which correspond to diffusion processes, in a domain with a reflecting boundary. More specifically, we want to construct a stochastic process $\xi(t)$ that continuously reflects when it hits the boundary of the domain $\mathbb{R}^d_+$. The form of the SDE with normal reflection at the boundary that we will consider is the following:

$$\mathrm{d}\xi(t) = a(t, \xi(t))\,\mathrm{d}t + \sum_k b_k(t, \xi(t))\,\mathrm{d}W_k(t) + n\mathrm{d}l(t), \tag{20}$$

where $a = a(t,x)\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $b_k = b_k(t,x)\colon \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are measurable functions, $\xi(0) = \xi_0$, where $\xi_0 \in \overline{\mathbb{R}^d_+}$ is an $\mathcal{F}_0$-measurable random variable, and all the conditions of the Skorokhod problem are satisfied taking $g(t) = \xi(t)$ and

$$f(t) = \xi_0 + \int_0^t a(s, \Gamma f(s))\,\mathrm{d}s + \sum_k \int_0^t b_k(s, \Gamma f(s))\,\mathrm{d}W_k(s) \qquad \text{a.s.,} \tag{21}$$

making the identification $\xi(t) = \Gamma f(t)$, and assuming all the integrals in the equation are well-defined. If $f(t)$ is a solution of equation (21), then $\xi(t) = \Gamma f(t)$ is a solution of equation (20).

Definition 5.2. We say that equation (20) has a *weak solution* if there exists a pair $(\xi(t), l(t))$ of continuous $\mathcal{F}_t$-adapted processes that verify all the above conditions.

Theorem 5.1. *Let the functions $a = a(t,x)$ and $b_k = b_k(t,x)$ in equation* (20) *be continuous in (t,x)*

and bounded, i.e. let the following condition be satisfied, for some constant K:

$$\sup_t \sup_x \left(|a(t,x)| + \sum_k |b_k(t,x)| \right) \leq K. \qquad (22)$$

Then there exists a weak solution to equation (20)*.*

By the Lipschitz-continuity of the Skorokhod map and by theorem 5.1, we conclude that equation (20) has a unique solution.

We identify the function $l(t)$ in the definition of the Skorokhod problem and in definition (20) with the local time $L_t(x)$ defined below. The proof that $l(t)$ is a local time at the point zero of the process $\xi(t)$ on the domain $\mathbb{R}_+$ can be seen in [4, pp. 49–50].

Definition 5.3. The *local time* of the Wiener process W_t is a family of non-negative random variables $\{L_t(x,\omega)\colon (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \omega \in \Omega\}$ such that the map $(t,x) \longmapsto L_t(x,\omega)$ is continuous for almost all $\omega \in \Omega$, and

$$L_t(x) = \lim_{\varepsilon \to 0} \frac{1}{4\varepsilon} \int_0^t \mathbb{1}_{]x-\varepsilon, x+\varepsilon[}(W_s)\,\mathrm{d}s \qquad \text{a.s.} \quad (23)$$

for $x \in \mathbb{R}$. This limit is finite and always exists.

The concept of local time corresponds to the measurement of the amount of time spent by a path of the Wiener process in the neighbourhood of a point $x \in \mathbb{R}$. The integral on the right-hand side of (23) corresponds to the Lebesgue measure of the set $\{0 \leq s \leq t\colon |W_s - x| \leq \varepsilon\}$. The introduction of this definition was motivated by the discovery that, for every $x \in \mathbb{R}$, the set $\{t \in \mathbb{R}_+\colon W_t(\omega) = x\}$ has Lebesgue measure zero (the proof can be found in [3, pp. 105–106]), which gives us no information about the time spent by a path in the neighbourhood of x. Although we have been discussing specifically the concept of the local time of the Wiener process, it is also possible to define the local time of any continuous semimartingale X. In that case we substitute $\mathrm{d}s$ in equation (23) by $\mathrm{d}\langle X, X\rangle_s$.

Note that the local time depends on a time and on a space variable, and that integration can be defined with respect to the two variables or with respect to each one individually, keeping the other fixed (more details on the problem of integration with respect to local time can be found in [4, pp. 63–64]).

6 Derivation of the Navier–Stokes Equations from a Stochastic Variational Principle

This section is based on [2, pp. 2–5] (and more details can be found in [4, pp. 51–60]). Assume that we have a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, and let $\mathbb{T}$ be a flat torus of dimension d, which we identify with $[0, 2\pi]^d$ (corresponding to periodic boundary conditions). Let ξ be a semimartingale defined, in differential form, as

$$\mathrm{d}\xi_t(x,\omega) = \mathrm{d}M_t(x,\omega) + \mathrm{D}_t\xi_t(x,\omega)\,\mathrm{d}t \qquad (24)$$

for $x \in \mathbb{T}$, and such that $\xi_0(x) = x$. We call $\mathrm{d}M_t$ the martingale part of the decomposition of ξ_t and $\mathrm{D}_t\xi_t$ the drift. From now on we omit the parameter $\omega \in \Omega$.

Since martingales are not differentiable a.s. in time, we cannot compute their derivative with respect to time. Nevertheless, it is possible to define a time derivative after conditional expectation of the paths.

Definition 6.1. The *generalized derivative* D_t of a martingale $F\colon [0,T] \times \mathbb{T} \to \mathbb{R}$ is defined as follows:

$$\mathrm{D}_t F(t, \xi_t(x)) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathrm{E}\left[F(t+\varepsilon, \xi_{t+\varepsilon}(x)) - F(t, \xi_t(x)) | \mathcal{F}_t\right] \qquad (25)$$

a.s. when the limit exists.

Observe that when $F(t,x) = x$, the generalized derivative corresponds to the above semimartingale's drift.

Consider now the following SDE, a particular case of semimartingale (24),

$$\mathrm{d}g_t(x) = \sqrt{2\nu}\,\mathrm{d}W_t + u(t, g_t(x))\,\mathrm{d}t, \qquad (26)$$

for $x \in \mathbb{T}$, with $g_0(x) = x$, and where $g_t(\cdot)$ are diffeomorphisms, which represent diffusions on the torus. As above, for the case of $F(t,x) = x$, we have $\mathrm{D}_t g_t(x) = u(t, g_t(x))$. For the case of $F(t,x) = u(t,x)$, using the Itô formula in d dimen-

sions we obtain

$$\mathrm{d}u^i(t, g_t(x)) = \sqrt{2\nu}\sum_{j=1}^{d} \partial_j u^i \,\mathrm{d}W_t^j + (\partial_t u^i + u^i \cdot \nabla u^i + \nu\Delta u^i)\,\mathrm{d}t \quad (27)$$

for $i = 1, \ldots, d$.

Let us denote by $\mathbb{H}$ a linear subspace dense in $L^2([0,T]\times\mathbb{T})$ and by E the expectation with respect to the underlying probability measure P. Let $p \in \mathbb{H}$ be a function. We define the following action functional

$$\begin{aligned} S(g_t, p) &:= \frac{1}{2}\,\mathrm{E}\int_0^T \int |\mathrm{D}_t g_t(x)|^2 \,\mathrm{d}x\mathrm{d}t \\ &\quad + \mathrm{E}\int_0^T \int p(t, g_t(x))(\det \nabla g_t(x) - 1)\,\mathrm{d}x\mathrm{d}t \\ &:= S^1(g_t) + S^2(g_t, p), \end{aligned} \quad (28)$$

where $S^1(g_t)$ can be interpreted as corresponding to an action integral of the Lagrangian composed solely of the kinetic energy and $S^2(g_t, p)$ as corresponding to the part of a modified Lagrange function which includes the Lagrange multiplier expressed in terms of the pressure p. In fact, since $\mathrm{D}_t g_t(x) = u(t, g_t(x))$, we can interpret $S^1(g_t)$ as corresponding to the total energy functional of the fluid. Observe that $p(t, g_t(x))$ corresponds to the Lagrange multiplier and $\det \nabla g_t(x) - 1$ to the restrictions, which ensure that the Lebesgue measure is conserved by the transformation $g_t(x)$ for all t.

We consider the following functional variations of g_t and p:

$$g_t(\cdot) \longrightarrow g_t^\varepsilon(\cdot) = g_t(\cdot) + \varepsilon h(t, g_t(\cdot)), \quad (29)$$

$$p(t,\cdot) \longrightarrow p^\varepsilon(t,\cdot) = p(t,\cdot) + \varepsilon\varphi(t, g_t(\cdot)), \quad (30)$$

where $h(t,x)$ and $\varphi(t,x)$ are smooth functions in $x, \varphi \in \mathbb{H}$ such that $h(T,\cdot) = h(0,\cdot) = 0$. We assume also that we have periodic boundary conditions with period 2π for all x.

Theorem 6.1. *The diffusions $g_t(\cdot)$ and a function $p \in \mathbb{H}$ verify $\delta S(g_t, p) = 0$, i.e. they are critical to the action functional S, if and only if the drift $u(t,\cdot)$ of $g_t(\cdot)$ satisfies the Navier–Stokes equations* (13) *and* (14) *for $x \in \mathbb{T}$ and $t \in [0,T]$, taking $\rho = 1$ and $f = 0$.*

7 Diffusions on the Multidimensional Half-Space With Reflection at the Boundary

In the previous section we assumed that the Navier–Stokes equations were defined in the domain $[0,T]\times\mathbb{T}$, where $\mathbb{T}$ is the d-dimensional flat torus. In this section we consider the Navier–Stokes equations with Neumann boundary conditions in the domain $[0,T]\times\mathbb{R}^d_+$, or more specifically, we consider the system composed of equations (13), (14) and (15), taking $\rho = 1$ and $f = 0$. We will see how the results expounded in the previous section change when we consider the domain $\mathbb{R}^d_+$, and take the SDE (26) and write it in the form of the SDE (20) with normal reflections at the boundary $\partial\mathbb{R}^d_+ = \mathbb{R}^{d-1}_+ \times \{0\}$. The former equation becomes

$$\mathrm{d}g_t(x) = u(t, g_t(x))\,\mathrm{d}t + \sqrt{2\nu}\sum_{k=1}^{d} \mathrm{d}W_k(t) + n\,\mathrm{d}l(t), \quad (31)$$

where $g_t(\cdot)$ are diffeomorphisms (representing diffusions on the half-space) such that $g_0(x) = x$, $n = (0, \ldots, 0, 1)$, and l is a function verifying the conditions in definition 5.1.

Writing $f(t)$ in a similar fashion to equation (21), by the Lipschitz-continuity of the Skorokhod map and by theorem 5.1, we conclude that equation (31) has a unique solution.

Proceeding now as in the previous section, we calculate the generalized derivative of $g_t(x)$ obtaining

$$\mathrm{D}_t g_t(x) = u(t, g_t(x)) + nl(t) \qquad \text{a.s.} \quad (32)$$

Using the Itô formula in d dimensions we obtain the stochastic differential of $\mathrm{D}_t g_t(x)$, which is

$$\begin{aligned} \mathrm{d}\mathrm{D}_t g_t(x) &= (\partial_t u^i + n\partial_t l(t) + u^i \cdot \nabla u^i + \nu\Delta u^i)\,\mathrm{d}t \\ &\quad + \sqrt{2\nu}\sum_{k=1}^{d} \nabla u\,\mathrm{d}W_k(t) \\ &\quad + n\nabla u\,\mathrm{d}l(t) \end{aligned} \quad (33)$$

for $i = 1, \ldots, d$. We can simplify this result by observing that $\partial_t l(t) = 0$, since $l(t)$ is constant on

each interval in the complement of the closed set $\{t \in \mathbb{R}_+ \colon W_t(x) = x\}$ (the source of this result is [3, p. 202]).

We consider again the definition of the action functional (28) and the functional variations of g_t and p in (29) and (30), respectively, in whose definitions the smooth functions h and φ are used. The main part of the proof of theorem 6.1 consists of the computation of the variation of the action (28), using the SDE (26). When we perform the same computation, now using the SDE (31) we obtain

$$\begin{aligned}\delta S = &- \mathrm{E} \int_0^T \int (\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p) \cdot h \, \mathrm{d}x \mathrm{d}t \\ &- \mathrm{E} \int_0^T \int n \nabla u \, l(t) \cdot h \, \mathrm{d}x \mathrm{d}t. \end{aligned} \tag{34}$$

This variation would be zero (as in theorem 6.1) if and only if u satisfied the Navier–Stokes equations and

$$\nabla u \cdot \boldsymbol{n} \, l(t) = 0 \qquad \text{on } [0,T] \times \partial \mathbb{R}^d_+. \tag{35}$$

Since we are assuming that u verifies the Navier–Stokes equations with the Neumann boundary conditions, we conclude that this variation is indeed zero. Therefore we deduce that theorem 6.1 is valid when we consider diffusions with reflection on the domain $\mathbb{R}^d_+$, provided that we assume that the Navier–Stokes equations satisfy the Neumann boundary conditions.

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