Complex flows and geodesics in the space of Kähler metrics

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Declaration

I declare that this document is an original work of my own authorship and that it fulfils all the requirements of the Code of Conduct and Good Practices of the University of Lisbon.

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Resumo

Nesta dissertação, mostramos como a equação das geodésicas para a métrica de Mabuchi, com condições iniciais analíticas, no espaço das estruturas Kähler $\mathcal{H}$ numa variedade Kähler compacta, pode ser resolvida utilizando uma noção apropriada de evolução complexa para um campo vetorial Hamiltoniano e analítico. Começamos por rever a geometria da variedade Fréchet de dimensão infinita $\mathcal{H}$, por estabelecer a equação das geodésicas para a métrica de Mabuchi e por provar que esta equação é equivalente a um problema de Dirichlet para a equação de Monge-Ampère Complexa e Homogénea. Prosseguimos com um resumo de [**MN15**]. Depois de definirmos evolução complexa através da teoria das Séries de Lie, utilizamos o fluxo complexo de um campo vetorial Hamiltoniano analítico para agir na estrutura Kähler the uma variedade compacta. Provamos que a restrição desta acção a tempo imaginário define um caminho geodésico em $\mathcal{H}$. Finalmente, estudamos dois exemplos importantes. O primeiro, em $\mathbb{R}^2$, é o fluxo complexo do campo vetorial Hamiltoniano de um oscilador harmónico em que uma fração da sua energia cinética é imaginária. O segundo, em S^2, é o fluxo em tempo imaginário do campo vetorial Hamiltoniano associado a metade do quadrado do mapa de momentos de uma acção de S^1. Estudamos tanto o caso de métricas iniciais invariantes pela ação de S^1 como o caso de métricas não invariantes. Mostramos que, sob uma escolha adequada do valor mínimo do Hamiltoniano, as geodésicas de Mabuchi existem durante tempo geodésico infinito.

Palavras-chave: Geometria Kähler; Geodésicas de Mabuchi; Dinâmica Complexa; Fluxo Hamiltoniano complexo;

Abstract

In this dissertation, we show how the geodesic equation for the Mabuchi metric, with analytic initial conditions, on the space of Kähler structures $\mathcal{H}$ on a compact Kähler manifold, can be solved by using an appropriate notion of complex evolution for analytic Hamiltonian vector fields. We start by reviewing the geometry of the infinite-dimensional Fréchet manifold $\mathcal{H}$, by establishing the geodesic equation for the Mabuchi metric and by proving that this equation is equivalent to a Dirichlet problem for a Complex Homogeneous Monge-Ampère equation. We proceed by presenting an overview of [**MN15**]. After defining complex evolution through the theory of Lie Series, we use the complex flow of an analytic hamiltonian vector field to act on the Kähler structure of a compact manifold. We prove that the restriction of this action to imaginary time defines a geodesic path in $\mathcal{H}$. Finally, we study two important examples. The first one, on $\mathbb{R}^2$, is the complex flow of the Hamiltonian vector field of a harmonic oscillator, where a fraction of its kinetic energy is imaginary. The second one, on S^2, is the imaginary time flow of the hamiltonian vector field of half the square of the moment map for an S^1-action. In this case, we study both S^1-invariant and non-invariant starting metrics, and show that, under a particular choice of the minimum of the Hamiltonian, the Mabuchi geodesics exist for infinite geodesic time.

Keywords: Kähler geometry; Mabuchi geodesics; Complex dynamics; Complex Hamiltonian flow;

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Notation

$\mathcal{T}^l_k(M)$	(l,k)-tensor fields on M
$\Omega^k(M)$	differential forms of degree k on M
$\Omega^{k,l}(M)$	differential forms of bidegree (k,l) on M
ω	a symplectic form on M (also a real parameter)
J	a complex structure on M
g	a Riemannian metric on M
$\mathcal{S}(M)$	the set of symplectic forms on M
$\mathcal{J}(M)$	the set of complex structures on M
$\mathcal{R}(M)$	the set of Riemannian metrics on M
$\mathcal{K}(M)$	the set of K¨ähler triples for M
$\mathcal{H}(\omega, J)$	the set of distortion potentials from ω to any other cohomologous form
$\sharp$	the raising index operator
ι_X	contraction with a vector field X
$\mathbb{K}$	the field $\mathbb{R}$ or $\mathbb{C}$
$C^\infty(U,\mathbb{K})$	smooth functions from U to $\mathbb{K}$
$C^\omega(U,\mathbb{K})$	real analytic functions from U to $\mathbb{K}$
$\mathfrak{X}(M)$	vector fields on M
$\mathfrak{X}^\omega(M,\mathbb{K})$	$\mathbb{K}$-valued real analytic vector fields on M
$\langle\cdot,\cdot\rangle$	the mabuchi metric
$\mathcal{E}$	the metric energy functional
ω_u	$\omega + i\partial\overline{\partial}u$
∇^u	the gradient with respect to the metric $\omega_u(\cdot, J\cdot)$
$\lvert\cdot\rvert_u$	the norm with respect to the metric $\omega_u(\cdot, J\cdot)$
Z	the cylinder $[1,e]\times S^1$
$e^{\tau X}\cdot f$	the Lie series for the vector field X and the function f at time $\tau\in\mathbb{C}$
φ^T_τ	the complex flow of a vector field defined for complex time τ such that $\lvert\tau\rvert\le T$
∇^{g_τ}	the gradient with respect to the metric g_τ
$\lvert\cdot\rvert_{g_\tau}$	the norm with respect to the metric g_τ
Ω	the symplectic form in matrix form
G	the metric tensor in matrix form
S	the scalar curvature

Introduction

The goal of this work is to study the notion of complex Hamiltonian dynamics. This is an extension of the usual evolution in which both time and the Hamiltonian itself are considered to be complex quantities. At first sight, it is not clear how can one give meaning to such an evolution, so this is an interesting problem by itself. It turns out that, after being appropriately defined, complex dynamics has deep connections with the geometry and analysis of Kähler manifolds, and this constitutes another strong reason for its study.

The text is divided into three chapters. On the first one, we explore the connections between complex evolution and Kähler Geometry. Working on a general compact manifold, we start by introducing the structure of an infinite-dimensional manifold on the space of Kähler forms with fixed cohomology class. Equipping it with the Mabuchi metric, we proceed to study the geodesic curves in this space. By relating the geodesic equation with a Dirichlet problem for the Complex Homogeneous Monge-Ampère equation, we argue that a geodesic path is a kind of complexified version of a Hamiltonian flow. This ends chapter 1 and the presentation of the problem that will guide us throughout the text.

In chapter 2, we define complex dynamics on a compact complex manifold. To do so we rely on an evolution operator acting on the algebra of local complex-valued real-analytic functions, locally defined through the theory of Lie Series. If the manifold has a complex structure, it will be possible, by restricting the operator to the proper subalgebra of holomorphic functions, to build a family of diffeomorphisms, indexed by a complex time parameter, that extends the usual flow. After that, we introduce a compatible symplectic form and study how the corresponding Kähler structure is changed by the family of diffeomorphisms. Finally, we show how this family corresponds in fact to a geodesic in the space introduced in chapter 1.

We finish with chapter 3, where we study interesting examples of complex Hamiltonian dynamics on (real) surfaces. We begin by explicitly deriving the coordinate expressions for relevant tensors, such as the complex structure or the metric, in the case of surfaces. For $M = \mathbb{R}^2$, we consider a version of the Hamiltonian for the harmonic oscillator, where a fraction of its kinetic energy is an imaginary quantity. We focus our analysis on the properties of the evolution. For $M = S^2$, we study a Hamiltonian function that is half of the square of the moment map of an S^1 action. We consider the complex dynamics with non-invariant starting Kähler geometry, and conclude that, for an appropriately chosen minimum of that function, the geodesic exists for infinite geodesic time. In that case, the corresponding Kähler geometry approaches exponentially fast an axially symmetric one.

Throughout we make use of some standard notions in Kähler Geometry that we assume the reader is familiar with. They can be recalled by consulting [**Mor07**] or [**Bal06**].

CHAPTER 1

Geodesics in the space of Kähler metrics

1. The Space of Kähler Structures

Kähler Geometry is a wonderful subject where the Symplectic, the Complex, and the Riemannian theories meet. To study the intricate interactions between these three types of geometry let us consider each one of them separately. Denote by $\mathcal{S}(M)$ the set of symplectic structures of a differentiable manifold, that is,

$$\mathcal{S}(M) = \{\, \omega \in \Omega^2(M) \mid d\omega = 0 \text{ and } \omega \text{ is non-degenerate} \,\}.$$

Similarly, let $\mathcal{J}(M)$ and $\mathcal{R}(M)$be the sets of complex and Riemannian structures on M, meaning that

$$\mathcal{J}(M) = \{\, J \in \mathfrak{T}^1_1(M) \mid J^2 = -id \text{ and } J \text{ is integrable} \,\}, \text{ and}$$

$$\mathcal{R}(M) = \{\, g \in \mathfrak{T}^0_2(M) \mid g \text{ is symmetric and positive definite} \,\}.$$

An element (ω, J, g) of $\mathcal{S}(M) \times \mathcal{J}(M) \times \mathcal{R}(M)$ is called a triple for M. A Kähler manifold is then a manifold M together with a compatible triple, i.e., a triple for which the symplectic form ω, the complex structure J, and the Riemannian metric g are all compatible [**CdS01**].

The compatibility conditions read:

$$\text{(i) } \omega(X,Y) = g(JX,Y); \qquad \text{(ii) } J(X) = (\iota_X\omega)^\sharp; \qquad \text{(iii) } g(X,Y) = \omega(X,JY),$$

for each $X, Y \in \mathfrak{X}(M)$. Recall that two structures are compatible if they define the third one, as given from the respective expression above. For example, a symplectic 1-form ω and a complex structure J are compatible if g as defined by (iii) is indeed a Riemannian metric. As so, one only needs two of the three elements of the compatible triple to fully determine a Kähler structure. Throughout the text we will consider $\mathcal{K}(M)$ to be parametrized by the symplectic form and the complex structure, and will resort to (iii) to calculate the corresponding Riemannian metric. We will often write $(\omega, J) \in \mathcal{K}(M)$ and call it a *Kähler pair*. The form ω is referred to as the Kähler form.

Example 1.1. *Consider $\mathbb{R}^{2n} \cong \mathbb{C}$ with the standard differential structure. Let it carry the canonical compatible triple (Ω, J, G). In local coordinates $\{x_1, \dots, x_n, y_1, \dots, y_n\}$, these structures read*

$$\Omega = \sum_{j=1}^n dx_j \wedge dy_j \qquad J = \sum_{j=1}^n \frac{\partial}{\partial x_j} \otimes dy_j - \frac{\partial}{\partial y_j} \otimes dx_j \qquad G = \sum_{j=1}^n dx_j^2 + dy_j^2.$$

We identify them with the matrices

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \text{ and } G = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}.$$

The compatibility conditions are then rewritten as

$$\textit{(i) } \Omega = J^T G; \qquad \textit{(ii) } J = G^{-1}\Omega^T; \qquad \textit{(iii) } G = \Omega J.$$

These relations hold true not just for the canonical triple but also for translational invariant triples, under the same identifications of tensors with matrices.

2. Restricting $\mathcal{K}(M)$

When it is not empty, the space $\mathcal{K}(M)$ is, in general, very big. As so, we will consider a proper subset $\mathcal{H}_0 \subset \mathcal{K}(M)$, that is relevant in two different contexts, namely geometric quantization and the stability of Kähler metrics, even though we will not study any of them. This subset will be obtained by imposing two constraints. Firstly, we will fix a complex structure J on M. This corresponds to the study of a subset of $\mathcal{K}(M)$ which is only parametrized by the symplectic forms compatible with the fixed J. Secondly, we will only let these forms vary within a fixed cohomology class. That is, we will put a particular Kähler form ω on M and consider only the forms that belong to same cohomology class as ω. So, if (ω, J) is a Kähler pair for a manifold M, we define

$$\mathcal{H}_0(\omega, J) = \{\, \alpha \mid (\alpha, J) \text{ is a Kähler pair for } M \text{ and } [\alpha] = [\omega] \,\}.$$

From now on, we will restrict our set-up even further by considering M to be a compact manifold. This restriction is extremely important for the motivation problem and the geometric description of $\mathcal{H}_0$ we are providing. In spite of this, we will see that the developed formalism will sometimes make sense for non-compact manifolds, and we will even study with more detail an example of a complex flow defined on $\mathbb{C}$ later in chapter 3. When M is compact we have a classical lemma.

LEMMA 1.2 (Global $\partial\overline{\partial}$-Lemma **[Huy05]**). *Let $\alpha \in \Omega^{1,1}(M)$ be a closed $(1,1)$-form on a compact Kähler manifold M. Then α is exact if and only if there exists a function $u \in C^\infty(M, \mathbb{R})$ such that $\alpha = \partial\overline{\partial}u$.*

One usually states the lemma by saying that α is exact if and only if it is $\partial\overline{\partial}$-exact. The fact that the Kähler form is itself closed and of type (1,1) provides the following assertion.

LEMMA 1.3 ($i\partial\overline{\partial}$-Lemma). *Let (M, ω, J) be a compact Kähler manifold and let $\widetilde{\omega}$ be other Kähler form such that $[\widetilde{\omega}] = [\omega]$. Then there exists a function $u \in C^\infty(M, \mathbb{R})$ such that*

$$\widetilde{\omega} = \omega + i\partial\overline{\partial}u.$$

PROOF. As $(\widetilde{\omega}, J)$ is a Kähler pair for M, $\widetilde{\omega}$ is closed and of type (1,1). Then, the difference $\alpha := \widetilde{\omega} - \omega$ is also closed and of type (1,1). Moreover, it is an exact form, because ω and $\widetilde{\omega}$ are cohomologous. By linearity, $\alpha' := -i\alpha$ is also an exact (1,1)-form. By the $\partial\overline{\partial}$-Lemma, there exists a function $u \in C^\infty(M, \mathbb{R})$ such that $\alpha' = \partial\overline{\partial}u \Leftrightarrow -i\alpha = \partial\overline{\partial}u \Leftrightarrow \alpha = i\partial\overline{\partial}u$. This is the same as $\widetilde{\omega} = \omega + i\partial\overline{\partial}u$. $\square$

REMARK 1.4. The function u is called a *distortion potential* from ω to $\widetilde{\omega}$. We will denote $\widetilde{\omega}$ by ω_u.

The $i\partial\overline{\partial}$-Lemma allows us to identify the space $\mathcal{H}_0(\omega, J)$, for a compact manifold M, as

$$\mathcal{H}_0(\omega, J) \cong \mathcal{H}(\omega, J)/\mathbb{R},$$

$$\mathcal{H}(\omega, J) = \{\, u \in C^\infty(M, \mathbb{R}) \mid \omega_u := \omega + i\partial\overline{\partial}u \text{ is compatible with } J \,\}.$$

The reason for the quotient by $\mathbb{R}$ stems from the fact that two distortion potentials will define the same Kähler form if they differ by a constant.

Recall that ω_u being compatible with J means that the tensor field given by $g(X,Y) = \omega_u(X, JY) = \omega(X, JY) + i\partial\overline{\partial}u(X, JY)$ for each $X, Y \in \mathfrak{X}(M)$ is a Riemannian metric. We call such ω_u a *positive* $(1,1)$-form [**Mor07**], and write $\omega_u \succ 0$. With this notation,

$$\mathcal{H}(\omega, J) = \{\, u \in C^\infty(M, \mathbb{R}) \mid \omega_u = \omega + i\partial\overline{\partial}u \succ 0 \,\}.$$

If κ is a local Kähler potential for ω and $\hat{u}$ is the local representative of the function $u \in C^\infty(M, \mathbb{R})$, the *Levi matrix* of ω_u is $\left(\frac{\partial^2(\kappa+\hat{u})}{\partial z_j \partial \bar{z}_k}\right)$. For $u \in C^\infty(M, \mathbb{R})$ to be in $\mathcal{H}$, the positivity condition is locally equivalent to asking for the Levi matrix of ω_u to be positive definite. We call such functions u *strictly ω-plurisubharmonic*, and $\mathcal{H}$ is just the set of those functions.

As the set $C^\infty(M, \mathbb{R})$ is a Fréchet vector space (actually a Fréchet algebra), it carries a natural structure of an infinite-dimensional manifold. The subset $\mathcal{H}(\omega, J)$ is defined by an open condition and so it is open in $C^\infty(M, \mathbb{R})$. This makes $\mathcal{H}(\omega, J)$ into a Fréchet manifold. The tangent space at any $u \in \mathcal{H}(\omega, J)$ can be naturally identified with $C^\infty(M, \mathbb{R})$, $T_u\mathcal{H}(\omega, J) \cong C^\infty(M, \mathbb{R})$. Some geometric properties of Fréchet manifolds are reviewed in Appendix A.

3. The Mabuchi metric

We will now introduce a metric on the manifold $\mathcal{H}(\omega, J)$. Throughout (ω, J) will denote a fixed Kähler pair for the compact $2n$-dimensional manifold M, and so we will suppress it from the notation.

Definition 1.5. Given two tangent vectors $F, G \in T_u\mathcal{H} \cong C^\infty(M, \mathbb{R})$ the *Mabuchi metric* is defined by the expression

$$\langle F, G\rangle_u = \int_M FG\, d\mu_u, \text{ with } d\mu_u = \frac{\omega_u^n}{n!},$$

where $\omega_u = \omega + i\partial\overline{\partial}u, u \in \mathcal{H}$. This is a weak Riemannian metric.

4. The geodesic equation in $\mathcal{H}$

The standard approach to study geodesic curves involves defining the notion of covariant differentiation by introducing a connection on the tangent bundle. Although this is possible in the inifinite-dimensional setting, the usual arguments that lead to the existence of the Levi-Civita connection do not hold, and have to be reformulated [**Kol12**]. Because of this, we will work instead with the definition of geodesics as stationary curves of the energy functional associated to the metric.

Definition 1.6. The *metric* (or free-particle) *energy functional* is the map $\mathcal{E} : C^\infty([0,1], \mathcal{H}) \to \mathbb{R}$ that in each curve u_t reads

$$\mathcal{E}(u_t) = \frac{1}{2}\int_0^1 \langle \dot{u}_t, \dot{u}_t\rangle_{u_t}\, dt = \frac{1}{2}\int_0^1 \int_M \dot{u}_t{}^2\, d\mu_{u_t}\, dt.$$

A *geodesic* is a stationary curve of the restriction of $\mathcal{E}$ to paths with fixed endpoints.

Proposition 1.7. [**Bal06**] *The geodesic equation for the Mabuchi metric is*

(1) $$\ddot{u}_t = \frac{1}{2}|\nabla^{u_t}\dot{u}_t|^2_{u_t},$$

where both the usual gradient ∇^{u_t} and the vector norm $|\cdot|_{u_t}$ are calculated with respect to the Riemannian metric g_{u_t} on M corresponding to the compatible Kähler pair (ω_{u_t}, J), i.e., $g_{u_t}(\cdot,\cdot) = \omega_{u_t}(\cdot, J\cdot)$.

REMARK 1.8. Note that $\dot{u} \in T_u\mathcal{H} \cong C^\infty(M,\mathbb{R})$, and so its gradient $\nabla^u \dot{u}$ is a well-defined element of $\mathfrak{X}(M)$.

For the proof we will need some auxiliary results.

LEMMA 1.9. *Let α and β be one-forms on the symplectic manifold (M,ω). Then*

$$\alpha \wedge \beta \wedge \omega^{n-1} = \frac{1}{n}\omega(a,b)\omega^n,$$

where the vectors a and b are such that $\alpha = \iota_a\omega$ and $\beta = \iota_b\omega$.

PROOF. As ω^n is a volume form, we have

$$\begin{aligned} 0 &= \iota_b(\alpha \wedge \omega^n) \\ &= \alpha(b)\omega^n - \alpha \wedge (\iota_b\omega^n) \\ &= \alpha(b)\omega^n - n\alpha \wedge (\iota_b\omega) \wedge \omega^{n-1} \\ &= \omega(a,b)\omega^n - n\alpha \wedge \beta \wedge \omega^{n-1}, \end{aligned}$$

from which the result follows. □

Defining the trace of a two-form Ω with respect to a symplectic form ω as the tensor contraction, locally given by

$$\mathrm{Tr}_\omega(\Omega) = \sum_{j,k} \omega^{jk}\Omega_{jk},$$

we have [**Bal06**] that the laplacian Δ of a function $f \in C^\infty(M,\mathbb{R})$ is

$$\mathrm{Tr}_\omega(i\partial\bar{\partial} f) = \Delta f. \tag{2}$$

LEMMA 1.10. *Let Ω be a two-form on the symplectic manifold (M,ω). Then*

$$\Omega \wedge \omega^{n-1} = \frac{1}{2n}\,\mathrm{Tr}_\omega(\Omega)\omega^n.$$

PROOF. By linearity, it suffices to prove the case $\Omega = \alpha \wedge \beta$, for α and β one-forms. But then this is a consequence of Lemma 1.9 and the computation that shows

$$\mathrm{Tr}_\omega(\alpha \wedge \beta) = 2\omega(a,b),$$

where a and b are the vectors such that $\alpha = \iota_a\omega$ and $\beta = \iota_b\omega$. □

LEMMA 1.11. *Consider a variation of the path u_t in $\mathcal{H}$, such that $u_{0,t} = u_t, u_{s,0} = u_0$ and $u_{s,1} = u_1$. Write $\delta u_t = \frac{\partial u}{\partial s}\big|_{s=0}$ i.e., $u_{s,t} = u_t + s\,\delta u_t + O(s)$. The first variation of the volume form $d\mu_{u_t} = \omega_{u_t}^n/n!$ is*

$$\delta\left(d\mu_{u_t}\right) = \frac{1}{2}\Delta(\delta u_t)\,d\mu_{u_t}.$$

PROOF. Let $\omega_s = \omega + is\partial\bar{\partial}(\delta u)$. We have, by Lemma 1.10

$$\left.\frac{d}{d_s}\right|_{s=0}(\omega_s^n) = n[i\partial\bar{\partial}\delta u] \wedge \omega^{n-1} = \frac{1}{2}\,\mathrm{Tr}_\omega\left[i\partial\bar{\partial}(\delta u)\right]\omega^n = \frac{1}{2}\Delta(\delta u)\omega^n.$$

The result then follows by linearity. □

Proof of Theorem 1. Let us find the extrema of the metric energy functional $\mathcal{E}$. Fixing the endpoints as in Lemma 1.11, the first variation of $\mathcal{E}$ is

$$\delta\mathcal{E}(u_{s,t}) = \frac{1}{2}\int_0^1 \int_M \left(2\dot{u}_t \delta\dot{u}_t \, d\mu_{u_t} + \frac{1}{2}\dot{u}_t{}^2 \, \Delta(\delta u_t) \, d\mu_{u_t} \right) dt.$$

Using integration by parts and the fact that the variation has fixed endpoints we calculate

$$\begin{aligned}\delta\mathcal{E} &= \frac{1}{2}\int_0^1 \int_M \left(-\delta u_t [2\ddot{u}_t + \dot{u}_t \Delta \dot{u}_t] + \frac{1}{2}\delta u_t \Delta(\dot{u}_t^2) \right) d\mu_{u_t} \, dt \\ &= -\int_0^1 \int_M \delta u_t \left(\ddot{u}_t - \frac{1}{2}|\nabla^{u_t}\dot{u}_t|_{u_t}^2 \right).\end{aligned}$$

As we are looking for stationary curves, $\delta\mathcal{E} = 0$, and the geodesic equation holds. □

5. The Monge-Ampère equation

We will now give a more geometric interpretation of the geodesic equation. This section is included for motivation only, and most of the results are only cited without proof. Let u_t, $t \in [0,1]$, be a path in $\mathcal{H}$. We can regard u_t as a function on $M \times [0,1]$ and extend it trivially to become a function on $M \times Z$, where $Z = [1,e] \times S^1$. In other words, we define the axially symmetric function

$$\begin{aligned} v : M \times [1,e] \times S^1 &\to \mathbb{R} \\ (p, e^t, e^{is}) &\mapsto u_t(p), \quad t \in [0,1]. \end{aligned} \tag{3}$$

In this way we can consider the cylinder $Z = [1,e] \times S^1$ as a complex manifold with boundary, with standard coordinates $z = t + is$. We can pull back the Kähler form ω from M to $M \times Z$ via the projection map $\pi : M \times Z \to M$. With these definitions we have the following proposition.

Proposition 1.12. *Let $u_t \in C^\infty([0,1], \mathcal{H})$ be a path in $\mathcal{H}$, and let v be its the extension to $M \times Z$ given by (3). Then u_t is a geodesic in $\mathcal{H}$ if and only if v is a solution of the restricted Complex Homogeneous Monge-Ampère equation:*

$$\begin{cases} \Omega_v^{n+1} = \left(\Omega + i\partial\overline{\partial} v\right)^{n+1} = 0, \\ \Omega_v|_{M\times\{z\}} \succ 0, \quad \forall z \in Z, \end{cases} \tag{4}$$

where $\Omega = \pi^*\omega$ is the pull-back of the Kähler form on M, and ∂ and $\overline{\partial}$ are the Dolbeault operators on $M \times Z$.

Proof. Let ∂_M and $\overline{\partial}_M$ be the Dolbeault operators on M. We will simplify the expression $\Omega_v^{n+1} = 0$ until it is clear its dependence on $\dot{u}_t$. We have

$$\partial v = \partial_M v + \frac{\partial v}{\partial z} dz \qquad \overline{\partial} v = \overline{\partial}_M v + \frac{\partial v}{\partial \overline{z}} d\overline{z},$$

and so

$$\partial\overline{\partial} v = \partial_M \overline{\partial}_M v - \overline{\partial}_M \left(\frac{\partial v}{\partial z}\right) \wedge dz + \partial_M \left(\frac{\partial v}{\partial \overline{z}}\right) \wedge d\overline{z} + \left(\frac{\partial^2 v}{\partial z \partial \overline{z}}\right) dz \wedge d\overline{z}.$$

This means that Ω_v can be written as $\Omega_v = \omega_u + i\eta$, with

$$\eta = -\alpha \wedge dz + \beta \wedge d\overline{z} + c dz \wedge d\overline{z},$$

where

$$\alpha = \overline{\partial}_M \left(\frac{\partial v}{\partial z} \right), \qquad \beta = \partial_M \left(\frac{\partial v}{\partial \overline{z}} \right), \qquad c = \left(\frac{\partial^2 v}{\partial z \partial \overline{z}} \right).$$

As $\eta^3 = 0$, we find

$$\Omega_v^{n+1} = (\omega_u + i\eta)^n = \omega_u^{n+1} + i(n+1)\omega_u^n \wedge \eta - \frac{n(n+1)}{2}\omega_u^{n-1} \wedge \eta \wedge \eta.$$

As ω_u^n is a top form on M, and

$$\omega_u^n \wedge \eta = c\,\omega_u^n \wedge (dz \wedge d\overline{z})$$
$$\omega_u^{n-1} \wedge \eta \wedge \eta = 2\left(\alpha \wedge \beta \wedge \omega_u^{n-1}\right) \wedge (dz \wedge d\overline{z}),$$

$\Omega_v^{n+1} = 0$ if and only if

$$ic\,\omega_u^n \wedge (dz \wedge d\overline{z}) - n\left(\alpha \wedge \beta \wedge \omega_u^{n-1}\right) \wedge (dz \wedge d\overline{z}) = 0.$$

Now, for every smooth real-valued function $f \in C^\infty(M, \mathbb{R})$ it is true that

$$\overline{\partial}_M f = \omega_u(-iP\nabla^u f, \cdot), \qquad \partial_M(f) = \omega_u(iQ\nabla^u f, \cdot),$$

where the gradient ∇^u is taken with respect to the metric $g_u(\cdot,\cdot) = \omega_u(\cdot, J\cdot)$, and the tensors P and Q are

$$P = \frac{1}{2}(I - iJ), \qquad Q = \frac{1}{2}(I + iJ),$$

with I the identity. Moreover,

$$\omega_u(-iP\nabla^u f, iQ\nabla^u f) = \frac{i}{2}|\nabla^u \dot{u}|_u^2,$$

with $|X|_u^2 = g_u(X, X)$, for a vector field $X \in \mathfrak{X}(M)$. As v is a trivial extension of u, we have that

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial \overline{z}} = \dot{u}, \qquad \frac{\partial^2 v}{\partial z \partial \overline{z}} = \ddot{u}.$$

Using Lemma 1.9

$$\alpha \wedge \beta \wedge \omega_u^{n-1} = \frac{i}{2n}|\nabla^u \dot{u}|_u^2 \omega_u^n.$$

As so, $\Omega_v^{n+1} = 0$ if and only if

$$\ddot{u} = \frac{1}{2}|\nabla^u \dot{u}|_u^2.$$

□

REMARK 1.13. Note the condition in (4) which forces Ω_v to be strictly positive in the M-slices, $M \times \{z\}$, is exactly what is necessary for the corresponding u_t to be an ω-plurisubharmonic function, i.e., for u_t to belong to $\mathcal{H}$. Observe also that, even though it is crucial for M to be compact so that Riemannian structure on $\mathcal{H}$ makes sense, the equation (1) is still well defined for non-compact manifolds. In that case, we will continue calling it geodesic equation and its solutions geodesics. We will see an example in chapter 3 where we solve this equation on the manifold $\mathbb{C}$.

The Proposition 1.12 establishes an equivalence between the geodesic problem and the rotationally invariant solutions of the Complex Homogeneous Monge-Ampère problem (4).

6. Motivation for complexified Hamiltonian flows

Let us now provide some motivation for the solution of the geodesic problem in $\mathcal{H}$ as a complex Hamiltonian flow. We will prove later that some complex flows will be, in fact, solutions of the geodesic equation or give rise to solutions of the Complex Homogenous Monge-Ampère Equation. Here, we will only sketch the construction that is formally presented in [**Don99**]. We begin by stating a result that establishes a geometry on the domain of a Monge-Ampère operator.

PROPOSITION 1.14. *Suppose Ω_v is a form on $M \times Z$ that satisfies the Complex Homogeneous Monge-Ampère equation (4). Then there is a foliation $\mathcal{F}_v$ of $M \times Z$ whose leaves are Riemann Surfaces transverse to the slices $M \times \{z\}, z \in Z$.*

SKETCH OF PROOF. We will only show how the foliation is defined. If Ω_v is a form that satisfies the Monge-Ampère equation (4), $\ker(\Omega_v)$ is a distribution of real dimension 2. As Ω_v is closed, this distribution is integrable. This means there is a foliation $\mathcal{F}_v$ of $M \times Z$ by surfaces. The nontrivial fact that the leaves are Riemann surfaces with the desired properties can be found in [**Kli91**]. □

REMARK 1.15. For the proposition to hold we only need Ω_v to have constant rank. The condition that it is positive on the M-slices is slightly stronger. Also, the foliation depends on the solution v. More precisely, it depends on the boundary conditions only. This follows from the uniqueness of the solution to the Monge-Ampère problem, which is derived from a maximum principle [**Don99**].

The foliation described in the proposition is called the *Monge-Ampère foliation* of $M \times Z$. The interaction between the solutions of the equation and this foliation is established in the next result [**Don02**].

PROPOSITION 1.16. *Let $\Omega_v = \Omega + i\partial\overline{\partial}v$ be a form on $M \times Z$ that satisfies the Complex Homogeneous Monge-Ampère equation (4), with axially symmetric boundary conditions, that is, $v|_t$ is the trivial extension (3) of some u_t on M. Then the graph of the trajectory of each point $p \in M$ under the Hamiltonian flow of $\dot{u}_t$ lies in a single leaf of the Monge-Ampère foliation.*

Let us now recast the process of finding geodesics as a problem of complexified Hamiltonian dynamics. Suppose we have a geodesic path u_t in $\mathcal{H}$, with $t \in [0, 1]$, and such that $u_0 = 0$. As described above, we have a corresponding Complex Homogenous Monge-Ampère equation in $M \times Z$, where $Z = [1, e] \times S^1$ is the cyllinder. The Proposition 1.14 then gives a foliation of $M \times Z$, transverse to the M-slices. We will use this foliation to find a two-parameter map on M which will be interpreted as a complexified flow.

Regard the derivative $\dot{u}_0 \in T\mathcal{H} \cong C^\infty(M, \mathbb{R})$ as a Hamiltonian function on M. As usual, its Hamiltonian vector field $X_{\dot{u}_0}$ defines a flow $e^{sX_{\dot{u}_0}} : M \to M$, where $s \in \mathbb{R}$, because M is compact. Fix a point $p \in M$. We can consider its trajectory under the flow, i.e., the curve $\gamma_p : \mathbb{R} \to M$ such that $\gamma_p(s) = e^{sX_{\dot{u}_0}}(p)$. Let us now lift the curve γ_p to a map Γ_p in $M \times Z$. Observe that, by the Propositon 1.16, the graph of γ_p lies in a single leaf S_α of the Monge-Ampère foliation $\mathcal{F}_v = \{S_\alpha\}$ of $M \times Z$. Recalling that the leaves are Rieman surfaces, we can produce, for each $p \in M$, an analytic continuation $\tilde{\gamma}_p : S_\alpha \to M$ of γ_p. We thus obtain a family of holomorphic maps $\{\tilde{\gamma}_p\}$ one on each leaf of the foliation. These maps depend on two real variables, or equivalently, on a single complex one, that gives a local coordinate on S_α. Gathering together these maps, we obtain an holomorphic function Γ_p to the manifold M:

$$\begin{aligned}\Gamma_p : [0,1] \times \mathbb{R} &\to M\\ (t,s) &\mapsto \tilde{\gamma}_p(s,t).\end{aligned}$$

This function is such that $\Gamma_p(0,s) = \gamma_p(s)$ and Γ_p restricts to the Hamiltonian flow of $\dot{u}_t$ in the Kähler manifold (M, ω_{u_t}). In particular, the map $\phi_t : M \to M$ given by $\phi_t(p) = \Gamma_p(t,0)$ takes the Hamiltonian $\dot{u}_0$ in (M,ω) to $\dot{u}_t$ in (M,ω_{u_t}). This allows one to understand de study of geodesics of $\mathcal{H}$ as a sort of analytic continuation of Hamiltonian Dynamics, where the time parameter $\tau = t + is$ is allowed to be complex. This is the main motivation, coming from the geometry of $\mathcal{H}$, for one to worry about defining a notion of complex Hamiltonian flow. That is what we will do in chapter 2.

CHAPTER 2

Complex Hamiltonian Dynamics

1. Introduction

In chapter 1, we have seen how the problem of finding geodesics in the infinite-dimensional manifold of Kähler forms, with fixed cohomology class and fixed complex structure, can be related to a problem of complex Hamiltonian evolution. So, to solve the geodesic problem, we are faced with the question of giving meaning to this type of evolution. If we manage to do that, we will be able to use complex Hamiltonian dynamics to evolve an initial Kähler structure in such a way that the resulting path of Kähler structures is a geodesic. Thus, we will be effectively reducing, for appropriately chosen analytic initial conditions, the non-linear partial differential equation (4) to a Hamiltonian system of ordinary differential equations. In the present chapter, we will first answer the question of what is a complex flow of a complex vector field, for general complex manifolds. We then introduce the symplectic structure and refine the analysis for Hamiltonian vector fields.

As a guiding principle, let us briefly describe the procedure for establishing the evolution in complex time for a real vector field, on a compact manifold M. If we consider the usual time flow ϕ_t of a fixed $X \in \mathfrak{X}(M, \mathbb{R})$ there is a linear action by pullback on local functions:

$$\begin{aligned} \boldsymbol{\phi}_t : C^\infty(U, \mathbb{R}) &\to C^\infty(\phi_t^{-1}(U), \mathbb{R}) \\ f &\mapsto f \circ \phi_t, \end{aligned}$$

where $U \subset M$ is an open set. This means that a flow ϕ_t defines an isomoprhism $\boldsymbol{\phi}_t$ of algebras of functions. Then, our approach would be to extend both the domain of this operator $\boldsymbol{\phi}_t$, so it can act on complex functions, and its time parameter, so it is defined for values of complex time τ, thus obtaining an operator $\boldsymbol{\varphi}_\tau$. The way to construct this operator in a real analytic setting is to consider the power series form of $\boldsymbol{\phi}_t$, given by e^{tX}, and use analytic continuation in time to extend it to $\boldsymbol{\varphi}_\tau = e^{\tau X}$. The convergence of such a power series must then be established, and we will employ the theory of Lie Series, due to Grobner [**Gro67**], to do so.

The problem of defining complex evolution then consists of finding a diffeomorphism φ_τ on the manifold, such that the extended operator $\boldsymbol{\varphi}_\tau$ acts as the pull-back by φ_τ, i.e., $\boldsymbol{\varphi}_\tau = \varphi_\tau^*$ as operators. This cannot be done considering the local action of $\boldsymbol{\varphi}_\tau = e^{\tau X}$ for all local functions, because, for non-real time, $\boldsymbol{\varphi}_\tau$ would take real-valued functions to complex-valued ones, and no diffeomorphism can do that. So, instead of extending the operator $\boldsymbol{\phi}_t$ to the full algebra of local complex functions, we consider a complex manifold (M, J_0), and only build the extension $\boldsymbol{\varphi}_\tau$ on the algebras of local *holomorphic* functions.

This way, we will be able to build a diffeomorphism φ_τ, such that $\boldsymbol{\varphi}_\tau = \varphi_\tau^*$ as operators on local holomorphic functions. This map φ_τ will be unique, but will not, in general, preserve the global complex structure J_0 on the manifold. Nonetheless it will be possible to define a new global one, J_τ, and make φ_τ a biholomorphism between (M, J_0) and (M, J_τ). This is a generic idea in complex evolution where most of the times there will be a change in the complex structure.

The sections proceed in steps of increasing structure. Firstly, we define the operator $\varphi_\tau = e^{\tau X}$ for general real-analytic manifolds using the theory of Lie Series, whose results we state. We then study how this operator behaves locally on a complex manifold M, restrict it to the subalgebra of local holomorphic functions, and find the sought after diffeomorphism φ_τ. We will have constructed our complex flow. Then we restrict our analysis to the symplectic case and to Hamiltonian vector fields, investigating how the map φ_τ interacts with the symplectic structure. We proceed further to Kähler case, and establish an evolution of Kähler structures. Finally, we use the formalism to choose this evolution in a geodesic fashion, solving the motivation problem of chapter 1. The full approach is described in [**MN15**].

2. Lie Series and the evolution operator

Throughout this section M will be a real analytic compact manifold. We denote the algebra of complex-valued real analytic functions on some open subset $U \subset M$ as $C^\omega(U,\mathbb{C})$. A real analytic vector field is an analytic section of $TM\otimes\mathbb{C}$. We write $\mathfrak{X}^\omega(M,\mathbb{C})$ for the set of all such vector fields, which forms a Lie sub-algebra of the algebra of all vector fields under the usual commutator. Consider $X \in \mathfrak{X}^\omega(M,\mathbb{C})$. If $f \in C^\omega(U,\mathbb{C})$, we have $X^k(f) \in C^\omega(U,\mathbb{C}), \forall k \in \mathbb{N}$. This means that we can try to define the evolution operator as an exponential series, where, at least, each term in the expansion will belong to $C^\omega(U,\mathbb{C})$.

Definition 2.1. Consider the real analytic complex-valued vector field $X \in \mathfrak{X}^\omega(M,\mathbb{C})$ and function $f \in C^\omega(U,\mathbb{C})$. The *Lie series for X and f, at complex time $\tau \in \mathbb{C}$*, is the series

$$\sum_{k=0}^{\infty} \frac{\tau^k}{k!} X^k(f). \tag{5}$$

By letting τ vary in some subset of the complex numbers, we get an extra variable. Viewing the series in this way we call it the *Lie series for X and f* or, if there is no risk of confusion, simply the *Lie series*.

The next result shows that the Lie Series is more than a formal series, it converges for small values of complex time [**Gro67**].

Proposition 2.2. *Let M be a compact real analytic manifold, $X \in \mathfrak{X}^\omega(M,\mathbb{C})$ and $f \in C^\omega(U,\mathbb{C})$, for an open set $U \subset M$. There exists a $T_f > 0$ such that the Lie series for X and f is absolutely and uniformly convergent on compact sets of $U \times D_{T_f}$, where $D_{T_f} = \{ \tau \in \mathbb{C} \mid |\tau| < T_f \}$.*

Corollary 2.3. *The Lie series for $X \in \mathfrak{X}^\omega(M,\mathbb{C})$ and $f \in C^\omega(U,\mathbb{C})$ defines a real analytic function on $U \times D_{T_f}$. In particular, for fixed $\tau \in D_{T_f}$ the Lie series for X and f at time τ defines a function in $C^\omega(U,\mathbb{C})$,*

Remark 2.4. The real analytic structure on D_{T_f} comes from the usual complex structure on $\mathbb{C}$.

We will now define the evolution operator $e^{\tau X}$ associated with $X \in \mathfrak{X}^\omega(M,\mathbb{C})$, for a general $\tau \in \mathbb{C}$.

Definition 2.5. Fix a vector field $X \in \mathfrak{X}^\omega(M,\mathbb{C})$ and an open subset $U \subset M$. For each $\tau \in \mathbb{C}$, let $\mathrm{Dom}_U(e^{\tau X}) \subset C^\omega(U,\mathbb{C})$ be the set of functions $f \in C^\omega(U,\mathbb{C})$ such that the Lie series for X and f at time τ converges uniformly on compact subsets of U. The *evolution operator for* X is

$$\begin{aligned} e^{\tau X} : \mathrm{Dom}_U(e^{\tau X}) &\to C^\omega(U, \mathbb{C}) \\ f &\mapsto \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X^k(f). \end{aligned}$$

If we have a function $f \in \mathrm{Dom}_U(e^{\tau X})$ we say that $e^{\tau X}$ can be applied to f and write $e^{\tau X} \cdot f$ for the image of f under the operator.

REMARK 2.6. Note firstly that $\mathrm{Dom}_U(e^{\tau X})$ cannot be empty, as the zero function $f = 0$ is always there. With this in mind, by the Corollary 2.3, the image of the evolution map is indeed a subspace of $C^\omega(U, \mathbb{C})$ and so $e^{\tau X}$ is well defined. Secondly, if f is a function in $C^\omega(U, \mathbb{C})$, there will always be a family of evolution operators which can be applied to it. This is because Proposition 2.2 guarantees the convergence of the Lie series for all complex τ such that $|\tau| < T_f$ for some T_f, so $f \in \mathrm{Dom}_U(e^{\tau X})$ for all such τ. For the particular case $\tau = 0$, the Lie series is trivial, and we have $e^{0X} \equiv id|_{C^\omega(U,\mathbb{C})}$. Then, again by Corollary 2.3, $e^{\tau X} \cdot f$ is homotopic to f.

If $f, g \in C^\omega(U, \mathbb{C})$ we have that $e^{\tau X}$ can be applied to f for $|\tau|$ up to some T_f and it can be applied to g for $|\tau|$ up to T_g. By choosing a τ such that $|\tau| < \min\{T_f, T_g\}$ one shows that

$$e^{\tau X} \cdot (fg) = (e^{\tau X} \cdot f)(e^{\tau X} \cdot g), \tag{6}$$

and so $e^{\tau X}$ acts as a local (in time) automorphism of the algebra $C^\omega(U, \mathbb{C})$ **[Gro67]**. This argument can be extended to any finite number of functions.

THEOREM 2.7 (Commutation Theorem). **[Gro67]** *Let U be an open subset of a manifold M with local coordinates $(x^1, x^2, \ldots, x^n)$. Consider a complex-valued real analytic function $f \in C^\omega(U, \mathbb{C})$ and a vector field $X \in \mathfrak{X}^\omega(U, \mathbb{C})$. Suppose the local representation of f is $\widehat{f} : \widehat{U} \to \mathbb{C}$, with $\widehat{U} \subset \mathbb{R}^n$, and consider its unique analytic continuation $\widehat{F} : \widehat{V} \to \mathbb{C}$ to some open subset $\widehat{V} \subset \mathbb{C}^n$. If $e^{\tau X}$ can be applied to $f, x^1, x^2, \ldots, x^{n-1}$ and x^n, there exists an open set $V \subset U$ such that*

$$\widehat{e^{\tau X} \cdot f}(x^1, x^2, \cdots, x^n) = \widehat{F}(e^{\tau X} \cdot x^1, e^{\tau X} \cdot x^2, \ldots, e^{\tau X} \cdot x^n),$$

where $\widehat{e^{\tau X} \cdot f}$ is the local version of $e^{\tau X} \cdot f$ on V with respect to the coordinates $(x^1, x^2, \ldots, x^n)$.

REMARK 2.8. Note that this theorem simply states that, locally, the action of the evolution operator on a function $\widehat{f}$ can be calculated by acting on the coordinates first and then applying the analytic continuation $\widehat{F}$ to the result. Writing $x = (x^1, x^2, \ldots, x^n)$ for the coordinate map and if we have $\widehat{f} : \widehat{U} \subset \mathbb{R}^n \to \mathbb{C}^n$ and $\widehat{F} : \widehat{V} \subset \mathbb{C}^n \to \mathbb{C}$, the subset V is just $V = x^{-1}\left[\hat{\imath}^{-1}\left(\widehat{V} \cap \hat{\imath}(\widehat{U})\right)\right]$, where $\hat{\imath}$ is the inclusion of $\widehat{U}$ on $\mathbb{C}^n$.

The evolution operator defined in this way is an extension of the usual flow. In particular, if both the vector field X and the time parameter $\tau = t$ are real, as a result of the Commutation Theorem, $e^{tX} \cdot f = f \circ \varphi_t$, where φ_t is the flow of X. In fact, for a coordinate $x^j, j = 1, 2, \ldots, n$,

$$\frac{d}{dt}\left(e^{tX} \cdot x^j\right) = \sum_{k=1}^{\infty} k \frac{t^{k-1}}{k!} X^k(x^j) = X\left(e^{tX} \cdot x^j\right).$$

Writing $x = (x^1, x^2, \dots, x^n)$ and $e^{tX} \cdot x = (e^{tX} \cdot x^1, e^{tX} \cdot x^2, \dots, e^{tX} \cdot x^n)$, we have $\frac{d}{dt}\left(e^{tX} \cdot x\right) = X(e^{tX} \cdot x)$. Provided that e^{tX} can be applied to the coordinates, we see that $e^{tX} \cdot x$ satisfies the differential equations for the flow of X, and the uniqueness follows from the Picard-Lindelöf theorem.

The evolution operator $e^{\tau X}$ is defined for compact real analytic manifolds M. While this might seem a restriction it is not. A theorem attributed to Whitney states that a differentiable manifold (and in fact any C^k manifold with $k > 1$) is (C^k) diffeomorphic to a unique real analytic manifold, up to real analytic diffeomorphism [**Hir94**]. This means that $e^{\tau X}$ can be defined for any compact manifold M considering the real analytic structure provided by the theorem. For non-compact real analytic manifolds the results hold, in general, only locally.

3. Evolution on complex manifolds

Throughout this section consider J_0 to be an integrable almost complex structure on a $2n$-dimensional compact manifold M. This extra structure will allow us to extract a map φ_τ from the restriction of the evolution operator $e^{\tau X}$ to the algebra of locally holomorphic functions, as described in the introduction.

THEOREM 2.9. [**MN15**] *Consider a compact complex manifold (M, J_0). Let $\{z^j\}_{1 \le j \le n}$ be a system of coordinates in an open neighbourhood U of $p \in M$, such that z^j is J_0-holomorphic for every $j = 1, ...n$. Then there exists $T > 0$ such that, for every $\tau \in D_T = \{\, \tau \in \mathbb{C} \mid |\tau| < T \,\}$, the functions are well defined*

$$z^j_\tau = e^{\tau X} \cdot z^j, \quad j = 1, 2, \dots n$$

and form a coordinate chart on a neighbourhood $V \subset U$ of p. This coordinate chart defines a unique complex structure J_τ on V for which the functions $\{z^j_\tau\}_{1 \le j \le n}$ are holomorphic.

PROOF. Let V' be a neighbourhood of p that is compactly contained in U, that is, whose closure is a compact subset of U. As we have observed in equation 6, we can find $T > 0$ such that $e^{\tau X}$ can be applied to all functions z^j, $j = 1, 2, \dots, n$, for every complex τ such that $|\tau| < T$. This and Theorem 2.3 assure each z_j is a well defined continuous function of τ. By taking a smaller T if necessary, the continuity in τ and the fact that $dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n \neq 0$ at every point in V', guarantee that $dz^1_\tau \wedge \cdots \wedge dz^n_\tau \wedge d\bar{z}^1_\tau \wedge \cdots \wedge d\bar{z}^n_\tau \neq 0$ at every point of V' for $|\tau| < T$. The, the inverse function theorem implies that there is an open neighbourhood $V \subset V'$ of p for which the real and imaginary parts of z_j form a coordinate system. ☐

This theorem shows that even locally the complex evolution changes the complex structure. The global result that extends this theorem and provides the existence and uniqueness of the map φ_τ, which acts on J_0-holomorphic functions by pullback the same way $e^{\tau X}$ does, is the content of the next statement.

THEOREM 2.10. [**MN15**] *Consider the vector field $X \in \mathfrak{X}^\omega(M, \mathbb{C})$. There exists a $T > 0$ such that, for $\tau \in D_T = \{\, \tau \in \mathbb{C} \mid |\tau| < 0 \,\}$, there is a global complex structure J_τ on M and a unique biholomorphism*

$$\varphi_\tau : (M, J_\tau) \to (M, J_0)$$

which acts by pullback on J_0-holomorphic coordinates as $e^{\tau X}$.

PROOF. As M is compact there is a finite atlas $\mathscr{A} = \{U_\alpha, z_\alpha\}_{\alpha=1,\dots,K}$ of J_0-holomorphic local charts. Consider the values $T_{\alpha,p} > 0$ and the open neighbourhoods $V_{\alpha,p} \subset U_\alpha$ of $p \in M$

for which the functions

$$z_{\alpha,p,\tau} = e^{\tau X} \cdot z_{\alpha,p} = \left(e^{\tau X} \cdot z^1_{\alpha,p}, \cdots, e^{\tau X} \cdot z^n_{\alpha,p}\right)$$

are holomorphic functions of τ, for $\tau \in D_{T_{\alpha,p}}$, and with $z_{\alpha,p,\tau}(V_\alpha, p) \subset z_\alpha(U_\alpha)$, where we are writing $z_{\alpha,p}$ for the restriction $(z_\alpha)|_{V_{\alpha,p}}$. Extract from the open cover $\{V_{\alpha,p}\}_{\alpha=1,\cdots,K,p\in M}$ a finite subcover $\{V_{\alpha_j,p_j}\}_{j=1,\cdots,N}$. Set $\widetilde{T} = \min_j\{T_{\alpha_j,p_j}\}$. Consider the transition functions ϕ_{α_j,α_k}, which coincide with the transitions functions for $\mathscr{A}$ restricted to the appropriate subsets. It follows from Theorem 2.10 that there exists a T, with $0 < T \leq \widetilde{T}$ such that

$$z_{\alpha_j,p_j,\tau} = e^{\tau X} \cdot z_{\alpha_j,p_j} = e^{\tau X} \cdot (\phi_{\alpha_j,\alpha_k} \circ z_{\alpha_k,p_k}) = \phi_{\alpha_j,\alpha_k} \circ z_{\alpha_k,p_k,\tau}, \tag{7}$$

for $\tau \in D_T$, and the $z_{\alpha_j,p_j,\tau}$ is coordinate map for V_{α_j,p_j}. Therefore we have a new atlas

$$\mathscr{A}' = \{(V_{\alpha_j,p_j}, z_{\alpha_j,p_j,\tau})\}_{j=1,\cdots,N}$$

on M, with the same transition functions restricted to the open sets $V_{\alpha_j,p_j} \cap V_{\alpha_k,p_k}$. For $\alpha = 1, \cdots, K$ denote by $\phi_\alpha : z_\alpha(U_\alpha) \subset \mathbb{C} \to U_\alpha$ the inverse of the coordinate function z_α. Define the maps

$$\varphi_{\tau,j} = \phi_{\alpha,j} \circ z_{\alpha_j,p_j,\tau} : V_{\alpha_j,p_j} \to U_\alpha$$

for $\tau \in D_T$. The equation (7) shows that the maps $\{\varphi_{\tau,j}\}j = 1, \cdots, N$ glue together to give a global map $\varphi_\tau : M \to M$. This map φ_τ is surjective because it is a local diffeomporhism near every point and it is homotopic to the identity so it maps each connected component of M onto itself. On the other hand, the inverse map φ_τ exists. It is given on the charts $\varphi_\tau(V_{\alpha_j,p_j,\tau})$ by

$$e^{-\tau X} \cdot z_{\alpha_j,p_j,\tau} = z_{\alpha_j,p_j}.$$

The map φ_τ is clearly a biholomorphism from (M, J_τ) to (M, J_0), and it is the unique one such that $z_{\alpha_j,p_j,\tau} = z_{\alpha_j,p_j} \circ \varphi_\tau$. So the atlas $\mathscr{A}'$ defines a new complex structure J_τ on M, equivalent to J_0. □

Definition 2.11. The biholomorphism $\varphi_\tau : (M, J_\tau) \to (M, J_0)$ whose existence is asserted in Theorem 2.10 is called the *complex flow* of $X \in \mathfrak{X}^\omega(M, \mathbb{C})$. The complex structure J_τ is the induced structure by φ_τ, and it is given by $J_\tau = \varphi_\tau^* J_0 := (\varphi_\tau)_*^{-1} \circ J_0 \circ (\varphi_\tau)_*$.

Remark 2.12. The complex flow is only defined for $\tau \in \mathbb{C}$ such that $|\tau| < T$, for some $T > 0$ whose existence is also asserted by Theorem 2.10. When we want to stress this dependence we will write φ^T_τ. Moreover, have in mind that, in general, the complex flow φ_τ is *not* a flow. This is $\varphi_{\tau+\sigma} \neq \varphi_\tau \circ \varphi_\sigma$ even when all the maps are defined. The problem stems from the dependence of φ_τ on the starting complex structure J_0 present on the manifold. If we explicitly write it $\varphi^{J_0}_\tau$ we have, for $|\tau|, |\sigma|, |\tau+\sigma| < T$, the following commutative diagram.

$$\begin{tikzcd}
& (M, J_{\tau+\sigma}) \arrow[dl, "\varphi^{J_\tau}_\sigma"'] \arrow[dr, "\varphi^{J_\sigma}_\tau"] \arrow[dd, "\varphi^{J_0}_{\sigma+\tau}"] & \\
(M, J_\tau) \arrow[dr, "\varphi^{J_0}_\tau"'] & & (M, J_\sigma) \arrow[dl, "\varphi^{J_0}_\sigma"] \\
& (M, J_0) &
\end{tikzcd}$$

The group property will hold if all the time parameters and the vector field are real valued, because in that case the (complex) flow is J_0-independent.

COROLLARY 2.13. *Let φ_τ be the complex flow of a vector field $X \in \mathfrak{X}^\omega(M,\mathbb{C})$ on a compact complex manifold (M, J_0). Consider J_τ the induced complex structure by φ_τ. Then φ_τ^* maps local J_0-holomorphic functions to J_τ-holomorphic ones, i.e.,*

$$\varphi_\tau^* \left[\mathcal{O}_{J_0}(U)\right] = \mathcal{O}_{J_\tau}(U),$$

for any U open set in M, with $\mathcal{O}_J(U)$ the set of J-holomorphic functions on U.

The theorem shows clearly that the problem of complex evolution can only be given meaning by letting the complex structure vary. This is one of the reasons why complex dynamics is useful to solve the geodesic problem in the space of Kähler structures, because the evolution itself changes the complex structure present in the Kähler pair. We will in the next sections study this in greater detail.

Even though we are considering compact manifolds, the local theorem 2.9 is true for general complex manifolds. The existence of a global complex flow suffers from the same type of problems the existence of the usual global flow encounters, plus the additional problems regarding the global existence of the new complex structure. Thus, we have to carefully check wether global existence holds. In fact, there are examples of local complex flows in non-compact manifolds that do not extend to global ones [**MN15**]. Taking this into consideration, we present the simplest example of a complex evolution in $\mathbb{R}^2 \cong \mathbb{C}$. This can be seen as an example of the local result or an instance of the global one, as the transformations involved are all linear, and so, globally defined.

EXAMPLE 2.14 (The free particle in $\mathbb{R}^2 \cong \mathbb{C}$). *Consider $\mathbb{R}^2 \cong \mathbb{C}$ with the usual complex structure given by the coordinate $z = x+iy$, or, equivalently, by the integrable almost complex $J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $X = y\frac{\partial}{\partial x} \in \mathfrak{X}^\omega(\mathbb{R}^2,\mathbb{R}) \subset \mathfrak{X}^\omega(\mathbb{R}^2,\mathbb{C})$. This vector derives z as $X(z) = y$ and $X^k(z) = 0, \forall k \geq 2$. The evolution operator for $\tau = t + is$, with $t, s \in \mathbb{R}$, acts on the complex coordinate as*

$$z_\tau = e^{\tau X} \cdot z = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} X^k(z) = z + \tau y = x + ty + i(1+s)y.$$

For $s \neq -1$, this is a (global) diffeomorphism $\varphi_\tau : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y)^T \mapsto (u(x,y), v(x,y))^T$, with

$$u(x,y) = x + ty,$$
$$v(x,y) = (1+s)y.$$

In the case $s = -1$ this complex flow degenerates. This is not a counterexample for the previous results, because they are only valid for small values of $|\tau|$.

When $s = 0$, the map φ_τ is just the flow of X, i.e., a horizontal motion with constant velocity. For pure imaginary time $t = 0, s > 0$ the complex flow actually has vertical trajectories. For fixed values of $\vartheta = \arg \tau$, this is $\tau = re^{i\vartheta}$, with varying r, the complex flow is a skewed motion parallel to the half-line defined by the angle $\theta = \vartheta$, cf. Figure 1.

For $s \neq 0, -1$, φ_τ is not J_0-holomorphic. In fact, as J_0 is the standard complex structure, for φ_τ to be J_0-holomorphic means simply φ_τ obeys the Cauchy-Riemann equations. One easily checks that

$$\frac{\partial u}{\partial x} = 1 \neq 1 + s = \frac{\partial v}{\partial y}.$$

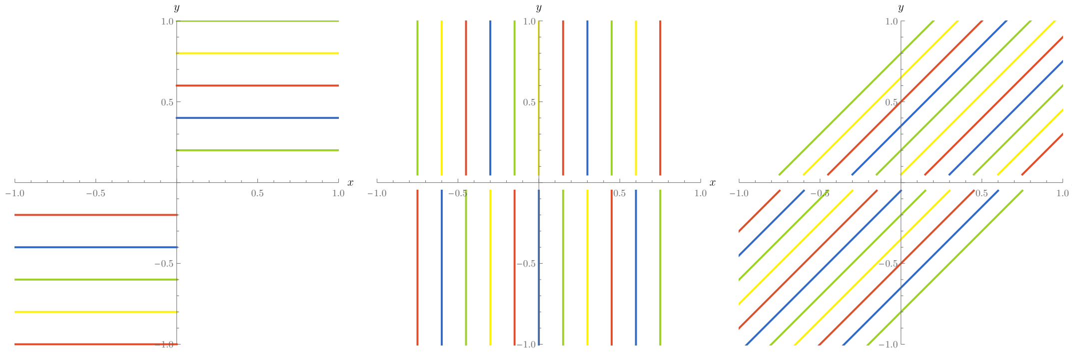


FIGURE 1. Some trajectories under the complex flow φ_τ for $\tau = re^{i\vartheta}$, parameterized by $r \in \mathbb{R}$, for $\vartheta = 0, \frac{\pi}{2}, \frac{\pi}{4}$, respectively. The complexified trajectories flow outwards the horizontal axis.

On the other hand, one can find the complex structure J_τ which makes $\varphi_\tau : (\mathbb{R}^2, J_\tau) \to (\mathbb{R}^2, J_0)$ into a holomorphic map. Recall that being holomorphic means $d\varphi_\tau \circ J_\tau = J_0 \circ d\varphi_\tau$, which for $J_\tau = J_0$ reduces to the previous Cauchy-Riemann equations. As so, J_τ is precisely given by

$$J_\tau = d\varphi_\tau^{-1} \circ J_0 \circ d\varphi_\tau.$$

Calculating the differential of φ_τ we get

$$d\varphi_\tau = \begin{pmatrix} 1 & t \\ 0 & 1+s \end{pmatrix}, \quad d\varphi_\tau^{-1} = \frac{1}{1+s} \begin{pmatrix} 1+s & -t \\ 0 & 1 \end{pmatrix},$$

which provides the complex structure

$$J_\tau = \frac{1}{1+s} \begin{pmatrix} -t & -(t^2+s+1) \\ 1 & t \end{pmatrix}.$$

Note that exactly when $\tau = 0$ we get $J_\tau = J_0$, and when $s = -1$ the structure is not defined. In a more general case, for instance when the vector field X is not linear, the complex structure depends not only on time τ, but also on the x and y variables.

To attest how deceiving the name complex flow is, let us show that the group property does not hold, as stated in Remark 2.12. Let $\tau_1 = t_1 + is_1$ and $\tau_2 = t_2 + is_2$ for $t_1, t_2, s_1, s_2 \in \mathbb{R}$, such that all the complex flows φ_{τ_1}, φ_{τ_2} and $\varphi_{\tau_1+\tau_2}$ are well defined. The maps

$$\varphi_{\tau_1+\tau_2} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + (t_1+t_2)y \\ (1+s_1+s_2)y \end{pmatrix}$$

and

$$\varphi_{\tau_1} \circ \varphi_{\tau_2} \begin{pmatrix} x \\ y \end{pmatrix} = \varphi_{\tau_1} \begin{pmatrix} x + t_2 y \\ (1+s_2)y \end{pmatrix} = \begin{pmatrix} x + [(1+s_1)t_1 + t_2]y \\ (1+s_1)(1+s_2)y \end{pmatrix}$$

are different diffeomorphisms of $\mathbb{R}^2$.

After the complex flow φ_τ is defined, we can use it to act not only on local J_0-holomorphic functions but on all differentiable functions. This raises the question of how the action of φ_τ^* relates to the action of $e^{\tau X}$ on functions that are *not* J_0-holomorphic. Remember that $e^{\tau X}$ is defined for some functions in $\mathrm{Dom}_U(e^{\tau X}) \subset C^\omega(U, \mathbb{C})$. It is possible that a given a function $g \in C^\infty(U, \mathbb{C})$ that is not J_0-holomorphic lies in $\mathrm{Dom}_U(e^{\tau X})$ (for this to happen, g has to be real analytic, of course). In this case, we can write g in local complex coordinates $\{z_j\}_{j=1,\dots,n}$ and we have the result.

Corollary 2.15. *Let $g \in C^\infty(M,\mathbb{C})$ be a differentiable function on a compact complex manifold (M,J_0), and φ_τ the complex flow of a vector field $X \in \mathfrak{X}^\omega(M,\mathbb{C})$. If $\widehat{g}$ and $\widehat{\varphi_\tau^* g}$ are the local representatives of g and of $\varphi_\tau^* g$ in local J_0-holomorphic coordinates $\{z_j\}_{1\leq j\leq n}$, we have*

$$\widehat{\varphi_\tau^* g}(z^1\dots,z^n,\overline{z}^1,\dots,\overline{z}^n) = \widehat{g}(z^1_\tau\dots,z^n_\tau,\overline{z}^1_\tau,\dots,\overline{z}^n_\tau),$$

where, for each $j = 1,\dots,n$,

$$z^j_\tau = e^{\tau X}\cdot z^j$$
$$\overline{z}^j_\tau = e^{\overline{\tau}\overline{X}}\cdot \overline{z}^j.$$

Remark 2.16. Notice the action on the complex conjugates $\overline{z}^j$ of the local coordinates is made by the conjugate vector field $\overline{\tau}\overline{X}$. This means that the action of $e^{\tau X}$ is, in general, *different* from the action of φ_τ^*, even though they agree on $\mathcal{O}_{J_0}(U)$, for any open set $U \subset M$. In conclusion, for a compact complex manifold M and a complex-valued real-analytic vector field $X \in \mathfrak{X}^\omega(M,\mathbb{C})$ we have the operators

$$e^{\tau X} : Dom(e^{\tau X}) \subset C^\omega(U,\mathbb{C}) \to C^\omega(U,\mathbb{C}), \qquad \varphi_\tau^* : C^\infty(U,\mathbb{C}) \to C^\infty(U,\mathbb{C})$$

where $e^{\tau X}$ is the evolution operator and φ_τ the complex flow. These operators satisfy

$$e^{\tau X}|_{\mathcal{O}_{J_0}(U)} = \varphi_\tau^*|_{\mathcal{O}_{J_0}(U)}.$$

4. Complex Hamiltonian Dynamics in Kähler Geometry

After having established complex evolution for general complex manifolds, we are in conditions to introduce a compatible symplectic form ω and study the dynamics on Kähler manifolds. In this case, we will consider Hamiltonian vector fields and observe that, for small enough times, their complex flow will induce a new complex structure that is still compatible with ω. This is the fundamental theorem that connects complex evolution and Kähler Geometry.

Theorem 2.17. **[MN15]** *Let M be a compact Kähler manifold with a real analytic Kähler pair (ω,J_0). Consider the real analytic complex valued function $h \in C^\omega(M,\mathbb{C})$ and its Hamiltonian vector field $X_h \in \mathfrak{X}^\omega(M,\mathbb{C})$. If φ_τ^T is the complex flow of X_h and J_τ the induced complex strucutre, for all $\tau \in D_T = \{\,\tau \in \mathbb{C} \mid |\tau| < T\,\}$ the following hold:*

i) (ω,J_τ) is a Kähler pair for M;
ii) A (local) Kähler potential for (ω,J_τ) is given by κ_τ, defined from the initial Kähler potential κ_0 for (ω,J_0) as

$$\kappa_\tau = -2\,\mathrm{Im}\,\psi_\tau$$

where

$$\psi_\tau = \tau h - \frac{i}{2}e^{\tau X}\cdot\kappa_0 - r_\tau,$$

r_τ is the analytic continuation in t of

$$r_t = \int_0^t e^{t'X}\cdot\theta(X_h)\,dt',$$

and $\theta = \frac{i}{2}(\partial_0 - \overline{\partial}_0)\kappa_0$, with ∂_0 and $\overline{\partial}_0$ Dolbeault operators relative to the initial complex structure J_0.

Remark 2.18. In general, φ_τ will not be a symplectomorphism. This means that the Kähler pairs (ω, J_0) and (ω, J_τ) will, in general, be non-equivalent, and we have a dynamic motion of non-equivalent triples inside $\mathcal{K}(M)$, the space of Kähler structures.

The action on Kähler structures as defined above fixes the symplectic structure, but the geometries considered in $\mathcal{H}$ assume instead a fixed complex structure on the manifold M. To get an evolution that respects this restriction, we need to use the complex flow φ_τ to move ω from (M, J_τ) back to (M, J_0). As φ_τ is not, in general, a symplectomorphism, the two-form obtained this way $\omega_\tau = (\varphi_\tau^{-1})^*\omega$ will be different from ω. That is, we will have built a Kähler equivalence, i.e., a holomorphic and symplectic diffeomorphism, between the pairs (J_τ, ω) and (J_0, ω_τ).

Proposition 2.19. *Let $\omega_\tau = (\varphi_\tau^{-1})^*\omega$. The Kähler pairs (ω, J_τ) and (ω_τ, J_0) are equivalent.*

Definition 2.20. The form ω_τ that is Kähler with respect to J_0 is called the *form induced by the complex flow*. The pair (ω_τ, J_0) is called *induced Kähler structure* or *evolved Kähler structure*.

Remark 2.21. The induced Kähler structure by the complex flow is (ω_τ, J_0), with modified symplectic form and the *starting* complex structure. It is *not* the pair (ω, J_τ), even though they are equivalent. We call the description of evolution in terms of the first pair the *complex picture*, because the complex structure J_0 is fixed. To the description with respect to the second pair we give the name of *symplectic picture*, as the symplectic form is kept the same.

To summarise, the evolution of the Kähler structure occurs as follows. We start with a compact manifold M with a Kähler pair (ω, J_0). The pullback by the complex flow can change the complex structure on M. So, we apply it to J_0 to get $J_\tau = \varphi_\tau^* J_0$. We do *not* apply it to ω, otherwise we would have equivalent Kähler structures. As the new complex structure is compatible with the unchanged symplectic form, for small enough $|\tau|$, we now have the pair (ω, J_τ). We apply the pullback $(\varphi_\tau^{-1})^*$ to both ω and J_τ, getting $(\varphi_\tau^{-1})^*\omega = \omega_\tau$ and $(\varphi_\tau^{-1})^* J_\tau = (\varphi_\tau^{-1})^*\varphi_\tau^* J_0 = J_0$. This is an equivalent structure. The complex evolution is thus the correspondence $(\omega, J_0) \mapsto (\omega_\tau, J_0)$. We have the following diagram where the dashed arrow represents evolution with fixed ω, and the squiggly one portraits the dynamics with fixed J_0.

$$\begin{tikzcd}
(\omega, J_0, g_0) \arrow[r, dashed, "\varphi_\tau^* \text{ only on } J_0"] \arrow[dr, squiggly, "\text{evolution}"'] & (\omega, J_\tau, g_\tau) \arrow[d, "(\varphi_\tau^{-1})^*"] & \text{(symplectic picture)} \\
& (\omega_\tau, J_0, g'_\tau) & \text{(complex picture)}
\end{tikzcd}$$

Even though the Kähler form changes by diffeomorphism, the corresponding Kähler metrics are always calculated from the compatibility condition. This means that, in general, the metrics do not evolve by diffeomorphism, i.e., $g'_\tau \neq (\varphi_\tau^{-1})^* g_0$.

Example 2.22 (The free particle and the Kähler structure of $\mathbb{R}^2 \cong \mathbb{C}$). *Consider again the case of Example 2.14, where $X = y\frac{\partial}{\partial x}$ represents the free particle motion on $\mathbb{R}^2 \cong \mathbb{C}$ with standard Kähler structure (Ω, J). Let φ_τ be the complex flow of X at time $\tau = t + is$. We* *have calculated*

$$\varphi_\tau \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ty \\ (1+s)y \end{pmatrix} \qquad d\varphi_\tau = \begin{pmatrix} 1 & t \\ 0 & 1+s \end{pmatrix}, \qquad d\varphi_\tau^{-1} = \frac{1}{1+s} \begin{pmatrix} 1+s & -t \\ 0 & 1 \end{pmatrix}.$$

The Kähler form induced by φ_τ is

$$\Omega_\tau = (\varphi_\tau^{-1})^* \Omega = (d\varphi_\tau^{-1})^T \Omega \, d(\varphi_\tau^{-1}) = \begin{pmatrix} 0 & \frac{1}{1+s} \\ -\frac{1}{1+s} & 0 \end{pmatrix}.$$

The compatibility condition between J and Ω_τ is the assertion that the following G_τ is a Riemannian metric

$$G_\tau = \Omega_\tau J_\tau = \begin{pmatrix} \frac{1}{1+s} & 0 \\ 0 & \frac{1}{1+s} \end{pmatrix}.$$

The induced Kähler structure that is the result of the complex evolution is (Ω_τ, J, G_τ).

5. Solving the geodesic problem in $\mathcal{H}$

We are now in conditions to show that pure imaginary time evolution of analytic real vector fields gives an analytic solution to the geodesic problem.

Theorem 2.23. *Let (ω, J_0) be a Kähler pair for the compact manifold M. Consider the complex flow φ_τ^T of the analytic and real Hamiltonian vector field $X \in \mathfrak{X}^\omega(M, \mathbb{R})$. Let ω_τ be the induced symplectic form, and consider all such forms for $\tau = it$, $0 \leq t < T$. Then $\{\omega_{it}\}_{t\in[0,T)}$ is a geodesic path in $\mathcal{H}(\omega, J_0)$.*

Proof. By Proposition 2.19, (ω_{it}, J_0) is a Kähler pair, and, as $\omega_{it} = \varphi_{it}^{-1}\omega$, and the complex flow is homotopic to the identity, the path $\{\omega_{it}\}_{t\in[0,T)}$ lies in $\mathcal{H}$. We only need to show that the path $\{\omega_{it}\}_{t\in[0,T)}$ satisfies the geodesic equation. If we write ∂_{it} and $\overline{\partial}_{it}$ for the Dolbeault operators with respect to the induced complex structure J_{it}, we have

$$\omega = i\partial_0\overline{\partial}_0\kappa_0$$
$$\omega = i\partial_{it}\overline{\partial}_{it}\kappa_{it}.$$

Observe that $\omega_{it} = \left(\varphi_{it}^{-1}\right)^* \omega = \omega + i\partial_0\overline{\partial}_0 u_t$ and

$$\left(\varphi_{it}^{-1}\right)^* \omega = \left(\varphi_{it}^{-1}\right)^* i\partial_{it}\overline{\partial}_{it}\kappa_{it} = i\partial_0\overline{\partial}_0\kappa_{it} \circ \varphi_{it}^{-1},$$

as the complex flow is a biholomorphism between (M, J_{it}) and (M, J_0). This means that

$$u_t = \kappa_{it} \circ \varphi_{it}^{-1} - \kappa_0. \tag{8}$$

Recall that, for complex $\tau \in D_T = \{\, \tau \in \mathbb{C} \mid |\tau| < T \,\}$, by Theorem 2.17, we have

$$\kappa_\tau = -2 \operatorname{Im} \psi_\tau,$$

with

$$\psi_\tau = \tau h - \frac{i}{2} e^{\tau X} \cdot \kappa_0 - r_\tau,$$

r_τ the analytic continuation in t of

$$r_t = \int_0^t e^{t' X} \cdot \theta(X_h) \, dt',$$

and $\theta = \frac{i}{2}(\partial_0 - \overline{\partial}_0)\kappa_0$.

Observing that we can write r_τ as a series, by expanding $e^{t'X}$ and performing term by term integration, we calculate

$$\begin{aligned}
\frac{dr_{it}}{dt} &= \frac{d}{dt}\left(\sum_{k=1}^{\infty}\frac{(it)^k}{k!}X_h^{k-1}(\theta(X_h))\right)\\
&= iX_h\left(\sum_{k=1}^{\infty}\frac{(it)^k}{k!}X_h^{k-1}(\theta(X_h))\right)+i\theta(X_h)\\
&= iX_h(r_{it})+i\theta(X_h).
\end{aligned}$$

The derivative of the Kähler potential is

$$\begin{aligned}
\frac{d\kappa_{it}}{dt} &= \frac{1}{2}X_h\left(ie^{itX_h}\cdot\kappa_0\right)+\frac{1}{2}X_h(ie^{-itX_h}\cdot\kappa_0)-2h-i\frac{dr_{it}}{dt}+i\frac{dr_{-it}}{dt}\\
&= -X_h\left(-\frac{i}{2}e^{itX_h}\cdot\kappa_0-\frac{i}{2}e^{-itX_h}\cdot\kappa_0-r_{it}-r_{-it}\right)-2h+2\theta(X_h)\\
&= -X_h(\psi_{it}+\overline{\psi}_{it})-2h+2\theta(X_h).
\end{aligned}$$

Recalling the action of the complex flow in coordiantes, equation (8) is written

$$u_t(z,\overline{z}) = \kappa_{it}\left(e^{-itX_h}z, e^{itX_h}\overline{z}\right)-\kappa_0(z,\overline{z}).$$

Decomposing the differential of κ_{it} into its J_{it}-holomorphic and anit-holomorphic parts, we get

$$\dot{u}_t = \dot{\kappa}_{it}+\partial_{it}\kappa_{it}(-iX_h)+\overline{\partial}_{it}\kappa_{it}(iX_h).$$

Substituting back $\dot{\kappa}_{it}$ and using $X_h(\psi_{it}) = d\psi_{it}(X_h) = \partial_{it}\psi_{it}X_h+\overline{\partial}_{it}\psi_{it}X_h$ and $\theta = \partial_{it}\overline{\psi}_{it}+\overline{\partial}\psi_{it}$, we obtain

$$\dot{u}_t = -2h\circ\varphi_{it}^{-1}. \tag{9}$$

The time derivative satisfies the expression

$$\ddot{u}_t\circ\varphi_{it} = 2i\partial_{it}h(X_h)-2i\overline{\partial}_{it}h(X_h) = 2dh(J_{it}X_h) = 2\omega(X_h,J_{it}X_h) = 2|X_h|^2_{g'_{it}} = 2|dh|^2_{g'_{it}},$$

with $g'_{it}(\cdot,\cdot) = \omega(\cdot,J_{it}\cdot)$ such that $g'_{it} = \varphi_{it}^* g_{it}$, where $g_{it}(\cdot,\cdot) = \omega_{it}(\cdot,J_0\cdot)$. Note, that the last equality holds for any hamiltonian vector field, on a Kähler manifold. On the other hand, taking the gradient with respect to g_{it} of equation (9) we get

$$\nabla^{g_{it}}\dot{u}_t = -2\nabla^{g_{it}}\left(h\circ\varphi_{it}^{-1}\right) = -2\nabla^{g'_{it}}h\circ\varphi_{it}^{-1}.$$

Considering again that $|\nabla^{g'_{it}}h|^2_{g'_{it}} = |dh|^2_{g'_{it}}$, we get the equality

$$\ddot{u}_t\circ\varphi_{it} = \frac{1}{2}|\nabla^{g_{it}}\dot{u}_t|_{g_{it}}\circ\varphi_{it},$$

from which the geodesic equation follows. □

6. Complex evolution as a system of differential equations

Throughout the chapter we have seen how to transform the motivation problem of chapter 1 into a question on complexified dynamics. We did that by introducing the evolution operator $e^{\tau X}$ and using its action on holomorphic functions to build a diffeomorphism of the manifold. We will now briefly present another approach found in [**BLU13**]. This point of view complements the one developed here, and provides it further flexibility.

THEOREM 2.24. *Let (ω, J_0) be a Kähler pair for a compact manifold M. Consider the complex valued real-analytic Hamiltonian $h \in C^\omega(M, \mathbb{C})$, and the Hamiltonian vector fields $X_{\operatorname{Re} h}$ and $X_{\operatorname{Im} h}$ of the real and imaginary parts of h. Then there exists a family of complex structures J_t on M, for t in some open interval $E \in \mathbb{R}$, such that*

$$\dot{J}_t = -\mathcal{L}_{X_t} J_t, \qquad X_t = X_{\operatorname{Re} h} + J_t X_{\operatorname{Im} h}. \tag{10}$$

Also, the flow ϕ_t of the time dependent vector field X_t is holomorphic for all $t \in E$ if seen as a map $\phi_t : (M, J_0) \to (M, J_t)$, i.e.

$$\phi_t^* J_t = J_0, \qquad \dot{\phi}_t \circ \phi_t^{-1} = X_t.$$

The proof of this theorem is done by explicitly building the diffeomorphism ϕ_t. Firstly, the authors introduce a complexification $M_\mathbb{C}$ of the manifold M. The map ϕ_t is then a projection of an appropriate flow in $M_\mathbb{C}$. Although the local existence of J_t is guaranteed by the Cauchy-Kovalevsky theorem, because (10) is a non-linear first-order system of partial differential equations with analytic coefficients, they use the geometry of $M_\mathbb{C}$ to prove the (global) existence of J_t. It is also possible to write the equations in local coordinates.

THEOREM 2.25. *Let J_t denote the complex structure of Theorem 2.24 and also its matrix representation in local coordinates $\{x_k\}_{1 \le k \le 2n}$. Consider the Hamiltonian $h \in C^\omega(M, \mathbb{C})$, the Hamiltonian vector fields $X_{\operatorname{Re} h}$ and $X_{\operatorname{Im} h}$, and $X_t = X_{\operatorname{Re} h} + J_t X_{\operatorname{Im} h}$. In local coordinates, the equation (10) is*

$$\dot{J}_t(x) = [X_t'(x), J_t(x)] - \sum_{k=1}^{2n} X_t^k(x) \frac{\partial J_t(x)}{\partial x_k},$$

where $X_t^k(x)$ are the components of X_t and $X_t'(x) = D_x X_t = \left(\frac{\partial X_t^j(x)}{\partial x_k}\right)_{jk}$.

It is possible to simplify the expression, and, in the case of quadratic Hamiltonians in $\mathbb{R}^{2n}$ this system of partial differential equations becomes a system of ordinary differential equations.

COROLLARY 2.26. *If $M = \mathbb{R}^{2n}$, $h \in C^\omega(M, \mathbb{C})$ is quadratic, and J_t is independent of the x coordinates, the evolution of J_t in Theorem 2.25 is written*

$$\dot{J}_t = [\Omega(\operatorname{Re} h)'' + J_t \Omega(\operatorname{Im} h)'', J_t],$$

where Ω is the standard symplectic form and $(\operatorname{Re} h)''$ and $(\operatorname{Im} h)''$ are the hessians of the real and the imaginary parts of the Hamiltonian.

It was proved in [**MN15**] that this approach is equivalent to the one developed throughout the chapter, for real-valued vector fields and pure imaginary time. The complex flow φ_{it} as we have defined it is exactly the inverse map ϕ_t^{-1} of Theorem 2.24.

CHAPTER 3

Complex evolution on (real) surfaces

1. Local Kähler geometry of complex evolution

In this chapter we will study with some detail examples of complex evolution in surfaces. We start by recalling some relations for Kähler manifolds of real dimension $2n$, and how they simplify in the case of surfaces. Remember that the classical local relations in Kähler Geometry all come from the fact that the metric is completely specified by the Kähler potential. This means that the Kähler structure is extremely more rigid than a typical Riemannian one. In general, a Riemannian metric depends on $n(2n+1)$ local functions, having much more freedom than a Kähler structure.

Let M be a $2n$-dimensional real manifold with Kähler structure (ω, J, g). Consider the local holomorphic coordinates $\{z_j\}_{1\leq j\leq n}$. Recall that the J tensor is diagonal in these coordinates,

$$J = \sum_{j=1}^{n}\left(i\frac{\partial}{\partial z^j}\otimes dz^j - i\frac{\partial}{\partial \overline{z}^j}\otimes d\overline{z}^j\right).$$

If the metric tensor is locally $g = \sum_{j,k=1}^{n} g_{jk}\, dz^j d\overline{z}^k$, by the compatibility condition, the Kähler form is simply

$$\omega = i\sum_{j\leq k=1}^{n} g_{jk}\, dz^j\wedge d\overline{z}^k. \tag{11}$$

If now we let (ω, J_0, g_0) be an initial Kähler structure on M, $X\in\mathfrak{X}^\omega(M,\mathbb{C})$ be a vector field and assume its complex flow exists φ_τ^T for $|\tau|<T$, for some $T>0$, we can get the induced complex structure as $J_\tau = \varphi_\tau^* J_0$ by remembering the action of φ_τ on coordinates. That is, in J_0-holomorphic coordinates $\{z_0^j\}_{1\leq j\leq n}$, we have, by Corollary 2.15, for each $1\leq j\leq n$

$$\begin{aligned} z_\tau^j &= \varphi_\tau^* z_0^j = e^{\tau X}\cdot z_0^j \\ \overline{z}_\tau^j &= \varphi_\tau^* \overline{z}_0^j = e^{\overline{\tau}\overline{X}}\cdot \overline{z}_0^j. \end{aligned} \tag{12}$$

As the complex structure J_τ is precisely the one for which these functions are holomorphic coordinates, J_τ is diagonal when written with respect to them, i.e.,

$$J_\tau = \sum_{j=1}^{n}\left(i\frac{\partial}{\partial z_\tau^j}\otimes dz_\tau^j - i\frac{\partial}{\partial \overline{z_\tau}^j}\otimes d\overline{z}_\tau^j\right). \tag{13}$$

Using the expressions (12) above we can get the expression of J_τ in the J_0-holomorphic coordinates. We first find the forms $\{dz_\tau^j, d\overline{z}_\tau^j\}_{1\leq j\leq n}$. This set is, at every point p, a basis for $T_p^*M\otimes\mathbb{C}$. We then find its dual basis $\{\frac{\partial}{\partial z_\tau^j}, \frac{\partial}{\partial \overline{z}_\tau^j}\}_{1\leq j\leq n}$, and calculate J_τ. For the purposes of this section we will work in the symplectic picture. This means that the symplectic form is kept fixed. For $|\tau|<T$, we can write it in J_0-holomorphic coordinates and use

the compatibility condition to get the metric g_τ. We have then locally characterised the compatible triple (ω, J_τ, g_τ) which is the result of the complex evolution we are considering.

Let us work through the special case of Riemann surfaces (which are, in fact, all oriented two-manifolds). All such surfaces are Kähler. So, consider z_0 to be a J_0-holomorphic coordinate on some open subset U of the two-dimensional manifold M, with Kähler strucuture (ω, J_0). Let φ_τ be the complex flow of a vector field $X \in \mathfrak{X}^\omega(M, \mathbb{C})$. The induced complex strucutre J_τ has z_τ as J_τ-holomorphic coordiante. In J_0-holomorphic coordinates we write $z_\tau = \varphi_\tau^* z_0 = f_0(z_0, \overline{z}_0)$ and $\overline{z}_\tau = \varphi_\tau^* \overline{z}_0 = \overline{f}_0(z_0, \overline{z}_0)$. With his in mind, we get the expressions

$$
\begin{aligned}
z_\tau &= f_0(z_0, \overline{z}_0), \qquad & dz_\tau &= \frac{\partial f_0}{\partial z_0} dz_0 + \frac{\partial f_0}{\partial \overline{z}_0} d\overline{z}_0, \\
\overline{z}_\tau &= \overline{f}_0(z_0, \overline{z}_0), \qquad & d\overline{z}_\tau &= \frac{\partial \overline{f}_0}{\partial z_0} dz_0 + \frac{\partial \overline{f}_0}{\partial \overline{z}_0} d\overline{z}_0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z_\tau} &= F_0^{-1}\left(\frac{\partial}{\partial z_0}\right) = \frac{1}{\det(F_0)}\left(\frac{\partial \overline{f}_0}{\partial \overline{z}_0}\frac{\partial}{\partial z_0} - \frac{\partial \overline{f}_0}{\partial z_0}\frac{\partial}{\partial \overline{z}_0}\right), \\
\frac{\partial}{\partial \overline{z}_\tau} &= F_0^{-1}\left(\frac{\partial}{\partial \overline{z}_0}\right) = \frac{1}{\det(F_0)}\left(-\frac{\partial f_0}{\partial \overline{z}_0}\frac{\partial}{\partial z_0} + \frac{\partial f_0}{\partial z_0}\frac{\partial}{\partial \overline{z}_0}\right),
\end{aligned}
$$

where F_0 is the (complex) Jacobian of f_0 in $\{z_0, \overline{z}_0\}$ coordinates

$$
F_0 = \begin{pmatrix} \frac{\partial f_0}{\partial z_0} & \frac{\partial f_0}{\partial \overline{z}_0} \\ \frac{\partial \overline{f}_0}{\partial z_0} & \frac{\partial \overline{f}_0}{\partial \overline{z}_0} \end{pmatrix}.
$$

The J_τ tensor can then be found by using expression (13). In matrix notation this is (recall that $J_\tau = \varphi_\tau^* J_0$)

$$
J_\tau = F_0^{-1} J_0 F_0 = F_0^{-1} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} F_0.
$$

Having the symplectic form ω written in J_0-holomorphic coordinates we can use the compatibility condition to get the metric g_τ. If $(\omega_{j,k}) = \Omega$ and $(g_{\tau j,k}) = G_\tau$ this is simply,

$$
G_\tau = \Omega J_\tau.
$$

We have found a detailed description of the triple (ω, J_τ, g_τ) in J_0-coordinates.

Of course, one might simply be interested in the metric properties of g_τ, or one might be wanting to work with real coordinates. In these cases there is a simple trick. Choose Darboux coordinates $\{x, y\}$ for (M, ω). Pretty much all of the previous analysis is the same with z_0 and $\overline{z}_0$ replaced by x and y. The exception is that the J_0 is no longer diagonal in these coordinates, but instead we have $\omega = dx \wedge dy$. Writing the new coordinates as $z_\tau = f_0(z_0, \overline{z}_0) = f(x, y)$, we then have

$$
dz_\tau \wedge d\overline{z}_\tau = \det(F)\, dx \wedge dy = \det(F)\, \omega, \qquad F = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial \overline{f}}{\partial x} & \frac{\partial \overline{f}}{\partial y} \end{pmatrix}.
$$

We can now invert the expression to get ω in J_τ-holomorphic coordinartes. But, in this system of coordinates, the compatibility condition (11) gives the tensor g_τ directly,

$$
\omega = \frac{1}{\det(F)} = i\frac{-i}{\det(F)} \Longrightarrow g = \frac{-i}{\det(F)} dz_\tau \otimes d\overline{z}_\tau + \frac{i}{\det(\overline{F})} d\overline{z}_\tau \otimes dz_\tau.
$$

It is now only necessary to expand the tensor products, written in terms of Darboux coordinates.

$$dz_\tau \otimes d\overline{z}_\tau = \left|\frac{\partial f}{\partial x}\right|^2 dx \otimes dx + \left|\frac{\partial f}{\partial y}\right|^2 dy \otimes dy + \frac{\partial f}{\partial x}\frac{\partial \overline{f}}{\partial y} dx \otimes dy + \frac{\partial \overline{f}}{\partial x}\frac{\partial f}{\partial y} dy \otimes dx.$$

Noting that $d\overline{z}_\tau \otimes dz_\tau = \overline{dz_\tau \otimes d\overline{z}_\tau}$ and that $\det \overline{F} = -\det F$, we calculate

$$g_\tau = \frac{2i}{\det(F)}\left(\left|\frac{\partial f}{\partial x}\right|^2 dx^2 + \left|\frac{\partial f}{\partial y}\right|^2 dy^2 + 2\operatorname{Re}\left(\frac{\partial f}{\partial x}\frac{\partial \overline{f}}{\partial y}\right) dxdy\right),$$

or, in matrix form,

$$G_\tau = \frac{2i}{\det(F)}\begin{pmatrix} \left|\frac{\partial f}{\partial x}\right|^2 & \operatorname{Re}\left(\frac{\partial f}{\partial x}\frac{\partial \overline{f}}{\partial y}\right) \\ \operatorname{Re}\left(\frac{\partial f}{\partial x}\frac{\partial \overline{f}}{\partial y}\right) & \left|\frac{\partial f}{\partial y}\right|^2 \end{pmatrix}, \tag{14}$$

where, recall, F is the Jacobian of $z_\tau = f(x, y)$. The J_τ tensor can now be recovered from the compatibility condition.

With these set of expressions we are now ready to study some examples.

2. A damped harmonic oscilator

Recall that the theory developed above for complex flows does hold locally. The compactness hypothesis on the manifold ensures mainly that the objects, such as the flow or the evolved structures, are globally defined. We will now study a situation in $\mathbb{R}^2 \cong \mathbb{C}$ where the constructions will happen to be globally defined. This is the case of a complex flow for a particular complex Hamiltonian vector field. We will work with real time $t \in \mathbb{R}$. Throughout the rest of the chapter we make use of the software *Mathematica*®[**WR18**] for symbolic and graphic manipulation. This example will mostly serve to illustrate complex evolution. As so, we will work with the complex structure even when it is not compatible with the symplectic form.

Consider then $\mathbb{R}^2 \cong \mathbb{C}$ with standard symplectic structure $\omega = dx \wedge dy$ and the complex structure J_0, defined by the holomorphic coordinate $z = \alpha x + \beta y$, with $\alpha\overline{\beta} - \beta\overline{\alpha} \neq 0$. Note that this is a globally defined function (because x, y are global coordinates). Its differential is

$$F = \begin{pmatrix} \alpha & \beta \\ \overline{\alpha} & \overline{\beta} \end{pmatrix}.$$

The condition on α and β is precisely for the determinant of the differential to be non vanishing, so the function that defines z is a linear diffeomorphism. This means that z is indeed a complex coordinate and the J_0 tensor in $\{x, y\}$ coordinates is

$$J_0 = F^{-1}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} F = \frac{2i}{\det F}\begin{pmatrix} |\alpha|^2 & \operatorname{Re}(\alpha\overline{\beta}) \\ -\operatorname{Re}(\alpha\overline{\beta}) & -|\beta|^2 \end{pmatrix}. \tag{15}$$

Observe that the condition for the coordinates to be well defined, i.e, $\det F = \alpha\overline{\beta} - \beta\overline{\alpha} \neq 0$, implies that $\alpha \neq 0$. If we rescale the coordinates by a factor of $1/\alpha$ we get the same complex structure J_0, as one can see from equation (15). By this argument, a complex structure of the type considered will only depend on $\rho_0 = \frac{\beta}{\alpha}$. Let us then set $z_0 = x + \rho_0 y$, $\operatorname{Im}(\rho_0) \neq 0$. Notice that (ω, J_0) is a compatible Kähler pair only if $\operatorname{Im}(\rho_0) > 0$. We will, nevertheless, consider both the cases $\operatorname{Im}(\rho_0) > 0$ and $\operatorname{Im}(\rho_0) < 0$.

Consider now the Hamiltonian function

$$h(x,y) = \frac{\omega^2}{2}x^2 + \frac{\delta^2}{2}y^2$$

with $\omega \in \mathbb{R}$ and $\delta = a + ib \in S^1$ a unit modulus complex number, with non-zero imaginary part, $b \neq 0$. This is a Hamiltonian function which was briefly considered in [**GS12**], in the context of quantum evolution of coherent states. It is a harmonic oscillator where a fraction of its kinetic energy is considered to be an imaginary quantity. One expects that the complex flow will have some resemblance with the usual dynamics. The imaginary part of h will, in principle and for appropriate sign of $\operatorname{Im}\delta$, act as a kind of friction or dissipation term. The associated Hamiltonian vector field is

$$X_h = \delta^2 y\frac{\partial}{\partial x} - \omega^2 x\frac{\partial}{\partial y}.$$

As the vector field is linear, the Lie Series $e^{\tau X_h}$ acts just as the usual exponential series for the matrix

$$H = \begin{pmatrix} 0 & \delta^2 \\ -\omega^2 & 0 \end{pmatrix},$$

which can be calculated by the usual closed expression [**EN00**]. For $\tau = t \in \mathbb{R}$ we have

$$e^{tH} = \begin{bmatrix} \cos(\omega\delta t) & \frac{\delta}{\omega}\sin(\omega\delta t) \\ -\frac{\omega}{\delta}\sin(\omega\delta t) & \cos(\omega\delta t) \end{bmatrix}.$$

The action of e^{tX_h} on each of the coordinate functions is just

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto e^{tH}\begin{pmatrix} x \\ y \end{pmatrix},$$

reading

$$\begin{aligned} x &\mapsto \cos(\omega\delta t)x + \frac{\delta}{\omega}\sin(\omega\delta t)y \\ y &\mapsto -\frac{\omega}{\delta}\sin(\omega\delta t)x + \cos(\omega\delta t)y \end{aligned}$$

as a map on functions. So, the new complex coordinate is

$$z_t = e^{tX_h}\cdot z_0 = e^{tX_h}\cdot(x+\rho_0 y),$$

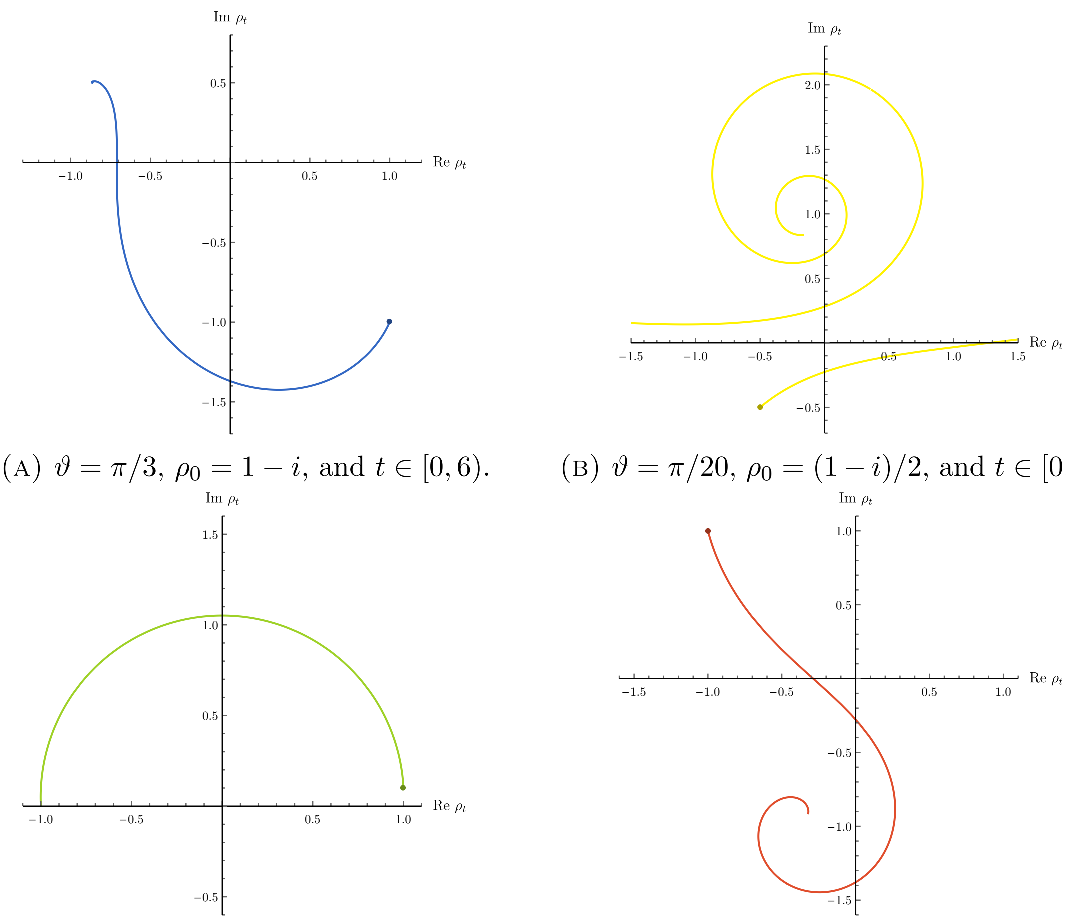
$$z_t = \left(\cos(\omega\delta t) - \rho_0\frac{\omega}{\delta}\sin(\omega\delta t)\right)x + \left(\frac{\delta}{\omega}\sin(\omega\delta t) + \rho_0\cos(\omega\delta t)\right)y. \tag{16}$$

The fact that the vector field is linear, which is a consequence of the Hamiltonian being quadratic, guarantees that the coordinate z_t is of the form $z_t = \alpha_t x + \beta_t y$, with $\alpha_t = \cos(\omega\delta t) - \rho_0\frac{\omega}{\delta}\sin(\omega\delta t)$ and $\beta_t = \frac{\delta}{\omega}\sin(\omega\delta t) + \rho_0\cos(\omega\delta t)$. As so, we could calculate the tensor J_t directly from (15), with α replaced by α_t and β by β_t. Then one could get the Kähler metric, for small values of t and starting compatible J_0, either by the compatibility condition or by using (14) directly. Note that none of these structures will depend explicitly on the coordinates x and y. Under the interpretation of the complex evolution as a system of differential equations, that we briefly presented at the end of chapter 2, this statement is obvious. This is because the Hamiltonian is quadratic and the starting J_0 does not depend on x and y. Corollary 2.26 states precisely that, in this case, the evolution is given by a system of ordinary differential equations.

The equations for the tensors give intricate expressions. Another approach to study the complex structure would be to normalise the coordinate by the same method used for z_0. This can still be done, because for small enough t, $\alpha_t \neq 0$, as we have $\alpha_0 \neq 0$ and α_t is a continuous function of t. We will abstain from doing that explicitly, and keep our coordinate as defined. Nonetheless it is still relevant to study the quotient $\rho_t = \frac{\beta_t}{\alpha_t}$, because, as we have noticed, it completely defines the complex structures of the type considered. The expression for ρ_t is

$$\rho_t = \frac{\frac{\delta}{\omega}\sin(\omega\delta t) + \rho_0\cos(\omega\delta t)}{\cos(\omega\delta t) - \rho_0\frac{\omega}{\delta}\sin(\omega\delta t)}. \tag{17}$$

As this complex number alone contains all the information about the complex structure, it is a remarkable fact that the evolution of the tensor J_t can be *visualized*. This is achieved by plotting ρ_t as a curve in the complex plane as in Figure 2. Have in mind that *a priori* the curve explicitily depends on the starting point ρ_0 that corresponds to the initial complex structure.



(A) $\vartheta = \pi/3$, $\rho_0 = 1 - i$, and $t \in [0, 6)$.

(B) $\vartheta = \pi/20$, $\rho_0 = (1-i)/2$, and $t \in [0, 10)$.

(C) $\vartheta = \pi/2$, $\rho_0 = 1 - 0.1i$, and $t \in [0, 10)$.

(D) $\vartheta = 7\pi/8$, $\rho_0 = i - 1$, and $t \in [0, 5)$.

FIGURE 2. The evolution of the structures defined by ρ_t under the complex dynamics. The parametric description is given by (17). The starting complex structure ρ_0 is marked by a darker point. The oscillator frequence is $\omega = 1$ and the damping parameter is $\delta = e^{i\vartheta}$.

Note that when $\mathrm{Im}(\rho_t) = 0$, i.e., when the curves drawn cross the horizontal axis, the coordinate becomes real valued. In this case, it no longer defines a complex structure. Moreover, we are not always considering values of ρ_t that define complex structures that are compatible with the symplectic form. In particular, after a curve crosses the x-axis, a sign change happens in the differential of z_t. This means that the metric tensor (14) changes from positive definite to negative definite. For instance, making $\delta = e^{i\pi/4}$, and looking at the imaginary part, one has, after expansion, the expression (24) in Appendix B, whose graphs for some values of ρ_0 are presented below.

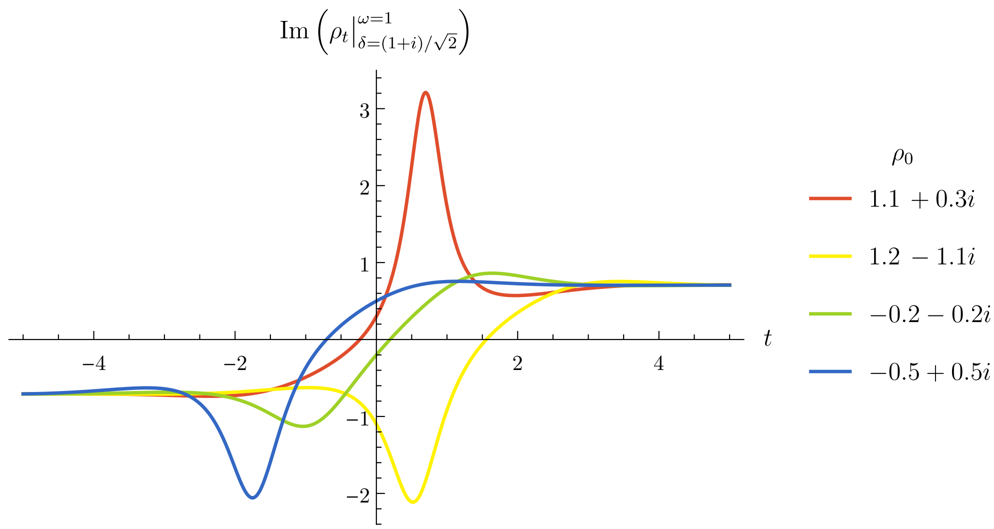


FIGURE 3. Graphs of the imaginary parts of ρ_t for some starting complex structures ρ_0. The pictures were drawn for $\omega = 1$ and $\delta = (1+i)/\sqrt{2}$.

Observe that there are similarities between the appearances of $\mathrm{Im}\,(\rho_t)$ for different values of ρ_0. In particular, it seems that the time evolution takes different starting structures to closer ones. One can see this by noting the asymptotic behaviour for both positive and negative time. It is worth asking if this is a property induced by the particular choice of ω and δ, or a global feature of the curves ρ_t, common to bigger parameter domains. For that, let us trace several curves with different starting complex structures ρ_0, cf. Figure 4.

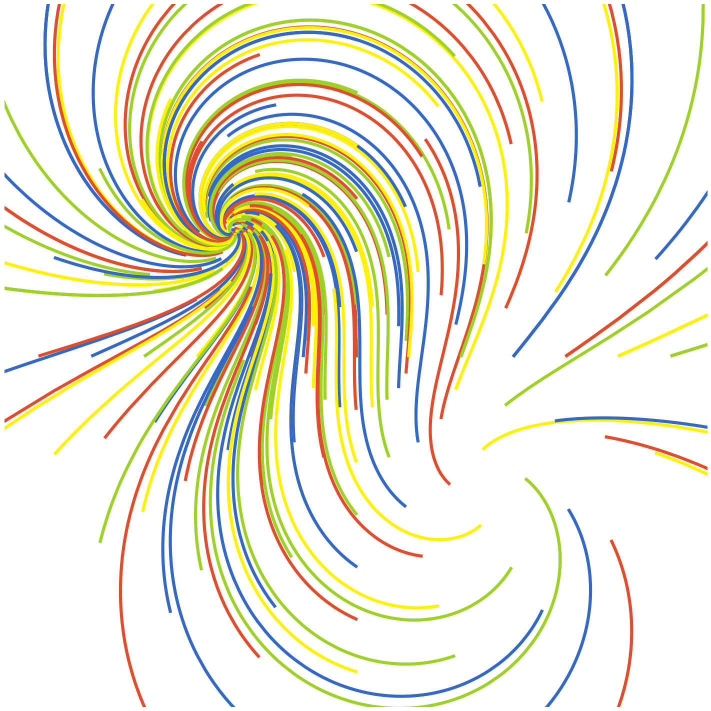
A closer look at these pictures hints at a strange behaviour near two points, one on the second quadrant and the other on the fourth. Lets call them $\rho_0^\times$. These seem to work by attracting or repelling the trajectories. In particular, the points itself seem to be stationary for the complex dynamics. We can use the fixed point condition to search for them. Setting,

$$\rho_t = \rho_0^\times \quad\iff\quad \frac{\frac{\delta}{\omega}\sin(\omega\delta t) + \rho_0^\times\cos(\omega\delta t)}{\cos(\omega\delta t) - \rho_0^\times\frac{\omega}{\delta}\sin(\omega\delta t)} = \rho_0^\times$$

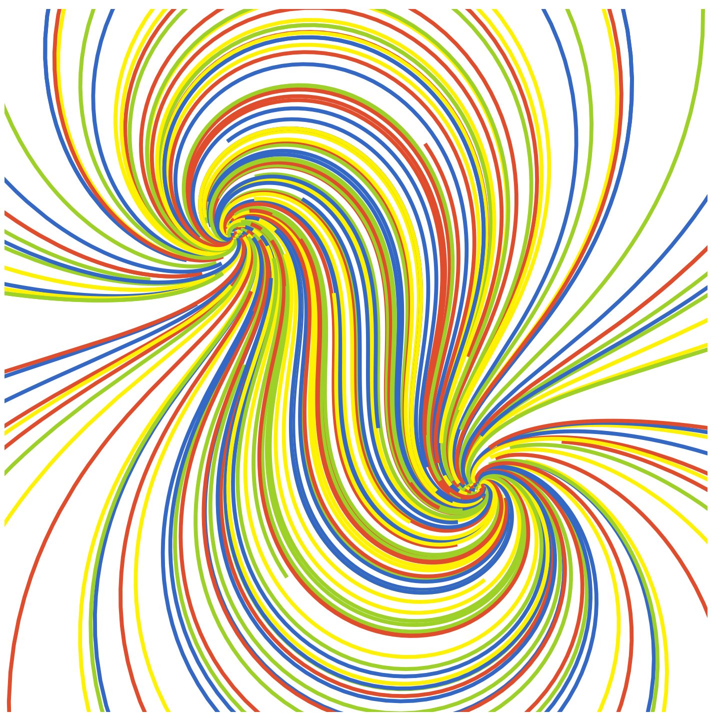
for arbitrary parameters, one finds the values

$$\rho_0^\times = \pm i\frac{\delta}{\omega}.$$

If the starting condition is precisely $\rho_0 = \rho_0^\times$, the complex structure is kept the same! This is a remarkable property of the complex flow.



(A) $\vartheta = \pi/4$ and $t \in [0, 2)$.



(B) $\vartheta = \pi/4$ and $t \in [-2, 2)$.

FIGURE 4. Evolution of several complex strucutres by the complex flow. Each curve is given by (17), with $\omega = 1$, and $\delta = e^{i\vartheta}$.
The starting complex structures ρ_0 take values on the radial grid $\Lambda = \{\, re^{i\eta} \mid 0 \le r = 0.3n < 2 \text{ and } 0 \le \eta = m\pi/10 < 2\pi, n, m \in \mathbb{N}_0 \,\}$.
Both pictures are plots of the square $[-2, 2] \times i[-2, 2]$, whose center is the origin.

Locally the action on the complex coordinate is given by equation (16) with $\rho_0 = \pm i\frac{\delta}{\omega}$, i.e.,

$$z_t^\times = e^{\mp i\omega\delta t} z_0^\times = e^{\mp i\omega\delta t}\left(x + \rho_0^\times y\right).$$

This action is just a rescaling by a complex factor which preserves the complex structure tensor $J_t = J_0^\times$. In this case, the complex evolution takes the Kähler pair $(\omega, J_0^\times)$, for $\operatorname{Im}\rho_0^\times > 0$, to itself. This is similar to what happens with the usual flow of a real vector field, it is independent of the complex structure. As the expression for $\varphi_t^\times$ is intricate, we will instead present some of its trajectories.

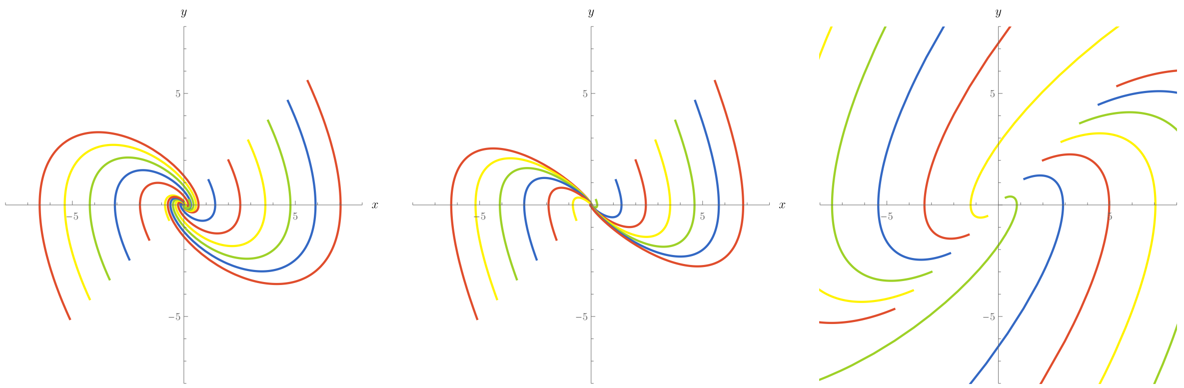


FIGURE 5. Some trajectories under $\varphi_t^\times$, for $\omega = 1$, $\delta = e^{i\vartheta}$ with $\vartheta = -\frac{\pi}{5}, -\frac{\pi}{3}$ and $\frac{\pi}{4}$, and $t \in [0, 7)$.The first two sets of curves flow inwards to the origin, while the last one flows outwards.

Recall that under the usual flow of the harmonic oscillator the trajectories are ellipses. So, in the fixed point, the complex flow $\varphi_t^\times$ behaves as expected. The imaginary part of δ controls the dissipation of energy. This can be seen in the pictures, where a change in δ altered the properties of the curves. The more negative $\operatorname{Im}\delta$, the stronger the damping, and the faster the trajectories approach the origin. On the other hand, a sign reversion in $\operatorname{Im}\delta$ turns the system, from a damped oscillator, to a continuously forced one. The trajectories in this case move away from the origin.

Under the interpretation of the complex flow as a solution of a system of differential equations, one can use Corollary 2.26 to show that $\varphi_t = x(t)$ satisfies exactly the equation

$$\ddot{x} - 2\omega \operatorname{Im}\delta\, \dot{x} + \omega^2 x = 0,$$

as first observed in [**GS12**]. Also using this interpretation, in particular Theorem 2.24, one finds that $X_t = X_0 = \operatorname{Re} X_h + J_0^\times \operatorname{Im} X_h$ is a symmetry of the tensor $J_0^\times$, in the sense that $\mathcal{L}_{X_0} J_0^\times = 0$. A matrix expression for $J_0^\times$ is found using (15)

$$J_0^\times = \begin{pmatrix} \frac{b}{a} & \mp\frac{1}{\omega a} \\ \pm\frac{\omega}{a} & \frac{b}{a} \end{pmatrix},$$

where $a = \operatorname{Re}(\delta)$ and $b = \operatorname{Im}(\delta)$.

3. The generator of rotations on the sphere

Let us now study a complex flow on the sphere. Consider the two-sphere S^2 as a subset of $\mathbb{R}^3$, $S^2 = \{\, (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \,\}$, with standard Kähler structure. We will use $\{\hat{x}, \hat{y}\}$ as the coordinates for stereographic projection from the north pole, and $\{z, \theta\}$ as the standard cylindrical coordinates. The standard complex structure has holomorphic coordinate

$$\mathfrak{z} = \hat{x} + i\hat{y} = \sqrt{\frac{1+z}{1-z}}e^{i\theta}$$

defined on the appropriate open sets. Consider the Hamiltonian given by the height function

$$h(z, \theta) = z.$$

Recalling that $\{z, \theta\}$ are Darboux coordinates for the sphere, i.e., $\omega = dz \wedge d\theta$, we calculate the corresponding Hamiltonian vector field

$$X_h = -\frac{\partial}{\partial \theta}.$$

As $X_h^k(\theta) = 0$ for $k > 2$, the Lie Series for the complex coordinate $\mathfrak{z}$ acts as

$$\mathfrak{z}_\tau = e^{\tau X_h} \cdot \mathfrak{z} = \sqrt{\frac{1+z}{1-z}}e^{i(\theta-\tau)} = e^{-i\tau}\mathfrak{z},$$

defining the complex flow φ_τ as usual. We see that $\mathfrak{z}_\tau$ is a holomorphic function of $\mathfrak{z}$ and therefore, using the same arguments as before, one concludes that this is also a complex flow that keeps the complex structure constant. For pure imaginary time $\tau = it$ this flow acts as

$$\sqrt{\frac{1+z}{1-z}}e^{i\theta}e^t,$$

or, using the stereographic projection

$$\begin{aligned} \varphi_{it} &: S^2 \to S^2 \\ &\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \mapsto e^t \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}. \end{aligned}$$

The trajectories of this flow in pure imaginary time are straight lines in the $\hat{x}\hat{y}$-plane, which are meridians of the sphere. We encounter, once again, the idea that a complex flow is sometimes a rotated version of the usual flow, whose trajectories in this case would be parallels, i.e, circles of latitude, on the sphere.

4. Another Hamiltonian on the sphere

Consider now the same exact setup, but with starting coordinate

$$\mathfrak{z}_0 = \mathfrak{z} + \alpha(1-z)\overline{\mathfrak{z}} = \sqrt{\frac{1+z}{1-z}}\left[e^{i\theta} + \alpha(1-z)e^{-i\theta}\right],$$

where $\alpha \in \mathbb{R}$ is a real parameter such that $|\alpha| < 1/2$. If $\alpha = 0$ we recover the usual complex structure on the sphere given by $\mathfrak{z}$. Note that, in the previous examples, we were using standard coordinate systems, or, in the case of $\mathbb{R}^2 \cong \mathbb{C}$, linear ones. In this situation it is not even clear that the function $\mathfrak{z}_0$ defines a complex coordinate system, let alone a system that is compatible with the coordinates $\{z, \theta\}$. So, let us start by proving that this is indeed the case. Observe that, using the usual coordinates $\{\hat{x}, \hat{y}\}$ for stereographic projection from the north pole $N \in S^2$ we can write $\mathfrak{z}_0$ as

$$\begin{aligned} \mathfrak{z}_0 &= \mathfrak{z} + \alpha(1-z)\overline{\mathfrak{z}} \\ &= (\hat{x} + i\hat{y}) + \alpha(1-z)(\hat{x} - i\hat{y}), \end{aligned} \qquad \text{on } S^2 \setminus N \tag{18}$$

With $\mathfrak{w}$ the complementary coordinate of $\mathfrak{z}$, we can set

$$\mathfrak{w}_0 = \frac{1}{\mathfrak{z}_0} = \frac{|\mathfrak{w}|^2}{\mathfrak{w} + \alpha(1-z)\mathfrak{w}}, \qquad \text{on } S^2 \setminus \{S, N\},$$

where $S \in S^2$ is the south pole. We can then try to extend $\mathfrak{w}_0$ in a smooth fashion to N. This is

$$\begin{aligned} \mathfrak{w}_0 &= \frac{|\mathfrak{w}|^2}{\mathfrak{w} + \alpha(1-z)\mathfrak{w}} \\ &= \frac{\breve{x}^2 + \breve{y}^2}{(\breve{x} + i\breve{y}) + \alpha(1-z)(\breve{x} - i\breve{y})}, \end{aligned} \qquad \text{on } S^2 \setminus S,$$

where $\{\breve{x}, -\breve{y}\}$ are the coordinates for stereographic projection from the south pole. We thus want to prove that $\mathscr{A} = \{(\mathfrak{z}_0, S^2 \setminus N), (\mathfrak{w}_0, S^2 \setminus S)\}$ is a complex atlas that is C^∞-compatible with the standard one. Note, that, by the definition of $\mathfrak{w}_0$, the transition function is the biholomorphism $\frac{1}{\mathfrak{z}_0}$, so we only need to check that $\mathfrak{z}_0$ and $\mathfrak{w}_0$ are diffeomorphisms, in particular that $\mathfrak{w}_0$ can be in fact extended to the north pole. Let us write $\mathfrak{z}_0 = \hat{x}_0 + i\hat{y}_0$. With this notation, equation (18) is equivalent to

$$\begin{cases} \hat{x}_0 = [1 + \alpha(1-z)]\,\hat{x} \\ \hat{y}_0 = [1 - \alpha(1-z)]\,\hat{y}. \end{cases}$$

This establishes a diffeomorfism $\psi : \mathbb{R}^2 \to \mathbb{R}^2$, $(\hat{x}, \hat{y}) \mapsto (\hat{x}_0, \hat{y}_0)$, that for $|\alpha| < 1/2$ has inverse

$$\psi^{-1} : \begin{cases} \hat{x}_0 = \frac{1}{1+\alpha(1-z)}\hat{x} \\ \hat{y}_0 = \frac{1}{1-\alpha(1-z)}\hat{y}. \end{cases}$$

The coordinate $\mathfrak{z}_0$ is thus given by the composition of this diffeomorphism ψ and the standard chart chart for $\mathfrak{z}$. Because of this, it is C^∞-compatible with the standard atlas.

As $\mathfrak{w}_0 = \frac{1}{\mathfrak{z}_0}$, on $S^2 \setminus \{S, N\}$, we have that it is also a diffeomorphism in this region. We only need to show that it can be smoothly extended to the north pole. Writing $\mathfrak{w}_0$ using stereographic projection from the south, this is equivalent to say that the function

$$\mathfrak{w}_0 = \breve{x}_0 + i\breve{y}_0 = \frac{\breve{x}^2 + \breve{y}^2}{(\breve{x} + i\breve{y}) + \alpha(1-z)(\breve{x} - i\breve{y})}$$

is smooth at the origin. Simplifying $\mathfrak{w}_0$, one gets

$$\begin{aligned} \mathfrak{w}_0 &= \nu(\breve{x}, \breve{y})\left[\breve{x}\,(1 + \alpha(1-z)) + i\breve{y}\,(1 - \alpha(1-z))\right], \\ \nu(\breve{x}, \breve{y}) &= \frac{\breve{x}^2 + \breve{y}^2}{\breve{x}^2\,(1 + \alpha(1-z))^2 + \breve{y}^2\,(1 - \alpha(1-z))^2}. \end{aligned}$$

Noting that

$$z = \frac{1 - (\breve{x}^2 + \breve{y}^2)}{1 + \breve{x}^2 + \breve{y}^2}$$

we can simplify $\nu(\breve{x},\breve{y})$, obtaining

$$\nu(\breve{x},\breve{y}) = \frac{1}{1+\alpha^2(1-z)+4\alpha\frac{\breve{x}^2-\breve{y}^2}{1+\breve{x}^2-\breve{y}^2}}.$$

We can write $\mathfrak{w}_0$ as

$$\begin{cases} \breve{x}_0 = \frac{1+\alpha(1-z)}{1+\alpha^2(1-z)+4\alpha\frac{\breve{x}^2-\breve{y}^2}{1+\breve{x}^2-\breve{y}^2}}\breve{x} \\ \breve{y}_0 = \frac{1-\alpha(1-z)}{1+\alpha^2(1-z)+4\alpha\frac{\breve{x}^2-\breve{y}^2}{1+\breve{x}^2-\breve{y}^2}}\breve{y}. \end{cases}$$

This is a smooth function on every point of its domain. Its differential $\frac{D(\breve{x}_0,\breve{y}_0)}{D(\breve{x},\breve{y})}$ can be calculated and satisfies

$$\left.\frac{D(\breve{x}_0,\breve{y}_0)}{D(\breve{x},\breve{y})}\right|_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{19}$$

By the inverse function theorem, the inverse of $\mathfrak{w}_0$ is also a smooth function. We have established that $\mathscr{A}$ is indeed a smoothly compatible atlas for the standard structure.

Consider now the Hamiltonian function

$$h(z,\theta) = \frac{(z+a)^2}{2},$$

where $a > 1$. Calculating the Hamiltonian vector field, one gets

$$X_h = -(z+a)\frac{\partial}{\partial\theta}.$$

Note that the a parameter is set so this vector field has no zeros on the sphere, except on the poles. Calculating the action of $e^{\tau X_h}$ on θ and z, by observing that $X^k(\theta) = 0$, for $k \geq 2$, one finds that,

$$\mathfrak{z}_\tau = e^{\tau X_h}\cdot\mathfrak{z}_0 = \sqrt{\frac{1+z}{1-z}}\left[e^{i[\theta-(z+a)\tau]} + b(1-z)e^{-i[\theta-(z+a)\tau]}\right].$$

We can now use equation (14) to obtain of the metric tensor. This is the intricate expression (25) in Appendix B. It is an example of how the computations get more involved from now on, so we will rely heavily on computer-aided symbolic manipulation. We will work in pure imaginary time $\tau = it$, so that our complex flow will induce a geodesic in the space of Kähler structures. Our goal will be to describe the properties of the metric tensor G_t^α, namely the scalar curvature S_t^α. We will start by studying the metric for $t = 0$ and proceed to the case $t \neq 0$ to understand how the complex flow evolves the structure.

First of all let $\alpha = 0$.For $t = 0$, the complex coordinate $\mathfrak{z}_0$ is simply the standard complex coordinate on the sphere. This means that we have the usual metric, and the scalar curvature is just $S = 2$. For $\alpha = 0$, but $t \neq 0$, using (14) one finds the metric tensor in z, θ coordinates

$$G_t^0 = \begin{pmatrix} \frac{1-z^2}{1+t(1-z^2)} & 0 \\ 0 & \frac{1+t(1-z^2)}{1-z^2} \end{pmatrix}. \tag{20}$$

The scalar curvature reads,

$$S_t^0 = 2\frac{1+t(1+3z^2)}{[1+t(1-z^2)]^3}. \tag{21}$$

Observe that this metric is independent of θ meaning it is S^1-invariant. A contour plot for S at several times is presented in Figure 6.

(A) $t = 0.1$ (B) $t = 0.5$ (C) $t = 1.0$

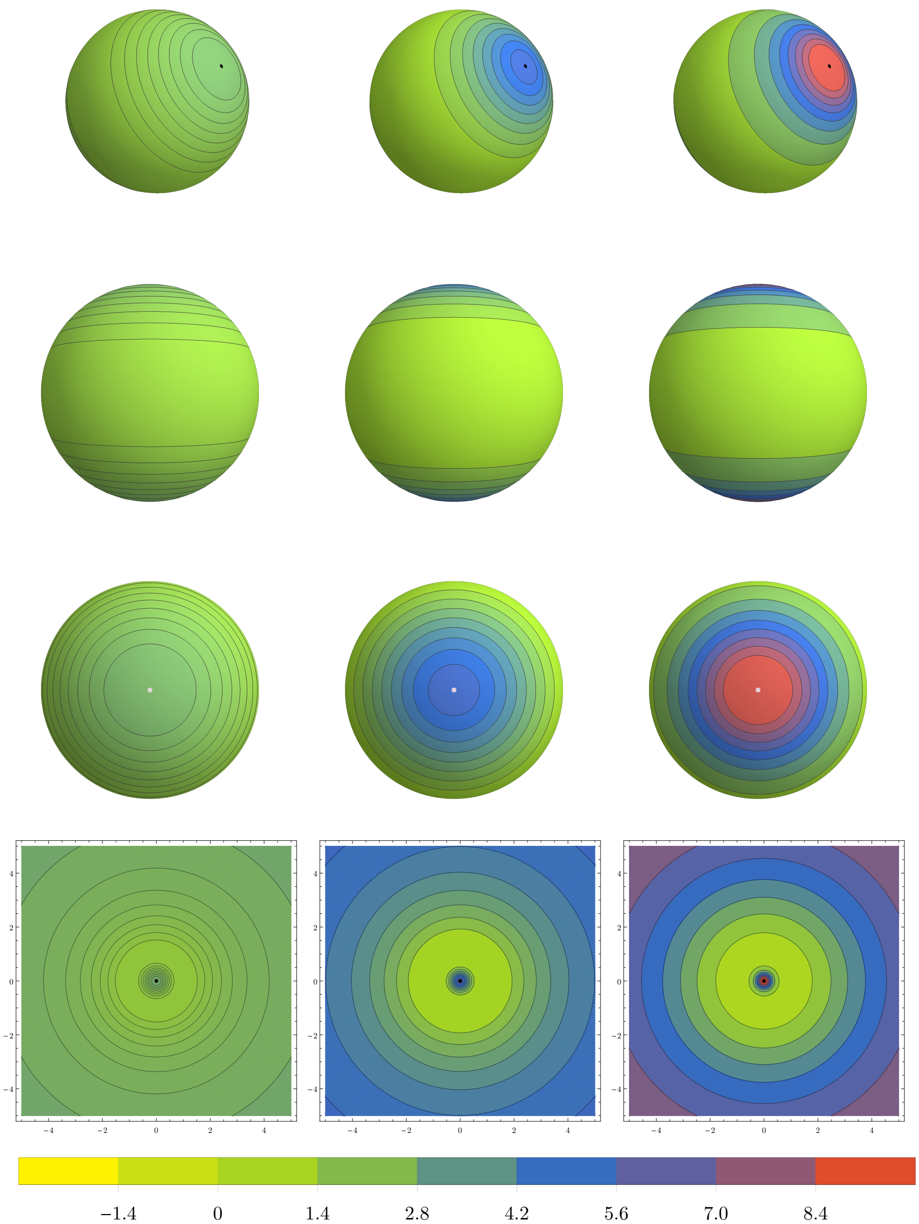


FIGURE 6. Contour plot for the scalar curvature S_t^0, for $\alpha = 0$, with $a = 2$, at different times t. The lines connect points with the same value of S_t^0. The extremal colors on the legend identify clipping regions, i.e., regions where the curvature is bigger than 8.4 or lower than -1.4. The north and the south pole are marked with a dark and a light spot, respectively. The first row presents a tilted view of the sphere. The second is a frontal representation and the third shows a perspective from below the manifold. The fourth row is the contour plot after stereographic projection from the south pole.

The time evolution alters the symmetry of the metric tensor. Starting from the standard O(3)-symmetric structure, with constant scalar curvature $S^0_0 = 2$, the complex dynamics produces a metric that is only O(2)-invariant. Looking at the graphs, we see that the complex flow makes the curvature grow near the poles.

Recall that by the Gauss-Bonet theorem, the total curvature is a topological invariant, and has to be kept the same by the flow. This means that the curvature must be decreasing somewhere else on the sphere, namely near the equator, as one can check in the pictures. By calculating the limit of S^0_t as t grows, we confirm the observation,

$$\lim_{t\to\infty} S^0_t = \begin{cases} \infty \text{ if } z = \pm 1 \\ 0 \text{ if } |z| < 1. \end{cases}$$

Intuitively, the sphere is being stretched from the poles, acquiring a prolate form that gets thinner and thinner as time passes. In the limit, the sphere degenerates into a slim spike. Recalling that ω is an area form, and using the Gauss-Bonnet theorem,

$$\int_{S^2} S\,\omega = 4\pi\chi(S^2),$$

where $\chi(S^2)$ is the Euler characteristic of the sphere, one can find the degenerate scalar curvature. In the sense of distributions with respect to the measure defined by ω this is

$$S^0_\infty = \lim_{t\to\infty} S^0_t = 4\pi\delta_N + 4\pi\delta_S,$$

where δ_N and δ_S are Dirac delta distributions supported on the north and south pole, respectively.

Let us now study what happens when the starting complex structure is not the standard-one, i.e, $\alpha \neq 0$. In the beginning $t = 0$ the metric tensor is

$$G^\alpha_0 = b^\alpha\, L^\alpha,$$

$$L^\alpha = \begin{pmatrix} (1-z^2)\left[1+(1-z)^2\alpha^2 - 2(1-z)\cos(2\theta)\alpha\right] & -(1-z)^2\alpha\sin(2\theta) \\ -(1-z)^2\alpha\sin(2\theta) & \frac{1+(1-z)^2z^2\alpha^2-2(1-z)z\cos(2\theta)\alpha}{1-z^2} \end{pmatrix},$$

$$b^\alpha = \frac{1}{1-\alpha\,(1-z^2)\cos(2\theta)+\alpha^2(z-1)^2z}.$$

The scalar curvature is

$$S^\alpha_0 = \frac{c^\alpha}{d^\alpha},$$

$$\begin{aligned} c^\alpha &= 2 - 6\alpha\left(1-z^2\right)\cos(2\theta)\left(\alpha^4(1-z)^5 + 6\alpha^2(1-z)^2+1\right) - 2\alpha^6(1-z)^6 \\ &\quad - 6\alpha^4(1-z)^4\left(z(1-2z)+3(1+z)^2\cos(4\theta)+4\right) - 6\alpha^2(5z+4)(1-z)^2, \\ d^\alpha &= \left(-\alpha\left(z^2-1\right)\cos(2\theta)+\alpha^2\left(-(z-1)^2\right)z-1\right)^3. \end{aligned}$$

This is consistent with the previous calculations in the sense that, for $\alpha = 0$, $c = 1$, and the metric and the corresponding scalar curvature are those presented in (20) and in (21) for $t = 0$. As in the previous case, we present a contour plot for the scalar curvature in Figure 7. Observe that the curvature is no longer S^1-invariant. It has two distinct lateral features that reach negative curvature, represented with a yellow shade.

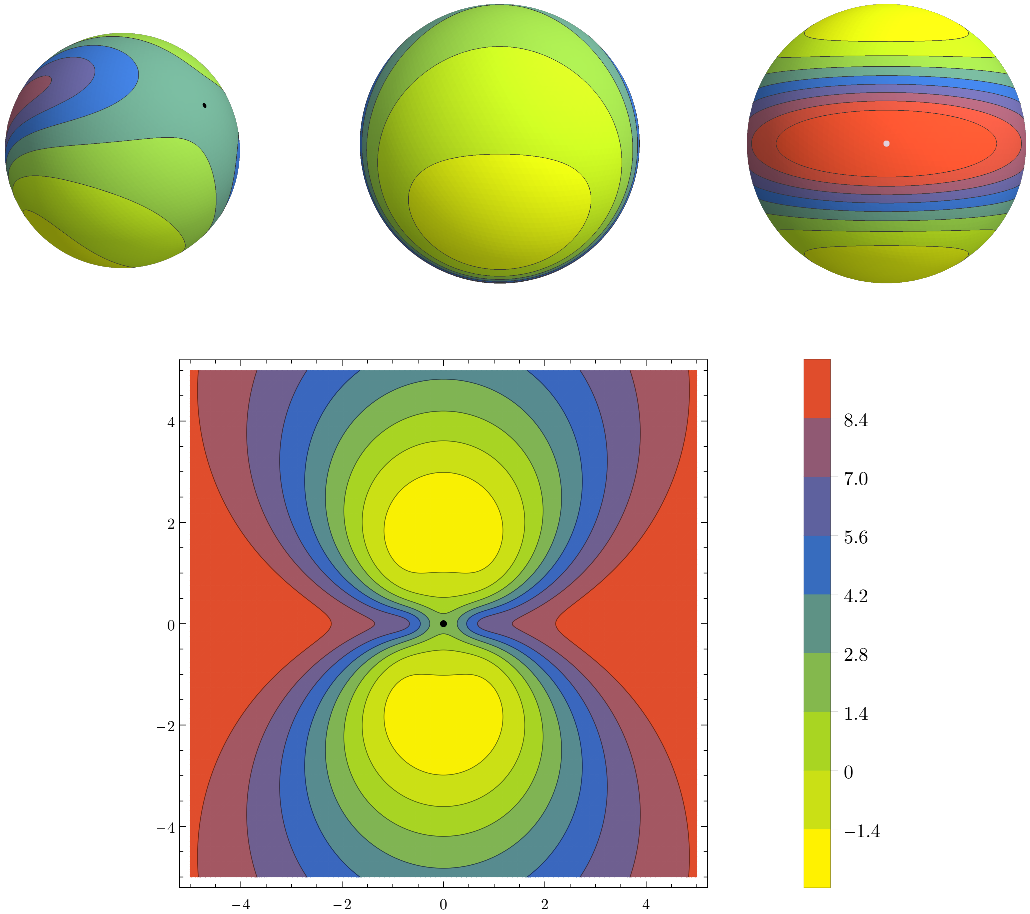


FIGURE 7. Contour plot for the initial scalar curvature S_0^α.The lines connect points with the same value of S_t^0. The extremal colors on the legend identify clipping regions. The north and the south pole are marked with a dark and a light spot, respectively. The views presented are in order: tilted, frontal, and from below the sphere. The boxed image is the contour plot after stereographic projection from the south pole.

To study the evolution of this metric with time we should look at the metric tensor and scalar curvature, for both $t \neq 0$ and $\alpha \neq 0$. The expressions for them are presented in Appendix B. Their properties can be studied more easily by inspection of the contour plots for different times, presented in Figure 8.

(A) $t = 0.1$ (B) $t = 0.5$ (C) $t = 1.0$ (D) $t = 2.0$

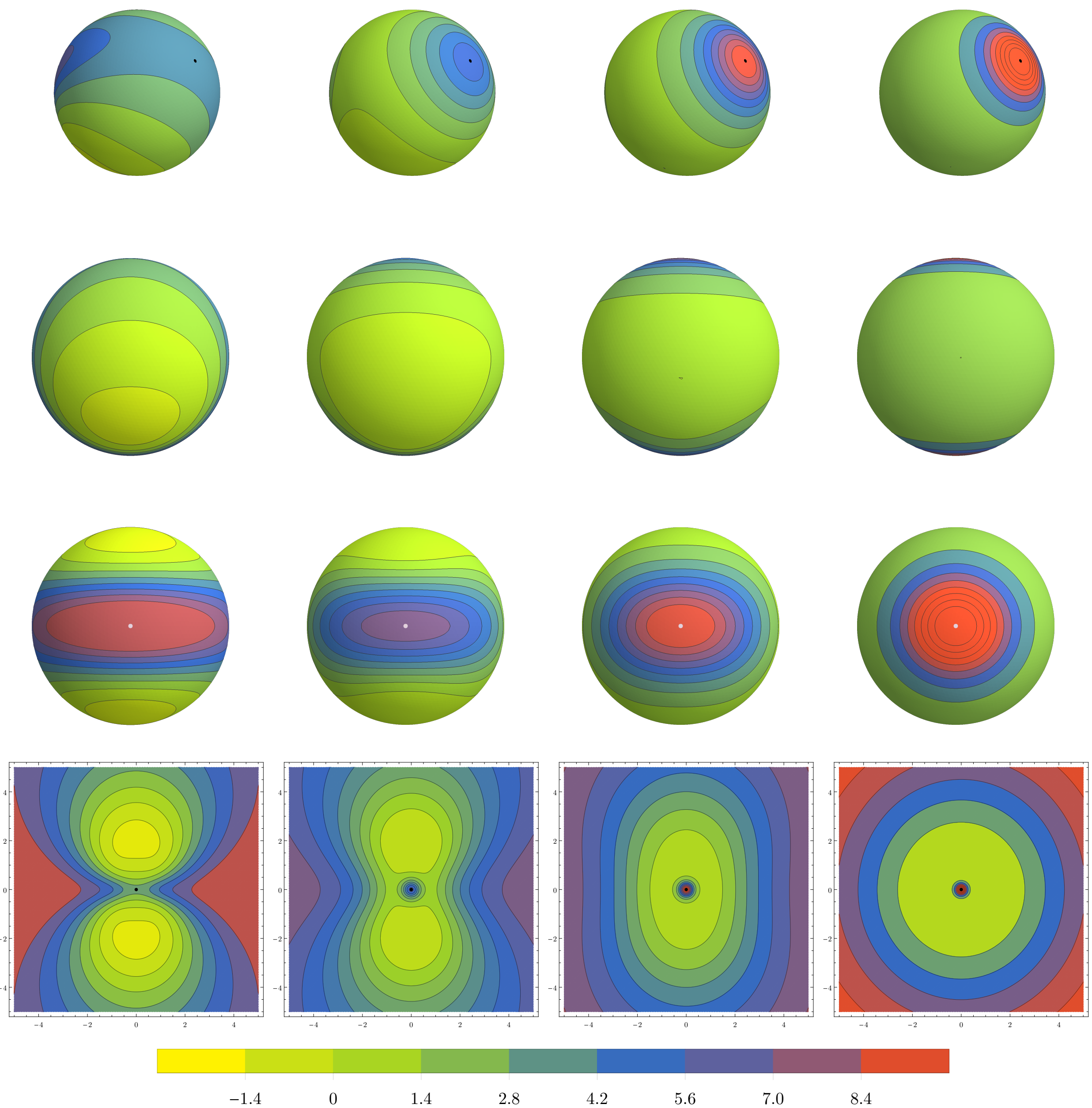


FIGURE 8. Contour plot for the scalar curvature S_t^α, for $\alpha = 0.3$, with $a = 2$, at different times t. The lines connect points with the same value of S_t^0. The extremal colors on the legend identify clipping regions. The north and the south pole are marked with a dark and a light spot, respectively. The first row presents a tilted view of the sphere. The second is a frontal representation and the third shows a perspective from below the manifold. The fourth row is a view after stereographic projection from the south pole

Looking at the pictures the time evolution works the same way as in the symmetric case, with $\alpha = 0$, by increasing the curvature near the poles and decreasing it in the rest of the sphere. On the other hand, there is an interesting phenomenon that only becomes visible when considering a non S^1-invariant metric. The complex flow restores axial symmetry. Even though, at the initial time, the metric has asymmetric features, the flow changes it, approaching the prolate metric of the case $\alpha = 0$, restoring the S^1-invariance. It is possible to make this phenomenon evident by expanding the expressions for G_t^α and S_t^α in powers of $\epsilon = e^{-2t(a+z)}$.

$$G_t^\alpha = L_t^\alpha + O\left(\epsilon^2\right), \quad \epsilon = e^{-2t(a+z)},$$

$$L_t^\alpha = \begin{pmatrix} \dfrac{1-z^2}{1+t\left(1-z^2\right)} - \dfrac{(1-z)^3(1+z)\alpha\cos(2\theta)}{\left[1+t(1-z^2)\right]^2}\epsilon & -\dfrac{(1-z)^2\alpha\sin(2\theta)}{1+t(1-z^2)}\epsilon \\ -\dfrac{(1-z)^2\alpha\sin(2\theta)}{1+t(1-z^2)}\epsilon & \dfrac{1+t(1-z^2)}{1-z^2} + \dfrac{(1-z)\alpha\cos(2\theta)}{1+z}\epsilon \end{pmatrix}.$$

$$S_t^\alpha = -2\frac{1+t(1+3z^2)}{\left[1+t(1-z^2)\right]^3} + s_t^\alpha\epsilon + O\left(\epsilon^2\right),$$

$$s_t^\alpha = \frac{4\alpha\left(1-z^2\right)\cos(2\theta)\left\{t(z-1)\left[2t(z-1)\left[2t\left(tz^4-2(t+2)z^2+t+4\right)+3(z+4)\right]+3(z-5)\right]+3\right\}}{\left[1+t(1-z^2)\right]^4}.$$

One sees that the leading terms of this metric, and the corresponding scalar curvature, agree with the ones for the standard metric on the sphere. As ϵ goes exponentially fast to zero, the θ-dependent components vanish quickly. This clarifies the symmetrization effect of the complex flow.

APPENDIX A

Fréchet Manifolds

In this appendix we state some results and definitions from the theory of Fréchet vector spaces and manifolds. The concepts involved are usually of interest only on the infinite-dimensional case, for instance when considering spaces of maps or tensors. In that setting, it is natural to study even more general topological vector spaces or manifolds modelled on them [**KM97**], and to consider the results presented here as a particular case of the general theory. Throughout we consider the field $\mathbb{K}$ to be $\mathbb{R}$ or $\mathbb{C}$.

Definition A.1. A *seminorm* on a vector space F over a field $\mathbb{K}$ is a function $p : F \to \mathbb{R}_0^+$ such that, for all $f, g \in F$ and $c \in \mathbb{K}$

$$\text{a) } p(f+g) \leq p(f) + p(g) \qquad \text{(subadditivity)}$$

$$\text{b) } p(cf) = |c| p(f) \qquad \text{(absolute homogeneity).}$$

A *norm* is a seminorm that is also positive definite, i.e., $p(f) = 0 \iff f = 0$ for $f \in F$.

A countable family of seminorms $\mathscr{P} = \{p_k\}_{k \in \mathbb{N}}$ defines a topology on the underlying vector space F. This is the topology for which, given a sequence $(f_m)_{m \in \mathbb{N}}$, one has

$$f_m \to f \iff \forall k \in \mathbb{N},\, p_k(f_m - f) \to 0.$$

When this topology is Hausdorff it is called *locally convex topology*. A sequence $(f_m)_{m \in \mathbb{N}}$ is *Cauchy* with respect to this topology if for all k

$$p_k(f_{m_1} - f_{m_2}) \to 0 \text{ for } m_1, m_2 \to \infty.$$

Definition A.2. A vector space F with a countable family of seminorms $\mathscr{P} = \{p_k\}_{k \in \mathbb{N}}$ that is complete with respect to the locally convex topology is called a *Fréchet (vector) space*. If F is an algebra and its multiplication is continuous, F is said to be a *Fréchet algebra*.

Example A.3. *Every Banach space, with norm $||\cdot||$, is a Fréchet space where the countable family of seminorms is just $\mathscr{P} = \{||\cdot||\}$.*

Example A.4. *The space $C^\infty(U, \mathbb{R})$ of complex smooth functions f on an domain of a chart $\psi : U \to \mathbb{R}^n$ of n-dimensional compact manifold M is a Fréchet space, with family of seminorms given by $\mathscr{P} = \{p_{m,K_j}\}_{m \in \mathbb{N}_0^n}$,*

$$p_{m,K_j} = \sup\left\{ \left| D^m \left(f \circ \psi^{-1}\right)(x) \right| : x \in \psi(K_j) \right\},$$

with $\{K_j\}_{j \in \mathbb{N}}$ an exhaustion of U by compact sets, i.e, $K_{j-1} \subset K_j$ and $\cup_{j=1}^\infty K_j = U$, and where D^m denotes the partial derivative for each multiindex $m \in \mathbb{N}_0^n$. This space is actually a Fréchet algebra under pointwise product of functions. By covering the manifold M with a finite atlas we observe that $C^\infty(M, \mathbb{R})$ is also a Fréchet algebra.

Example A.5. *Let (ω, J) be a Kähler pair for the compact manifold M. The space*

$$\mathcal{H}(\omega, J) = \{\, u \in C^\infty(M, \mathbb{R}) \mid \omega_u = \omega + i\partial\overline{\partial} u \succ 0 \,\} \tag{22}$$

is an open set of the Fréchet algebra $C^\infty(M,\mathbb{R})$. In fact, consider a finite atlas $\{(U_k,\phi_k)\}_{k=1,\dots,N}$ for M. The set $\mathcal{H}_k$ of forms that are positive on a domain U_k is open in $C^\infty(M,\mathbb{R})$. This is because the positivity condition is equivalent to the Levi matrix being positive definite on $\phi_k(U_k)$, and this is an open condition. The set of forms that is positive on all M is the finite intersection $\cap_{k=1}^N \mathcal{H}_k$, which is open.

Definition A.6. Let F and G be topological vector spaces, $U \subset F$ an open set and $L : U \to G$ a map. The *derivative of L at $f \in U$ in the direction of $h \in F$* is

$$dL_f(h) = \lim_{s\to 0} \frac{L(f+sh) - L(f)}{s}$$

whenever this limit exists. We say L is *differentiable* at f if its derivative at f exists for all directions $h \in F$. We call L *continuously differentiable* if it is differentiable at all points of U and

$$dL : U \times F \to G \qquad (f,h) \mapsto dL_f(h)$$

is a continuous map. The map L is C^1 if it is continuously differentiable and continuous. We define a C^n map for $n \geq 2$ to be a C^1 map such that its differential is C^{n-1}. A map is called C^∞ or *smooth* if it is C^n for all $n \in \mathbb{N}$.

Definition A.7. Let $\mathcal{M}$ be a Hausdorff topological space and F a Fréchet space. A *Fréchet chart* on an open set $\mathcal{U} \subset \mathcal{M}$ is a homeomorphism $\Phi : \mathcal{U} \to \Phi(\mathcal{U}) \subset F$ onto an open subset $\Phi(\mathcal{U})$ of F. We denote a Fréchet chart by the pair $(\Phi,\mathcal{U})$. Two charts $(\Phi,\mathcal{U})$ and $(\Psi,\mathcal{V})$ are said to be *smoothly compatible* if

$$\Psi \circ \Phi^{-1}|_{\Phi(\mathcal{U}\cap\mathcal{V})} : \Phi(\mathcal{U}\cap\mathcal{V}) \to \Psi(\mathcal{U}\cap\mathcal{V})$$

is a smooth map, between open sets of Fréchet spaces. A *Fréchet atlas* is a family $\mathscr{A} = \{(\Phi_k,\mathcal{U}_k)\}_{k\in I}$ of pairwise smoothly compatible Fréchet charts, such that $\cup_{k\in I}\mathcal{U}_k = \mathcal{M}$. A *Fréchet structure* on $\mathcal{M}$ is a maximal Fréchet atlas $\mathscr{A}$. A Fréchet manifold is a pair $(\mathcal{M},\mathscr{A})$, where $\mathscr{A}$ is a Fréchet structure on $\mathcal{M}$.

Example A.8. *Any open set V of a Fréchet vector space F is a Fréchet manifold with trivial atlas $\{(V, id|_V)\}$. In particular, for M a compact Kähler manifold, with Kähler pair (ω, J), the space $\mathcal{H}(\omega,J) = \{\, u \in C^\infty(M,\mathbb{R}) \mid \omega_u = \omega + i\partial\overline{\partial}u \succ 0 \,\}$ is a Fréchet manifold, because it is open, Example A.5.*

Definition A.9. Let $\mathscr{A} = \{(\Phi_k,\mathcal{U}_k)\}_{k\in I}$ be an atlas for the Fréchet manifold $\mathcal{M}$ with values in a Fréchet space F. On the disjoint union of the sets $\Phi(\mathcal{U}_k) \times F$ consider the equivalence relation

$$(f,v) \sim \big((\Phi_j \circ \Phi_k^{-1})(f), d(\Phi_j \circ \Phi_k^{-1})_f(v)\big)$$

for $f \in \Phi_k(\mathcal{U}_k \cap \mathcal{U}_j)$ and $v \in F$. Write $[f,v]$ for the equivalence class of (f,v). Considering $p \in \mathcal{U}_k$, we call *tangent vectors at p* to the classes of the form $[\Phi_k(p), v]$. The set $T_p\mathcal{M}$ of all such classes is called the *tangent space at p*, and it forms a vector space that is isomorphic to F via the map $F \to T_p\mathcal{M}$, $v \mapsto [f,v]$. The disjoint union of all tangent spaces is

$$T\mathcal{M} := \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}$$

which has a structure of a smooth vector bundle over $\mathcal{M}$ with local trivialisations given by

$$T\mathcal{U}_k := \bigcup_{p\in\mathcal{U}_k} T_p\mathcal{M} \to \Phi(\mathcal{U}_k) \times F, \qquad [\Phi(f), v] \mapsto (\Phi(f), v). \tag{23}$$

A smooth section of $T\mathcal{M}$ is a *(smooth) vector field*.

Example A.10. *If $\mathcal{M} = U$ is an open subset of a Fréchet space F, then $TU = U \times F$ with bundle projection $\pi_{TU} : U \times F \to U$, $(f, v) \mapsto f$. Each smooth vector field is of the form $X(f) = (f, \widetilde{X}(f))$ for some function $\widetilde{X} : U \to F$, and thus we can identify the space of vector fields with $C^\infty(U, F)$. In particular, for the space $\mathcal{H}(\omega, J)$ of the Example A.5 we have $T\mathcal{H} \cong \mathcal{H} \times C^\infty(M, \mathbb{R})$, which implies $T_u\mathcal{H} \cong C^\infty(M, \mathbb{R})$ for each $u \in \mathcal{H}$.*

Definition A.11. [**Bru16**] Let $\mathcal{M}$ be a Fréchet manifold. A *weak Riemannian metric* is a smooth map

$$\mathcal{G} : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$$

that at each point $p \in \mathcal{M}$ satisfies:

a) $\mathcal{G}_p(\cdot, \cdot)$ is bilinear

b) $\mathcal{G}_p(f, f) \geq 0$ for all $f \in T_p\mathcal{M}$ with equality only for $f = 0$.

A *strong Riemannian metric* is required to satisfy a), b) and the additional condition

c) The topology of the inner product space $(T_p\mathcal{M}, \mathcal{G}_p(\cdot, \cdot))$ coincides with the topology $T_p\mathcal{M}$ inherits from the manifold $\mathcal{M}$.

Remark A.12. Observe that when the Fréchet atlas for $\mathcal{M}$ takes values on a finite dimensional Fréchet space (i.e. $\mathbb{R}^n$), condition c) is automatically satisfied. Therefore, the distinction between weak and strong Riemannian metrics is only relevant in the infinite-dimensional setting. Also, note that c) is a vey restrictive condition, because it is possible to show that if a Fréchet manifold admits a strong Riemannian metric, then its atlas takes values on a Hilbert space.

APPENDIX B

Explicit formulas

Here we present some expressions that are not included in the main body of the text.

$$\mathrm{Im}(\rho_t\big|_{\delta=i}^{\omega=(1+i)/\sqrt{2}}) = \frac{R_1}{R_2} \tag{24}$$

$$\begin{aligned} R_1 = {} & \sqrt{2}({\rho_0^{(1)}}^2 + {\rho_0^{(2)}}^2 - 1)\sin(\sqrt{2}t) - \sqrt{2}({\rho_0^{(1)}}^2 + {\rho_0^{(2)}}^2 + 1)\sinh(\sqrt{2}t) \\ & -2(\rho_0^{(1)} + \rho_0^{(2)})\cos(\sqrt{2}t) + 2(\rho_0^{(1)} - \rho_0^{(2)})\cosh(\sqrt{2}t) \\ R_2 = {} & 2(({\rho_0^{(1)}}^2 + {\rho_0^{(2)}}^2 - 1)\cos(\sqrt{2}t) - ({\rho_0^{(1)}}^2 + {\rho_0^{(2)}}^2 + 1)\cosh(\sqrt{2}t) \\ & +\sqrt{2}((\rho_0^{(1)} + \rho_0^{(2)})\sin(\sqrt{2}t) + (\rho_0^{(1)} - \rho_0^{(2)})\sinh(\sqrt{2}t))), \end{aligned}$$

where $\rho_0 = \rho_0^{(1)} + i\rho_0^{(2)}$.

$$G_t^\alpha = \begin{pmatrix} G_{zz} & G_{\theta z} \\ G_{z\theta} & G_{\theta\theta} \end{pmatrix} \tag{25}$$

$$G_{zz} = \frac{(z-1)(z+1)((z-1)^2\alpha^2 + 2e^{2t(a+z)}(z-1)\cos(2\theta)\alpha + e^{4t(a+z)})}{(z-1)^2(t(z^2-1)-z)\alpha^2 + e^{2t(a+z)}(2t(z-1)-1)(z^2-1)\cos(2\theta)\alpha + e^{4t(a+z)}(t(z^2-1)-1)}$$

$$G_{\theta z} = G_{z\theta} = \frac{\alpha}{\alpha\frac{(2t(z-1)-1)(z+1)\cot(2\theta)}{z-1} + e^{-2t(a+z)}((t(z^2-1)-z)\alpha^2 + \frac{e^{4t(a+z)}(t(z^2-1)-1)}{(z-1)^2})\csc(2\theta)}$$

$$G_{\theta\theta} = \frac{e^{4t(a+z)}(-tz^2+t+1)^2 + (z-1)^2(-tz^2+z+t)^2\alpha^2 + 2e^{2t(a+z)}(z-1)(t(z^2-1)-1)(t(z^2-1)-z)\alpha\cos(2\theta)}{(z-1)(z+1)((z-1)^2(t(z^2-1)-z)\alpha^2 + e^{2t(a+z)}(2t(z-1)-1)(z^2-1)\cos(2\theta)\alpha + e^{4t(a+z)}(t(z^2-1)-1))}$$

$$S_t^\alpha = -\frac{S_1}{S_2} \tag{26}$$

$$\begin{aligned} S_1 = {} & 2(\alpha^2 e^{8t(a+z)}(3(z-1)^2(t(8t^2(z+1)^2(z-1)^3 - 2t(z+1)(z+11)(z-1)^2 + z(25z+4) - 17) - 5z - 4) \\ & -2t(z^2-1)^2\cos(4\theta)(2t(z-1)^2(2t(z-1)-3)-3)) \\ & -\alpha(z^2-1)\cos(2\theta)e^{2t(a+z)}((2t(-4t(z-1)^2(t(z^2-1)-3) - 15z + 9) + 3)e^{8t(a+z)} \\ & -9\alpha^2(z-1)^2(t(4t(z-1)^3 - (z-10)z - 5) - 2)e^{4t(a+z)} \\ & +\alpha^4(z-1)^4(2t(4t^2(z+1)(z-1)^3 - 12tz(z-1)^2 + 3z(2z-5) + 3) - 3z + 6)) \\ & +\alpha^4(z-1)^4 e^{4t(a+z)}(3(-8t^3(z+1)^2(z-1)^3 + 2t^2(z+1)(11z+1)(z-1)^2 \\ & +t(z((10-11z)z+19)-6) + z(2z-1) - 4) + (z+1)^2\cos(4\theta)(2t(2t(z-1)^2(2t(z-1)-3) \\ & +6z-3)-3)) + \alpha^3 t(z^2-1)^3\cos(6\theta)e^{6t(a+z)} + (3tz^2+t+1)e^{12t(a+z)} + \alpha^6(z-1)^6(tz(z^2+3)-1)) \\ S_2 = {} & (\alpha(z^2-1)\cos(2\theta)(2t(z-1)-1)e^{2t(a+z)} + (t(z^2-1)-1)e^{4t(a+z)} + \alpha^2(z-1)^2(t(z^2-1)-z))^3 \end{aligned}$$

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