On quantum corrections to the kinetic equations

João O'Neill Cortes

June 2018

Abstract

Plasma phenomena as well as many other many-body systems of technological interest have been studied mostly as purely classical fields. However, in dense plasmas, in some semiconductor devices, metallic nanostructures and thin metal films, when the de Broglie wavelength of the charge carriers is comparable to the inter-particle distance, quantum effects come into play.

The classical kinetic equations being phase space equations with positions and momenta non commuting quantum variables, kinetic equations are not directly applicable to quantum plasmas. Therefore most treatments consider a full quantum particle problem in Hilbert space and then, by reduction to global variables, obtain the quantum version of the kinetic equations

However, quantum mechanics may also be directly formulated in phase-space with a modification of the Poisson algebra to a new deformed algebra and the classical kinetic equations may be deformed to the corresponding quantum versions. This is the approach followed here and applied to the Poisson-Vlasov, Maxwell-Vlasov and Boltzmann equations.

Keywords Kinetic Theory, Quantum plasmas, Theory of deformations, Quantum Boltzmann equation, Quantum Vlasov equation

1 Introduction

In the past plasma physics phenomena have been studied mostly as a purely classical field. However, if the de Broglie wavelength of the charge carriers is comparable to the inter-particle distance, quantum effects come into play [1, 2, 3, 4, 5]. This is the case of dense plasmas, as are present in white dwarfs, the atmosphere of neutron stars or intense laser-solid plasma interactions. In general many-body charged particle systems cannot be treated by purely classical physics equations when there is considerable overlap of the wave functions. This is the case not only of dense plasmas but also of semiconductor devices for which kinetic equations might also be used. Quantum plasma effects are also relevant for the physics of metallic nanostructures and thin metal films.

The classical kinetic equations relevant to plasma physics are phase space equations and because in quantum mechanics positions and momenta are non commuting variables, kinetic equations have not been considered as immediately applicable to quantum plasmas. Therefore most treatments (see for example [1]) consider a full quantum particle problem in Hilbert space and then, by reduction to global variables, obtain the quantum version of the kinetic equations. However, quantum mechanics may also be directly formulated in phase-space with a modification of the Poisson algebra to a new deformed algebra. This suggests that the quantum version of the kinetic equations might also be obtained by deformation of the classical kinetic equations. It turns out that this is possible and simpler than the traditional approaches. Of particular interest for the applications are the leading quantum corrections to the kinetic equations which, for example, change the stability conditions of the solutions [4].

2 Quantum mechanics and deformation theory

The phase space of classical mechanics is a symplectic manifold $W = (T^*M, \boldsymbol{\omega})$ where T^*M is the cotangent bundle over the configuration space M and $\boldsymbol{\omega}$ is a symplectic form. In local (Darboux) coordinates (p_i, q_i) the symplectic form is

$$d\,\boldsymbol{\omega} = \sum dp_i \wedge dq_i \tag{1}$$

The Poisson bracket gives a Lie algebra structure to the $C^\infty-$functions on W, namely

$$\{f, g\} = \sum_i \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}_i}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{p}_i} - \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{p}_i}\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}_i} \tag{2}$$

in local coordinates.

The transition to quantum mechanics is now regarded as a deformation of this Poisson algebra [6]. Let for example $T^*M = \boldsymbol{R}^{2n}$. Then,

$$\boldsymbol{\omega} = \sum_{1\leqslant \boldsymbol{i},\boldsymbol{j}\leqslant \boldsymbol{n}} \boldsymbol{\omega}_{ij} dx^i \wedge dx^j = \sum_{1\leqslant \boldsymbol{i}\leqslant \boldsymbol{n}} dx^i \wedge dx^{i+n} \tag{3}$$

Consider the following bidifferential operator

$$P^r(f,g) = \sum_{\substack{i_1\cdots i_r \\ j_1\cdots j_r}} \boldsymbol{\omega}^{i_1 j_1}\cdots\boldsymbol{\omega}^{i_r j_r} \boldsymbol{\partial}_{i_1}\cdots\boldsymbol{\partial}_{i_r}(f)\boldsymbol{\partial}_{j_1}\cdots\boldsymbol{\partial}_{j_r}(g) \tag{4}$$

$P^1(f,g)$ is the Poisson bracket. $P^3(f,g)$ is a non-trivial $2-$cocycle and, barring obstructions, one therefore expects the existence of non-trivial deformations of the Poisson algebra. Existence of non-trivial deformations have indeed been proved in a very general context [7] [8] [9] [10]. They always exist if W is finite-dimensional and for a flat Poisson manifold they are all equivalent to the Moyal [11] bracket

$$[f,g]_M = \frac{2}{\hbar} sin(\frac{\hbar}{2}P)(f,g) = \{f,g\} - \frac{\hbar^2}{4.3!}P^{\boldsymbol{3}}(f,g) + \cdots \tag{5}$$

Moreover $[f,g]_M = \frac{1}{i\hbar}(f *_\hbar g - g *_\hbar f)$ where $f *_\hbar g$ is an associative star-product

$$f *_\hbar g = \exp\left(i\frac{\hbar}{2}P(f,g)\right) \tag{6}$$

Correspondence with quantum mechanics formulated in Hilbert space is obtained by the Weyl quantization prescription. Let $f(p,q)$ be a function in phase space and $\widetilde{f}$ its Fourier transform. Then, if to the function f we associate the Hilbert space operator

$$\boldsymbol{\Omega}(f) = \int \widetilde{f}(x_i, y_i) e^{-\frac{i}{\hbar}\sum x_i Q_i + y_i P_i}\, dx_i dy_i \tag{7}$$

where $Q_i \boldsymbol{\psi} = \boldsymbol{x}_i \boldsymbol{\psi}$ and $P_i = -\,i\hbar \frac{\boldsymbol{\partial}}{\boldsymbol{\partial} x_i}\boldsymbol{\psi}$, one finds

$$[\boldsymbol{\Omega}(f), \boldsymbol{\Omega}(g)] = -\,i\hbar\boldsymbol{\Omega}([f,g]_M) \tag{8}$$

with, in the left-hand side, the usual commutator for Hilbert space operators and in right hand side the Moyal bracket. Therefore quantum mechanics may be described either by associating self-adjoint operators in Hilbert space to the observables or, equivalently, staying in the classical setting of phase-space functions but deforming their product to a $*_\hbar-$ product and their Poisson brackets to Moyal brackets.

3 Kinetic equations and quantum corrections

A kinetic equation deals with a probability density $f(t,x,p)$ of particles in phase space. The typical (Boltzmann) form is

$$\frac{\partial}{\partial t}f + \frac{p}{m}\cdot\nabla_x f + F_{ext}\cdot\nabla_p f = S(f) \tag{9}$$

the left hand side being a drift term defining the characteristics along which the particles move between collisions and the right hand side a collision term. It is therefore a equation involving a probability distribution in the (x,p) phase space. In quantum mechanics $f(x,p)$ cannot be a classical probability distribution because x and p are non-commuting variables. However $f(x,p)$ may be interpreted as a functional of elements in an algebra with a deformed product and, as discussed before, this leads to the correct quantum results.

It is tempting, to obtain the quantum corrections to Eq.(9), by simply replacing all products by a deformed product. However, recalling that at the basis of this interpretation of quantum mechanics is the deformation of a Poisson algebra, it is more appropriate to deform the kinetic equation when their (non-canonical) Hamiltonian structure is exhibited. This is the approach that will be followed.

3.1 The Poisson-Vlasov equation

The Poisson-Vlasov equation describing a collisionless plasma with purely electrostatic interactions is

$$\frac{\partial f}{\partial t}+\frac{p}{m}\cdot\frac{\partial f}{\partial x}-e\frac{\partial \phi}{\partial x}\cdot\frac{\partial f}{\partial p}=0 \tag{10}$$

with

$$\triangle\phi=-e\int dpf\left(x,p,t\right) \tag{11}$$

It is a non-canonical Hamiltonian system [12], with Hamiltonian,

$$H_{PV}=\frac{1}{2}\int\left|\frac{p}{2m}\right|^{2}f\left(x,p,t\right)dxdp+ \frac{e}{2}\int dx\phi\left(x\right)\int f\left(x,p,t\right)dp \tag{12}$$

the time evolution of arbitrary phase-space functions given by

$$\frac{dF}{dt}=\left[F,H_{PV}\right] \tag{13}$$

the Poisson structure $[\cdot,\cdot]$ being

$$\left[F,G\right]=\int f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}dxdp \tag{14}$$

where $\{\cdot,\cdot\}$ stands for the usual Poisson bracket for functions of x and p

$$\left\{A,B\right\}=\sum_{i}\left(\frac{\partial A}{\partial x_i}\frac{\partial B}{\partial p_i}-\frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x_i}\right) \tag{15}$$

and the functional derivative $\frac{\delta F}{\delta f}$ being related to the Fréchet derivative by

$$\left(D_f F\right)\cdot f^{'}=\int\frac{\delta F}{\delta f}f^{'}dxdv \tag{16}$$

Taking into attention that

$$\begin{aligned}\frac{\delta f\left(y,\mu,t\right)}{\delta f\left(x,p,t\right)}&=\delta^{3}\left(y-x\right)\delta^{3}\left(\mu-p\right)\\ \frac{\delta H_{PV}}{\delta f\left(x,p,t\right)}&=\frac{1}{2m}\left|p\right|^{2}+e\phi\left(x\right)\end{aligned} \tag{17}$$

and using Eq.(14) one obtains the classical Poisson-Vlasov equation

$$\frac{df}{dt}=\left[f,H_{PV}\right]=-\frac{p}{m}\cdot\nabla_x f+e\nabla_x\phi\cdot\nabla_p f \tag{18}$$

For the quantum version all one has to do is to replace in Eq.(14) the Poisson (15) by the Moyal bracket (5).

$$\frac{df}{dt}=\int d^{3}xd^{3}pf\left(x,p,t\right)\frac{2}{\hbar}\sin\left(\frac{\hbar}{2}P\right)\left(\frac{\delta f}{\delta f},\frac{\delta H_{PV}}{\delta f}\right) \tag{19}$$

P being the bidifferential operator in (4).

Of special interest is the leading quantum correction. The 6−dimensional ω matrix in the symplectic form (3) has $\omega_{i,i+3}=-\omega_{i+3,i}=1$ with all the other elements being zero. Because $\frac{\delta H_{PV}}{\delta f(x,p,t)}$ is quadratic in p, all terms in $\omega_{i,i+3}\omega_{j,j+3}\omega_{k,k+3}$ vanish. Finally one obtains in leading $\hbar^2$ order,

$$\frac{df}{dt}=\left[f,H_{PV}\right]_M=-\frac{p}{m}\cdot\nabla_x f+e\nabla_x\phi\cdot\nabla_p f- e\frac{\hbar^2}{24}\sum_{i,j,k=1}^{3}\frac{\partial^3 f}{\partial p_i\partial p_j\partial p_k}\frac{\partial^3\phi}{\partial x_i\partial x_j\partial x_k}+O\left(\hbar^4\right) \tag{20}$$

3.2 The Maxwell-Vlasov equation

The Maxwell-Vlasov equation,

$$\frac{\partial f}{\partial t}+v\cdot\nabla_x f+\frac{e}{m}\left(E+\frac{v\times B}{c}\right)\cdot\nabla_v f=0 \tag{21}$$

describing a classical collisionless plasma in an electromagnetic field, is also a non-canonical Hamiltonian system. There are several variational formulations of the Maxwell-Vlasov system, the most complete one being probably the one by Marsden and Weinstein [13]. However, in their formulation, part of the dynamics is coded on the Poisson structure rather than on the Hamiltonian and to apply the deformation theory for the transition to quantum mechanics, one would also need to handle the deformation of the electromagnetic fields dynamics, not just the replacement of the Poisson bracket involving position and momentum of the particles. Hence, because here one only wants to obtain the quantum corrections to the f dynamics, it is more convenient to use the Low [14] Hamiltonian,

$$H_{MV} = \int d^3x d^3p \left\{ \frac{1}{2m} \left(p - \frac{e}{c} A \right)^2 + e\phi(x) \right\} f(x,p,t) + \int d^3x \left(E^2 + B^2 \right) \quad (22)$$

where $E = -\nabla_x \phi - \frac{1}{c}\frac{\partial A}{\partial t}$, $B = \nabla \times A$ in terms of the independent variables (ϕ, A).

The Poisson structure is the same as in (14) for the f dynamics. With this Hamiltonian

$$\frac{\delta H_{MV}}{\delta f(x,p,t)} = \frac{1}{2m} \left(p^2 - \frac{e}{c}(p \cdot A + A \cdot p) + \frac{e^2}{c^2} A^2 \right) + + e\phi(x) \quad (23)$$

Then, using (14) and (15) one obtains for the classical equation

$$\frac{\partial f}{\partial t} = -\frac{1}{m}\left(p - \frac{e}{c}A \right) \cdot \nabla_x f + \quad (24)$$
$$+ \left(\frac{e}{m} \nabla_x \phi - \frac{e}{mc} p \cdot \nabla_x A + \frac{e^2}{2mc^2} \nabla_x A^2 \right) \cdot \nabla_p f$$
$$= -\frac{1}{m}\left(p - \frac{e}{c}A \right) \cdot \nabla_x f + \quad (25)$$
$$+ \left(-eE - \frac{e}{c} v \times B - \frac{e}{c}\frac{dA}{dt} \right) \cdot \nabla_p f \quad (26)$$

with

$$\begin{aligned} \frac{dA}{dt} &= \frac{\partial A}{\partial t} + v \cdot \nabla_x A \\ v &= \frac{1}{m}\left(p - \frac{e}{c}A \right) \end{aligned} \quad (27)$$

Eq.(26) is the same as (21) written in the variables (x, p) instead of (x, v). The first set is the more convenient one because the Moyal bracket deformation acts on these variables. Then, the quantum Maxwell-Vlasov equation becomes[1]

$$\frac{\partial f}{\partial t} = \int d^3x d^3p f(x,p,t) \frac{2}{\hbar} \sin\left(\frac{\hbar}{2} P \right) \left(\frac{\delta f}{\delta f}, \frac{\delta H_{MV}}{\delta f} \right) \quad (28)$$

and, computing the leading quantum corrections, one obtains

$$\begin{aligned} \frac{\partial f}{\partial t} = \\ -\frac{1}{m}\left(p - \frac{e}{c}A \right) \cdot \nabla_x f + \left(-eE - \frac{e}{c} v \times B - \frac{e}{c}\frac{dA}{dt} \right) \cdot \nabla_p f \\ - \frac{e\hbar^2}{24} \sum_{i,j,k=1}^{3} \frac{\partial^3 f}{\partial p_i \partial p_j \partial p_k} \frac{\partial^3}{\partial x_i \partial x_j \partial x_k} \left(\phi + \frac{e}{2mc^2} A^2 \right) \\ + \frac{e\hbar^2}{24} \sum_{i,j,k=1}^{3} \frac{\partial^3 f}{\partial p_i \partial p_j \partial p_k} p \cdot \frac{\partial^3 A}{\partial x_i \partial x_j \partial x_k} + \\ + 3 \frac{\partial^2 f}{\partial p_i \partial p_j} \frac{\partial^3 A_k}{\partial x_i \partial x_j \partial x_k} + O\left(\hbar^4 \right) \end{aligned}$$

3.3 The Boltzmann equation

Here we will be concerned with the kinetic dynamics of a low density chargeless gas described by the following classical Boltzmann equation [15] for $t \in [0, T)$

$$\left(\partial_t + \frac{p}{m} \cdot \nabla_x \right) f_1 = Q(f_1, f_1) \quad (29)$$

the collision term $Q(f, f)$ being

$$Q(f_1, f_1)(x,p) = \int d^3p_1 \int d\overrightarrow{n} B\left(\overrightarrow{n}, |p - p_1| \right) \times \left(f_1(x,p') f_1(x,p_1') - f_1(x,p) f_1(x,p_1) \right) \quad (30)$$

with p' and p_1' the outgoing momenta after the collision and p and p_1 the incoming momenta.

$$\begin{aligned} p' &= p - \left((p - p_1) \cdot \overrightarrow{n} \right) \overrightarrow{n} \\ p_1' &= p_1 + \left((p - p_1) \cdot \overrightarrow{n} \right) \overrightarrow{n} \end{aligned} \quad (31)$$

$\overrightarrow{n} \in S^2$ is the impact parameter and the integration in (30) is restricted to $(p - p_1) \cdot \overrightarrow{n} \geqslant 0$.

[1] Notice that in Hamiltonian, products should also be replaced by $*-$products. However $p * A + A * p = 2p \cdot A$

$B(\overrightarrow{n},|p-p_1|)$ is the collision kernel (cross section) which depends on the particles interacting potential. For a gas of hard spheres it is [16] [17]

$$B(\overrightarrow{n},|p-p_1|)=\overrightarrow{n}\cdot(p-p_1) \quad (32)$$

The Boltzmann equation does not follow directly from an Hamiltonian, its irreversible nature arising mostly from the choice of incoming configurations to represent the collisions [18]. To obtain the quantum version of the Boltzmann equation basically two approaches have been followed. The first is just to solve the scattering problem in quantum mechanics and then to replace, in the classical Boltzmann equation, the classical cross section by the quantum cross section. For weakly coupled gases the Born approximation has been used. The second, more sophisticated approach, starting from the Schrödinger equation for a many-body problem writes the evolution equation for the Wigner function and then goes through limiting steps analogous to the classical ones to obtain an equation for the one-particle marginal of the Wigner function (see [19] and references therein).

The quantum mechanical computation of the cross section is always, of course, a necessary step. It will depend on the particular interaction potential and we will have nothing to say about it. Our attention will focus on the geometric term $(f_1(x,p')f_1(x,p_1')-f_1(x,p)f_1(x,p_1))$ and on eventual quantum corrections to this term. Because the quantum deformation is only defined for Hamiltonian systems, our approach will be to proceed as far as possible in a reversible Hamiltonian framework and only in the final stage, after the quantum deformation has been taken into account, do we make use of the specific approximations leading to the Boltzmann equation. Consider a system of N particles interacting by a short-range two-body potential ϕ. The Hamiltonian which drives the evolution of the $N-$particles density is

$$H=\int\prod_{i=1}^{N}dx_i^3dp_i^3f^N(x_i,p_i)\left(\sum_{i=1}^{N}\frac{p_i^2}{2m}+\sum_{i<j}\phi(\overrightarrow{x_i}-\overrightarrow{x_j})\right) \quad (33)$$

Taking into account the quantum deformation, the time evolution of the $N-$particle density f_N is, as before, given by

$$[f_N,H]=\int f_N\left\{\frac{\delta f_N}{\delta f_N},\frac{\delta H}{\delta f_N}\right\}_M\prod_{i=1}^{N}dx_i^3dp_i^3 \quad (34)$$

which, in leading order, is

$$\frac{\partial f_N}{\partial t}=-\sum_{i=1}^{N}\frac{p_i}{m}\cdot\nabla_x f_N+\sum_{i<j}\nabla\phi(\overrightarrow{x_i}-\overrightarrow{x_j})\cdot(\nabla_{p_i}-\nabla_{p_j})f_N- -\frac{\hbar^2}{24}\sum_{i,j,k,a,b,c}\frac{\partial^3}{\partial x_i^a\partial x_j^b\partial x_k^c}\left(\sum_{l<m}\phi(\overrightarrow{x_l}-\overrightarrow{x_m})\right)\frac{\partial^3 f_N}{\partial p_i^a\partial p_j^b\partial p_k^c}+O(\hbar^4) \quad (35)$$

Integrating over the $N-1$ identical particles to obtain the one-particle marginal f_1

$$\left(\partial_t+\frac{p}{m}\cdot\nabla_x\right)f_1(x,p)= (N-1)\int d^3p_1dx_1^3\nabla_{x_1}\phi(\overrightarrow{x}-\overrightarrow{x_1})\cdot\nabla_p f_2(x,p,x_1,p_1) -\frac{\hbar^2}{24}(N-1)\int d^3p_1dx_1^3\sum_{a,b,c}\frac{\partial^3}{\partial x_1^a\partial x_1^b\partial x_1^c}\phi(\overrightarrow{x}-\overrightarrow{x}_1)\frac{\partial^3 f_2}{\partial p^a\partial p^b\partial p^c} \quad (36)$$

In spherical coordinates centered at x, $dx_1^3=r^2drd\overrightarrow{n}$. Make now the assumption that the two-body potential is a hard core potential, with

$$\nabla_x\phi(\overrightarrow{x}-\overrightarrow{x}_1)=\mu(\overrightarrow{n})\delta(r-\varepsilon)\overrightarrow{n} \quad (37)$$

$\mu(\overrightarrow{n})$ coding both for the intensity and the impact parameter dependence of the potential. The equation becomes

$$\left(\partial_t+\frac{p}{m}\cdot\nabla_x\right)f_1(x,p)= -(N-1)\varepsilon^2\int d^3p_1d\overrightarrow{n}\mu(\overrightarrow{n})\overrightarrow{n}\cdot\nabla_p f_2(x,p,x+\varepsilon\overrightarrow{n},p_1) +\frac{\hbar^2}{24}(N-1)\varepsilon^2\sum_{a,b}\frac{\partial^2}{\partial x^a\partial x^b}\frac{\partial^2}{\partial p^a\partial p^b} \int d^3p_1d\overrightarrow{n}\mu(\overrightarrow{n})\overrightarrow{n}\cdot\nabla_p f_2(x,p,x+\varepsilon\overrightarrow{n},p_1) +O(\hbar^4) \quad (38)$$

and in the (Boltzmann-Grad) limit $(N-1)\,\varepsilon^2 \to \lambda^{-1}$

$$\left(\partial_t + \frac{p}{m}\cdot\nabla_x\right) f_1(x,p) = -\lambda^{-1}\int d^3p_1 d\overrightarrow{n}\mu(\overrightarrow{n})\,\overrightarrow{n}\cdot\nabla_p f_2(x,p,x,p_1) + \frac{\hbar^2}{24}\lambda^{-1}\sum_{a,b}\frac{\partial^2}{\partial x^a\partial x^b}\frac{\partial^2}{\partial p^a\partial p^b}\int d^3p_1 d\overrightarrow{n}\mu(\overrightarrow{n})\,\overrightarrow{n}\cdot\nabla_p f_2(x,p,x,p_1) + O\left(\hbar^4\right) \quad (39)$$

Up to this point we have stayed in the Hamiltonian framework and even the Boltzmann-Grad limit is simply a scaling low density limit. Incidentally this shows that this limit can only have a marginal effect on the irreversibility of the Boltzmann equation, a fact also noticed by other authors [18]. Rather the irreversible collision terms of the Boltzmann equation [20] [21] [22] are obtained by the restriction to two-body collisions, factorization of the two-particle marginal and the restriction of the integration to the incoming particles. The last two choices are actually related, because factorization implies independence and after the collision the particles are certainly correlated.

All the collision term contributions must be contained in the term

$$\int d^3p_1\int_{S^2} d\overrightarrow{n}\mu(\overrightarrow{n})\,\overrightarrow{n}\cdot\nabla_p f_2(x,p,x,p_1) \qquad (40)$$

which appears both in the classical and the deformed part of Eq.(39). Therefore using the results of Refs. [21] [22] for short-range potentials one may directly obtain the final result, Eq.(43). In these works the short-range potential result is obtained by restricting the calculation to the free space outside the potential range, which gives a contribution analogous to the hard spheres problem and then showing that the inner space contribution vanishes in the Boltzmann-Grad limit. Here we show that a simple direct approach leads to the same result.

In the evolution equation (39) the term $\overrightarrow{n}\cdot\nabla_p f_2(x,p,x,p_1)$ stands for a continuous rate of variation of the density f_2 when the momentum p changes under the action of the force $\nabla_x\phi$. However, in the limit of the hard core potential (37) the variation of the momentum before and after the collision is discontinuous (as in a hard spheres collision) and we replace $\overrightarrow{n}\cdot\nabla_p f_2(x,p,x,p_1)$ by

$$\overrightarrow{n}.\nabla_p f_2(x,p,x,p_1) \simeq \frac{(p-p_1)\cdot\overrightarrow{n}}{\left((p-p_1)\cdot\overrightarrow{n}\right)^2}\left(f_2(x,p',x,p_1') - f_2(x,p,x,p_1)\right) \qquad (41)$$

Making this replacement in both terms of the right hand side of (39) and factorizing the two-body marginal leads to

$$\left(\partial_t + \frac{p}{m}\cdot\nabla_x\right) f_1(x,p) = \int d^3p_1\int_{S_+^2} d\overrightarrow{n}\lambda^{-1}\mu(\overrightarrow{n})\frac{(p-p_1)\cdot\overrightarrow{n}}{\left((p-p_1)\cdot\overrightarrow{n}\right)^2}\left\{\left(f_1(x,p')f_1(x,p_1') - f_1(x,p)f_1(x,p_1)\right) - \frac{\hbar^2}{24}\sum_{a,b}\frac{\partial^2}{\partial p^a\partial p^b}f_1(x,p')\frac{\partial^2}{\partial x^a\partial x^b}f_1(x,p_1') - \frac{\partial^2}{\partial p^a\partial p^b}f_1(x,p)\frac{\partial^2}{\partial x^a\partial x^b}f_1(x,p_1)\right\} + O\left(\hbar^4\right) \quad (42)$$

Generalizing to a general scattering kernel $B\left(\overrightarrow{n},|p-p_1|\right)$,

$$\begin{aligned}\left(\partial_t+\frac{p}{m}\cdot\nabla_x\right)f_1\left(x,p\right)\\ =\int d^3p_1\int_{S_+^2}d\overrightarrow{n}B\left(\overrightarrow{n},|p-p_1|\right)\Bigg\{\left(f_1\left(x,p'\right)f_1\left(x,p_1'\right)-f_1\left(x,p\right)f_1\left(x,p_1\right)\right)\\ -\frac{\hbar^2}{24}\sum_{a,b}\left(\frac{\partial^2}{\partial p^a\partial p^b}f_1\left(x,p'\right)\frac{\partial^2}{\partial x^a\partial x^b}f_1\left(x,p_1'\right)-\frac{\partial^2}{\partial p^a\partial p^b}f_1\left(x,p\right)\frac{\partial^2}{\partial x^a\partial x^b}f_1\left(x,p_1\right)\right)\Bigg\}\\ +O\left(\hbar^4\right)\end{aligned} \quad (43)$$

In the usual construction of the collision term in the classical Boltzmann equation, terms of order $1/N$ are neglected. For practical finite N configurations these terms might indeed be larger than the quantum corrections. Nevertheless, the essential point of our result is that, if all the Boltzmann assumptions are valid, then not only the cross section but also the geometric collision term must have quantum corrections for non-uniform densities. Another relevant remark concerns the fact that we have assumed Maxwell-Boltzmann statistics for the identical particles. Bosonic or fermionic statistics would correspond to the replacement of the term $\left(f_1\left(x,p'\right)f_1\left(x,p_1'\right)-f_1\left(x,p\right)f_1\left(x,p_1\right)\right)$ by [23]

$$\begin{aligned}&\{f_1\left(x,p'\right)f_1\left(x,p_1'\right)\left(1+\theta f_1\left(x,p\right)\right)\left(1+\theta f_1\left(x,p_1\right)\right)\\ &-f_1\left(x,p\right)f_1\left(x,p_1\right)\left(1+\theta f_1\left(x,p'\right)\right)\left(1+\theta f_1\left(x,p_1'\right)\right)\}\end{aligned}$$

with a similar replacement in the deformation term and $\theta=+1$ or $\theta=-1$ for either Bose-Einstein or Fermi-Dirac statistics. In any case, for the low density limit, where the Boltzmann equation is most relevant, the particles are too rare to make statistical correlations a dominant effect.

References

[1] F. Haas; *Quantum Plasmas: an Hydrodynamic Approach,* Springer, New York, 2011.

[2] P.K. Shukla and B. Eliasson; *Nonlinear collective interactions in quantum plasmas with degenerate electron fluids,* Rev. Mod. Phys. 83 (2011) 885-906.

[3] P. K. Shukla and B. Eliasson; *Nonlinear aspects of quantum plasma physics*, Physics - Uspekhi 53 (2010) 51 - 76.

[4] F. Haas, G. Manfredi and J. Goedert; *Nyquist method for Wigner-Poisson quantum plasmas,* Phys. Rev. E 64 (2001) 026413.

[5] J. Dufty and J. Wrighton; *Kinetic theory for strongly coupled Coulomb systems,* Phys. Rev. E 97 (2018) 012149.

[6] F. Bayen, M. Flato, C. Fronsdal, C. Lichnerowicz and D. Sternheimer; *Deformation theory and quantization*, Ann. Phys (NY) 111 (1978) 61-151.

[7] J. Vey; *Déformation du crochet de Poisson sur une variété symplectique*, Comment. Math. Helv. 50 (1975), 421–454.

[8] M. De Wilde and P. Lecomte; *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. 7 (1983) 487-496.

[9] O. M. Neroslavsky and A. T. Vlassov; *Sur les deformations de l'algebre des fonctions d'une*

variété symplectique, C. R. Acad. Sci. Paris 292 I (1981) 71-73.

[10] M. Cahen and S. Gutt; *Regular star-representations of Lie algebras*, Lett. Math. Phys. 6 (1982) 395-404.

[11] J. Moyal; Proc. *Quantum mechanics as a statistical theory*, Proc. Cambridge Phil. Soc. 45 (1949) 99-124.

[12] P. J. Morrison; *Hamiltonian and action principle formulations of plasma physics*, Physics of Plasmas 12 (2005) 058102.

[13] J. E. Marsden and A. Weinstein; *The Hamiltonian structure of the Maxwell-Vlasov equations*, Physica D 4 (1982) 394-406.

[14] F. E. Low; *A Lagrangian formulation of the Boltzmann-Vlasov equation for plasmas*, Proc. R. Soc. London Ser. A 248 (1958) 282-287.

[15] L. Boltzmann; *Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen*, Wiener Berichte 66 (1872) 275–370.

[16] O. Lanford III; *Time evolution of large classical systems*, Lecture Notes in Physics, vol. 38, pp. 1-111, E. J. Moser (Ed.), Springer 1975.

[17] R. Illner and M. Pulvirenti; *Global validity of the Boltzmann equation for a two-dimensional rare gas in the vacuum*, Comm. Math. Phys. 105 (1986) 189-203, 121 (1989) 143-146.

[18] V. Ardourel; *Irreversibility in the derivation of the Boltzmann equation*, Foundations of Physics 47 (2017) 471-489.

[19] M. Pulvirenti; *On the quantum Boltzmann equation*, in Multiscale methods in quantum mechanics, Ph. Blanchard and G. Dell' Antonio (Eds.) pp. 129-138, Birkhäuser, Boston 2004.

[20] H. Grad; *Principles of the kinetic theory of gases*, Handbuch der Physik XII, pp. 205-294, Springer Berlin 1958.

[21] M. Pulvirenti, C. Saffirio and Simonella; *On the validity of Boltzmann equation for short range potentials*, Reviews in Math. Phys. 26 (2014) 1450001.

[22] I. Gallagher, L. Saint-Raymond, B. Texier; *From Newton to Boltzmann: Hard sphere and short range potentials*, Zurich Lectures in Advanced Mathematics, European Math. Soc. 2014.

[23] E.A. Uehling, G.E. Uhlembeck; *Transport Phenomena in Einstein-Bose and Fermi-Dirac Gases* I, Phys. Rev. 43 (1933) 552-561.