On ARL-unbiased c-charts and two-sided CUSUM schemes for i.i.d. and INAR(1) Poisson counts

Mafalda Filipa Vicente Clara
Instituto Superior Técnico – Universidade de Lisboa
July, 2019

Abstract

In Statistical Process Control (SPC), we usually assume that counts of defects in a fixed size random sample have a Poisson distribution with parameter $\lambda$. The non-negative, discrete and asymmetrical character of this distribution and the value of its target mean $\lambda_0$ may prevent the quality control engineer to deal with a chart to monitor $\lambda$ with: a pre-specify in-control average run length (ARL); the ability to swiftly detect not only increases but also decreases in $\lambda$ and, thus, use the chart as a tool for continuous improvement. Furthermore, as far as we have investigated, the charts for $\lambda$ proposed in the SPC literature fail to have an in-control ARL larger than any out-of-control ARL, i.e., they are ARL-biased. There are a few honourable exceptions in Paulino (2015), Paulino et al. (2016a), Paulino et al. (2016b) and Morais et al. (2018). The main goals of this article are to: propose ARL-unbiased cumulative sum (CUSUM) schemes to speed up the detection of small-to-moderate shifts in the mean of i.i.d. and first-order integer-valued autoregressive (INAR(1)) Poisson counts; use the R statistical software to provide instructive illustrations of all these charts/schemes and to assess their in-control and out-of-control performance.

Keywords: Average run length; Poisson counts; statistical process control; R statistical software.

1 Introduction

The use of quality control charts generally requires an upper control limit ($UCL$), a lower control limit ($LCL$), a centre line (CL) representing the target value of the parameter we are monitoring, and a control statistic. It is desirable that the observed values of the control statistics are between the two control limits.

Shewhart control charts are the most commonly used process monitoring tools due to the simplicity of their design and the evaluation of their performance. Among these, the c-chart with 3-σ control limits used to monitor the expected number of defects ($\lambda$), in a fixed size random sample, assuming that the counts of defects $X_n$ have a Poisson distribution. However, this Shewhart quality control chart fails to swiftly detect small-to-moderate size shifts from the target value $\lambda_0$ because it only uses the most recent information, ignoring the remaining data.

A way to increase the speed of detection of these shifts is to consider all the information in successive samples. In 1954, Page originally proposed CUSUM charts as a way to quickly detect shifts in the expected value of a normally distributed quality characteristic and of the fraction defective. As expertly put by Lucas (1985), two-sided CUSUM schemes can be used to detect either an increase or a decrease in the process mean are obtained by running simultaneously two one-sided charts for this parameter.

The performance of CUSUM charts or schemes is usually evaluated by determining their average run length (ARL) profile. The ARL is the most popular performance metrics and represents the average number of samples taken before a signal is triggered by the chart/scheme. Unfortunately, the two-sided CUSUM schemes for the process mean $\lambda$ are ARL-biased in the sense that they take longer (in average) to detect some shifts in $\lambda$ than to trigger a false alarm. Moreover, due to the discrete character of the control statistics, the two-sided CUSUM scheme cannot be set in such way that its in-control ARL coincides with a pre-specified in-control ARL.

The main goal of this article is to describe how to obtain ARL-unbiased versions of two-sided
CUSUM schemes for the mean of independent and identically distributed (i.i.d.) and first order integer-valued autoregressive (INAR(1)) Poisson counted output.

But before we proceed, we briefly discuss the use of c-charts in the monitoring of the mean of such processes.

2 Revisiting the c-charts for the mean of i.i.d. Poisson counts

Many researchers looked for ways to improve the poor performance of the c-chart. They are mainly aiming at an ARL-unbiased c-chart. Paulino et al. (2016a) describe some of the alternatives to the standard c-chart. These alternatives were originally reported in detail by Aebtarm and Bouguila (2011) and have the purpose of controlling the expected value of i.i.d. Poisson counts.

Inspired by the UMPU tests, Paulino et al. (2016a) decided to combine the upper and lower control limits, LCL and UCL, with randomization probabilities, γ_{LCL} and γ_{UCL}. Indeed, the ARL-unbiased c-chart proposed by these authors triggers a signal at a sample with:

- probability one if the sample number of defects is below LCL or above UCL;
- probabilities γ_{LCL} and γ_{UCL} if the sample number of defects is equal to LCL and UCL, respectively.

These randomization probabilities are obtained by solving a system of two linear equations and are given by:

\[
\gamma_{LCL} = \frac{de - bf}{ad - bc} \quad (1)
\]
\[
\gamma_{UCL} = \frac{af - ce}{ad - bc} \quad (2)
\]

where \(a = P_{\lambda_0}(X = LCL)\), \(b = P_{\lambda_0}(X = UCL)\), \(c = LCL \times P_{\lambda_0}(X = LCL)\), \(d = UCL \times P_{\lambda_0}(X = UCL)\), \(e = a - 1 + \sum_{x=LCL}^{UCL} P_{\lambda_0}(X = x)\) and \(f = \alpha\lambda_0 - \lambda_0 + \sum_{x=LCL}^{UCL} x P_{\lambda_0}(X = x)\), with \(P_{\lambda_0}(x) = e^{-\lambda_0}\lambda_0^x/x!\), \(x = 0, 1, \ldots\).

As for the control limits LCL and UCL and as far we have investigated, there are three possible search procedures. The one proposed by Paulino et al. (2016a) which involves a suitable grid of non-negative integer numbers.

It is also possible to resort to an adaptation of the search procedure suggested by Morais (2016) to obtain the control limits of an ARL-unbiased np-chart used to monitor the fraction defective. Finally, we can also capitalize on Geyer and Meeden (2005), who resort to the notion of fuzzy set instead of the concept of randomization. Geyer and Meeden (2005) is associated with the ump package of the statistical software R Geyer and Meeden (2017). This package returns, for example, the critical function of an UMPU test solely for the binomial distribution. However, the fuzzy set concept can be applied to any distribution belonging to the exponential family, such as the Poisson distribution. Thus, we adapted the code to obtain the critical function of an UMPU test for \(\lambda\) and therefore we were able to provide the control limits and the associated randomization probabilities defining the ARL-unbiased c-chart for i.i.d. Poisson counts.

Expectedly, the three search procedures led to the same control limits and randomization probabilities that differ only in the last decimal places, as illustrated by Clara (2019, Tabela 2.1).

The statistical software R also provides a package that allows you to view quality control charts. The package in question, qcc, is described in detail by Scrucca (2004).

What follows is an illustration of the use of the qcc command, properly adapted and making use of simulated data.

A set of 30 observations has been considered: the first 5 (resp. last 25) refer to the in-control (resp. out-of-control) process mean \(\lambda = \lambda_0 = 7\) (resp. \(\lambda = \lambda_0 + 1\)).

In Figure 1 we can find the ARL-unbiased c-chart for this data along with the values of the control limits (LCL and UCL) and the associated randomization probabilities (\(\gamma_{LCL}\) and \(\gamma_{UCL}\)).

Figure 1: ARL-unbiased c-chart obtained by qcc command.
comprises the in-control process mean $\lambda_0$.

There are two observations marked in red and a third one marked in yellow in Figure 1. The first two refer to signals because the observed value of the control statistic is beyond the control limits. The third observation corresponds to an observation responsible for a signal because it is equal to the lower control limit $LCL$ and the corresponding generated pseudo-random number from the Bernoulli distribution with parameter $\gamma_L$ is equal to 1, thus a signal has been triggered.

3 Modified $c$-chart for the mean of a Poisson INAR(1) process

As Paulino et al. (2016b) and several other authors emphasize, counts of nonconformities are often autocorrelated, severely restricting the use of control charts depending on the assumption that those counts refer to independent and identically distributed (i.i.d.) output.

Paulino et al. (2016b) go on to add that when we are dealing with processes of counts with large values, the time series in question can be studied by using continuous-valued models. Regrettably, when the observed values of the time series are small, processes such as $p$-th order autoregressive models (AR(p)) are of limited use.

This limitation arises from the fact that the simple procedure of multiplying an integer-valued r.v. by a real constant may lead to a non-integer r.v. A possible solution is to replace the (scalar) multiplication by a random operation such as the thinning operation can be defined as follows. Let $X$ be a discrete r.v. with range $\mathbb{N}_0$ and $\beta$ a scalar in $(0, 1)$. Then the binomial thinning operation on $X$ results in another r.v. defined as $\beta \circ X = \sum_{i=1}^{X} Y_i$, where $\circ$ represents the binomial thinning operator and \{ $Y_i : i \in \mathbb{N}$ \} is a sequence of i.i.d. Bernoulli($\beta$) r.v. independent of $X$.

The first-order integer-valued autoregressive, INAR(1), process was introduced by McKenzie (1985). The definition of an INAR(1) process is also taken from Paulino et al. (2016b). Let \{ $\epsilon_n : n \in \mathbb{Z}$ \} be a sequence of nonnegative integer-valued i.i.d. r.v., with mean $\lambda$, and variance $\sigma^2_\epsilon$, and $\beta \in (0, 1)$. Then \{ $X_n : n \in \mathbb{Z}$ \} is said to be an INAR(1) process if

$$X_n = \beta \circ X_{n-1} + \epsilon_n, \quad (3)$$

where: $\epsilon_n \in \mathbb{Z}$ and $X_{n-1}$ are assumed to be independent r.v.; all thinning operations are performed independently of each other and of \{ $\epsilon_n : n \in \mathbb{Z}$ \}; the thinning operations at time $n$ are independent of \{ $\ldots, X_{n-2}, X_{n-1}$ \}. In particular, if $\epsilon_n \overset{i.i.d.}{\sim} \text{Poisson}(\lambda), \ n \in \mathbb{Z}$, then \{ $X_n = \beta \circ X_{n-1} + \epsilon_n : n \in \mathbb{Z}$ \} is said to be a Poisson INAR(1) process. As referred by Paulino et al. (2016b), this process is a homogeneous Markov chain with state space $\mathbb{N}_0$ and (one-step) transition probabilities $p_{ij} \equiv p_{ij}(\lambda, \beta)$ equal to

$$p_{ij} = \sum_{m=0}^{\min(i,j)} \left( \frac{\lambda}{m!} \right) \beta^m (1-\beta)^{i-m} \times e^{-\lambda} \frac{\lambda^{j-m}}{(j-m)!}. \quad (4)$$

Citing Knoth and Schmid (2004) and Knoth et al. (2009), Paulino et al. (2016b) refers that there are three approaches to monitor the mean of an autocorrelated process: (i) ignore the autocorrelation structure of the output and use a standard control chart that assumes independent output; (ii) plot the original time series in a standard chart, however, with readjusted control limits to account for the autocorrelation – leading to what is termed a modified chart; (iii) plot the residuals of the time series (instead of the original data) in a standard chart – this type of chart is usually called a residual chart.

Controlling the expected value of a Poisson INAR(1) process implies detecting changes from its target value, $\frac{\lambda_0}{1-\beta}$, to some out-of-control value, $\frac{\lambda}{1-\beta}$. Paulino et al. (2016b) emphasized some limitations of the use of a modified $c$-chart with $3\sigma$ control limits: when the $LCL$ is null, the chart takes in average longer to detect any decreases in the process mean (due to downward shifts in either $\lambda$ or $\beta$) than to trigger a false alarm; when the $LCL$ is positive, the ARL function of the chart does not achieve its maximum value when the process mean is equal to its target value, thus we are dealing with an ARL-biased chart.

Paulino et al. (2016b) proposed what is probably the first ARL-unbiased modified $c$-chart. As its counterpart for i.i.d. output, the ARL-unbiased modified $c$-chart relies on two control limits, say $L$ and $U$, and two randomization probabilities, $\gamma_L$ and $\gamma_U$. Furthermore, it also triggers a signal at sample $n$ with: probability one if the observed number of defects $x_n$ is beyond the control limits $L$ and $U$; probability $\gamma_L$ (resp. $\gamma_U$) if $x_n$ is equal to $L$ (resp.
The modified c-chart has dependent control statistics, thus obtaining the control limits, \( L \) and \( U \), and the randomization probabilities, \( \gamma_L \) and \( \gamma_U \), requires the use of a non-trivial iterative search procedure distinct from the case i.i.d. The purpose of this procedure is to eliminate the bias of the ARL function, as well as to obtain an in-control ARL which is equal to a pre-specified value, \( ARL^* \). The search procedure is described in detail by Paulino et al. (2016a). For instance, for \( \lambda_0 = 9, \beta_0 = 0.8 \) and \( ARL^* = 500 \), the search procedure leads to \([L, U] = [27, 66]\) and \((\gamma_L, \gamma_U) = (0.056483, 0.183917)\).

The ARL function of the modified c-chart used to monitor the mean of a Poisson INAR(1) process essentially depends on two parameters: \( \lambda \) and \( \beta \). This is the reason why Paulino et al. (2016b) refers to \( \lambda \)-design and \( \beta \)-design. In the first design, the ARL function achieves a maximum value at \( \lambda = \lambda_0 \) and in the second design at \( \beta = \beta_0 \). In Figure 2, it is possible to see how the target value \( \beta_0 \) influences the ARL profile of the ARL-unbiased modified c-chart: larger values of \( \beta_0 \) lead to smaller out-of-control ARL compared to the same change in \( \lambda \).

What follows is a brief study of the impact of falsely assume i.i.d. Poisson counts.

It is indeed helpful to assess the impact on the performance of the ARL-unbiased c-chart proposed by Paulino et al. (2016a) of assuming that the process is i.i.d. when dealing effectively with a Poisson INAR(1) process.

Let us consider \( \lambda_0 = 20 \). The control limits and randomization probabilities of the ARL unbiased c-chart for i.i.d. Poisson counts, when \( ARL^* = 370.4 \), are \( L = 8, U = 35, \gamma_L = 0.566150 \) and \( \gamma_U = 0.549842 \). The estimated ARL in-control and out-of-control values were obtained through Monte Carlo simulation assuming that we are dealing with i.i.d. Poisson counts when in fact the output process is Poisson INAR(1) with: \( \lambda = \lambda_0 + \delta_\lambda, \delta_\lambda = -1, 0, +1 \) and \( \beta_0 = 0(0.1)0.9 \). Figure 3 leads to the conclusion that the estimates of ARL in-control and out-of-control increase with the increase of the parameter \( \beta \). Consequently, the true in-control ARL is underestimated. It is important to notice that underestimating ARL means on the one hand less frequent false alarms, but on the other hand a slower detection of changes in the process mean.

4 CUSUM schemes for the mean of i.i.d. Poisson counts

Shewhart control charts have well known limitations, in particular their slowness in detecting small-to-moderate shifts. In contrast, CUSUM charts and schemes ensure a more rigorous monitoring of processes, because they not only use the most recent information but also all past information.

One-sided CUSUM charts are associated with sequential probability ratio tests (SPRT), as mentioned by Johnson and Leon (1962). Upper and lower one-sided CUSUM charts can be useful to speed up the detection of increases and decreases in the parameter we want to monitor. These charts make use of the control statistics

\[
S^+_n = \max\{0, S^+_{n-1} + (X_n - k^+)\}, \quad (5)
\]

\[
S^-_n = \max\{0, S^-_{n-1} + (k^- - X_n)\}, \quad (6)
\]

where: \( X_n \) represents the count of defects in the \( n \)-th sample; \( k^+ \) (resp. \( k^- \)) is the reference value of

Figure 2: ARL curves for the ARL unbiased c-chart for different values of \( \beta_0 = 0.3 \) (solid line), 0.5 (dashed line), 0.8 (dotted line).

Figure 3: ARL function and ARL estimates in-control (solid line, \( \delta_\lambda = 0, \lambda_0 = 20 \)) and out-of-control (dashed line, \( \delta_\lambda = -1, \lambda = 19 \); dotted line, \( \delta_\lambda = +1, \lambda = 21 \)).
the upper (resp. lower) one-sided CUSUM chart. The reference value \( h^+ \) (resp. \( h^- \)) is chosen in order to ensure a fast detection of a shift from the target value, \( \lambda_0 \), to \( \lambda^+ > \lambda_0 \) (resp. \( \lambda^- < \lambda_0 \)). These control statistics can be found in Lucas (1985), who also suggests that the reference values should chosen as close as

\[
k^* = \frac{\lambda_0 - \lambda^*}{\ln(\lambda) - \ln(\lambda_0)},
\]

where \(* = +, -\) depending if we are dealing with an upper or a lower one-sided CUSUM chart, respectively. In (5) and (6), \( S_{n+}^+ = S_{n-}^- = 0, \{X_n : n \in \mathbb{N}\} \) is a sequence of i.i.d. Poisson r.v. with mean \( \lambda \).

### 4.1 The ARL function

The performance of these charts is also evaluated in terms of ARL. When \( S_{n}^+ = S_{n}^- = 0 \), is above the control limit \( h^*, * = +, - \), a signal is emitted. The value of \( h^+ \) should give an appropriately large ARL when the process is in-control. As in Gan (1993), we shall assume that the reference values and the control limits are either integers or positive fractions: \( k^* = a^* / b^* \) and \( h^* = c^* / b^* \), where \( a, b, c \) are positive integer values and \(* = +, -\). Under these conditions, \( \{S_{n}^+, S_{n}^- : n \in \mathbb{N}\} \) is a bivariate Markov chain with state space in

\[
\{0, 1/b^+, 2/b^+, \ldots, c^+/b^+, (c^+ + 1)/b^+, \ldots\} \times \{0, 1/b^-, 2/b^-, \ldots, c^-/b^-, (c^- + 1)/b^-, \ldots\}.
\]

The two-sided CUSUM scheme is set by running simultaneously the two one-sided CUSUM charts we have just defined. It triggers a signal as soon as one of its constituent charts does it so. Expectedly, the ARL of the two-sided CUSUM scheme, for the mean value of i.i.d. Poisson counts, can be obtained in terms of the ARL values of the one-sided CUSUM charts. According to Lucas (1985), if

\[
\begin{align*}
    h^+ + k^+ - k^- & \geq S_{n}^+ + S_{n}^- \\
    h^- + k^+ - k^- & \geq S_{n}^+ + S_{n}^- \\
    k^- - k^+ & \geq h^+ - h^- \\
    k^- - k^- & \geq h^+ - h^+.
\end{align*}
\]

then

\[
L^+ (S_{n}^+) L^- (0) + L^+ (0) L^- (S_{n}^-) - L^+ (0) L^- (0) \frac{L^+ (0) + L^- (0)}{L^+ (0) + L^- (0)},
\]

where \( L^* (S_{n}^*), * = +, -\), is the ARL of the one-sided CUSUM chart whose control statistic takes the initial value \( S_{n}^*, * = +, -\).

There is also another method of obtaining the ARL of the two-sided CUSUM scheme. It involves the transition probabilities matrix of the bivariate Markov chain, as suggested by Lucas (1985). The entries of this matrix are from now on represented by \( p(i^+, i^-) (j^+, j^-) = P (S_{n}^+ = j^+, S_{n}^- = j^- | S_{n-1}^+ = i^+, S_{n-1}^- = i^-) \), and the following four cases have to be considered: \( j^+ = 0 \) and \( j^- = 0 \); \( j^+ = 0 \) and \( j^- > 0 \); \( j^+ > 0 \) and \( j^- = 0 \); \( j^+ > 0 \) and \( j^- > 0 \). These probabilities can be found in Clara (2019, p. 29).

The ARL function of the two-sided CUSUM scheme is given by

\[
ARL(\lambda) = e_{i^+}^T (I - Q)^{-1} 1, \tag{10}
\]

where \( Q = \{p(i^+, i^-) (j^+, j^-) (i^+, j^-) \in \mathcal{T}\} \) is a sub-stochastic matrix matrix, with \( \mathcal{T} = T^+ \times T^- = \{0, 1/b^+, 2/b^+, \ldots, c^+/b^+, \ldots\} \times \{0, 1/b^-, 2/b^-, \ldots, c^-/b^-, (c^- + 1)/b^-, \ldots\} \); \( e_{i^+}^T \) is the first vector of the orthogonal basis for \( \mathbb{R}^{(c^+ + 1) \times (c^- + 1)} \); \( I \) represents an identity matrix with rank \( (c^+ + 1) \times (c^- + 1) \); \( 1 \) is a column-vector with \( (c^+ + 1) \times (c^- + 1) \) ones.

The ARL results we have obtained led to the conclusion that we deal with a ARL-biased two-sided CUSUM scheme. Furthermore, the discrete nature of the control statistics \( S_{n}^+ \) and \( S_{n}^- \) prevents the value of in-control ARL to coincide with the prespecified ARL.

### 4.2 Deriving an ARL-unbiased two-sided CUSUM scheme

We adapted the algorithm described by Paulino et al. (2016b) and managed to derive an ARL-unbiased two-sided CUSUM scheme for i.i.d. Poisson counts.

Needless to say, this ARL-unbiased scheme triggers a signal with:

- probability one as soon as the control statistic \( S_{n}^+ \) (resp. \( S_{n}^- \)) exceeds the control limit \( h^+ \) (resp. \( h^- \));
- with probability \( \gamma^+ \) (resp. \( \gamma^- \)) if the observed value of \( S_{n}^+ \) (resp. \( S_{n}^- \)) coincides with \( h^+ \) (resp. \( h^- \)).

The control statistics \( S_{n}^+ \) and \( S_{n}^- \) are obviously the ones defined in (5) and (6), and the reference values are calculated rounding (7) to an integer or a fraction.

Randomizing the emission of a signal means considering a modified sub-stochastic matrix \( Q \) in (10). Indeed, when there is a transition from state \( (i^+, i^-) \in \mathcal{T} \) to state:
\[ (j^+, h^-), j^+ \in \{0, 1/b^+, \ldots, (c^+ - 1)/b^+ \}, \text{the entry is equal to} \ (1 - \gamma h^-) \times p(i, j^+)(j^+, h^-); \]

\[ (h^+, j^-), j^- \in \{0, 1/b^-, \ldots, (c^- - 1)/b^- \}, \text{the entry is} \ (1 - \gamma h^+) \times p(i, j^-)(h^+, j^-); \]

\[ (h^+, h^-), \text{the entry is given by} \ (1 - \gamma h^+) \times p(i, j^-)(h^+, h^-). \]

The remaining entries of the matrix \( Q \) remain equal to the transition probabilities \( p(i, j^-)(j^+, j^-) \), where \( (j^+, j^-) \in \{0, 1/b^-, \ldots, (c^- - 1)/b^- \} \times \{0, 1/b^-, \ldots, (c^- - 1)/b^- \} \).

The two-sided CUSUM scheme will use randomization probabilities for the signal emission, which occurs with probability \( \gamma^- \) (resp. \( \gamma^+ \)) if the observed value of statistic \( S_n \) (resp. \( S_h \)) coincides with control limit \( h^- \) (resp. \( h^+ \)).

Only the first step of this adaptation differs from the one of the original algorithm, thus the sole description of that step.

Identifying the grid of control limits: If the in-control ARL is smaller than the desired value then it is futile to complement the control limits with randomization probabilities to bring it precisely to \( ARL^* \). Consequently, we start this step by obtaining pairs of control limits leading to a two-sided CUSUM scheme without randomization probabilities and such that its in-control ARL satisfies

\[ ARL(\lambda_0) > ARL^*. \tag{11} \]

To accomplish this, the algorithm is initialized by searching for values of \( h^+ \) and \( h^- \) that ensure that

\[ ARL^+ (\lambda_0) > 2 \times ARL^*; \]
\[ ARL^- (\lambda_0) > 2 \times ARL^*. \tag{12} \tag{13} \]

We still need to check if the \( ARL(\lambda) \) function achieves a maximum at a point \( \lambda^* \) to the left or right of the target value \( \lambda_0 \). Therefore, the search for other possible values for the control limits has to be divided into two distinct cases:

- \( \lambda^* > \lambda_0; \)
- \( \lambda^* < \lambda_0. \)

In both cases we want to get the largest (resp. smallest) possible value of \( h^- \) (resp. \( h^+ \)) so that the ARL function achieves a maximum point to the right (resp. left) of \( \lambda_0 \). It is also important to realise that if one considers a fixed value for \( h^+ \), when one increases (resp. decrease) the value of \( h^- \), the argmax of the \( ARL(\lambda) \) function moves to the left (resp. right). While searching for these tentative control limits, we must never forget that they should be integer (resp. fractional) if the reference values were chosen to be integer (resp. fractional). If the value is fractional, we must consider three different cases: the decimal place is five, an even digit or an odd digit distinct from five.

For these cases, the possible values for the control limits are: \( \{0, 0.5, 1.0, 1.5, \ldots\}, \{0, 0.2, 0.4, 0.6, \ldots\}, \{0, 0.1, 0.2, 0.3, \ldots\} \), respectively. As a consequence, fractional control limits can be defined in three different ways: \( a \times 0.5, a \times 0.2 \in a \times 0.1, \) where \( a \in \mathbb{N}_0 \). Thus, the fractional and integer control limits are given by \( a \times l \) where \( l = 0.1, 0.2, 0.5, 1 \).

Let \( h^+_0 \) and \( h^-_0 \) be the starting values of the control limits.

- For \( h^- = h^-_0, h^+_0 + l, \ldots \), consider a value of \( h^+ \) large enough, \( h^+ \geq h^+_0 + l \), such that (11) is satisfied. Then one should search for a value of \( h^- \) such that the maximum of the function \( ARL(\lambda) \) moves to the left side (resp. right) of the \( \lambda_0 \). Let this value be \( h_{\min} \) (resp. \( h_{\max} \)). So \( h_{\min} \) and \( h_{\min} - l \) (resp. \( h_{\max} \) and \( h_{\max} + l \)) are candidates values for \( h^- \) and for each of them get the lowest possible value of \( h^+ \) also satisfying (11).

The admissible control limits are all pairs that are within the candidate limits described above.

### 4.3 Illustrations

The adapted algorithm was implemented in R and the reference values, control limits and randomization probabilities of several ARL-unbiased two-sided CUSUM schemes, for the mean of i.i.d. Poisson counts, can be found in Table 1. In Figure 4, we can find the ARL-unbiased profile of a two-sided CUSUM scheme for the mean of i.i.d. Poisson counts, for \( \lambda_0 = 3 \) and \( ARL^* = 150 \). The reference values are \( k^- = 2 \) and \( k^+ = 4 \).

Monte Carlo simulation (with 30000 replications) was used to provide ARL estimates and verify the results obtained for the ARL function. These ARL estimates virtually coincide with the values of the ARL function, as shown by Figure 4. Designing an ARL-unbiased two-sided CUSUM scheme requires choosing values \( \lambda^+ > \lambda_0 \) and \( \lambda^- < \lambda_0 \), upon which the reference values \( k^+ \) and \( k^- \) will depend. As a consequence, we decided to briefly study the impact of the choice of \( \lambda^*, \ast = +, - \) on the ARL curve.

Let us assume that \( \lambda^+ = \lambda_0 + \Delta \) and \( \lambda^- = \lambda_0 - \Delta \) and that \( k^+ \) and \( k^- \) are obtained using (7) in order
Table 1: Reference values, control limits and randomization probabilities of ARL-unbiased two-sided CUSUM scheme for the mean of i.i.d. Poisson counts — $\text{ARL}^* = 370.4$, $\Delta = 1$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$k^-$</th>
<th>$k^+$</th>
<th>$h^-$</th>
<th>$h^+$</th>
<th>$\gamma^-$</th>
<th>$\gamma^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>25</td>
<td>26</td>
<td></td>
<td>0.026935</td>
<td>0.052192</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td></td>
<td>0.844986</td>
<td>0.587951</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>11</td>
<td>0.452273</td>
<td>0.395165</td>
</tr>
<tr>
<td>5</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>13</td>
<td>0.281217</td>
<td>0.304541</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>7</td>
<td>12</td>
<td>15</td>
<td>0.193924</td>
<td>0.274863</td>
</tr>
<tr>
<td>7</td>
<td>6</td>
<td>8</td>
<td>14</td>
<td>17</td>
<td>0.159580</td>
<td>0.293418</td>
</tr>
<tr>
<td>8</td>
<td>7</td>
<td>9</td>
<td>16</td>
<td>19</td>
<td>0.168556</td>
<td>0.357005</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>10</td>
<td>18</td>
<td>21</td>
<td>0.218569</td>
<td>0.467140</td>
</tr>
<tr>
<td>10</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>23</td>
<td>0.311161</td>
<td>0.628757</td>
</tr>
<tr>
<td>11</td>
<td>10</td>
<td>12</td>
<td>22</td>
<td>25</td>
<td>0.450755</td>
<td>0.850187</td>
</tr>
<tr>
<td>12</td>
<td>11</td>
<td>13</td>
<td>24</td>
<td>26</td>
<td>0.525565</td>
<td>0.918847</td>
</tr>
<tr>
<td>13</td>
<td>12</td>
<td>14</td>
<td>26</td>
<td>28</td>
<td>0.643992</td>
<td>0.992573</td>
</tr>
<tr>
<td>14</td>
<td>13</td>
<td>15</td>
<td>27</td>
<td>30</td>
<td>0.742668</td>
<td>0.992573</td>
</tr>
<tr>
<td>15</td>
<td>14</td>
<td>16</td>
<td>29</td>
<td>32</td>
<td>0.844986</td>
<td>0.992573</td>
</tr>
<tr>
<td>16</td>
<td>15</td>
<td>17</td>
<td>31</td>
<td>33</td>
<td>0.918847</td>
<td>0.992573</td>
</tr>
<tr>
<td>17</td>
<td>16</td>
<td>18</td>
<td>32</td>
<td>35</td>
<td>0.992573</td>
<td>0.992573</td>
</tr>
<tr>
<td>18</td>
<td>17</td>
<td>19</td>
<td>34</td>
<td>37</td>
<td>0.992573</td>
<td>0.992573</td>
</tr>
<tr>
<td>19</td>
<td>18</td>
<td>20</td>
<td>36</td>
<td>38</td>
<td>0.992573</td>
<td>0.992573</td>
</tr>
<tr>
<td>20</td>
<td>19</td>
<td>21</td>
<td>37</td>
<td>40</td>
<td>0.992573</td>
<td>0.992573</td>
</tr>
</tbody>
</table>

Figure 4: ARL curve and estimates for the ARL-unbiased two-sided CUSUM scheme for the mean of i.i.d. Poisson counts — $\lambda_0 = 4$, $\text{ARL}^* = 370.4$.

Figure 5 allows us to conclude that considering lower values of $\Delta$ leads to the decrease of the out-of-control ARL values, suggesting a better performance of the ARL-unbiased two-sided CUSUM scheme for the mean of i.i.d. Poisson counts.

5 CUSUM schemes for the mean of a Poisson INAR(1) processes

As we have illustrated in Section 3, the violation of the independence hypothesis can lead to the degradation of the performance of any control charts or schemes that are planned disregarding the autocorrelation structure of the process. For example, false alarm rates are generally much higher than desired.

Weiß and Testik (2009) proposed upper one-sided CUSUM charts to detect increases in the mean of Poisson INAR (1) processes. More recently, Yontay et al. (2013) proposed a two-sided CUSUM scheme. Although Yontay et al. (2013) did not do it, we can consider fractional reference values and control limits. Yontay et al. (2013) suggest that some of the illustrated two-sided CUSUM schemes are associated with ARL unbiased profiles if the in-control ARL of both one-sided CUSUM charts coincides. This statement should have been made with some caution by the authors because the closest values to $\lambda_0$ differ from this target value by 5%.

5.1 The overall ARL function

Obtaining an ARL-unbiased curve requires the adoption of a procedure similar to the one we used to derive an ARL-unbiased two-sided CUSUM scheme for the mean of i.i.d. Poisson counts.

The derivation of an ARL-unbiased two-sided CUSUM scheme in this particular setting requires the calculation of the ARL function of the two one-sided CUSUM charts for the mean of a Poisson INAR(1) processes. This is achieved by using ARL formula in Section 4.1 and considering the sub-stochastic matrix $Q$ related to the bivariate Markov chain $\{(S_n^*, X_n) : n \in \mathbb{N}_0\}$ with transition probabilities given by $P((i^*, j^*)| (i^*, a) = P(S_n^* = j^*, X_n = a | S_{n-1}^* = i^*, X_{n-1} = d)$, where $*=+,−$. The detailed calculation of these probabilities can be found in Clara (2019, p. 38).
According to Yontay et al. (2013), the combined use of one-sided CUSUM charts to monitor Poisson INAR(1) processes results in a trivariate Markov chain, \( \{(X_n, S_n^+, S_n^-) : n \in \mathbb{N}_0\} \). Considering that reference values and control limits take integer or fractional values, \( k^+ = a^+/b^+ \), \( h^+ = c^+/b^+ \), \( k^- = a^-/b^- \) and \( h^- = c^-/b^- \), the trivariate Markov chain has state space \( \{0, 1, \ldots\} \times \{0, 1/b^+, 2/b^+, \ldots\} \times \{0, 1/b^-, 2/b^-, \ldots\} \) and the transition probabilities are given by \( p_{(a^+, i^+, i^-)} = P(X_n = a, S_n^+ = j^+, S_n^- = j^- | X_{n-1} = b, S_{n-1}^+ = i^+, S_{n-1}^- = i^-) \), (see Clara (2019, p. 39)). We should notice that the admissible in-control values belong to a finite set defined by

\[
\{(x, i, j) : \text{max} \left(0, i + k^+ - h^+ + 1\right) \leq x \leq i + k^+, \text{max} \left(0, k^- - j\right) \leq x, x < h^+ + k^+ - j \text{ iff } j > 0\}.
\]

where \( i = 0, \ldots, h^+ \) and \( j = 0, \ldots, h^- \). Since we are dealing with a Markov chain it is possible to use Brook and Evans (1972) approach in order to obtain the ARL function of this scheme as the expected value of time until the absorbing of a Markov chain with transient states \( \{0, 1, \ldots\} \times \{0, 1/b^+, \ldots, c^+/b^\} \times \{0, 1/b^-, \ldots, c^-/b^\} \) and a unique absorbent state that combines all the states \((a, i^+, i^-)\) of the original chain with \( i^+ > h^+ \) or \( i^- > h^- \). Since the set of transient states is infinite, it has to be limited to \( x \) using the set described above.

Since the value of \( X_0 \) is usually not known, it is plausible to rely on \( X_1 \equiv X_1(\lambda, \beta) \sim \text{Poisson}(\lambda/(1 - \beta)) \) and follow Weiβ and Testik (2009) who recommended the use of what the authors called overall ARL. In this case, the overall ARL function of the ARL-unbiased two-sided CUSUM scheme for the mean of a Poisson INAR(1) process is given by:

\[
1 + \sum_{(x, u^+, u^-) : u^+ \neq h^+, u^- \neq h^-} ARL^{x, u^+, u^-} \\
\times p(0, 0, 0)(x, u^+, u^-) \\
+ (1 - \gamma_{h^-}) \times \sum_{(x, u^+, h^-) : u^+ \neq h^+} ARL^{x, u^+, h^-} \\
\times p(0, 0, 0)(x, u^+, h^-) \\
+ (1 - \gamma_{h^+})(1 - \gamma_{h^-}) \times \sum_{(x, h^+, h^-)} ARL^{x, h^+, h^-} \\
\times p(0, 0, 0)(x, h^+, h^-) \\
+ (1 - \gamma_{h^+}) \times \sum_{(x, h^+, u^-) : u^- \neq h^-} ARL^{x, h^+, u^-} \\
\times p(0, 0, 0)(x, h^+, u^-),
\]

with \((x, u^+, u^-) \in \{0, 1, \ldots\} \times \{0, 1/b^+, \ldots, c^+/b^\} \times \{0, 1/b^-, \ldots, c^-/b^\} \) and the transition probabilities \( P(X_1 = x, S_1^+ = u^+, S_1^- = u^- | S_0^+ = 0, S_0^- = 0) \) are derived in the same way and can be found in Clara (2019, p. 41).

### 5.2 Illustrations

The search procedure to derive the parameters of the ARL-ubiased two-sided CUSUM scheme for the mean of a Poisson INAR(1) process is similar to the one used in the i.i.d. case. However, it is rather computer-intensive and due to memory limitations of the personal computer we used, we adopted \( ARL^* = 150 \) and \( ARL^* = 100 \), an in-control ARL value substantially smaller than 370.4.

Figure 6 shows a plot of the overall ARL curve of the ARL-ubiased two-sided CUSUM scheme for the mean of a Poisson INAR(1) process with \( \lambda_0 = 3 \) and \( \beta_0 = 0.05 \).

Some ARL estimates were obtained using Monte Carlo simulation with 30000.

![Figure 6: Overall ARL curves and ARL estimates of the ARL-unbiased two-sided CUSUM scheme for the mean of a Poisson INAR(1) process — \( \lambda_0 = 3, \beta_0 = 0.05 \).](image-url)
where $ARL^+$ and $ARL^-$ are the ARL functions of the one-sided CUSUM charts.

Finally, we ought to note that we were unable to obtain a two-sided CUSUM scheme whose ARL function achieves a maximum value at $\beta_0$, thus only results for the $\lambda-$design are mentioned in this section.

5.3 The impact of falsely assume i.i.d. Poisson counts

Now, we briefly assess the impact of falsely assuming i.i.d. Poisson counts on the performance of an ARL-unbiased two-sided CUSUM scheme.

With this purpose in mind, we obtained an ARL-unbiased two-sided CUSUM scheme for the mean of i.i.d. Poisson counts with $\lambda_0 = 4$ and $ARL^* = 150$, $k^- = 3$, $k^+ = 4$, $h^- = 6$, $h^+ = 43$, $\gamma^- = 0.422905$, and $\gamma^+ = 0.167864$. Through Monte Carlo simulation, the in-control and out-of-control ARL values of this scheme were estimated when we are actually dealing with a Poisson INAR(1) process, with $\lambda = \lambda_0 + \delta\lambda$, $\delta\lambda = -1, 0, +1$ and $\beta_0 = 0(0.2)0.8$. Figure 7 suggests that the in-control and out-of-control ARL decrease with the parameter $\beta$. Consequently, the true ARL is underestimated. These results are expected because the autocorrelation structure of the process was ignored in the design of the ARL-unbiased two-sided CUSUM scheme.

6 Shewhart charts vs. CUSUM schemes

So far we have focused on Shewhart charts and CUSUM schemes to monitor i.i.d. and INAR(1) Poisson counts. As we know, the main difference between these two approaches is that CUSUM schemes take into account not only current but also past information. Thus, it is relevant to assess the difference between the ARL performance of these two types of SPC tools.

7 Conclusions and further work

The main purpose of this article was to obtain ARL-unbiased two-sided CUSUM schemes for the
mean of i.i.d Poisson counts and of a Poisson INAR(1) process.

To derive the control limits and the randomization probabilities of these schemes, we had to adapt the search procedure originally proposed by Paulino et al. (2016b). The ARL results were confirmed using Monte Carlo simulation.

Our research led to the conclusion that the existence of autocorrelation requires extra care in the design of such CUSUM schemes.

It is important to point that we deal with ARL-unbiased two-sided CUSUM scheme for the mean of a Poisson INAR(1) process, the search procedure is not efficient. Moreover, obtaining such schemes was only possible when we considered ARL smaller than the values we have taken while monitoring the mean of i.i.d. Poisson counts.

Therefore a direction for future work would be optimizing the search procedure to allow the derivation of ARL-unbiased two-sided CUSUM schemes for the mean of Poisson INAR(1) counts, regardless of the value we may consider for ARL∗.

References


