Distributed Source Coding Based on Integer-Forcing

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***Abstract*—One of the research areas in communications with significant development in recent years is distributed source coding (DSC). The main idea behind DSC is to exploit the existing spatial correlation among the observations of non-cooperating encoders. Integer-forcing source coding (IFSC) is a specific case of lossy DSC, in which all encoders employ the same nested lattice codebook to code their observations and send them individually to the decoder, which instead of directly retrieving the individual signals, first recovers a set of integer linear combinations of those signals and then inverts it to obtain the final estimates within some predefined distortion measure. A comprehensive study of this scheme is provided in this paper, which also addresses the underlying optimization problem of finding the appropriate integer coefficients to perform the integer linear combinations. As a first approach, this problem is solved with the LLL lattice reduction algorithm, which at best, yields an approximate solution, and so, an alternative algorithm that returns the exact solution based on the successive minima problem (SMP) is also explored. Further, the IFSC scheme is applied to a situation where the correlation among the sources belongs to a finite set of possible correlation models, each of which with a given known probability. Finally, a simpler version of IFSC is analysed in order to allow an easier implementation at the cost of performance degradation. All the work herein presented is supported by appropriate illustrations and simulation results in the form of rate-distortion curves.**

***Index Terms*—Lossy distributed source coding, integer-forcing source coding, spatial correlation, nested lattices, modulo-lattice reduction, successive minima problem.**

I. Introduction

In recent years, one of the research areas in communications with particular growth is the one involving wireless sensor networks. In contrast to other network elements with enough computational power to sustain all sort of applications, these sensors must work under a set of constraints like energy consumption and processing power. To address these limitations, one of the existing technologies is distributed source coding (DSC), that presents an effective source compression scheme based on the exploitation of the existing correlation between the information to be collected and transmitted. The research on DSC was born in the realms of information theory with the breakthroughs of Slepian and Wolf [1], and Wyner and Ziv [2], but recently both communication theory and signal processing have also contributed to this research [3]. Furthermore, since DSC is proven to provide efficient solutions in scenarios where a source correlation exists, it has been applied not only in wireless sensor networks but also in a wide range of applications. DSC can be divided into lossless and lossy DSC. The focus of this paper is on the latter, where the compression scheme requires a minimum rate to successfully recover the source at the decoder within some maximum distortion measure.

Fundamental to DSC is the *binning process* that consists on the partition of all possible outcomes of the source into disjoint sets or bins. With this partition, the decoder can infer about the information sent by one encoder that only sends the index corresponding to the bin its observation belongs to by exploiting the existing correlation this information has with the one received from a different encoder. This binning process was proven to provide a significant rate reduction and motivated researchers to propose various forms of codes for efficient binning in lossy DSC. One particular proposition was the contribution made by Zamir *et al.* in [4], where the use of nested lattice codes is suggested to perform this binning process in a lossy DSC context.

The concept of integer-forcing (IF) was first introduced for a new communication architecture based on IF linear receivers in [5]. In that work, a new type of linear receivers is presented for the Gaussian multiple-input multiple-output (MIMO) channel, based on the decoding of multiple integer linear combinations of the incoming symbols. Provided that the integer coefficients are chosen appropriately, these integer linear combinations can be solved for the original codewords. Based on the IF framework and on the use of nested lattice codes to perform the binning process, Ordentlich and Erez proposed in [6] a new scheme for lossy DSC: integer-forcing source coding (IFSC). This novel compression scheme is the groundwork for this paper that is organized as follows: section II exposes the lattice-based IFSC scheme; in section III the new algorithm proposed to solve the underlying optimization problem of IFSC and the results obtained are presented; section IV presents the proposed solution to the application of IFSC in a scenario with a set of possible models for the correlation of the sources; in section V the low-complexity version of IFSC is defined and implemented to attain the minimum performance degradation; finally, section VI concludes the paper.

II. Lattice-Based Integer-Forcing Source Coding

A. Overview of the Integer-Forcing Source Coding scheme

The IFSC scheme follows the ideia behind IF, where the receiver decodes integer linear combinations of the transmitted codewords in order to estimate the original signals. This is only possible if all the encoders and the decoder have a shared knowledge of the same nested lattice codebook $\mathcal{C} = \Lambda_f \cap \mathcal{V}(\Lambda_c)$, where Λ_f is the fine lattice and $\mathcal{V}(\Lambda_c)$ the Voronoi region of the coarse lattice Λ_c. Λ_f is good for MSE quantization and Λ_c is good for channel coding (cf. appendix A of the thesis document). Λ_f has a second moment $\sigma^2(\Lambda_f) = d$, where d can be seen as the variance of the quantization distortion irreversibly added in the quantization process by the encoders.

All the K encoders in this scheme employ the same encoding operation. The kth encoder uses a dither vector $\mathbf{d}_k$ statistically independent of the sources $\mathbf{x}_k \in \mathbb{R}^{1\times n}$ (n being the number of time realizations) and uniformly distributed over the basic Voronoi region of Λ_f, $\mathcal{V}(\Lambda_f)$, and performs dithered quantization of $\mathbf{x}_k$ onto Λ_f. Next, it reduces the obtained point modulo-Λ_c to obtain

$$\left[Q_{\Lambda_f}(\mathbf{x}_k + \mathbf{d}_k)\right] \bmod \Lambda_c,$$

which belongs to the codebook $\mathcal{C}$. This modulo reduction amounts to a binning operation, and, consequently, the encoder only transmits the least significant bits (LSBs) of the dithered quantization of the original source $\mathbf{x}_k$. These nR bits that describe the modulo reduced point are then sent to the decoder in the other terminal.

At the other terminal, the decoder, having full knowledge about vector $\mathbf{d}_k$ added at the encoder's side, subtracts it back and reduces the result modulo-Λ_c resulting in

$$\begin{aligned}\tilde{\mathbf{x}}_k &= \left[\left[Q_{\Lambda_f}(\mathbf{x}_k + \mathbf{d}_k)\right] \bmod \Lambda_c - \mathbf{d}_k\right] \bmod \Lambda_c \\ &= \left[Q_{\Lambda_f}(\mathbf{x}_k + \mathbf{d}_k) - \mathbf{d}_k\right] \bmod \Lambda_c \\ &= \left[\mathbf{x}_k + Q_{\Lambda_f}(\mathbf{x}_k + \mathbf{d}_k) - (\mathbf{x}_k + \mathbf{d}_k)\right] \bmod \Lambda_c \\ &= \left[\mathbf{x}_k + \mathbf{u}_k\right] \bmod \Lambda_c,\end{aligned} \tag{1}$$

where the second equality follows from the distributive law (cf. appendix B of the thesis document) and $\mathbf{u}_k$ is the estimation error. This estimation error is, as a consequence of the Crypto Lemma [7], uniformly distributed over the Voronoi region of the fine lattice Λ_f and statistically independent of $\mathbf{x}_k$ for all $k = 1, ..., K$.

The main feature of IFSC is applied as follows. If the elements of $\mathbf{x} = [x_1 \cdots x_K]^T \in \mathbb{R}^{K\times 1}$ (the column vector containing each encoder's observation at a given time slot) present a correlation expressed by a covariance matrix obtained through

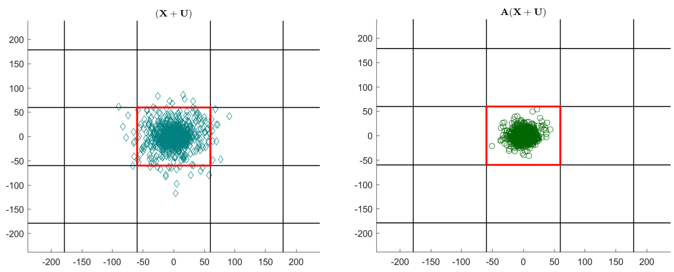
$$\mathbf{K_{xx}} \triangleq \mathbb{E}(\mathbf{x}\mathbf{x}^T), \tag{2}$$

then linear combinations of the vectors $\mathbf{x}_k + \mathbf{u}_k$, for $k = 1, ..., K$, with integer-valued coefficients may have a smaller variance than the original $\mathbf{x}_k + \mathbf{u}_k$. These integer-valued coefficients take a form of an integer-valued full-rank matrix $\mathbf{A} \in \mathbb{Z}^{K\times K}$ that depends on the covariance matrix $\mathbf{K_{xx}}$. If chosen appropriately, the integer-valued matrix $\mathbf{A}$ combined with $\mathbf{X}+\mathbf{U}$, where $\mathbf{X} = [\mathbf{x}_1^T \cdots \mathbf{x}_K^T]^T$ and $\mathbf{U} = [\mathbf{u}_1^T \cdots \mathbf{u}_K^T]^T$ are both $K \times n$ matrices, has the effect of "enclosing" $\mathbf{X}+\mathbf{U}$ in the basic Voronoi region of Λ_c. This effect is exemplified with the plots presented in figure 1.

With the precalculated full-rank integer-valued matrix $\mathbf{A}$, the decoder computes

$$\begin{aligned}\widehat{\mathbf{AX}} &\triangleq [\mathbf{A}\tilde{\mathbf{X}}] \bmod \Lambda_c \\ &= \left[\mathbf{A}\big([\mathbf{X}+\mathbf{U}] \bmod \Lambda_c\big)\right] \bmod \Lambda_c \\ &= \left[\mathbf{A}(\mathbf{X}+\mathbf{U})\right] \bmod \Lambda_c\end{aligned} \tag{3}$$

where the third equality follows from the "general" distributive law for matrices (cf. appendix B of the thesis document). One considers that $\mathbf{a}_k^T$ is the kth row of the matrix $\mathbf{A}$, and that



(a) (b)

Fig. 1. (a) The 2-dimensional samples of $\mathbf{X}$ are correlated through a covariance matrix $\mathbf{K_{xx}}$, and the sum $(\mathbf{X}+\mathbf{U})$ is plotted. (b) The integer linear combinations between the elements of $(\mathbf{X}+\mathbf{U})$ are enclosed in $\mathcal{V}(\Lambda_c)$ highlighted in red.

$r_{eff}(\Lambda_c)$ is the effective radius of Λ_c. It can be proven that since Λ_f is good for MSE quantization and Λ_c is good for channel coding, if

$$\max_{k=1,...,K} \frac{\mathbb{E}(\left\|\mathbf{a}_k^T(\mathbf{X}+\mathbf{U})\right\|^2)}{n} < \frac{r_{eff}^2(\Lambda_c)}{n}, \tag{4}$$

then

$$\left[\mathbf{A}(\mathbf{X}+\mathbf{U})\right] \bmod \Lambda_c = \mathbf{A}(\mathbf{X}+\mathbf{U}) \tag{5}$$

with high probability, and (3) turns into

$$\widehat{\mathbf{AX}} = \mathbf{A}(\mathbf{X}+\mathbf{U}). \tag{6}$$

The variance of the integer linear combinations between $\mathbf{X}+\mathbf{U}$ can be expressed by

$$\frac{\mathbb{E}(\left\|\mathbf{a}_k^T(\mathbf{X}+\mathbf{U})\right\|^2)}{n} = \mathbf{a}_k^T(\mathbf{K_{xx}} + d\mathbf{I})\mathbf{a}_k, \tag{7}$$

and so it is sufficient to set $r_{eff}^2(\Lambda_c)$ as the maximum of $\mathbf{a}_k^T(\mathbf{K_{xx}} + d\mathbf{I})\mathbf{a}_k$ for all $k = 1, ..., K$ to achieve (6). By the definition for the lattice's goodness for MSE quantization (cf. appendix A of the thesis document), and recalling that $\sigma^2(\Lambda_f) = d$ one has

$$\frac{r_{eff}^2(\Lambda_f)}{n} \to d$$

when n approaches infinity. Then, the attainable rate R for this scheme becomes

$$\begin{aligned}R &= \frac{1}{2}\log_2\left(\frac{r_{eff}^2(\Lambda_c)}{r_{eff}^2(\Lambda_f)}\right) \\ &= \frac{1}{2}\log_2\left(\max_{k=1,...,K} \mathbf{a}_k^T(\mathbf{I} + \frac{\mathbf{K_{xx}}}{d})\mathbf{a}_k\right).\end{aligned} \tag{8}$$

Finally, the decoder computes the estimates of $\mathbf{X}$ by applying the inverse matrix of $\mathbf{A}$ to (6) obtaining

$$\begin{aligned}\widehat{\mathbf{X}} &= \mathbf{A}^{-1}\widehat{\mathbf{AX}} \\ &= \mathbf{X}+\mathbf{U},\end{aligned} \tag{9}$$

and a final MSE

$$\frac{1}{n}\mathbb{E}(\|\hat{\mathbf{x}}_k - \mathbf{x}_k\|^2) = d, \tag{10}$$

which is obtained when n approaches infinity. The above described decoding process can be seen from a general perspective as the decoder using the correlation between the sources through $\mathbf{A}$ to return a best guess of the bin in which the quantized source originally was, and by that process estimating the most significant bits (MSBs) of that quantized source.

B. Performance of the Integer-Forcing Source Coding scheme

The performance of the IFSC scheme is analysed with respect to the rate-distortion vector (R, d) it achieves for the symmetric rate setup. For any distortion $d > 0$ and any full-rank integer-valued matrix $\mathbf{A}$ there exists a nested lattice pair such that this scheme can achieve any rate R that is greater than

$$R_{IFSC}(\mathbf{A}, d) \triangleq \frac{1}{2}\log_2\left(\max_{k=1,\ldots,K} \mathbf{a}_k^T(\mathbf{I} + \frac{\mathbf{K_{xx}}}{d})\mathbf{a}_k\right). \quad (11)$$

By choosing the full-rank integer-valued matrix $\mathbf{A}$ appropriately, the quadratic form inside the logarithm in (11) is minimized due to the "enclosing" effect, and $R_{IFSC}(\mathbf{A}, d)$ is also minimized. Therefore, one wants to choose an $\mathbf{A}$ such that (11) translates to

$$R_{IFSC}(d) \triangleq \frac{1}{2}\log_2\left[\min_{\substack{\mathbf{A}\in\mathbb{Z}^{K\times K}\\ \det(\mathbf{A})\neq 0}}\left(\max_{k=1,\ldots,K} \mathbf{a}_k^T(\mathbf{I} + \frac{\mathbf{K_{xx}}}{d})\mathbf{a}_k\right)\right]. \quad (12)$$

As a first approach, this matrix $\mathbf{A}$ is obtained with the LLL algorithm [8] by solving the problem detailed in section III.

C. Replication of Results

In order to obtain the rate-distortion curve given by (12), an example is required to provide a correlation model for the encoder's observations. The example considered is the one of a Gaussian network with M transmitters and K encoders, where the problem is that of compressing in a distributive way a K-dimensional Gaussian source $\mathbf{x}$ with zero mean and a covariance matrix given by

$$\mathbf{K_{xx}} = \mathrm{SNR}\,\mathbf{H}\mathbf{H}^T + \mathbf{I}, \quad (13)$$

where $\mathbf{H} \in \mathbb{R}^{K\times M}$ is the channel matrix between the M transmitters and the K relays which has entries i.i.d. with a Gaussian distribution $\mathcal{N}(0, 1)$.

To compare the performance of the IFSC scheme, a benchmark is defined as the rate-distortion vector attained by the Berger-Tung (BT) scheme [9], the best known achievable scheme for the problem of distributed lossy compression of jointly Gaussian random variables under a quadratic distortion measure [6], given by

$$R_{BT}(d) \triangleq \frac{1}{2K}\log_2\det\left(\mathbf{I} + \frac{1}{d}\mathbf{K_{xx}}\right). \quad (14)$$

It will also be considered in this comparison the rate-distortion vector obtained with a scheme that compresses each source without exploiting the correlation between the sources, the *naive scheme*. The rate for this scheme is given by

$$R_{Naive}(d) \triangleq \max_{k=1,\ldots,K} \frac{1}{2}\log_2\left(1 + \frac{\mathbf{K_{xx}}(k,k)}{d}\right), \quad (15)$$

where $\mathbf{K_{xx}}(k, k)$ is the kth diagonal entry of the matrix $\mathbf{K_{xx}}$. A distortion interval of $[-40; 20]$ dB where the step between distortion values is 2 dB was considered, and for each one a different covariance matrix was computed 1000 times so that the final rate is an average of 1000 rate values obtained for different covariance matrices.

Two setups were considered as in [6], the setup with $K = 4$ relays and $M = 4$ transmitters, and the setup with $K = 8$ relays and $M = 2$ transmitters. For both setups a SNR of 20 dB was considered. The rate-distortion curves for the three coding schemes in these two setups are presented in figures 2 and 3.

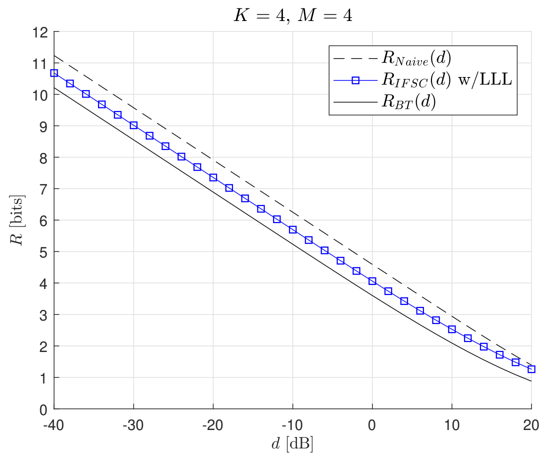


Fig. 2. Integer-forcing source coding rate-distortion curve for the setup with $K = M = 4$.

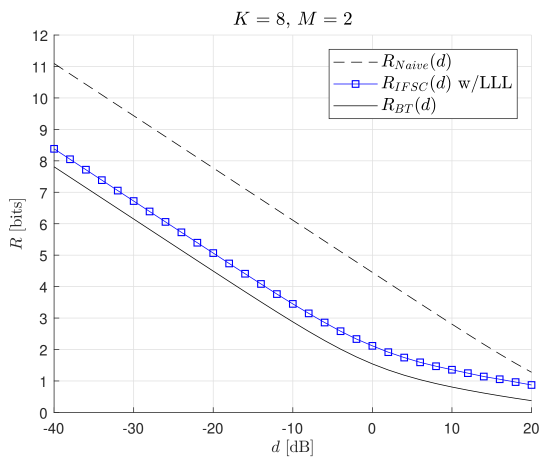


Fig. 3. Integer-forcing source coding rate-distortion curve for the setup with $K = 8$, $M = 2$.

The first conclusion that can be drawn from this results is that there is a significant gap between the schemes that take advantage of the existing correlation between sources and the one that does not. The second conclusion is that for both setups it is possible to see that $R_{IFSC}(d)$ follows the benchmark $R_{BT}(d)$ very closely with a gap of about half a bit for the target distortion interval considered.

III. Information-Theoretic Rate-Distortion of Integer-Forcing Source Coding

A. Underlying Optimization Problem of the IFSC Scheme

Successive decoding when using the IFSC scheme depends on an integer-valued full-rank matrix $\mathbf{A}$ to perform integer linear combinations between the signals received by the scheme's decoder. The choice of this matrix determines the rate-distortion vector attainable by the scheme, and therefore this matrix has to be chosen appropriately. The underlying optimization problem amounts to finding an optimal matrix $\mathbf{A}^*$ such that

$$\mathbf{A}^* = \underset{\substack{\mathbf{A}\in\mathbb{Z}^{K\times K}\\ \det(\mathbf{A})\neq 0}}{\arg\min} \left(\max_{k=1,...,K} \mathbf{a}_k^T(\mathbf{I} + \frac{\mathbf{K_{xx}}}{d})\mathbf{a}_k \right). \tag{16}$$

Noting that $\mathbf{G} = (\mathbf{I}+\frac{\mathbf{K_{xx}}}{d})$ is a symmetric and positive definite Gram matrix, it admits a Cholesky decomposition given by

$$(\mathbf{I} + \frac{\mathbf{K_{xx}}}{d}) = \mathbf{G} = \mathbf{R}^T\mathbf{R}, \tag{17}$$

where $\mathbf{R} \in \mathbb{R}^{K\times K}$ is an upper triangular matrix, and therefore (16) can be updated to

$$\mathbf{A}^* = \underset{\substack{\mathbf{A}\in\mathbb{Z}^{K\times K}\\ \det(\mathbf{A})\neq 0}}{\arg\min} \left(\max_{k=1,...,K} \|\mathbf{R}\mathbf{a}_k\|^2 \right). \tag{18}$$

The upper triangular matrix $\mathbf{R}$ can be seen as the generator matrix of a K-dimensional lattice $\Lambda(\mathbf{R})$ and each vector $\mathbf{a}_k$ a vector that spans a specific lattice point of $\Lambda(\mathbf{R})$ through the multiplication by $\mathbf{R}$. To solve the problem expressed by (18) one has to find an interger-valued full-rank matrix $\mathbf{A}^* = [\mathbf{a}_1^*, ..., \mathbf{a}_K^*] \in \mathbb{Z}^{K\times K}$ such that $\max_{k=1,...,K} \|\mathbf{R}\mathbf{a}_k\|^2$ is as small as possible. In a first approach, as mentioned before, this was solved with the LLL lattice reduction algorithm that returns a reduced generator matrix $\mathbf{R}_{red}$ that relates to $\mathbf{R}$ through

$$\mathbf{R}_{red} = \mathbf{R}\mathbf{T}, \tag{19}$$

where $\mathbf{T} \in \mathbb{Z}^{K\times K}$ is an integer-valued unimodular matrix and represents an approximate solution of the optimal matrix $\mathbf{A}^*$.

A different approach is to solve the SMP, that amounts to finding an integer-valued full-rank matrix $\mathbf{A}^*$ such that

$$\|\mathbf{R}\mathbf{a}_k^*\| = \theta_k \text{ for } k = 1, ..., K, \tag{20}$$

with θ_k being the kth successive minimum of a lattice $\Lambda(\mathbf{R})$.

B. Solving the Successive Minima Problem

To solve the SMP, the algorithm proposed in [10] is used and shall be henceforth referred to as the successive minima (SM) algorithm. Before tackling the SMP directly, this algorithm first uses a LLL reduction to pre-process the SMP. Specifically, the generator matrix $\mathbf{R}$ of the lattice of which one wants to find the K successive minima is first reduced using the LLL algorithm to obtain $\mathbf{R}_{red}$. $\mathbf{R}_{red}$ and $\mathbf{R}$ generate the same lattice. For the sake of efficiency of the algorithm, it is important that this reduced generator matrix is also an upper triangular matrix which is not the case after a LLL reduction. To guarantee that feature a QR decomposition is applied to $\mathbf{R}_{red}$ in order to obtain

$$\mathbf{R}_{red} = \bar{\mathbf{Q}}\bar{\mathbf{R}}, \tag{21}$$

where $\bar{\mathbf{Q}} \in \mathbb{R}^{K\times K}$ is an orthogonal matrix that when multiplied with $\bar{\mathbf{R}}$ amounts to rotating or reflecting the lattice generated by $\bar{\mathbf{R}}$, $\Lambda(\bar{\mathbf{R}})$, to obtain the lattice generated by $\mathbf{R}_{red}$, $\Lambda(\mathbf{R}_{red})$ [11].

Solving for the successive minima of lattice $\Lambda(\mathbf{R}_{red})$ is equivalent to solving the successive minima of lattice $\Lambda(\bar{\mathbf{R}})$ that results from a rotation of $\Lambda(\mathbf{R}_{red})$. With this in mind, the SMP can be updated to a reduced SMP (RSMP) where one wants to find an integer-valued full-rank matrix $\mathbf{C}^* = [\mathbf{c}_1^*, ..., \mathbf{c}_K^*] \in \mathbb{Z}^{K\times K}$ such that

$$\|\mathbf{R}_{red}\mathbf{c}_k^*\| = \left\|\bar{\mathbf{R}}\mathbf{c}_k^*\right\| = \theta_k, \tag{22}$$

for $k = 1, ..., K$. Since (20) and (22) are equivalent, one obtains the relationship between the solutions of the SMP and the RSMP

$$\mathbf{A}^* = \mathbf{T}\mathbf{C}^* \tag{23}$$

This first part of the algorithm can be summarized with the pseudocode presented in Algorithm 1.

Algorithm 1 Algorithm to solve the SMP

Input: A Gram matrix $\mathbf{G} \in \mathbb{R}^{K\times K}$.
Output: An integer-valued full-rank matrix $\mathbf{A}^* \in \mathbb{Z}^{K\times K}$.
1. Perform a Cholesky decomposition to $\mathbf{G}$ to obtain the nonsingular upper triangular matrix $\mathbf{R}$.
2. Perform a LLL reduction to $\mathbf{R}$ to obtain $\mathbf{R}_{red}$ and the unimodular matrix $\mathbf{T}$.
3. Apply a QR decomposition to $\mathbf{R}_{red}$ to get the nonsingular upper triangular matrix $\bar{\mathbf{R}}$.
4. Input $\bar{\mathbf{R}}$ in Algorithm 2 to find $\mathbf{C}^*$.
5. Compute $\mathbf{A}^* = \mathbf{T}\mathbf{C}^*$.

After this problem reduction, the algorithm starts with a suboptimal matrix $\mathbf{C}$ that corresponds to a $K \times K$ identity matrix with column permutations such that

$$\left\|\bar{\mathbf{R}}\mathbf{c}_1\right\| \leq \left\|\bar{\mathbf{R}}\mathbf{c}_2\right\| \leq \cdots \leq \left\|\bar{\mathbf{R}}\mathbf{c}_K\right\| \tag{24}$$

holds, and updates it until a solution to the RSMP is found. This update is made by finding new column vectors $\mathbf{c}_k$ such that at the end of the search procedure, $\mathbf{C}^*$ is an invertible integer-valued matrix and $\|\mathbf{c}_k^*\|$ are as short as possible, for $k = 1, ..., K$. An improved Schnorr-Euchner search algorithm proposed in [12] is used to solve this RSMP, where one starts by assuming that $\mathbf{c}$ is within the hyper-ellipsoid defined by

$$\left\|\bar{\mathbf{R}}\mathbf{c}\right\|^2 < \beta^2 \tag{25}$$

where β is a constant. If one considers that

$$q_i = -\frac{1}{\bar{r}_{ii}} \sum_{j=i+1}^{K} \bar{r}_{ij} c_j \text{ for } i = K-1, ..., 1, \tag{26}$$

then (25) can be updated to

$$\sum_{i=1}^{K} \bar{r}_{ii}^2 (c_i - q_i)^2 < \beta^2 \tag{27}$$

which in turn is equivalent to

$$\bar{r}_{ii}^2(c_i - q_i)^2 < \beta^2 - \sum_{j=i+1}^{K} \bar{r}_{jj}^2(c_j - q_j)^2 \tag{28}$$

for $i = K-1, ..., 1$, where i is designated the level index. The pseudocode of the algorithm that solves the RSMP is presented in Algorithm 2 where

$$\text{sgn}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}. \tag{29}$$

To update $\mathbf{C}^*$ in the Schnorr-Euchner search such that (24) holds, Algorithm 3 is also presented.

Algorithm 2 Algorithm to solve the RSMP

Input: An upper triangular matrix $\bar{\mathbf{R}} \in \mathbb{R}^{K\times K}$.
Output: A solution $\mathbf{C}^*$ to the RSMP.
1. Set the level index as $i = K$, the suboptimal matrix $\mathbf{C}$ as the $K \times K$ identity matrix with column permutations such that (24) holds, and $\beta = \left\|\bar{\mathbf{R}}\mathbf{c}_K\right\|$.
2. Set $c_i = \lfloor q_i \rceil$ where q_i is obtained through (26) and $s_i = \text{sgn}(q_i - c_i)$.
3. If (28) does not hold, then go to step 4. Else if $i > 1$, set $i = i-1$ and go to step 2. Else, i.e., $i = 1$, go to step 5.
4. If $i = K$ set $\mathbf{C}^* = \mathbf{C}$ and terminate. Else, set $i = i+1$ and go to step 6.
5. If $\mathbf{c} \neq \mathbf{0}$ use Algorithm 3 to update $\mathbf{C}$, set $\beta = \left\|\bar{\mathbf{R}}\mathbf{c}_K\right\|$ and $i = i+1$.
6. If $i = K$ or $\mathbf{c}_{i+1:K} = \mathbf{0}$, set $c_i = c_i + 1$. Else, set $c_i = c_i + s_i$, $s_i = -s_i - \text{sgn}(s_i)$. Go to step 3.

Algorithm 3 Algorithm to update $\mathbf{C}$

Input: A full column rank matrix $\mathbf{C} \in \mathbb{Z}^{K\times K}$, a nonzero column vector $\mathbf{c} \in \mathbb{Z}^{K\times 1}$ and a nonsingular upper triangular matrix $\bar{\mathbf{R}} \in \mathbb{R}^{K\times K}$.
Output: An invertible matrix $\bar{\mathbf{C}} \in \mathbb{Z}^{K\times K}$ whose columns are chosen from $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_K$ and $\mathbf{c}$ such that $\left\|\bar{\mathbf{R}}\bar{\mathbf{c}}_1\right\| \leq \left\|\bar{\mathbf{R}}\bar{\mathbf{c}}_2\right\| \leq \cdots \leq \left\|\bar{\mathbf{R}}\bar{\mathbf{c}}_K\right\|$ and $\left\|\bar{\mathbf{R}}\bar{\mathbf{c}}_k\right\|$ are as small as possible for all $1 \leq k \leq K$.
1. Find k and form $\tilde{\mathbf{C}} = [\mathbf{c}_1 \ \dots \ \mathbf{c}_k \ \ \mathbf{c} \ \ \mathbf{c}_{k+1} \ \dots \ \mathbf{c}_K]$ such that it satisfies $\left\|\bar{\mathbf{R}}\tilde{\mathbf{c}}_1\right\| \leq \cdots \leq \left\|\bar{\mathbf{R}}\tilde{\mathbf{c}}_{K+1}\right\|$.
2. Reduce $\tilde{\mathbf{C}}$ to its row echelon form by Gaussian elimination and denote it by $\mathbf{F}$ in order to efficiently find the index j of the column of $\tilde{\mathbf{C}}$ that is linearly dependent to $\tilde{\mathbf{c}}_i$ for some $1 \leq i \leq j-1$.
3. Check if it exists a j such that $f_{jj} = 0$ for $j = k+1, ..., K$. If it does, set $\bar{\mathbf{C}} = \tilde{\mathbf{C}}_{[\setminus j]}$. Otherwise if $f_{jj} \neq 0$ for all $k+1 \leq j \leq K$, set $\bar{\mathbf{C}} = \tilde{\mathbf{C}}_{[\setminus (K+1)]}$.

C. Complexity of the Successive Minima Algorithm

The proof of the optimality of the previously described algorithm to solve the SMP is provided in [10] as well as an expression for the expected value of its time complexity given by

$$\mathbb{E}\big[C(K)\big] = \mathcal{O}\big(K^{\frac{5}{2}}(2\pi e)^{\frac{K}{2}}\big), \tag{30}$$

where $C(K)$ denotes the time complexity with respect to the dimension of the problem K. Due to the use of the tree search algorithm, the SM algorithm has an exponential part that grows with $\frac{K}{2}$ which leads to an exponential overall running time. The complexity of the standard LLL algorithm used is provided in [13] as $\mathcal{O}(K^4 \log K)$. The LLL algorithm is then of polynomial complexity in K. In terms of complexity it is clear that the SM algorithm presents a higher complexity than the LLL algorithm since the former has an exponential running time, and the latter a polynomial running time.

D. Simulation of the Successive Minima Algorithm

In order to compare the rates obtained when using the LLL algorithm to solve the underlying optimization problem of the IFSC scheme with the rate obtained when using the SM algorithm, the example of section II was used to provide a correlation model. Since the SM algorithm yields the exact solution to the underlying optimization problem, the rate-distortion curve $R_{IFSC}(d)$ obtained with this algorithm for any setup considered corresponds to the minimum rate required to obtain a maximum distortion d attainable with the IFSC scheme. For comparison, the rate-distortion curve for the IFSC scheme using the LLL algorithm was plotted, as well as the rate-distortion curves attained with the Berger-Tung scheme and with the naive scheme.

For the setup with $K = M = 4$ the rate-distortion curves obtained for the IFSC scheme using the SM algorithm and using the LLL algorithm match as it is possible to observe in figure 4. From this it is possible to conclude that for $K = 4$, which is the dimension of the optimization problem, the LLL algorithm returns a solution that is optimal since it is equal to the one returned by the SM algorithm that is an exact one.

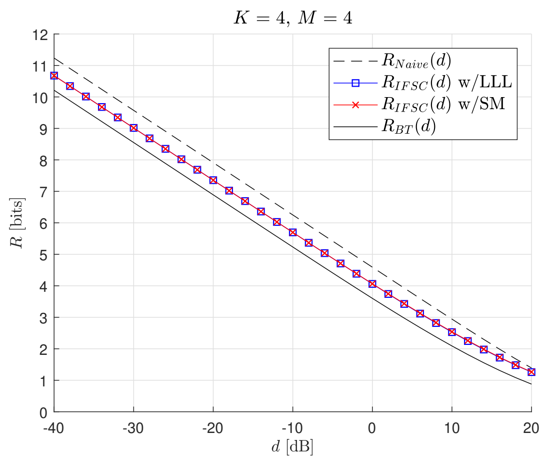


Fig. 4. Integer-forcing source coding rate-distortion curves with the SM algorithm for the setups $K = M = 4$.

For an increase of the dimension of the underlying optimization problem to $K = 25$, the approximate solution found by the LLL algorithm becomes a poor approximation of the exact solution as it is possible to observe in figure 5. An important conclusion is that there is a degradation in performance with respect to the optimum attainable rate for a given target distortion d for increasing K when using the LLL algorithm in the IFSC scheme.

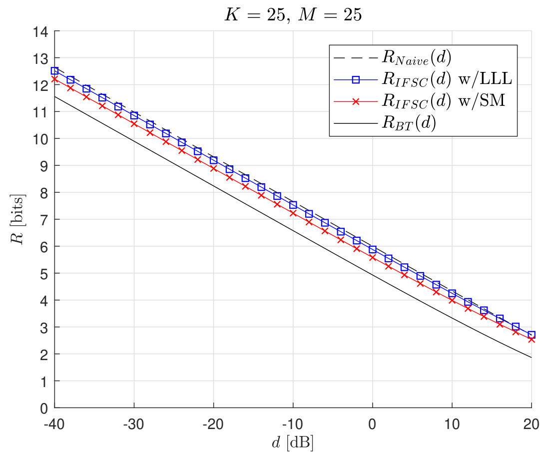


Fig. 5. Integer-forcing source coding rate-distortion curves with the SM algorithm for the setups $K = M = 25$.

IV. Semi-Blind Correlation Model

A. Problem Statement

The correlation model that is described by the covariance matrix $\mathbf{K_{xx}}$ can change over time and from situation to situation. To tackle this problem, one first considers the binary case where two covariance matrices that describe two different correlation models are possible. These covariance matrices are denoted by $\mathbf{K_{xx}^A}$ and $\mathbf{K_{xx}^B}$, and are both known by the decoder beforehand, which remains "blind" to which of these matrices best describes the correlation model for a given time slot. By having two possible covariance matrices to describe the correlation model between the sources, the problem that arises is the one of deciding which matrix $\mathbf{A}^*$ should be computed. If this computation is performed with the optimal matrix $\mathbf{A_A^*}$ obtained for $\mathbf{K_{xx}^A}$, this does not guarantee that the same optimal matrix $\mathbf{A_A^*}$ also spans the lattice's linearly independent points closest to the origin of a lattice $\Lambda(\mathbf{R_B})$.

B. Proposed Solution

One starts by assuming $\mathbf{K_{xx}^A}$ and $\mathbf{K_{xx}^B}$ have probabilities:

$$P(\mathbf{K_{xx}^A}) = p \tag{31}$$

and

$$P(\mathbf{K_{xx}^B}) = 1 - p \tag{32}$$

for some $p \in [0;1]$. For a random variable δ that follows a Bernoulli distribution, i.e., that takes the value 1 with probability p and takes value 0 with probability $1-p$, a covariance matrix is selected at a given time slot according to

$$\tilde{\mathbf{K}}_{\mathbf{xx}} = \delta \mathbf{K_{xx}^A} + (1-\delta)\mathbf{K_{xx}^B}. \tag{33}$$

This means that $\tilde{\mathbf{K}}_{\mathbf{xx}}$ is equal to either $\mathbf{K_{xx}^A}$ or to $\mathbf{K_{xx}^B}$ depending on the random binary variable δ. It is with this selected covariance matrix $\tilde{\mathbf{K}}_{\mathbf{xx}}$ that the rate $R_{IFSC}(d)$ attained by the IFSC scheme for a given target distortion d given by (12) is computed. The optimal matrix $\mathbf{A}^*$ is not obtained considering $\mathbf{R_A}$ or $\mathbf{R_B}$, but rather an extended generator matrix $\mathbf{R_{ext}} \in \mathbb{R}^{2K\times K}$ constructed by stacking the two in the following manner:

$$\mathbf{R_{ext}} = \begin{bmatrix} \sqrt{p}\,\mathbf{R_A} \\ --- \\ \sqrt{1-p}\,\mathbf{R_B} \end{bmatrix}_{2K\times K}. \tag{34}$$

This concatenation creates another lattice generating matrix that, when searching for the shortest vectors of the lattice it spans, it takes into account, in different sets of sub-dimensions, the two constituting lattice generating matrices "weighted" by the probability of the respective scenario. An important feature of this proposed solution is that the obtained $\mathbf{A_{ext}^*}$ is always a $K \times K$ matrix. This results from the fact that even though $\mathbf{R_{ext}}$ is a $2K \times K$ matrix (for this binary case), it generates a K-dimensional lattice since there are only K generating vectors (in the columns of $\mathbf{R_{ext}}$), even though they "live" in a $2K$ dimensional space. $\mathbf{R_{ext}}$ can be decomposed using a QR decomposition to obtain

$$\mathbf{R_{ext}} = \bar{\mathbf{Q}}\bar{\mathbf{R}} \tag{35}$$

where $\bar{\mathbf{Q}}$ is a $2K \times 2K$ orthogonal matrix and $\bar{\mathbf{R}}$ is a $2K \times K$ matrix. $\mathbf{R_{ext}}$ can also be expressed as

$$\mathbf{R_{ext}} = \bar{\mathbf{Q}}'\bar{\mathbf{R}}' \tag{36}$$

where $\bar{\mathbf{Q}}'$ is a $2K \times K$ orthogonal matrix that corresponds to the first K columns of $\bar{\mathbf{Q}}$. Expression (36) can be referred to as an "economy" QR decomposition of $\mathbf{R_{ext}}$ since it only considers the first K columns of $\bar{\mathbf{Q}}$ and the first K rows of $\bar{\mathbf{R}}'$, ignoring the remaining ones. Solving for the SM of lattice $\Lambda(\mathbf{R_{ext}})$ is the same as solving for the SM of lattice $\bar{\mathbf{R}}'$ and so, no matter the number of possible covariance matrices to describe the correlation model in the IFSC scheme, the underlying optimization problem of finding $\mathbf{A_{ext}^*}$ is always solved on a K-dimensional lattice.

It is possible to generalize beyond the binary case of two possible covariance matrices. Considering a set of N possible covariance matrices, these follow

$$\sum_{i=1}^{N} P(\mathbf{K_{xx}^i}) = 1 \tag{37}$$

and

$$P(\mathbf{K_{xx}^i}) = p_i \tag{38}$$

for $i = 1, ..., N$, and $p_i \in [0;1]$. The selected covariance matrix for which the rate is calculated is now given by

$$\tilde{\mathbf{K}}_{\mathbf{xx}} = \delta_1 \mathbf{K_{xx}^1} + ... + \delta_N \mathbf{K_{xx}^N}, \tag{39}$$

where δ_i takes value 1 with probability p_i and 0 with probability $1-p_i$ for $i = 1, ..., N$. Note that these Bernoulli random variables are restricted by $\delta_1 + ... + \delta_N = 1$. The extended lattice generator matrix $\mathbf{R_{ext}}$ is defined as

$$\mathbf{R_{ext}} = \begin{bmatrix} \sqrt{p_1}\,\mathbf{R_1} \\ --- \\ \vdots \\ --- \\ \sqrt{p_N}\,\mathbf{R_N} \end{bmatrix}_{NK\times K}, \tag{40}$$

where $\mathbf{R}_1, ..., \mathbf{R}_N$ are the lattice generator matrices that result from each corresponding covariance matrix $\mathbf{K}^{\mathbf{1}}_{\mathbf{xx}}, ..., \mathbf{K}^{\mathbf{N}}_{\mathbf{xx}}$. For a situation with three possible covariance matrices the row dimension of $\mathbf{R}_{\mathbf{ext}}$ is $3K$, for four is $4K$ and so on for increasing N. This linear increase in the dimension of the extended generator matrix, as stated before, does not represent an increase in the computational burden since the problem is always reduced to a problem of finding the K successive minima of a lattice generated by a $K \times K$ matrix obtained through the "economy" QR decomposition of $\mathbf{R}_{\mathbf{ext}}$.

C. Simulation of the Proposed Solution

The example used to simulate the previously exposed proposed solution to approach the semi-blind correlation model problem is the same as used in section II. This time however, instead of randomly generating only a channel matrix $\mathbf{H}$ that would lead to a covariance matrix $\mathbf{K}_{\mathbf{xx}}$, N channel matrices were randomly generated to compute N different possible covariance matrices. Each of these N possible covariance matrices had a probability associated with it that was changed from simulation to simulation. It is important to note that the algorithm used to solve this optimization problem was the SM algorithm, since it is the one that yields the exact solution. The rate for the IFSC scheme was obtained through expression (12) for the selected covariance matrix $\tilde{\mathbf{K}}_{\mathbf{xx}}$, the optimal matrix $\mathbf{A}^*_{\mathbf{ext}}$ and the given target distortion d. As before, this was performed for each target distortion 1000 times so that the final rate corresponds to an average of 1000 rates.

The size of the set of possible covariance matrices considered in the simulations presented in this paper was $N = 2$ (the binary case) for $K = 8$ encoders and $M = 2$ transmitters. These simulations are compared to the previously obtained results for this setup with the IFSC scheme using the SM algorithm, the Berger-Tung scheme and the naive scheme, where only a single possible covariance matrix was considered in each of the 1000 simulations for each target distortion d.

1) $N = 2$ Possible Covariance Matrices: The two possible covariance matrices $\mathbf{K}^{\mathbf{A}}_{\mathbf{xx}}$ and $\mathbf{K}^{\mathbf{B}}_{\mathbf{xx}}$ have a probability $P(\mathbf{K}^{\mathbf{A}}_{\mathbf{xx}}) = p$ and $P(\mathbf{K}^{\mathbf{B}}_{\mathbf{xx}}) = 1 - p$ as stated before. For $p = 0.8$ the proposed solution outperforms the naive scheme as observed in figure 6.

When considering $p = 0.5$ the worst performance among all values for p is obtained, but still leads to a rate-distortion curve below the one obtained with the naive scheme as observed in figure 7, which indicates a better performance. One can conclude from this result that even with total uncertainty about which of the two covariance matrices best describes the correlation model between encoders, it is advantageous to use the the IFSC scheme with the approach proposed when the number of encoders (K) outnumber the number of transmitters (M), which translates to introducing a higher correlation between the observations of the encoders.

2) Mismatch of the Correlation Models: To further assess the proposed solution another simulation was made, this time with a mismatch between the probability of a given covariance matrix and its *true* probability of becoming the selected covariance matrix for which the rate rate $R_{IFSC}(d)$

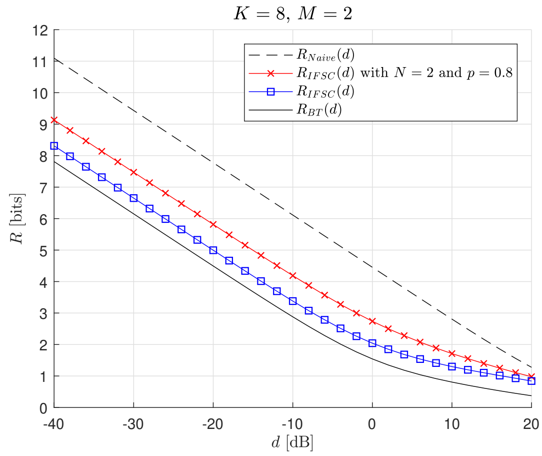


Fig. 6. Rate-distortion curves for $N = 2$ possible covariance matrices with $p = 0.8$ in the setup $K = 8$ and $M = 2$.

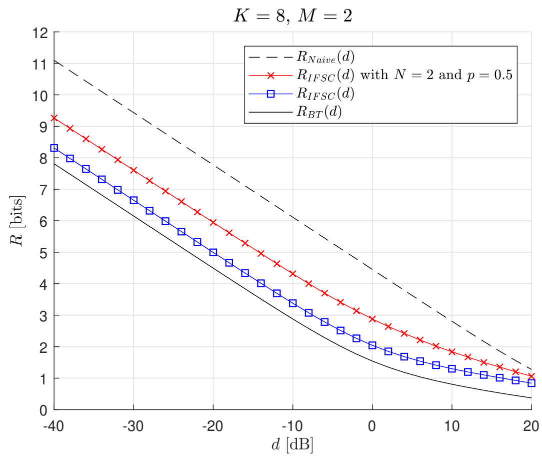


Fig. 7. Rate-distortion curves for $N = 2$ possible covariance matrices with $p = 0.5$ in the setup $K = 8$ and $M = 2$.

is computed. The mismatch is created by considering that the selected covariance matrix is given by

$$\tilde{\mathbf{K}}_{\mathbf{xx}} = (1 - \delta)\mathbf{K}^{\mathbf{A}}_{\mathbf{xx}} + \delta\mathbf{K}^{\mathbf{B}}_{\mathbf{xx}}. \tag{41}$$

instead of (33). As a consequence of this mismatch, the IFSC scheme with the solution proposed will compute, with the aid of the SM algorithm, an optimal matrix $\mathbf{A}^*_{\mathbf{ext}}$ that is more appropriate to a scenario with a lower probability. The compromise made is then unfair in the sense that it benefits a scenario that is not the most likely to occur.

For the case with $p = 0.8$, even with a performance degradation the IFSC scheme outperforms the naive coding scheme as depicted in figure 8, indicating that even with a mismatch situation it still is advantageous to exploit correlation provided that there is enough correlation between the observations of the encoders.

V. One-Shot Integer-Forcing Source Coding

In this approach, the IFSC process is applied individually to each one of the n time realizations of the K observations

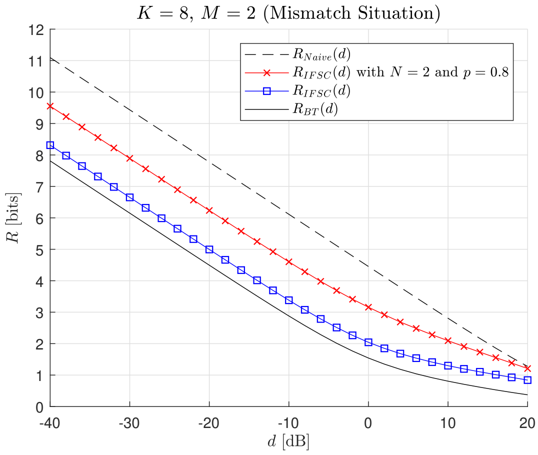


Fig. 8. Rate-distortion curves for $N = 2$ possible covariance matrices with $p = 0.8$ for the setup with $K = 8, M = 2$ in a mismatch situation.

retrieved by the K encoders. So, instead of using a n-D (n-dimensional) nested lattice pair, the OIFSC scheme requires only a 1-D nested lattice pair per dimension where both lattices are scaled versions of the integer lattice $\mathbb{Z}$. The reason for this decoupling of the dimensions comes from the orthogonality of both $\mathbb{Z}^n$ lattices.

A. 1-D Nested Lattice Pair

Given a quantization scheme with a quantization step size q and a quantization error e, the probability density function of the quantization error is typically uniform over the quantization interval,

$$\mathrm{P}(e) = \frac{1}{q}\mathrm{rect}\Big(\frac{e}{q}\Big), \tag{42}$$

where $\mathrm{rect}(\cdot)$ is the rectangular function. Let n_q^2 be the quantization noise power. Then,

$$n_q^2 \triangleq \mathbb{E}(e^2) = \int_{-\frac{q}{2}}^{\frac{q}{2}} e^2 \frac{1}{q}\, de = \frac{q^2}{12}, \tag{43}$$

and, denoting the MSE by d, which is equal to the quantization noise power n_q^2 in (43), one obtains the relation $q = \sqrt{12d}$. With the quantization step size q established so that the OIFSC scheme achieves an average MSE d, it is now possible to define the 1-D nested lattice pair, where the fine lattice is given by $\Lambda_f = \sqrt{12d}\,\mathbb{Z}$ and the coarse lattice by $\Lambda_c = 2^R\sqrt{12d}\,\mathbb{Z}$. As stated in [6], if 2^R is a positive integer then $\Lambda_c \subseteq \Lambda_f$ and the codebook $\mathcal{C} = \Lambda_f \cap \mathcal{V}_c$ with rate R, where $\mathcal{V}_c$ is the Voronoi region of the coarse lattice Λ_c, is a valid codebook for IFSC.

B. OIFSC Scheme Definition

In the OIFSC scheme the IFSC process is applied individually to each one of the n time realizations of the K observations retrieved by the K encoders. One can refer to this as a per dimension slicing.

Every encoder in the OIFSC scheme has the same 1-D nested lattice pair defined in the previous section, and so, the kth encoder starts the process by adding the dither value d_k to its observation x_k and quantizing the result on the fine lattice Λ_f. It then proceeds by reducing this quantized point modulo-Λ_c, obtaining the lattice point

$$[Q_{\Lambda_f}(x_k + d_k)] \bmod \Lambda_c.$$

x_k represents the observation of the kth encoder at a given time realization n, where the index n is dropped.

At the decoder, the dither values are subtracted from the corresponding received points and the result is reduced modulo Λ_c to obtain

$$\begin{aligned}\tilde{x}_k &= \big[[Q_{\Lambda_f}(x_k + d_k)] \bmod \Lambda_c - d_k\big] \bmod \Lambda_c \\ &= \big[Q_{\Lambda_f}(x_k + d_k) - d_k\big] \bmod \Lambda_c \\ &= \big[x_k + Q_{\Lambda_f}(x_k + d_k) - (x_k + d_k)\big] \bmod \Lambda_c \\ &= \big[x_k + u_k\big] \bmod \Lambda_c,\end{aligned} \tag{44}$$

where u_k is the estimation error that is uniformly distributed over the quantization interval. After that, the process performed by the decoder can be summarized by

$$\begin{aligned}\widehat{\mathbf{Ax}} &\triangleq [\mathbf{A}\tilde{\mathbf{x}}] \bmod \Lambda_c \\ &= \big[\mathbf{A}[\mathbf{x} + \mathbf{u}] \bmod \Lambda_c\big] \bmod \Lambda_c \\ &= \big[\mathbf{A}(\mathbf{x} + \mathbf{u})\big] \bmod \Lambda_c.\end{aligned} \tag{45}$$

Finally, the decoder computes

$$\hat{\mathbf{x}} = \mathbf{A}^{-1}\widehat{\mathbf{Ax}}, \tag{46}$$

to obtain the estimate of $\mathbf{x}$.

Taking into account the 1-D nested lattice pair used, this scheme takes advantage of the fact that the linear combinations between the elements of $\mathbf{x} + \mathbf{u}$ performed with the aid of an integer matrix $\mathbf{A}$ are more confined to the Voronoi region of Λ_c than the original observations. In other words, it explores the fact that

$$P(\mathbf{x} \notin \mathcal{V}_c) > P\big(\mathbf{A}(\mathbf{x} + \mathbf{u}) \notin \mathcal{V}_c\big), \tag{47}$$

and that $P\big(\mathbf{A}(\mathbf{x}+\mathbf{u}) \notin \mathcal{V}_c\big)$ is close to zero, meaning that the modulo-Λ_c reduction has no effect in (45). In these conditions, the estimate of $\mathbf{x}$ is given by

$$\hat{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{A}(\mathbf{x} + \mathbf{u}) = (\mathbf{x} + \mathbf{u}), \tag{48}$$

and the MSE attained by this scheme is

$$\mathbb{E}\big[(\mathbf{x} - \hat{\mathbf{x}})^2\big] = \mathbb{E}\big[(\mathbf{x} - (\mathbf{x} + \mathbf{u}))^2\big] = \mathbb{E}\big[\mathbf{u}^2\big] = d, \tag{49}$$

considering the estimation error $\mathbf{u}$ to be equal to the quantization error $\mathbf{e}$, and taking into account (43).

For the OIFSC scheme the probability that the linear combinations between the elements of $\mathbf{x} + \mathbf{u}$ are not confined to $\mathcal{V}_c$, $P\big(\mathbf{A}(\mathbf{x} + \mathbf{u}) \notin \mathcal{V}_c\big)$, is finite and its value depends on the linear size of the Voronoi region of the course lattice in each dimension: $2^R q = 2^R\sqrt{12d}$. The event where (48) does not hold can be referred to as the *overload* (OL) event that leads to an obtained MSE larger than d since there is loss of information when this event occurs. An upper bound for the probability of this event, $P_{ol} \in [0, 1]$, was given in [6] as

$$P_{ol} \le 2K \exp\Big[-\frac{3}{2}2^{2[R - R_{IFSC}(d)]}\Big]. \tag{50}$$

The rate penalty incurred when using the OIFSC scheme, i.e., in the difference $R - R_{IFSC}(d)$ that results from using a 1-D nested lattice pair, is derived from (50), coming as

$$\delta(P_{ol}) = \frac{1}{2}\log_2\Big(\frac{2}{3}\ln\frac{2K}{P_{ol}}\Big). \tag{51}$$

The question that arises at this point is the one of finding the required rate R and its corresponding probability P_{ol} such that the final MSE follows

$$\begin{aligned}\mathbb{E}\big[(x_k - \hat{x}_k)^2\big] &= \mathbb{E}\big[u_k^2\big] \\ &= \mathbb{E}\big[u_k^2 \,|\, OL\big]P_{ol} + \mathbb{E}\big[u_k^2 \,|\, \overline{OL}\big](1 - P_{ol}) \\ &\leq d\end{aligned} \tag{52}$$

for $k = 1, ..., K$. $\mathbb{E}\big[u_k^2 \,|\, OL\big]$ is the MSE attained provided there was an overload event, and $\mathbb{E}\big[u_k^2 \,|\, \overline{OL}\big]$ is the MSE attained when no overload event takes place and (48) holds. An upper bound for the latter is given in [6] as

$$\mathbb{E}\big[u_k^2 \,|\, \overline{OL}\big] \leq \frac{d}{1 - P_{ol}}, \tag{53}$$

and for the former a bound was derived (cf. main thesis document). The closed expression for the upper bound of $\mathbb{E}\big[u_k^2 \,|\, OL\big]$ from which the required rate R and the corresponding P_{ol} can be obtained such that (52) holds is given by

$$\mathbb{E}\big[u_k^2 \,|\, OL\big] \leq \frac{\max_{k=1,...,K}\|\mathbf{b}_k\|_1^2\left(\frac{2^R q}{2}\right)^2 + \max\big(\operatorname{diag}(\mathbf{K_{xx}})\big)}{\max_{k=1,...,K} 2\,\mathrm{Q}\Big(\frac{2^R q}{2\sqrt{\mathbf{a}_k^T(\mathbf{K_{xx}} + d\mathbf{I})\mathbf{a}_k}}\Big)}, \tag{54}$$

where $\mathbf{b}_k^T$ is the kth row of $\mathbf{A}^{-1}$, and $\mathrm{Q}(\cdot)$ the Q-function.

C. OIFSC Scheme Implementation

With the OIFSC scheme defined and the previous upper bound established, it is now possible to design the implementation of this scheme such that the final MSE attained is smaller or equal than a given target distortion d as expressed by (52). One can see $\mathbf{K_{xx}}$ and d as the input for the scheme's implementation from which the encoders and decoder have to agree on a number of parameters do design the pair of nested lattices. To find the rate R and the quantization step q such that (52) holds, the pre-process summarized in the form of pseudocode in Algorithm 4 was formulated. With the rate R and the quantization step q defined, the 1-D nested lattice pair (Λ_f, Λ_c) can be constructed.

D. OIFSC Scheme Performance

For the sake of consistency in the performance comparison between the OIFSC scheme and the original IFSC scheme, the simulation example used in section II was also applied here. As in all other simulations in this work, the Berger-Tung benchmark provided in [6] is used as a lower bound. Since one is interested in comparing the performance of the OIFSC scheme with its counterpart that does not explore the correlation across encoders, one has to compare the simulation results with the naive scheme metric plus the rate penalty required by the OIFSC scheme such that (52) holds, $R_{Naive}(d) + \delta(P_{ol})$.

Algorithm 4 Algorithm to find R and q such that (52) holds.

Input: A target distortion d, a covariance matrix $\mathbf{K_{xx}}$, an integer-valued full-rank matrix $\mathbf{A}^*$, and $R_{IFSC}(d)$.

Output: Rate R for the OIFSC scheme, quantization step q, and probability of an overload event P_{ol}.

1. Set $\hat{d} = 0.99\,d$ and $\mathbb{E}\big[u_k^2 \,|\, \overline{OL}\big] = \hat{d}$.
2. Set $q = \sqrt{12\hat{d}}$ and $R = R_{IFSC}(d)$.
3. Check if (52) holds. If it does, terminate and output R, q, and P_{ol}.
4. Increase R with 0.01 bits.
5. Update P_{ol} with (50) and the upper bound for $\mathbb{E}\big[u_k^2 \,|\, OL\big]$ with (54), and go to step 3.

Unlike the previous results, the simulation result of the OIFSC to be presented next was obtained considering only one $\mathbf{K_{xx}}$, randomly generated as before. Using this fixed covariance matrix, a source vector $\mathbf{x} \in \mathbb{R}^{K\times 1}$ was generated, and with it the OIFSC scheme was simulated to obtain the MSE. This process of generating a source vector $\mathbf{x} \in \mathbb{R}^{K\times 1}$ was repeated 100 times for each target distortion in the interval of $[-40; 20]$ dB, always with the same covariance matrix $\mathbf{K_{xx}}$, in order to simulate the performance of this scheme over 100 time realizations with the same correlation model for the observations of the encoders. With this, the final MSE attained by the OIFSC scheme with rate R such that (52) holds for a given target distortion d, corresponds to the average of the MSE attained for each of those 100 time realizations.

The setup with $K = 8$ and $M = 2$ was considered to simulate the OIFSC scheme. The obtained rate-distortion curve is depicted in figure 9. The rate penalty incurred by the use of a 1-D nested lattice pair instead of a high-dimensional nested lattice pair is about 2 bits, which is the required rate penalty in order to attain a final MSE equal to the specified target distortion d. This penalty rate of 2 bits allows, according to (50), a maximum probability of an overload event P_{ol} of about 6×10^{-10} , which can be neglected.

VI. Conclusions

The IFSC scheme presented and analyzed in this work exploits the correlation by means of integer linear combinations between the incoming signals at the decoder's side. By doing so, the required rate for the encoders to compress their observations within a maximum distortion value can be significantly reduced. The final MSE per dimension attained by this scheme approaches the maximum target distortion when n approaches infinity. This may become an inconvenient in the sense that the encoders have to quantize their observations in a n-dimensional nested lattice pair, which is not a low complexity operation.

One of the contributions of this paper is a clarification of the underlying geometry of the concept of dealing with integer linear combinations of lattice-quantized sources, thus making

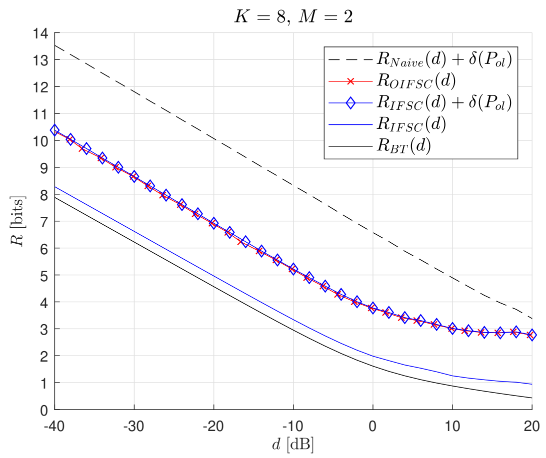


Fig. 9. One-shot integer-forcing source coding rate-distortion curves for the setup with $K = 8$, $M = 2$.

evident why that allows a rate reduction. Until now this had remained rather unclear in the literature.

To address the underlying optimization problem of IFSC, an alternative algorithm that has been very recently proposed in the literature, was presented. This algorithm yields the exact solution (i.e., finds the K successive minima of a lattice), at the cost of increasing the time complexity from $\mathcal{O}(K^4 \log K)$ to $\mathcal{O}\big(K^{\frac{5}{2}}(2\pi e)^{\frac{K}{2}}\big)$, i.e., from polynomial to exponential running time. With this exact algorithm, the information-theoretic rate-distortion attained by the IFSC scheme was provided for some values of K up to $K = 25$.

To address the problem of having uncertainty about the model that describes the correlation between the sources, a new technique for IFSC was devised that takes in consideration the probabilities of each one of the possible correlation models, outputting integer matrices that are more oriented towards the correlation model with higher probability. Even in the case of total uncertainty regarding the correlation model that best describes the existing correlation, it was seen by simulations that it is advantageous to use the IFSC scheme with the proposed solution rather than a naive scheme that does not exploit this correlation. Furthermore, it was seen that this is also the case even in a mismatch situation, where the true probability of a given correlation model does not match how appropriate the scheme is to that model, provided that there exists enough correlation between the sources to be compressed. An interesting and relevant feature of this proposed solution, is that the underlying optimization problem does not increase in complexity when the set of possible correlation scenarios increases, depending only on the number of encoders in the system.

Finally, to approach the limitation imposed on the encoders by the use of high-dimensional nested lattice pairs, the one-shot version of IFSC was defined and analyzed. Also, a suggestion for the pre-process of this scheme was made with the purpose of controlling the performance degradation imposed by the use of an 1-dimensional nested lattice pair. By analyzing the simulation results of this low-complexity scheme, it was possible to conclude that the suggested pre-process works and the scheme attains a final MSE smaller to or equal than the given target distortion. Furthermore, a rate penalty of 2 bits was found to be sufficient, leading to probabilities of overload than can be neglected.

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