On Cyclic Quandles with Several Fixed Points

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July, 2018

Abstract

In this thesis, we use some of the techniques developed in [5] to study quandles with a particular kind of profile, the cyclic quandles with several fixed points. We are interested in classifying these quandles and we do that for cyclic quandles of "higher orders". We prove, in particular, that there is only one such quandle, up to isomorphism. Furthermore, we identify this quandle.

Keywords: cyclic quandles with several fixed points; permutations; profiles; patterns; fixed points.

1 Introduction

The group-like structure known as quandle, which is under study in this thesis, was independently introduced in [4] and [6]. Consider, for example, the quandle Q_6^2 in Table 1, see also [2]. In the thesis, we try to find more examples of quandles with the same properties as Q_6^2, which we call cyclic quandles with several fixed points, and we propose to classify them. The results stated below are all proved in the thesis.

2 Preliminaries

The algebraic structure known as *quandle* is defined as follows.

Definition 2.1. *Let X be a set equipped with a binary operation denoted by $*$. The pair $(X, *)$ is said to be a* quandle *if, for each $a, b, c \in X$,*

1. $a * a = a$ *(idempotency);*
2. $\exists!\, x \in X : x * b = a$ *(right-invertibility);*
3. $(a * b) * c = (a * c) * (b * c)$ *(self-distributivity).*

We now present some simple examples of quandles.

Example 2.1. *Q_6^2, whose multiplication table is displayed in Table 1, is a quandle as it satisfies axioms 1. through 3. in Definition 2.1.*

Example 2.2. *Let G be a group and let $*$ be a binary operation on G given by $a * b = bab^{-1}$, for every $a, b \in G$, where the juxtaposition on the right-hand side denotes group multiplication. Then, the pair $(G, *)$ is a quandle;*

$*$	1	2	3	4	5	6
1	1	5	1	6	4	2
2	6	2	5	2	1	3
3	3	6	3	5	2	4
4	5	4	6	4	3	1
5	2	3	4	1	5	5
6	4	1	2	3	6	6

Table 1: Q_6^2 multiplication table.

Given a finite quandle $(X, *)$, we can easily check that axiom *2.* in Definition 2.1 is equivalent to saying that each element of X appears exactly once in each column of the quandle's multiplication table. Hence, each column of the multiplication table of a finite quandle $(X, *)$ can be regarded as a permutation of the elements of X. Specifically, if in $j * i$ we fix i and let j run over X, then we define a permutation, μ_i, of the elements of X, given by $\mu_i(j) = j * i$, for any $j \in X$. This gives rise to an alternative description of the structure of a quandle, which we state in the form of a theorem. This theorem, which is presented below, is based on a result from [1] and [3]. In fact, it is this description of the structure of a quandle that we use throughout the thesis and that allows us to classify a family of quandles.

Theorem 2.1. *Let $X = \{1, 2, \ldots, n\}$ be a set. Suppose a permutation $\mu_i \in S_n$ is assigned to each $i \in X$. Then, the expression $j * i := \mu_i(j), \forall j \in X$, yields a quandle structure if and only if $\mu_{\mu_i(j)} = \mu_i \mu_j \mu_i^{-1}$ and $\mu_i(i) = i$, $\forall i, j \in X$. This quandle structure is uniquely determined by the set of n permutations.*

Definition 2.2. *Given a quandle $(X, *)$, its permutations are the μ_i's referred to in the statement of Theorem 2.1, $\forall i \in X$. Unless otherwise stated in the sequel, a μ_i always refers to a permutation from the quandle under discussion.*

In the thesis, we study the properties of each quandle by analyzing the quandle structure uniquely determined by its set of n permutations. In particular, we use standard results on permutations in order to characterize and classify quandles.

Firstly, we note that any permutation can be decomposed into a set of disjoint cycles. The lengths of these cycles define the *pattern* of each permutation.

Definition 2.3. *The* pattern *of a permutation is the list of the lengths of the disjoint cycles making up the permutation.*

We can collect the information relative to the patterns of the n permutations defining a quandle of order n in order to define its *profile*.

Definition 2.4. *The* profile *of a quandle of order n is the list of the patterns of the n permutations defining the quandle.*

We now introduce the notion of *connected quandle* in order to state an important proposition.

Definition 2.5. *A finite quandle $(X, *)$ is said to be* connected *if, $\forall\, i, j \in X$, $\exists\, k_1, k_2, \ldots, k_n \in X$:*

$$j = (\cdots((i * k_1) * k_2) * \cdots * k_n) = \mu_{k_n} \circ \cdots \circ \mu_{k_2} \circ \mu_{k_1}(i).$$

Proposition 2.1. *Connected quandles have constant profiles.*

Finally, we introduce the key notion of *cyclic quandles with several fixed points.*

Definition 2.6. *Given $n, f \in \mathbb{N}$, $n-2 \geq f > 1$, a* cyclic quandle with f fixed points *of order n is a quandle of order n with constant profile given by* $\Big\{\{\underbrace{1,\ldots,1}_{f}, n-f\},\ldots,\{\underbrace{1,\ldots,1}_{f}, n-f\}\Big\}$.

The previous definition means that each one of the n permutations defining a cyclic quandle of order n with f fixed points has 1 cycle of length $n-f>1$ and f cycles of length 1.

In passing, we note, by inspection of Table 1, that Q_6^2 is a connected quandle. Hence, by Proposition 2.1, it has constant profile. But more than that, Q_6^2 is, in fact, a cyclic quandle of order 6 with 2 fixed points. Thus, it would be interesting to find more examples of cyclic quandles with several fixed points; in particular, we could try to classify them.

Our main goal in the thesis is, indeed, to classify cyclic quandles with several fixed points. Although we do not completely classify cyclic quandles with several fixed points, we prove that there is only one cyclic quandle of order n with f fixed points such that $n > 2f$, up to isomorphism. This unique cyclic quandle is precisely Q_6^2, up to isomorphism. The main theorem we prove in the thesis is then the following.

Theorem 2.2. *Given positive integers n, f such that $n > 2f > 2$, there is only one cyclic quandle of order n with f fixed points, up to isomorphism. This quandle is Q_6^2, up to isomorphism.*

We note that, for $n \leq f+1$, there are no cyclic quandles with f fixed points (otherwise, there would not be room for the cycle of length $n-f>1$). Therefore, the question of classifying cyclic quandles with several fixed points remains open only for the pairs (n,f) such that $f+2 \leq n \leq 2f$. In particular, for each f, we are left with the classification of cyclic quandles with several fixed points, whose orders are $f+2, f+3, \ldots, 2f-1$ and $2f$. We remark that for each f, these are finitely many orders; exactly $f-1$.

Throughout the thesis, we use the following notation.

Definition 2.7. *Let Q be a quandle of order n. As per Theorem 2.1, its n permutations are denoted μ_i, $i \in \{1,\ldots,n\}$. In particular, $\mu_i(i) = i, \forall i \in \{1,\ldots,n\}$. The set of fixed points of μ_i is denoted F_i, $i \in \{1,\ldots,n\}$. The set of points in the non-singular cycle of μ_i is denoted C_i, $i \in \{1,\ldots,n\}$. We note that $C_i \cap F_i = \emptyset$ and $C_i \cup F_i = \{1,\ldots,n\}$, $\forall i \in \{1,\ldots,n\}$.*

3 Properties of Cyclic Quandles with Several Fixed Points

After a few preliminary results, we state Theorem 3.1 which provides a number of conditions cyclic quandles of order n with f fixed points such that $n > 2f$ must verify. This theorem is key to proceed to a classification of cyclic quandles with several fixed points. It actually marks the beginning of the original research of the thesis.

From this point on, we restrict our study to cyclic quandles of order $n \geq 2f$. In fact, cyclic quandles with such an order satisfy a very useful property, which is a consequence of the following proposition.

Proposition 3.1. *Let Q be a cyclic quandle of order n with f fixed points such that $n \geq 2f$, and let $F_k = \{k, g_k^1, \ldots, g_k^{f-1}\}$ be the set of f fixed points of μ_k. Then $\mu_{g_k^i}(g) = g$, $\forall g \in F_k$, $\forall i \in \{1,\ldots,f-1\}$.*

Proposition 3.1 states that given $g \in F_k$, $F_g = F_k$. In fact, we can also prove that if there is another index g' such that $F_{g'} \cap F_k \neq \emptyset$, then $g' \in F_k$ and $F_{g'} = F_k$. This allows us to conclude that the sets of fixed points of two permutations are either equal or disjoint and thus the following result holds.

Corollary 3.1. *Suppose Q is a cyclic quandle of order n with f fixed points such that $n \geq 2f$. Then, n is a multiple of f.*

This result is quite useful as it restricts the orders these cyclic quandles can have, making them easier to classify.

We now introduce the notions of *associate indices* and *associate permutations*.

Definition 3.1. *Let Q be a quandle of order n with permutations denoted by μ_k, $k \in \{1, \ldots, n\}$.*

- *If i and j are different indices such that $\mu_i(j) = j$ and $\mu_j(i) = i$, we say that i and j are* associate indices*;*
- *If i and j are associate indices then μ_i and μ_j are said* associate permutations*;*

The following proposition is now an immediate consequence of our previous considerations.

Proposition 3.2. *For $n \geq 2f$, "i is associate to j" generates an equivalence relation.*

For Q_6^2, this equivalence relation generated by "i is associate to j" for $n \geq 2f$ is, indeed, a congruence relation, as it is compatible with the structure of the quandle.

From this point on, we assume the order n of any cyclic quandle to be greater than or equal to $2f$, unless otherwise stated. Therefore, Proposition 3.1 and Corollary 3.1 always apply.

The two following propositions tell us whether a cyclic quandle of order $n \geq 2f$ is connected or not.

Proposition 3.3. *If Q is a cyclic quandle of order n with f fixed points such that $n = 2f$, Q is not connected.*

Proposition 3.4. *Every cyclic quandle of order n with f fixed points such that $n > 2f$ is connected.*

From now on, we assume the order n of any quandle to be greater than $2f$, unless otherwise stated. In these conditions, all our cyclic quandles are connected.

The equalities $\mu_{\mu_i(j)} = \mu_i \mu_j \mu_i^{-1}$ from Theorem 2.1 can now be used to derive a number of conditions these quandles have to verify in order to be cyclic. These conditions are stated in the following theorem, which marks the beginning of the original research of the thesis. This theorem is key to prove Theorem 2.2.

Theorem 3.1. *Consider a cyclic quandle of order n with f fixed points such that $n > 2f$. Modulo isomorphism, its sequence of permutations satisfies the following conditions.*

1. $\mu_n = (1\ 2\ 3 \cdots n-f)(n-f+1) \cdots (n-1)(n)$;
2. *if h and h' are associate indices then $\mu_h = \mu_{h'}^{l_{h,h'}}$, where $GCD(n-f, l_{h,h'}) = 1$, $1 \leq l_{h,h'} < n-f$;*
3. $\mu_k = \mu_n^k \mu_{n-f} \mu_n^{-k}$, *for all $1 \leq k \leq n-f$;*
4. $\mu_{n-f} \mu_a \mu_{n-f}^{-1} = \mu_n^{\mu_{n-f}(a)} \mu_{n-f} \mu_n^{-\mu_{n-f}(a)}$, $\forall a \in F_n$;
5. $\mu_{n-f}^{-1} \mu_a \mu_{n-f} = \mu_n^{\mu_{n-f}^{-1}(a)} \mu_{n-f} \mu_n^{-\mu_{n-f}^{-1}(a)}$, $\forall a \in F_n$;
6. $\forall m \in \{1, \ldots, n-f\} \setminus \{\mu_{n-f}^{-1}(n-f+1), \ldots, \mu_{n-f}^{-1}(n)\}$, *there exists an integer $1 \leq k_m < n-f$ such that $\mu_n^{-\mu_{n-f}(m)} \mu_{n-f} \mu_n^m = \sigma \tau^{k_m}$, where σ is a permutation of F_{n-f} and τ is the cycle of length $n-f$ in μ_{n-f}.*

4 Classification of Cyclic Quandles with Several Fixed Points

We now go over some immediate consequences on the permutations of a cyclic quandle with several fixed points, implied by Theorem 3.1. Given a cyclic quandle of order n with f fixed points such that $n > 2f$, we can assume, without loss of generality, that $\mu_n = (1\ 2\ 3 \cdots n-f)(n-f+1)\cdots(n-1)(n)$, as stated in assertion *1.* in Theorem 3.1. Therefore, μ_{n-f} completely determines the entire quandle, since every other permutation in the quandle can be written as a function of μ_n and μ_{n-f}. To determine whether or not a given permutation μ_{n-f} generates a cyclic quandle, μ_{n-f} must verify assertions *2.* to *6.* in Theorem 3.1.

Furthermore, assertion *2.* in Theorem 3.1 states that associate permutations are powers of each other. Then, if two associate permutations satisfy $\mu_k(g') = \mu_{k'}(g')$, for a certain $g' \notin F_k$, we have $\mu_k(g) = \mu_{k'}(g)$, $\forall g \in C_k$. Otherwise, $\mu_k(g) \neq \mu_{k'}(g)$, $\forall g \in C_k$. This is actually a very important remark. We use it to show that for all but one (up to isomorphism) cyclic quandle of order n with f fixed points such that $n > 2f$, the associate permutations must be simultaneously equal to each other and different from each other, which is a contradiction. This idea is key to prove Theorem 2.2.

Finally, assertions *1.* to *6.* in Theorem 3.1 might seem innocuous at a first glimpse. However, they conflict with each other in such a way that there is only one cyclic quandle of order n with f fixed points such that $n > 2f$, up to isomorphism. This cyclic quandle is precisely Q_6^2. In order to conclude this, it is necessary to prove numerous propositions and corollaries, which lead to the two following results:

Proposition 4.1. *There are no cyclic quandles of order n with f fixed points such that $n = 3f$, for $f > 2$.*

Proposition 4.2. *There are no cyclic quandles of order n with f fixed points such that $n = cf$, for $c > 3$.*

Theorem 2.2 is now an immediate consequence of Propositions 4.1 and 4.2, together with Corollary 3.1 and the fact that there is only one cylic quandle of order 6 with 2 fixed points, up to isomorphism.

5 Conclusion

Our goal in the thesis is to classify cyclic quandles of order n with f fixed points such that $n > 2f$, something we successfully achieved by proving Theorem 2.2.

In fact, we prove that we can only have cyclic quandles of order n with f fixed points such that $n > 2f$ if $n = 6$ and $f = 2$, which is quite a strong result.

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