Cyclic Cosmologies

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To the 5-year-old me,
who will forever inspire my work.

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Resumo

A singularidade inicial pressuposta no modelo do *big bang* ainda é um problema da cosmologia actual. Uma solução é evitá-la completamente, considerando, por exemplo, universos cíclicos que colapsam num ressalto sem singularidade, apenas para se expandirem novamente. Existem várias propostas para este tipo de universos, sendo que todas concordam que se devem utilizar teorias alternativas à gravidade de Einstein. Nos primórdios do universo estamos na escala de Planck, a energias altíssimas, onde se devem considerar efeitos quânticos na descrição de qualquer evento cosmológico e a relatividade geral é uma teoria clássica, sendo inválida neste regime. Nesta tese, para encontrarmos uma solução de um universo com ressalto, usamos uma teoria que estende a relatividade geral, adicionando ao termo correspondente à curvatura do espaço R, na parte gravitacional da acção, uma correcção $f(G)$, onde G é o invariante de Gauss-Bonnet e f uma qualquer função do mesmo. Geralmente, teorias que modificam a gravidade, como esta, dão origem a equações diferenciais de ordem superior a dois na métrica, potencial da gravidade, tornando as suas soluções difíceis de interpretar. Para contornarmos esta dificuldade, recorremos a uma técnica que reduz a ordem das equações encontradas para segunda ordem, de modo a considerarmos apenas soluções que são perturbativamente próximas da relatividade geral. Ao reconstruirmos a acção, descobrimos que a parametrização de $f(G)$ tem de incluir os termos $G \ln G$ e $\sqrt[6]{G}$, para que a teoria obtida permita um universo com ressalto a determinada densidade crítica ρ_c.

Palavras-chave: relatividade geral, cosmologia cíclica, solução com ressalto, Gauss-Bonnet modificada, gravidade $f(G)$

Abstract

The big bang initial singularity is one of the outstanding problems in current theoretical cosmology. Universes that collapse with a bounce to expand back again, possibly in self-sustaining cycles, can avoid the initial singularity altogether and thus solve this problem. There are several proposals for bouncing universes, a common feature between them is that at such early times and so at such high energies, Planck scale quantum gravity phenomena must take hold of the physical processes and thus general relativity ceases to be valid there. In this thesis, to find a universe with a bouncing solution, we propose a theory that extends general relativity, by adding to the usual Einstein-Hilbert Ricci scalar term R in the gravitational action, an $f(G)$ term, where G is the Gauss-Bonnet invariant, and f some function of it. In general, the modified gravity theories contain higher order differential equations in the metric gravitational potentials making their solutions difficult to interpret. To bypass this complication, resort to a technique that reduces the order of the $f(G)$ theory to second order differential equations is necessary. This order reduction technique enables one to find solutions which are perturbatively close to general relativity. In building the covariant gravitational action it is found that $f(G)$ has to include the terms $G \ln G$ and $\sqrt[6]{G}$ to have a theory that yields a universe with a bounce at some critical density ρ_c.

Keywords: general relativity, cyclic cosmology, bouncing solution, modified Gauss-Bonnet, $f(G)$ gravity

Preface

The research included in this thesis has been carried out at Centro Multidisciplinar de Astrofísica (CENTRA) in the Physics Department of Instituto Superior Técnico. I declare that this thesis is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University and that no part of it has already been or is being concurrently submitted for any such degree or diploma or any other qualification.

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Contents

Chapter 1

Introduction

1.1 Motivation

1.1.1 General relativity, the big bang scenario and the singularity problem

Cosmology as the science that studies the beginning and development of our universe has always been the target of many interesting problems and respectively interesting solutions which fit the observations. Throughout history, this field of knowledge evolved from a purely philosophical and rudimentary observational state, to a mathematical and supported by advanced electronic observations one. The critical event that is thought to have boosted this change in the way we do cosmology, is considered to be the theory of general relativity developed and published by Albert Einstein [1]. This theory emerged as a natural and mathematically well-posed way to describe a symbiotic relationship between matter and spacetime curvature and the first real update on the gravitational interaction since the Newtonian description established a couple of centuries before.

With a strong physical framework provided by general relativity to describe gravity, the only non-negligible force when dealing with lengths of the order of the radius of the universe, various mathematical solutions arose as equations of motion for the cosmological evolution. To mimic such a breakthrough we start by the so-called Einstein-Hilbert action plus a cosmological constant, which reads as follows

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(R - 2\Lambda) + S_{\text{matter}}(g_{\mu\nu}, \psi), \tag{1.1}$$

where $\kappa = 8\pi G_N$, G_N is the Newton's constant, Λ the cosmological constant, and we use $c = 1$ for the vacuum velocity of light. Moreover, $g_{\mu\nu}$ is the metric and g its determinant, R the Ricci scalar or the scalar curvature, S_{matter} denotes the matter action and ψ collectively denotes the matter fields.

The equations of motion are obtained by varying the action of Eq. (1.1) with respect to the metric $g^{\mu\nu}$, resulting in the Einstein's field equations with a cosmological constant

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.2}$$

with $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ the Einstein's tensor, $R_{\mu\nu}$ the Ricci tensor, and $T_{\mu\nu}$ the stress-energy tensor.

Neglecting the effect of the cosmological constant, notice the linear relation established between matter (stress-energy tensor) and geometry (Einstein's tensor) in this set of equations, evidencing that matter is responsible for curving the spacetime, which in turn constrains the movement of matter through it. For the discussion of Einstein's equations, see Ref. [2].

One can evoke the cosmological principle, which states that when viewed on a sufficiently large scale, the properties of the universe are the same for all observers, and assume an homogeneous and isotropic universe. This assumption is equivalent to considering a Friedmann-Lemaître-Robertson-Walker (FLRW) line element [2]. In spherical coordinates (r, θ, ϕ) with t denoting the coordinate of time, it reads as

$$ds^2 = -dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right), \tag{1.3}$$

where k is the intrinsic curvature and $a = a(t)$ the scale factor.

It is also considered a perfect fluid description, with a stress-energy tensor given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \tag{1.4}$$

with $p = p(t)$ being the pressure and $\rho = \rho(t)$ the density. The assumption of a barotropic equation of state yields

$$p = w\rho, \tag{1.5}$$

being w a dimensionless parameter. In cosmology, it is not unusual to assume a perfect fluid description. Since any anisotropies that there may be in the universe can be neglected, at least to analyse its evolution through very big (billions of years) periods of time, where it is considered isotropic, as previously mentioned.

Computing Eqs. (4.1), (4.4) and (1.5) into the Einstein's field equations, an equation of motion for the evolution of the scale factor a, which measures the radius of the universe ($a = 1$ in the present time), is derived. This is the so-called first Friedmann equation for the Hubble parameter, $H = \frac{\dot{a}(t)}{a(t)} = \frac{\dot{a}}{a}$, and reads as follows, for general relativity plus a cosmological constant:

$$H^2 = \frac{\kappa}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{1.6}$$

A second Friedmann equation can be derived from the field equations of general relativity in Eq. (1.2), also known as the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{6}\rho(1 + 3w) + \frac{\Lambda}{3}, \tag{1.7}$$

where $\ddot{a} = \ddot{a}(t)$ denotes the acceleration of the scale factor a with time t. From Eqs. (1.6) and (1.7), it is possible to study the movement regarding the variation of the scale factor a, or rather, the variation of the radius of the universe. Here, with respect to time t and matter density ρ, parameterized by some constants κ, w, k and Λ. There is no unique mathematical solution, although one of the models that best fits cosmological observations concerning the beginning of the universe is the big bang scenario.

The big bang model was firstly developed to tackle the problem of the universe's expansion, a phenomenon that had recently been observed through the Doppler effect registered from observations of galaxies' velocities. In this first description, the evolution of the universe stated that all the spacetime firstly emerged from an initial singularity of infinite density and of an hot nature, such a phase was then followed by an expansion. The big bang model and the hot universe comprise the standard model accepted today, supported by experimental evidence, for example: the cosmic microwave background, the light elements' density distribution, the galaxies' form and disposition and the Doppler effect that evinced accelerated expansion.

The accelerated expansion phase that first took place right after the so-called big bang, was later named inflation by Alan Guth [3], who developed the theory concerning this stage in order to better explain the large-scale structure of the cosmos, in regards to some quantum fluctuations that might had happened. Inflation was orchestrated in order to solve the horizon and the flatness problems arising from the first big bang theories. The horizon problem is related to the fact that the cosmic microwave background reveals that separated regions of spacetime are causally disconnected although we assume in the big bang scenario the early universe to be homogeneous. The flatness problem refers to the fine-tuning that we must apply to the initial value of the Hubble constant (which provides us the timescale for the universe) in order for it to comply with the flatness that is today observed.

Concerning the nature of the singularity itself, this theory evolved to include recent advances that had been made with quantum field theory, parallelizing the initial singularity with an atomic description, given the similar scale lengths. The gravity theory provided by general relativity with a cosmological constant is a classical theory, meaning that its quantization is still a problem in fundamental physics and that any attempt at predicting quantum effects arising from this theory is therefore invalid. Moreover, an ultimate initial singularity would always carry problems regarding initial conditions and of fine-tuning for the moment of the first explosion.

1.1.2 Possible solutions: alternative gravity theories and cyclic universes

The physical description of the big bang initial singularity is still an open problem in cosmology and fundamental physics. A complete description of the universe must avoid spacetime singularities as their existence makes the future physically unpredictable. It is supposed that in a quantum gravity regime new physics sets in and spacetime singularities get a proper description. In such a regime, there are several possibilities, an intriguing one is to suppose that at some tiny scale, of the order of few Planck units, the universe undergoes a bounce, such that a previously collapsing universe expands back again originating our own visible universe, yet to possibly collapse again in a cyclic way, giving rise to a cyclic universe.

Cyclic models have always been appreciated as they avoid the problem of initial conditions. Even in general relativity there are indeed cyclic solutions for the universe, provided that we have a closed geometry ($k = +1$) and a null cosmological constant ($\Lambda = 0$) [4]. With recent observations regarding the acceleration and the flatness of the Universe ($k = 0$) [5], this solution was abandoned, since these new observations were contrary to the conditions needed for this model to work. The work of Hawking and Penrose [6] about singularity theorems were again a setback in cyclic models: they stated that a big crunch necessarily leads to a cosmic singularity where general relativity becomes invalid. Without another gravity theory, considerations of spacetime prior to the big bang were discouraged.

A theory that emerged to explain the universe was the Standard Cosmological Model (Refs. [7]). The latter, with an almost universal consensus amongst cosmologists, represents our best understanding of the observational data (Ref. [8]), the first time in the history of cosmology that such a consensus has existed. Nevertheless, it still carried some problems, especially regarding to the nature of the singularity itself. In the effort of trying to avoid this obstacle, theories concerning gravitational interaction alternative to general relativity started to emerge.

Since Einstein's general relativity is a classical theory, quantum gravity arose as a possible description for the gravitational interaction, that worked at very high energies (Planck's scales), and which behaved as general relativity in lower energy regimes. Many efforts have been made towards the possibility of joining quantum field theory with gravity. And many cosmological models supported on these alternative theories started to appear. Loop quantum gravity [9] is a non-perturbative and background independent attempt to tackle these issues, quantizing gravity in four dimensions and predicting a discrete quantum geometry for the spacetime continuum at quantum level. The application of loop quantum gravity to the universe leads to loop quantum cosmology [10], which yields an interesting scheme for having bouncing cosmological solutions.

Some modifications to the Friedmann dynamics presented by Eq. (1.6) naturally come out of this new cosmological model, more specifically it has been shown that at scales comparable to the Planck length, time evolution takes discrete steps and does not break down at zero volume, i.e. spacetime can be extended to a time that precedes the classical singularity event, erasing the singularity altogether in isotropic minisuperspaces [11]. Although the discreteness of the fundamental theory that describes loop quantum cosmology, this model admits an effective smooth differentiable geometry description which highlights quantum effects at high curvature and reduces to the classical theory at low curvatures.

Therefore, modified Friedmann equations such as

$$H^2 = \frac{\kappa}{3}\rho\left(1 - \frac{\rho}{\rho_c}\right), \tag{1.8}$$

where $c = 1$ and ρ_c is a critical energy density, that clearly describe a bouncing cosmology with the bounce at $\rho = \rho_c$, may be found by an effective action description in the framework of loop quantum cosmology. In Refs. [12, 13, 14],

$$\rho_c = \frac{\sqrt{3}}{16\pi^2\gamma^3}\rho_{Pl}, \tag{1.9}$$

with the Planck density being

$$\rho_{Pl} = \frac{1}{G_N^2 \hbar}, \tag{1.10}$$

$\hbar$ being the Planck constant, γ is the Barbero-Immirzi parameter and $\frac{\sqrt{3}}{16\pi^2\gamma^3}$ is a number of order one.

Similar solutions have been found in Ref. [15] where extensions of general relativity with higher curvature terms were used, see also [16] for cosmologies with a bounce using the ADM formalism in loop quantum cosmology. In addition, bouncing cosmologies are an integral part of ekpyrotic scenarios [17]. Contrary to the classical cyclic solutions previously mentioned in Subsec. 1.1.1, these non-singular bouncing scenarios which arose from alternative gravity theories that incorporate quantum behaviour, are compatible with current observations for the flatness of our universe and offer exotic solutions for the entropy problem, regarding the nature and interpretation of the second law of thermodynamics itself under such regimes distanced from the limits of classical physics, although it is still an open problem in this field of cosmology.

Given the rising interest in loop quantum cosmology, which has set to dismantle some of the most puzzling problems around early cosmology, questioning the covariance of such a theory has become a relevant obstacle to overcome.

1.2 Bouncing cosmologies in $f(R)$ and $f(G)$ gravities

1.2.1 Metric $f(R)$ gravity

Extending general relativity with the aim of finding a covariant action for bouncing solutions in the framework of loop quantum cosmology, constitutes a way to probe the covariance of such a theory. This framework has proved its cosmological worth since it is one of the few environments in $3+1$ dimensions where there was success in quantizing cosmological spacetime non-perturbatively [9].

In answering this question, Sotiriou [18] analysed $f(R)$ gravity as a possible extension of general relativity in order to find bouncing cosmological solutions. These $f(R)$ theories are extensions of general relativity in the sense that the Einstein-Hilbert action that does depend only on the Ricci scalar R, and possibly on the cosmological constant Λ, can have a more general form substituting R by a generic function $f(R)$ [19].

The $f(R)$ theories possess, in general, field equations with fourth-order derivatives, which in turn can generate non physical solutions. One way to deal with these solutions is to exclude them whenever they appear. A more interesting way is to devise a method in which these solutions do not appear at all. One such method states that the interesting physical solutions in $f(R)$ theories are the ones that can be found from general relativity in a perturbative way, the other solutions being considered artificial. This is the technique of order reduction [20, 21] which builds solutions perturbatively close to general relativity which are then second order differential equations by construction. Relying on this method it was possible to construct a Friedmann equation with a bounce [18].

Starting with the action for metric $f(R)$ gravity which reads as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}f(R) + S_{\text{matter}}(g_{\mu\nu},\psi), \tag{1.11}$$

and we vary Eq. (1.11) with respect to the metric $g^{\mu\nu}$ to get the field equations

$$f(R)'R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \left[\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box\right]f(R)' = \kappa T_{\mu\nu}, \tag{1.12}$$

where a prime denotes derivation with respect to R, ∇_μ is the covariant derivative and we define $\Box = \nabla_\mu\nabla^\mu$.

Since we want to treat metric $f(R)$ gravity, not as an exact but as an effective field theory whose solution ought to be perturbatively close to general relativity, we have to perform an order reduction to the field equation to discard the spurious degrees of freedom.

Without loss of generality we can parameterize f as:

$$f(R) = R + 2\Lambda + \epsilon\varphi(R), \tag{1.13}$$

where the parameter ϵ is dimensionless and marks the deviation from general relativity.

Due to the form of the field equations, the order reduction amounts to just replacing R and $R_{\mu\nu}$ in order ϵ terms with the expression one gets for them from the $\epsilon = 0$ version of the same equations.

Another useful expression consists of the contraction of the field equations depicted in Eq. (1.12) with the metric $g^{\mu\nu}$ to obtain the trace

$$f(R)'R - 2f(R) + 3\Box f(R)' = \kappa T, \tag{1.14}$$

where $T = g^{\mu\nu}T_{\mu\nu}$. We use Eq. (1.14) to help us express R to the lowest order, $R_T = -\kappa T - 4\Lambda$.

The reduced field equations are then:

$$G_{\mu\nu} - \Lambda g_{\mu\nu} + \epsilon\left[\varphi'(R_T)\left(\kappa T_{\mu\nu} - \frac{1}{2}\kappa T g_{\mu\nu} - \Lambda g_{\mu\nu}\right) - \frac{1}{2}\varphi(R_T)g_{\mu\nu} - [\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box]\varphi'(R_T)\right] = \kappa T_{\mu\nu} \tag{1.15}$$

Now, to derive the modified Friedmann equation corresponding to this order-reduced field equations: we start by assuming a FLRW line element and a perfect fluid description of matter. After some mathematical manipulation we set $w = 1$ (in the equation of state $w = p/\rho$) for a scalar field matter, $k = 0$ for a spatially flat spacetime and the cosmological constant Λ to zero as it does not appear in the Eq. (1.8), result that we are trying to mimic.

This procedure finally yields:

$$H^2 = \frac{1}{3}\kappa\rho - \frac{\epsilon}{3}\left[2\varphi_T'\kappa\rho + \frac{1}{2}\varphi_T + 12\varphi_T''\kappa^2\rho^2\right], \tag{1.16}$$

for simplicity φ_T denotes $\varphi(R_T)$ (the same applies for its derivatives), equation that we may compare to some Friedmann equation with a modified source of the type of Eq. (1.8) by matching its second term, a source term, to the second term that expresses deviation from the general relativity plus a cosmological constant (Eq. (1.6)) case:

$$\frac{\epsilon}{3}\left[2\varphi_T'\kappa\rho+\frac{1}{2}\varphi_T+12\varphi_T''\kappa^2\rho^2\right]=\frac{1}{3}\kappa\frac{\rho^2}{\rho_c}. \tag{1.17}$$

Solving the differential equation presented in Eq. (1.17), it is concluded that the general analytic solution is $\varphi(R)=(18\epsilon\kappa\rho_c)^{-1}R^2$ or

$$f(R)=R+\frac{\pi\gamma^3 l_P^2}{9\sqrt{3}}R^2+... \tag{1.18}$$

Sotiriou has, therefore, found a metric $f(R)$ action which, when treated as an effective action, leads to the desired Friedmann equation.

As a conclusion, it is stated that a covariant effective action, or at least a family of covariant effective actions, which lead to the effective Friedmann equation as predicted from the massless scalar field model of loop quantum cosmology, can be found in the framework of metric higher order theories of gravity.

1.2.2 Palatini $f(\mathcal{R})$ gravity

Another alternative to Einstein's theory that has been considered to solve this covariance problem was to use the Palatini $f(\mathcal{R})$ modified gravity [19] which differentiates itself when treating the metric and the connection as independent objects, i.e. we have to vary the action with respect to each of these to obtain the field equations. In this procedure the matter part of the action is considered independent of the connection.

In [22], G. J. Olmo and P. Singh derived the modified Friedmann equations from a covariant Palatini $f(\mathcal{R})$ action framework to attempt a correspondence with Eq. (1.8). They were trying to find whether the effective dynamics of loop quantum cosmology, which resulted in a non-singular evolution, corresponds to a covariant description, and they worked within the Palatini $f(R)$ gravity framework. They justify this choice by recalling that within the loop quantum gravity environment there is no assumption of compatibility between the metric and connection.

The generalized Palatini action is given by:

$$S(g,\Gamma,\psi)=\frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\mathcal{R},\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu},...)+S_{\text{matter}}(g_{\mu\nu},\psi), \tag{1.19}$$

with $\mathcal{R}:=g^{\mu\nu}\mathcal{R}_{\mu\nu}(\Gamma)$ the Ricci curvature scalar of the independent connection and Γ the Levi-Civita connection. For simplicity they consider the gravitational part only as function of $\mathcal{R}$:

$$S(g,\Gamma,\psi)=\frac{1}{2\kappa}\int d^4x\sqrt{-g}f(\mathcal{R})+S_{\text{matter}}(g_{\mu\nu},\psi). \tag{1.20}$$

The variation with respect to the metric and to the connection, for the action in Eq. (1.20), are given by, respectively:

$$f'(\mathcal{R})\mathcal{R}_{\mu\nu}(\Gamma) - \frac{1}{2}f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.21}$$

$$\nabla_\mu^\Gamma \left[\sqrt{-g}f'(\mathcal{R})g^{\alpha\beta}\right] = 0, \tag{1.22}$$

where the prime stands for differentiation with respect to $\mathcal{R}$. Another useful equation that will help us write $\mathcal{R} = \mathcal{R}(T)$ is the trace of Eq. (1.21),

$$\mathcal{R}f'(\mathcal{R}) - 2f(\mathcal{R}) = \kappa T. \tag{1.23}$$

Substituting the solution from Eq. (1.22) and the function $\mathcal{R}(T)$ obtained from Eq.(1.23), we write the field equations for the metric dependent part as

$$G_{\mu\nu}(g) = \frac{\kappa}{f'}T_{\mu\nu} - \frac{\mathcal{R}f' - f}{2f'}g_{\mu\nu} + \frac{1}{f'}(\nabla_\mu\nabla_\nu f' - g_{\mu\nu}\Box f') - \frac{3}{2f'^2}\left(\partial_\mu f'\partial_\nu f' - \frac{1}{2}g_{\mu\nu}(\partial f')^2\right), \tag{1.24}$$

notice that in this case there are no extra degrees of freedom, as the field equations stay at second-order. We recall the usual treatment where we consider a perfect fluid description, although with this approach the vacuum description plus a cosmological constant is hidden by noting that $\Lambda = \frac{\mathcal{R}f'-f}{2f'}$. After some mathematical manipulation we finally arrive to the first Friedmann equation for this theory

$$3H^2 = \frac{f'[2\kappa\rho + \mathcal{R}f' - f]}{2(f' + \frac{f''}{2}\frac{\dot{\mathcal{R}}}{H})^2}, \tag{1.25}$$

where a prime denotes derivative with respect to $\mathcal{R}$ and $\dot{\mathcal{R}}/H = -12\kappa\rho/(\mathcal{R}f'' - f')$. The problem of finding the effective action is thus reduced to finding an $f(\mathcal{R})$ satisfying:

$$\frac{f'[2\kappa\rho + \mathcal{R}f' - f]}{2(f' + \frac{f''}{2}\frac{\dot{\mathcal{R}}}{H})^2} = \kappa\rho\left(1 - \frac{\rho}{\rho_c}\right). \tag{1.26}$$

The numerical solution encountered reads as follows

$$f(\mathcal{R}) = \frac{\mathcal{R}}{12}\left(1 - \frac{1}{2}\ln\left[\left(\frac{\mathcal{R}}{12\mathcal{R}_c}\right)^2\right]\right) + \frac{\mathcal{R}(\mathcal{R} + 12\mathcal{R}_c)^2}{6500\mathcal{R}_c^2}, \tag{1.27}$$

where $\mathcal{R}_c \equiv \kappa\rho_c$. The first term dominates at higher curvatures and incorporates the non-perturbative quantum gravity effects that lead to the cosmic bounce in loop quantum cosmology when $\rho = \rho_c$. They conclude by celebrating their result as the beginning and motivation for more work regarding actions that extend general relativity and their applications as well as for more investigation on loop quantum cosmology.

1.2.3 $f(G)$ gravity: what makes it an interesting choice?

It is certainly of interest to see whether modified Gauss-Bonnet $f(G)$ gravity theories, where G is the Gauss-Bonnet term, can also yield bouncing solutions. The Gauss-Bonnet term G is given by

$$G = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}, \tag{1.28}$$

and has previously stated its cosmological value in higher than 4-dimensional theories such as the brane-world approach [23], as also naturally appears in the low energy effective action from string/M-theory, sometimes regarded as a string inspired modification of gravity [24].

The reasons to explore Gauss-Bonnet motivated solutions for higher order theories such as in brane-world cosmology or string cosmology are explicitly well-listed in [24]: the Gauss-Bonnet combination of curvature invariants, which is a topological invariant itself in 4 dimensions not making any contribution for the dynamical equations of motion, represents the unique 5-dimensional leading-order extension of the Einstein-Hilbert action and results in differential equations whose highest order derivative occurs linearly, guaranteeing a unique solution. However, the astrophysical evidence point to a current de Sitter phase, i.e. the universe is going through an accelerated expansion phase, which may be characterized by the existence of a small and positive cosmological constant, are contrary to the predictions made by some higher-dimensional model that uses an effective Gauss-Bonnet action, at least if we are considering the ones depicted by Refs.[23, 24], FRLW brane cosmology from Einstein-GB gravity and string cosmology with Gauss-Bonnet parameterization.

Although in four spacetime dimensions G is a topological invariant and does not contribute to the dynamics, $f(G)$ is non trivial even in four dimensions and the theory gets new degrees of freedom. These $f(G)$ theories had been earlier proposed [25] in the context of late time cosmology, in an attempt of overcoming the dark energy problem and providing a solution for the large structure of the universe. Furthermore, they found that a wide range of possible functions $f(G)$ pass solar system tests, can describe the transition from deceleration to acceleration as well as from the non-phantom phase to the phantom phase, and of course, regardless of the specific effective equation of state (cosmological constant, quintessence, phantom), modified Gauss-Bonnet theories are shown to being able to describe the late-time acceleration of the universe.

Since the theory predicts interesting results at late-time cosmological scales, it is just natural to study the phenomenological implications of this theory about early-time solutions. Therefore, our work will precisely focus on an alternative to the big bang scenario in the framework of $f(G)$ gravity. Different approaches have been taken to solve the same problem, in this sense interesting cosmological bouncing solutions in $f(G)$ gravity have been found in Ref. [26, 27, 28]. In Ref. [29] we may consult a review on bouncing cosmologies in $f(G)$ gravity and other theories.

In view of this, our objective is to extend general relativity with an $f(G)$ correction in the action in order to find a covariant action for bouncing cosmologies that fits the framework that we have been discussing. We believe that cyclic universes with bouncing solutions are candidates for solving the big bang initial singularity problem. General relativity without a cosmological constant yields the Friedmann equation with no bounce, although the Einstein-Hilbert action from which we get general relativity is one of the simplest covariant actions. Thus, seeking bouncing solutions one needs to resort to theories that extend in some way or another general relativity. Here we use a modified Gauss-Bonnet $f(G)$ gravity theory, where G is the Gauss-Bonnet term, and f some function of it, to find a universe with a bounce. Since general relativity is a theory whose equations are of second order, in finding such a solution we resort to a technique that reduces the order of the differential equations of the $f(G)$ theory to second order equations. This order reduction technique enables one to find solutions which are perturbatively close to general relativity.

Therefore, our aim is to find the function $f(G)$ that extends general relativity and builds a covariant action for bouncing cosmologies.

Through an effective Hamiltonian description and considering the action:

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(R + f(G)) + S_{\text{matter}}(g_{\mu\nu}, \psi), \tag{1.29}$$

where $G = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ is the Gauss-Bonnet term, a topological invariant in four dimensions, we intend to reach a modified Friedmann equation resembling the one in Eq. (1.8):

$$H^2 = \frac{1}{3}\kappa\rho\left(1 - \frac{\rho}{\rho_c}\right), \tag{1.30}$$

where there is clearly a bounce at $\rho = \rho_c$.

Loop quantum cosmology solves the singularity problem, doing away with it without the introduction of any degrees of freedom: the ρ^2 dependence in the solution portrayed in Eq. (1.30) is a result of a second order differential equation. To prove the covariance of such a theory, one only needs to find a covariant action to fit its results. Constraining our search to a covariant and second-order theory, one is obviously misled to the Einstein-Hilbert Lagrangian (plus or not a cosmological constant), theory which is not compatible with a non-singular Friedmann equation. We choose to overcome this theory by sticking to a covariant action, by adding an $f(G)$ correction in the Lagrangian in the action, which will give rise to a fourth-order Friedmann equation, although we will reduce its order by an order reduction technique.

To summarize, we shall develop the field equations associated with the action in Eq. (1.29), vary them with respect to $g^{\mu\nu}$ to obtain the respective first Friedmann equation, which we will compare with Eq. (1.30) and solve for $f(G)$. The result shall correspond to the specific function of the term G that when in the gravitational part of an action such as Eq. (1.29) will result in a first Friedmann equation as Eq. (1.30). This work will be done in the framework of isotropic models in $f(G)$ gravity motivated by loop quantum cosmology, for which some conditions will be set for concordance.

1.3 Thesis Outline

This thesis is organized as follows. In Chapter 2 we present the $f(G)$ theory, deriving from its action the full field equations through the principle of least action. To facilitate on the reading, we divide the chapter according to each of the variables to which we must deduce its variation before assembling the final equations resulting from this effort.

In Chapter 3 we explore the theoretical background of the order reduction technique that we want to apply to the field equations previously mentioned. We then proceed to use it, therefore deducing the order reduced field equations, i.e., the second order field equations that are perturbatively close to general relativity, that we shall work with from this point on. Since this process implies the calculation of each of the variables' lowest order form, we dedicate a subsection to each.

Chapter 4 is dedicated to implementing a method to solve these order reduced equations that we had just found in the previous Chapter 3. We then carefully explain the physical context of these equations, more specifically the choice of a Friedmann-Lemaître-Robertson-Walker metric, the perfect fluid assumption as well as how we computationally implement these factors to deduce a Friedmann equation for the evolution of the universe. Finally we assume the simplest possible model, i.e., a universe with zero cosmological constant, zero spatial curvature, and composed of a stiff fluid and derive the Lagrangian and the conditions obeyed by the $f(G)$ Gauss-Bonnet modified gravity to have a universe with a bounce.

In the last Chapter 5 we conclude by stating our achievements [30] and discuss possible future work.

Chapter 2

Field equations for $f(G)$ gravity theory

In this chapter we derive the field equations for the modified Gauss-Bonnet $f(G)$ gravity theory. The calculations are straightforward since we will just vary the action with respect to the inverse metric $g^{\mu\nu}$ in order to obtain the field equations. Although, as we will notice, the calculations are extensive, given the intricates of the G term itself and its dependencies.

2.1 The action and its variation

2.1.1 Varying the action with respect to $g^{\mu\nu}$

As we already saw in the previous Chapter 1, the action for the modified Gauss-Bonnet $f(G)$ gravity in four dimensions reads as

$$S = \frac{1}{2\kappa}\int dx^4\sqrt{-g}\,\mathcal{L}_{\rm grav} + S_{\sf matter}(g_{\mu\nu},\psi)\,, \tag{2.1}$$

where $\kappa = 8\pi G_{_N}$, g is the determinant of the metric $g_{\mu\nu}$, μ,ν are spacetime indices, $\mathcal{L}_{\rm grav}$ is the Lagrangian density for the gravity sector, and $S_{\sf matter}$ is the matter action defined as a function of $g_{\mu\nu}$ and of ψ which stands for matter fields. Further, we assume that the Lagrangian density for the gravity sector is given by

$$\mathcal{L}_{\rm grav} = R + f(G)\,, \tag{2.2}$$

where R is the Ricci scalar and $f(G)$ is a function of the Gauss-Bonnet term G defined as

$$G = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\,, \tag{2.3}$$

with $R_{\mu\nu\rho\sigma}$ being the Riemann tensor and $R_{\mu\nu}$ the Ricci tensor.

The action is a mathematical functional that takes the path of a dynamical system as its argument, to minimize the action is to choose the path corresponding to the minimum energy. This path is portrayed by the equations of motion, which are obtained by applying the principle of least action to the action, i.e.

requiring S to be stationary, the first-order change must be zero for any possible perturbation around the true evolution: $\delta S = 0$. Varying the action can be understood by varying each of its terms independently,

$$
\begin{aligned}
\delta S &= \frac{1}{2\kappa}\int dx^4 \delta(\sqrt{-g}(R+f(G))) + \delta S_{\mathrm{matter}} \\
&= \frac{1}{2\kappa}\int dx^4 \left[\delta(\sqrt{-g})(R+f(G)) + \sqrt{-g}(\delta R + \delta f(G))\right] + \delta S_{\mathrm{matter}} \\
&= \frac{1}{2\kappa}\int dx^4 \left[\delta(\sqrt{-g})(R+f(G)) + \sqrt{-g}\delta R + \sqrt{-g} f'(G)\delta G\right] + \delta S_{\mathrm{matter}},
\end{aligned} \tag{2.4}
$$

which facilitates our calculations.

2.1.2 Working on $\delta S_{\mathrm{matter}}$

The variation of the matter part of the action in Eq. (2.1), $\delta S_{\mathrm{matter}}$ as it reads in the last term of Eq. (2.4), is obtained through some useful definitions. First of all, note that the matter action reads as

$$
S_{\mathrm{matter}} = \int dx^4 \sqrt{-g}\mathcal{L}_{\mathrm{matter}}. \tag{2.5}
$$

The variation of the action depicted in Eq. (2.5) accounts to

$$
\begin{aligned}
\delta S_{\mathrm{matter}} &= \delta\left(\int dx^4 \sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right) \\
&= \int dx^4 \delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right) \\
&= -\frac{1}{2}\int dx^4 \left[\frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right)}{\delta g^{\mu\nu}}\right]\sqrt{-g}\delta g^{\mu\nu},
\end{aligned} \tag{2.6}
$$

where Eq. (2.6) was mathematically manipulated to evince a tensorial variable inside the integral. This quantity is defined as the stress-energy tensor,

$$
\begin{aligned}
T_{\mu\nu} &= \frac{-2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_{\mathrm{matter}}\right)}{\delta g^{\mu\nu}} \\
&= -2\frac{\delta\mathcal{L}_{\mathrm{matter}}}{\delta g^{\mu\nu}} + \mathcal{L}_{\mathrm{matter}} g_{\mu\nu},
\end{aligned} \tag{2.7}
$$

and is responsible for describing the density of energy and momentum as well as their flux in spacetime. Notice that in Eq. (2.7) we used Eq. (2.12) for the calculation of $\delta(\sqrt{-g})$. The stress-energy tensor shall include all the matter, radiation and any non-gravitational force field. In the general relativity case, the stress-energy tensor is the source term of the theory's full field equations.

2.1.3 Working on $\delta(\sqrt{-g})$

Focusing on the first term inside of the gravitational part of the action in Eq. (2.4), the term $\delta(\sqrt{-g})(R+f(G))$, its variation is only dependent on the variation of $\delta(\sqrt{-g})$, which starts by the differentiation of the function $\sqrt{-g}$ with respect to g,

$$\delta(\sqrt{-g}) = \frac{\partial(\sqrt{-g})}{\partial g}\delta g = -\frac{1}{2\sqrt{-g}}\delta g \tag{2.8}$$

where we evince δg. To help us treat $\delta g = \delta det(g_{\mu\nu})$, we recall the Jacobi's formula,

$$\delta g = \delta det(g_{\mu\nu}) = \frac{\partial det(g_{\mu\nu})}{\partial g_{\mu\nu}}\delta(g_{\mu\nu}) = g g^{\mu\nu}\delta(g_{\mu\nu}), \tag{2.9}$$

which help us express the differentiation of a determinant in terms of the inverse of the matrix and of the determinant itself.

Another useful expression allows us to write the variation of a matrix in terms of the variation of its inverse. This expression is derived from the fact that $\delta(g^{\mu\nu}g_{\alpha\beta}) = 0$ since the product $g^{\mu\nu}g_{\alpha\beta}$ corresponds to an invariant, which reads as follows

$$\begin{aligned}
\delta(\delta^{\mu}_{\alpha}) = 0 \quad &\Leftrightarrow \quad \delta(g^{\mu\nu}g_{\alpha\nu}) = 0 \\
&\Leftrightarrow \quad 0 = g^{\mu\nu}\delta(g_{\alpha\nu}) + g_{\alpha\nu}\delta(g^{\mu\nu}) \\
&\Leftrightarrow \quad g^{\mu\nu}\delta(g_{\alpha\nu}) = -g_{\alpha\nu}\delta(g^{\mu\nu}) \\
&\Leftrightarrow \quad g_{\mu\beta}g^{\mu\nu}\delta(g_{\alpha\nu}) = -g_{\mu\beta}g_{\nu\alpha}\delta(g^{\mu\nu}) \\
&\Leftrightarrow \quad \delta^{\nu}_{\beta}\delta(g_{\alpha\nu}) = -g_{\mu\beta}g_{\nu\alpha}\delta(g^{\mu\nu}) \\
&\Leftrightarrow \quad \delta(g_{\alpha\beta}) = -g_{\mu\beta}g_{\nu\alpha}\delta(g^{\mu\nu}),
\end{aligned} \tag{2.10}$$

for our case we may write Eq. (2.10) as

$$g^{\mu\nu}\delta(g_{\mu\nu}) = -g_{\mu\nu}\delta(g^{\mu\nu}). \tag{2.11}$$

Carrying on with our calculations, we substitute Eqs. (2.9) and (2.11) in Eq. (2.8), which after some mathematical manipulation results in

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}} g g^{\mu\nu}\delta g_{\mu\nu} = \frac{\sqrt{(-g)^{\cancel{2}}}}{2\cancel{\sqrt{(-g)}}} g^{\mu\nu}\delta g_{\mu\nu} = -\frac{\sqrt{-g}}{2} g_{\mu\nu}\delta g^{\mu\nu}, \tag{2.12}$$

it is important to notice that this expression for the variation of the square root of the determinant, $\delta(\sqrt{-g})$ will be useful for intermediate calculations when addressing other terms' variations.

After working on the variation of δg to evince its dependency on $\delta g^{\mu\nu}$, we can finally rewrite the first term of Eq. (2.4) as

$$\delta\sqrt{-g}(R+f(G)) = -\sqrt{-g}\frac{1}{2}g_{\mu\nu}(R+f(G))\delta g^{\mu\nu}. \tag{2.13}$$

2.1.4 Working on δR

We now turn to the second term of Eq. (2.4), $\sqrt{-g}\delta R$, which variation amounts to the understanding of δR and manipulating it in order to evince the variation on the inverse of the metric, $\delta g^{\mu\nu}$.

To commence, we must recognize that R is the Ricci scalar or scalar curvature and is obtained by the contraction of the Ricci tensor, $R_{\mu\nu}$ with the inverse metric $g^{\mu\nu}$ as in $R = g^{\mu\nu}R_{\mu\nu}$. With that said, we can write

$$\sqrt{-g}\delta R = \sqrt{-g}\delta(g^{\mu\nu}R_{\mu\nu}) = \sqrt{-g}(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}), \tag{2.14}$$

where in the first term in Eq. (2.14) on the right-hand-side of the equation is already evinced the variation of the inverse metric $\delta g^{\mu\nu}$.

Therefore, we focus on the second, which denotes a variation of the Ricci tensor, $\delta R_{\mu\nu}$. To work on this variation we must acknowledge that the Ricci tensor is a contraction of the Riemann tensor, $R_{\mu\nu\rho\sigma}$, with the inverse metric as in $R_{\nu\sigma} = g^{\mu\rho}R_{\mu\nu\rho\sigma} = R^{\rho}{}_{\nu\rho\sigma}$. Then, we may write

$$\delta R_{\mu\nu} = \delta(R^{\rho}{}_{\nu\rho\sigma}), \tag{2.15}$$

and before continuing it is important to know the intrinsic components that constitute the tensor $R^{\rho}{}_{\nu\rho\sigma}$.

We define the Riemann tensor, $R^{\mu}{}_{\nu\rho\sigma}$, through commutations of the covariant differentiation applied to any tensor, which result in

$$R^{\mu}{}_{\nu\rho\sigma} = \partial_\rho\Gamma^{\mu}_{\nu\sigma} - \partial_\sigma\Gamma^{\mu}_{\nu\rho} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\mu}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\Gamma^{\mu}_{\alpha\sigma}. \tag{2.16}$$

However, we will not be working with the Riemann tensor but rather with its variation. The Riemann tensor is a combination of a combination of metric connections, $\Gamma^{\mu}{}_{\nu\rho}$, and their derivatives, $\partial_\sigma\Gamma^{\mu}{}_{\nu\rho}$. The variation of the Riemann tensor as expressed in Eq. (2.16) amounts then to the following

$$\delta(R^{\mu}{}_{\nu\rho\sigma}) = \delta(\partial_\rho\Gamma^{\mu}_{\nu\sigma}) - \delta(\partial_\sigma\Gamma^{\mu}_{\nu\rho}) + \delta(\Gamma^{\alpha}_{\nu\sigma})\Gamma^{\mu}_{\alpha\rho} - \delta(\Gamma^{\alpha}_{\nu\rho})\Gamma^{\mu}_{\alpha\sigma} + \Gamma^{\alpha}_{\nu\sigma}\delta(\Gamma^{\mu}_{\alpha\rho}) - \Gamma^{\alpha}_{\nu\rho}\delta(\Gamma^{\mu}_{\alpha\sigma}). \tag{2.17}$$

If we look closely into Eq. (2.17), we may notice the resemblance between Eq. (2.17) and the difference between the two covariant derivatives of the variation of the metric connection that follows

$$\begin{aligned}
\nabla_\rho\delta\Gamma^{\mu}_{\nu\sigma} &= \partial_\rho\delta\Gamma^{\mu}_{\nu\sigma} + \delta\Gamma^{\alpha}_{\nu\sigma}\Gamma^{\mu}_{\rho\alpha} - \delta\Gamma^{\mu}_{\nu\alpha}\Gamma^{\alpha}_{\rho\sigma} - \delta\Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\rho\nu}, & (2.18)\\
\nabla_\sigma\delta\Gamma^{\mu}_{\nu\rho} &= \partial_\sigma\delta\Gamma^{\mu}_{\nu\rho} + \delta\Gamma^{\alpha}_{\nu\rho}\Gamma^{\mu}_{\sigma\alpha} - \delta\Gamma^{\mu}_{\nu\alpha}\Gamma^{\alpha}_{\sigma\rho} - \delta\Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\sigma\nu}, & (2.19)\\
\nabla_\rho\delta\Gamma^{\mu}_{\nu\sigma} - \nabla_\sigma\delta\Gamma^{\mu}_{\nu\rho} &= \partial_\rho\delta\Gamma^{\mu}_{\nu\sigma} - \partial_\sigma\delta\Gamma^{\mu}_{\nu\rho} + \delta\Gamma^{\alpha}_{\nu\sigma}\Gamma^{\mu}_{\alpha\rho} - \delta\Gamma^{\alpha}_{\nu\rho}\Gamma^{\mu}_{\alpha\sigma} + \Gamma^{\alpha}_{\nu\sigma}\delta\Gamma^{\mu}_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\delta\Gamma^{\mu}_{\alpha\sigma}, & (2.20)
\end{aligned}$$

and when comparing Eqs. (2.17) and (2.20) we should note that $\delta(\partial_\rho \Gamma^\mu_{\nu\sigma}) = \partial_\rho(\delta\Gamma^\mu_{\nu\sigma})$. As a result of this concordance, we can clearly write that

$$\delta(R^\mu{}_{\nu\rho\sigma}) = \nabla_\rho \delta\Gamma^\mu_{\nu\sigma} - \nabla_\sigma \delta\Gamma^\mu_{\nu\rho}, \tag{2.21}$$

which for our case in Eq. (2.15), when the first and the third indexes are the same, we get

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}\delta(R^\rho{}_{\mu\rho\nu}) &= g^{\mu\nu}\nabla_\rho \delta\Gamma^\rho_{\mu\nu} - g^{\mu\nu}\nabla_\nu \delta\Gamma^\rho_{\mu\rho} \\ &= \nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}) - \nabla^\mu \delta\Gamma^\rho_{\rho\mu} \\ &= \nabla_\rho(g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\sigma_{\sigma\mu}) \\ &= \nabla_\rho X^\rho, \end{aligned} \tag{2.22}$$

where we define $X^\rho = g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} - g^{\mu\rho}\delta\Gamma^\sigma_{\sigma\mu}$. To evaluate this term we need to contextualize it in the integral of the action of Eq. (2.4), where we can compute

$$\int_M dx^4 \sqrt{-g}\nabla_\rho X^\rho = \int_M dx^4 \partial_\rho (X^\rho \sqrt{-g}) = \int_{\partial M} dx^3 \sqrt{|h|} n_\rho X^\rho, \tag{2.23}$$

h being the determinant of the metric in the new manifold ∂M induced by pulling back the metric on the manifold M, and where we implicitly integrated by parts and used the results that $\nabla_\rho(\sqrt{-g}) = 0$, and for any tensor X^ρ it is true that $\nabla_\rho(X^\rho\sqrt{-g}) = \partial_\rho(X^\rho\sqrt{-g})$. If we assume that $\delta g_{\mu\nu}$ has support in a compact region that does not intersect ∂M this term will vanish. Considering the vanishing of $\delta g_{\mu\nu}$ and its derivative on ∂M implies that X^ρ will also vanish on ∂M, which amounts to considering that $\sqrt{-g} g^{\mu\nu}\delta R_{\mu\nu} = 0$ for the purpose of the calculation of the variation of the action, i.e. in the context of the integral form.

After the analysis carried on, we may finally rewrite the second term inside the integral that characterizes the gravitational part of Eq. (2.4) as

$$\sqrt{-g}\delta R = \sqrt{-g} R_{\mu\nu}\delta g^{\mu\nu}. \tag{2.24}$$

2.1.5 Variation of δG

Unfolding δG

We finally address the third term inside the integral of the gravitational part of the action from Eq. (2.4), and last term in deducing the full field equations for this $f(G)$ gravity theory. This term reads as $\sqrt{-g} f'(G)\delta G$, where G is defined by Eq. (2.3), meaning that we may unfold the variation of δG as

$$\begin{aligned} \delta G &= \delta(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}) \\ &= \delta(R^2) - 4\delta(R_{ab}R^{ab}) + \delta(R_{abcd}R^{abcd}). \end{aligned} \tag{2.25}$$

As we have been doing for the variation of the action S itself, in the following subsections of this Sec. 2.1.5 we shall address each of the terms which compose the variation depicted by Eq. (2.25) separately for simplicity.

Working on $\delta(R^2)$

We begin our analysis of the variation of the Gauss-Bonnet invariant, δG, by spreading out the first term of Eq. (2.25), $\delta(R^2)$, which reads as

$$\begin{aligned} \delta R^2 &= \frac{\partial R^2}{\partial R}\delta R = 2R\delta R \\ &= 2R\delta(g^{\mu\nu}R_{\mu\nu}) \\ &= 2RR_{\mu\nu}\delta g^{\mu\nu} + 2Rg^{\mu\nu}\delta R_{\mu\nu}. \end{aligned} \tag{2.26}$$

The first term in Eq. (2.26) already leaves $\delta g^{\mu\nu}$ in evidence, so we only need to work out the second, which has a dependence on $\delta R_{\mu\nu}$. From Eq. (2.21) we can write $\delta R_{\mu\nu}$ in terms of the metric connection as

$$\delta R_{\mu\nu} = \delta(R^{\rho}{}_{\mu\rho\nu}) = \nabla_\rho \delta\Gamma^{\rho}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\rho}_{\mu\rho}, \tag{2.27}$$

and the metric connection is defined as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right), \tag{2.28}$$

which helps us write that

$$\delta R_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\nabla_\rho\nabla_\mu\delta g_{\nu\sigma} + \nabla_\rho\nabla_\nu\delta g_{\mu\sigma} - \nabla_\rho\nabla_\sigma\delta g_{\mu\nu} - \nabla_\nu\nabla_\mu\delta g_{\rho\sigma} - \nabla_\nu\nabla_\rho\delta g_{\mu\sigma} + \nabla_\nu\nabla_\sigma\delta g_{\mu\rho}\right), \tag{2.29}$$

where it was implicitly used that $\nabla_\rho g^{\mu\nu} = 0$. It is important to note that contrarily to the latter, $\nabla_\rho \delta g^{\mu\nu}$ does not necessarily vanish.

To treat the long expression in Eq. (2.29) we will make use of the commutation relations on the covariant derivatives, more specifically for our case

$$\begin{aligned} \left[\nabla_\rho, \nabla_\nu\right]\delta g_{\mu\sigma} &= -R^{\alpha}{}_{\mu\rho\nu}\delta g_{\alpha\sigma} - R^{\alpha}{}_{\sigma\rho\nu}\delta g_{\mu\alpha}, & (2.30) \\ \nabla_\nu\nabla_\sigma\delta g_{\mu\alpha} &= \nabla_\sigma\nabla_\nu\delta g_{\mu\rho} - R^{\alpha}{}_{\mu\nu\sigma}\delta g_{\alpha\rho} - R^{\alpha}{}_{\rho\nu\sigma}\delta g_{\mu\alpha}, & (2.31) \end{aligned}$$

where the commutator operator is represented by rectangular parenthesis and is defined by $[\nabla_\rho, \nabla_\nu] = \nabla_\rho\nabla_\nu - \nabla_\nu\nabla_\rho$.

Using Eqs.(2.30) and (2.31) in Eq. (2.29) we are able to rewrite $\delta R_{\mu\nu}$ as

$$
\begin{aligned}
\delta R_{\mu\nu} &= \frac{1}{2}[g^{\rho\sigma}\left(\nabla_\rho\nabla_\mu\delta g_{\nu\sigma}+\nabla_\sigma\nabla_\nu\delta g_{\mu\rho}\right)-R^{\gamma}{}_{\mu\rho\nu}g^{\rho\sigma}\delta g_{\gamma\sigma}-R^{\gamma}{}_{\sigma\rho\nu}g^{\rho\sigma}\delta g_{\mu\gamma} \\
&\quad -\Box\delta g_{\mu\nu}+g_{\rho\sigma}\nabla_\nu\nabla_\mu\delta g^{\rho\sigma}-R^{\gamma}{}_{\mu\nu\sigma}g^{\rho\sigma}\delta g_{\gamma\rho}-R^{\gamma}{}_{\rho\nu\sigma}g^{\rho\sigma}\delta g_{\mu\gamma}] \\
&= \frac{1}{2}[-\left(\nabla^{\sigma}\nabla_\mu g_{\nu\alpha}g_{\sigma\beta}\delta g^{\alpha\beta}+\nabla^{\rho}\nabla_\nu g_{\mu\alpha}g_{\rho\beta}\delta g^{\alpha\beta}\right)+R^{\gamma}{}_{\mu\rho\nu}g_{\gamma\alpha}g_{\sigma\beta}g^{\rho\sigma}\delta g^{\alpha\beta}+R^{\gamma}{}_{\sigma\rho\nu}g_{\mu\alpha}g_{\gamma\beta}g^{\rho\sigma}\delta g^{\alpha\beta} \\
&\quad +g_{\mu\alpha}g_{\nu\beta}\Box\delta g^{\alpha\beta}+g_{\alpha\beta}\nabla_\nu\nabla_\mu\delta g^{\alpha\beta}+R^{\gamma}{}_{\mu\nu\sigma}g_{\gamma\alpha}g_{\rho\beta}g^{\rho\sigma}\delta g^{\alpha\beta}+R^{\gamma}{}_{\rho\nu\sigma}g_{\mu\alpha}g_{\gamma\beta}g^{\rho\sigma}\delta g^{\alpha\beta}] \\
&= \frac{1}{2}[-\nabla_\beta\left(\nabla_\mu g_{\nu\alpha}\delta g^{\alpha\beta}+\nabla_\nu g_{\mu\alpha}\delta g^{\alpha\beta}\right)+(\cancel{R_{\alpha\mu\beta\nu}}+\cancel{R_{\beta\sigma}{}^{\sigma}{}_{\nu}}+\cancel{R_{\alpha\mu\nu\beta}}+\cancel{R_{\beta}{}^{\sigma}{}_{\mu\sigma}})\delta g^{\alpha\beta} \\
&\quad +g_{\mu\alpha}g_{\nu\beta}\Box\delta g^{\alpha\beta}+g_{\alpha\beta}\nabla_\nu\nabla_\mu\delta g^{\alpha\beta}] \\
&= \frac{1}{2}\left[g_{\mu\alpha}g_{\nu\beta}\Box\delta g^{\alpha\beta}+g_{\alpha\beta}\nabla_\nu\nabla_\mu\delta g^{\alpha\beta}-\nabla_\beta\nabla_\mu g_{\nu\alpha}\delta g^{\alpha\beta}-\nabla_\beta\nabla_\nu g_{\mu\alpha}\delta g^{\alpha\beta}\right], \qquad (2.32)
\end{aligned}
$$

where the symmetries of the metric tensor ($g_{\mu\nu}$) the Ricci tensor ($R_{\mu\nu}$) and the Riemann tensor ($R_{\mu\nu\rho\sigma}$) were used, which read as follows

$$
g_{\mu\nu} = g_{\nu\mu}, \qquad (2.33)
$$

$$
R_{\mu\nu} = R_{\nu\mu}, \qquad (2.34)
$$

$$
R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho} = -R_{\nu\mu\rho\sigma} = R_{\rho\sigma\mu\nu}, \qquad (2.35)
$$

as well as the rule derived in Eq. (2.10). Substituting the results developed in Eq. (2.32) into the second term of Eq. (2.26) resumes to

$$
\begin{aligned}
2Rg^{\mu\nu}\delta R_{\mu\nu} &= R\left[-\nabla_\beta\left(\nabla\delta^{\mu}_{\alpha}\delta g^{\alpha\beta}+\nabla_\nu\delta^{\nu}_{\alpha}\delta g^{\alpha\beta}\right)+\delta^{\nu}_{\alpha}g_{\nu\beta}\Box\delta g^{\alpha\beta}+g_{\alpha\beta}\Box\delta g^{\alpha\beta}\right] \\
&= 2Rg_{\alpha\beta}\Box\delta g^{\alpha\beta}-2R\nabla_\alpha\nabla_\beta\delta g^{\alpha\beta}, \qquad (2.36)
\end{aligned}
$$

although none of these variations are calculated outside of the context of Eq. (2.4), so they are confined to an integral and accompanied by $\sqrt{-g}f'(G)$, so we must write

$$
2R\sqrt{-g}f'(G)g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{-g}\left(2Rg_{\alpha\beta}f'(G)\Box\delta g^{\alpha\beta}-2Rf'(G)\nabla_\alpha\nabla_\beta\delta g^{\alpha\beta}\right), \qquad (2.37)
$$

where in Eq. (2.37) $f'(G)$ is positioned right before the covariant derivatives on purpose. The difference between this term and the one solved in the Subsec. 2.1.5 is that when integrating by parts Eq. (2.37) in the context of the integral present in the action of Eq. (2.4), we must be careful with derivatives that may be applied to $f'(G)$ - these arithmetic calculations will actually be responsible for the appearance of higher order degrees of freedom.

Let us now return to our calculations and integrate by parts, implicitly, the first term of Eq. (2.37), which amounts to

$$
\begin{aligned}
2f'(G)Rg_{\alpha\beta}\nabla^\mu\nabla_\mu\delta g^{\alpha\beta} &= \nabla^\mu\left(2f'(G)Rg_{\alpha\beta}\nabla_\mu\delta g^{\alpha\beta}\right) - \nabla_\mu\delta g^{\alpha\beta}\nabla^\mu\left(2f'(G)Rg_{\alpha\beta}\right)\\
&= -\nabla_\mu\left(\delta g^{\alpha\beta}\nabla^\mu\left(2f'(G)Rg_{\alpha\beta}\right)\right) + \delta g^{\alpha\beta}\nabla_\mu\left(\nabla^\mu\left(2f'(G)Rg_{\alpha\beta}\right)\right)\\
&= 2g_{\alpha\beta}\delta g^{\alpha\beta}\nabla_\mu\left(R\nabla^\mu f'(G) + f'(G)\nabla^\mu R\right)\\
&= 2g_{\alpha\beta}\delta g^{\alpha\beta}\left(2\nabla_\mu R\nabla^\mu f'(G) + R\Box f'(G) + f'(G)\Box R\right), \qquad (2.38)
\end{aligned}
$$

where all the terms that were cancelled out are of the type $\nabla_\mu X^\mu$ and were so accordingly to the divergence theorem as we explicitly explained before for the case of δR, in Eq. (2.23).

We adopt a similar procedure for the second term of Eq. (2.37), resulting in

$$
\begin{aligned}
2f'(G)R\nabla_\alpha\nabla_\beta\delta g^{\alpha\beta} &= \nabla_\beta\left(2f'(G)R\nabla_\alpha\delta g^{\alpha\beta}\right) - \nabla_\alpha\delta g^{\alpha\beta}\nabla_\beta\left(2f'(G)R\right)\\
&= -\nabla_\alpha\left(\delta g^{\alpha\beta}\nabla_\beta\left(2f'(G)R\right)\right) + \delta g^{\alpha\beta}\nabla_\alpha\left(\nabla_\beta(2f'(G)R)\right)\\
&= 2\delta g^{\alpha\beta}(R(\nabla_\alpha\nabla_\beta f'(G)) + f'(G)(\nabla_\alpha\nabla_\beta R) + \nabla_\beta f'(G)\nabla_\alpha R + \nabla_\beta R\nabla_\alpha f'(G))\\
&= 2\delta g^{\alpha\beta}(R(\nabla_\alpha\nabla_\beta f'(G)) + f'(G)(\nabla_\alpha\nabla_\beta R) + 2\nabla_\beta f'(G)\nabla_\alpha R). \qquad (2.39)
\end{aligned}
$$

Finally, we substitute Eqs. (2.38) and (2.39) in Eq. (2.37), and afterwards Eq. (2.37) back to Eq. (2.26) to arrive at the final expression for the first term of Eq. (2.25) in the integral context of Eq. (2.4) that is

$$
\begin{aligned}
\sqrt{-g}f'(G)\delta R^2 &= \sqrt{-g}\delta g^{\mu\nu}[f'(G)(2RR_{\mu\nu} - 2\nabla_\mu\nabla_\nu R + 2g_{\mu\nu}\Box R) + 4g_{\mu\nu}\nabla_a R\nabla^a f'(G)\\
&\quad + 2g_{\mu\nu}R\Box f'(G) - 4\nabla_\nu f'(G)\nabla_\mu R - 2R(\nabla_\mu\nabla_\nu f'(G))], \qquad (2.40)
\end{aligned}
$$

with the dependence on $\delta g^{\mu\nu}$ in evidence.

Working on $\delta(R^{\mu\nu}R_{\mu\nu})$

Here, we shall address the second term of the variation of δG in Eq. (2.25). Respectively,

$$
\begin{aligned}
-4\delta(R^{\mu\nu}R_{\mu\nu}) &= -4\delta(R_{\mu\nu}g^{\mu\rho}g^{\nu\sigma}R_{\rho\sigma})\\
&= -4\delta(R^\gamma{}_{\mu\gamma\nu}g^{\mu\rho}g^{\nu\sigma}R^\eta{}_{\rho\eta\sigma})\\
&= -4\left(R^{\mu\nu}\delta R^\gamma{}_{\mu\gamma\nu} + R_\mu{}^\sigma R_{\rho\sigma}\delta g^{\mu\rho} + R^\rho{}_\nu R_{\rho\sigma}\delta g^{\nu\sigma} + R^{\rho\sigma}\delta R^\eta{}_{\rho\eta\sigma}\right)\\
&= -8R_\alpha{}^\mu R_{\beta\mu}\delta g^{\alpha\beta} - 8R^{\mu\nu}\delta R^\gamma{}_{\mu\gamma\nu}. \qquad (2.41)
\end{aligned}
$$

Looking at Eq. (2.41) we immediately notice the evinced dependence of the first term on the inverse metric $\delta g^{\alpha\beta}$, so that we may concentrate on working on the second one, more specifically on the variation $\delta R^\gamma{}_{\mu\gamma\nu} = \delta R_{\mu\nu}$ which we have already seen in Eq. (2.32).

Substituting the expression found for $\delta R_{\mu\nu}$ in Eq. (2.32) into the second term of Eq. (2.41), considering now the context of Eq. (2.4) where this term is multiplied by $\sqrt{-g}f'(G)$, we may write

$$\begin{aligned} -8\sqrt{-g}f'(G)R^{\mu\nu}\delta R^{\gamma}{}_{\mu\gamma\nu} &= -4\sqrt{-g}f'(G)(R_{\alpha\beta}\Box\delta g^{\alpha\beta} + g_{\alpha\beta}R^{\mu\nu}\nabla_{\nu}\nabla_{\mu}\delta g^{\alpha\beta} \\ &\quad -R^{\mu}{}_{\alpha}\nabla_{\beta}\nabla_{\mu}\delta g^{\alpha\beta} - R_{\alpha}{}^{\nu}\nabla_{\beta}\nabla_{\nu}\delta g^{\alpha\beta}) \\ &= \sqrt{-g}(-4f'(G)R_{\alpha\beta}\Box\delta g^{\alpha\beta} - 4f'(G)g_{\alpha\beta}R^{\mu\nu}\nabla_{\nu}\nabla_{\mu}\delta g^{\alpha\beta} \\ &\quad +8f'(G)R^{\mu}{}_{\alpha}\nabla_{\beta}\nabla_{\mu}\delta g^{\alpha\beta}). \end{aligned} \tag{2.42}$$

From here on the treatment goes in a similar way to the one we adopted for working out the derivatives in Eq. (2.37), we implicitly integrate by parts the expression in the context of the integral of Eq. (2.4) and use the divergence theorem explained in Eq. (2.23) to justify the vanishing of some terms of the type $\nabla_{\rho}X^{\rho}$.

Starting then with the first term of Eq. (2.42) we have

$$\begin{aligned} -4f'(G)R_{\alpha\beta}\nabla_{\mu}\nabla^{\mu}\delta g^{\alpha\beta} = &\ \cancel{\nabla_{\mu}(-4f'(G)R_{\alpha\beta}\nabla^{\mu}\delta g^{\alpha\beta})} - \nabla^{\mu}\delta g^{\alpha\beta}\nabla_{\mu}(-4f'(G)R_{\alpha\beta}) \\ = &\ \cancel{-\nabla^{\mu}(\delta g^{\alpha\beta}\nabla_{\alpha}(-4f'(G)R_{\alpha\beta}))} + \delta g^{\alpha\beta}\nabla^{\mu}(\nabla_{\mu}(-4f'(G)R_{\alpha\beta})) \\ = &\ -4\delta g^{\alpha\beta}\left(R_{\alpha\beta}\Box f'(G) + 2\nabla^{\mu}R_{\alpha\beta}\nabla_{\mu}f'(G) + f'(G)\Box R_{\alpha\beta}\right), \end{aligned} \tag{2.43}$$

in a similar way we can write for the second term of Eq. (2.42)

$$\begin{aligned} -4f'(G)g_{\alpha\beta}R^{\mu\nu}\nabla_{\nu}\nabla_{\mu}\delta g^{\alpha\beta} &= \cancel{\nabla_{\nu}(-4f'(G)g_{\alpha\beta}R^{\mu\nu}\nabla_{\mu}\delta g^{\alpha\beta})} - \nabla_{\mu}\delta g^{\alpha\beta}\nabla_{\nu}\left(-4f'(G)g_{\alpha\beta}R^{\mu\nu}\right) \\ &= \cancel{-\nabla_{\mu}(\delta g^{\alpha\beta}\nabla_{\nu}\left(-4f'(G)g_{\alpha\beta}R^{\mu\nu}\right))} + \delta g^{\alpha\beta}\nabla_{\mu}\left(\nabla_{\nu}\left(-4f'(G)g_{\alpha\beta}R^{\mu\nu}\right)\right) \\ &= -4g_{\alpha\beta}\delta g^{\alpha\beta}(R^{\mu\nu}(\nabla_{\mu}\nabla_{\nu}f'(G)) + f'(G)\left(\nabla_{\mu}\nabla_{\nu}R^{\mu\nu}\right) + \nabla_{\mu}f'(G)\nabla_{\nu}R^{\mu\nu} \\ &\quad +\nabla_{\nu}f'(G)\nabla_{\mu}R^{\mu\nu}) \\ &= -4g_{\alpha\beta}\delta g^{\alpha\beta}(R^{\mu\nu}(\nabla_{\mu}\nabla_{\nu}f'(G)) + f'(G)\left(\nabla_{\mu}\nabla_{\nu}R^{\mu\nu}\right) \\ &\quad +\nabla^{\mu}f'(G)\nabla_{\mu}R), \end{aligned} \tag{2.44}$$

where from the third line to the fourth we make use of the contracted Bianchi identity that follows in Eq. (2.45),

$$\nabla_{\mu}R^{\mu}{}_{\nu} = \frac{1}{2}\nabla_{\nu}R. \tag{2.45}$$

Ultimately, we attend to the same techniques, implicitly integrating by parts and using the divergence theorem expressed by Eq. (2.23) to let some terms vanish as well as using the contracted Bianchi identity from Eq. (2.45) to simplify the expressions obtained, that we have already used for the first and second terms of Eq. (2.42) to rewrite the third term as

$$
\begin{aligned}
8f'(G)R^{\mu}{}_{\alpha}\nabla_{\beta}\nabla_{\mu}\delta g^{\alpha\beta} &= \nabla_{\beta}(8f'(G)R^{\mu}{}_{\alpha}\nabla_{\mu}\delta g^{\alpha\beta}) - \nabla_{\mu}\delta g^{\alpha\beta}\nabla_{\beta}(8f'(G)R^{\mu}{}_{\alpha}) \\
&= -\nabla_{\mu}(\delta g^{\alpha\beta}\nabla_{\beta}(8f'(G)R^{\mu}{}_{\alpha})) + \delta g^{\alpha\beta}\nabla_{\mu}(\nabla_{\beta}(8f'(G)R^{\mu}{}_{\alpha})) \\
&= 8\delta g^{\alpha\beta}(R^{\mu}{}_{\alpha}(\nabla_{\mu}\nabla_{\beta}f'(G)) + f'(G)(\nabla_{\mu}\nabla_{\beta}R^{\mu}{}_{\alpha}) + \nabla_{\mu}f'(G)\nabla_{\beta}R^{\mu}{}_{\alpha} \\
&\quad +\nabla_{\beta}f'(G)\nabla_{\mu}R^{\mu}{}_{\alpha}) \\
&= 8\delta g^{\alpha\beta}\Big(R^{\mu}{}_{\alpha}(\nabla_{\mu}\nabla_{\beta}f'(G)) + f'(G)(\nabla_{\mu}\nabla_{\beta}R^{\mu}{}_{\alpha}) + \nabla_{\mu}f'(G)\nabla_{\beta}R^{\mu}{}_{\alpha} \\
&\quad +\frac{1}{2}\nabla_{\beta}f'(G)\nabla_{\alpha}R\Big). \qquad (2.46)
\end{aligned}
$$

We are now equipped to gather each of the terms that we unfolded with Eqs. (2.43), (2.44) and (2.46) and compute the second term of Eq. (2.25) in the context of the gravitational part of the integral of the action in Eq. (2.4), which reads as

$$
\begin{aligned}
-\sqrt{-g}f'(G)\delta(4R_{\mu\nu}R^{\mu\nu}) &= \sqrt{-g}\delta g^{\alpha\beta}[f'(G)(-8R_{\alpha}{}^{\mu}R_{\beta\mu} - 4\Box R_{\alpha\beta} - 4g_{\alpha\beta}\nabla_{\mu}\nabla_{\nu}R^{\mu\nu} \\
&\quad +8\nabla_{\mu}\nabla_{\beta}R^{\mu}{}_{\beta}) - 4R_{\alpha\beta}\Box f'(G) - 8\nabla^{\mu}R_{\alpha\beta}\nabla_{\mu}f'(G) \\
&\quad -4g_{\alpha\beta}R^{\mu\nu}(\nabla_{\mu}\nabla_{\nu}f'(G)) - 4g_{\alpha\beta}\nabla^{\mu}f'(G)\nabla_{\mu}R + 8R^{\mu}{}_{\alpha}(\nabla_{\mu}\nabla_{\beta}f'(G)) \\
&\quad +8\nabla_{\mu}f'(G)\nabla_{\beta}R^{\mu}{}_{\alpha} + 4\nabla_{\beta}f'(G)\nabla_{\alpha}R], \qquad (2.47)
\end{aligned}
$$

to facilitate we use the contracted Bianchi identity from Eq. (2.45) and the commutator of the covariant derivatives, to be able to respectively express

$$
-4g_{\alpha\beta}\nabla_{\mu}\nabla_{\nu}R^{\mu\nu} = -4g_{\alpha\beta}\nabla^{\mu}\nabla_{\nu}R_{\mu}{}^{\nu} = -2g_{\alpha\beta}\Box R \qquad (2.48)
$$

$$
\begin{aligned}
&8\nabla_{\mu}\nabla_{\beta}R^{\mu}{}_{\alpha} = 8\nabla_{\beta}\nabla_{\mu}R^{\mu}{}_{\alpha} + 8R^{\mu}{}_{\nu\mu\beta}R^{\nu}{}_{\alpha} - 8R^{\nu}{}_{\alpha\mu\beta}R^{\mu}{}_{\nu} \\
&\Leftrightarrow 8\nabla_{\mu}\nabla_{\beta}R^{\mu}{}_{\alpha} - 8\nabla_{\beta}\nabla_{\mu}R^{\mu}{}_{\alpha} = 8R^{\mu}{}_{\nu\mu\beta}R^{\nu}{}_{\alpha} - 8R^{\nu}{}_{\alpha\mu\beta}R^{\mu}{}_{\nu} \qquad (2.49)
\end{aligned}
$$

in order to substitute Eqs. (2.48) and (2.49) back in Eq. (2.47) to obtain the definite expression for the term $-\sqrt{-g}f'(G)\delta(4R_{ab}R^{ab})$, which goes as

$$
\begin{aligned}
-\sqrt{-g}f'(G)\delta(4R_{\mu\nu}R^{\mu\nu}) =& \sqrt{-g}\delta g^{\alpha\beta}[f'(G)\left(-4\Box R_{\alpha\beta} - 2g_{\alpha\beta}\Box R - 8R_{\mu\alpha\nu\beta}R^{\mu\nu} + 4\nabla_{\alpha}\nabla_{\beta}R\right) - 4R_{\alpha\beta}\Box f'(G) \\
&- 8\nabla^{\mu}R_{\alpha\beta}\nabla_{\mu}f'(G) - 4g_{\alpha\beta}R^{\mu\nu}(\nabla_{\mu}\nabla_{\nu}f'(G)) - 4g_{\alpha\beta}\nabla^{\mu}f'(G)\nabla_{\mu}R \\
&+ 8R^{\mu}{}_{\alpha}(\nabla_{\mu}\nabla_{\beta}f'(G)) + 8\nabla_{\mu}f'(G)\nabla_{\beta}R^{\mu}{}_{\alpha} + 4\nabla_{\beta}f'(G)\nabla_{\alpha}R]. \qquad (2.50)
\end{aligned}
$$

Working on $\delta(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})$

To evaluate the full variation of the Gauss-Bonnet term, as we might ratify with Eq. (2.25), we now only need to work on the expression for the variation of the contraction of two Riemann tensors, $\delta(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$. We may decompose this term as

$$\begin{aligned}
\delta(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) &= \delta(g_{\mu\gamma}R^{\gamma}{}_{\nu\rho\sigma}g^{\nu\eta}g^{\rho\zeta}g^{\sigma\xi}R^{\mu}{}_{\eta\zeta\xi}) \\
&= \delta g_{\mu\gamma}R^{\gamma}{}_{\nu\rho\sigma}R^{\mu\nu\rho\sigma} + \delta R^{\gamma}{}_{\nu\rho\sigma}R_{\gamma}{}^{\nu\rho\sigma} + \delta g^{\nu\eta}R_{\mu\nu\rho\sigma}R^{\mu}{}_{\eta}{}^{\rho\sigma} \\
&\quad + \delta g^{\rho\zeta}R_{\mu\nu\rho\sigma}R^{\mu\nu}{}_{\zeta}{}^{\sigma} + \delta g^{\sigma\xi}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\xi} + \delta R^{\mu}{}_{\eta\zeta\xi}R_{\mu}{}^{\eta\zeta\xi} \\
&= -g_{\mu\beta}g_{\gamma\alpha}\delta g^{\alpha\beta}R^{\gamma}{}_{\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 2\delta R^{\mu}{}_{\nu\rho\sigma}R_{\mu}{}^{\nu\rho\sigma} + 3\delta g^{\alpha\beta}R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho} \\
&= 2R_{\alpha\mu\nu\rho}R_{\beta}{}^{\mu\nu\rho}\delta g^{\alpha\beta} + 2R_{\mu}{}^{\nu\rho\sigma}\delta R^{\mu}{}_{\nu\rho\sigma}. \qquad (2.51)
\end{aligned}$$

The first term of Eq. (2.51) is already in the desired form with the variation only acting on the inverse metric $\delta g^{\mu\nu}$, so we turn to Eq. (2.21) to help us unfold the variation $\delta R^{\mu}{}_{\nu\rho\sigma}$ with Eq. (2.28) defining the metric connection so that we can write

$$\delta R^{\mu}{}_{\nu\rho\sigma} = \frac{1}{2}g^{\mu\gamma}(\nabla_\rho\nabla_\sigma\delta g_{\gamma\nu} + \nabla_\rho\nabla_\nu\delta g_{\gamma\sigma} - \nabla_\rho\nabla_\gamma\delta g_{\nu\sigma} - \nabla_\sigma\nabla_\rho\delta g_{\gamma\nu} - \nabla_\sigma\nabla_\nu\delta g_{\gamma\rho} + \nabla_\sigma\nabla_\gamma\delta g_{\rho\nu}). \quad (2.52)$$

Again, we follow the same process we have conducted to work on the terms containing $\delta(R^2)$ and $\delta(R_{\mu\nu}R^{\mu\nu})$ from the same Eq. (2.25). By applying the rule derived in Eq. (2.10) to pass from a metric variation to an inverse metric variation notation, as well as resourcing to the rules of commutation between covariant derivatives depicted in both Eqs. (2.30) and (2.31), we arrive at

$$\begin{aligned}
\delta R^{\mu}{}_{\nu\rho\sigma} &= \frac{1}{2}\Big[R^{\eta}{}_{\gamma\rho\sigma}g_{\eta\alpha}g_{\nu\beta}\delta g^{\alpha\beta} + R^{\eta}{}_{\nu\rho\sigma}g_{\gamma\alpha}g_{\eta\beta}\delta g^{\alpha\beta} + \nabla_c(-g_{\gamma\alpha}g_{\nu\beta}\nabla_\nu\delta g^{\alpha\beta} + g_{\sigma\beta}g_{\nu\alpha}\nabla_\gamma\delta g^{\alpha\beta}) \\
&\quad + \nabla_\sigma(-g_{\rho\alpha}g_{\nu\beta}\nabla_\gamma\delta g^{\alpha\beta} + g_{\gamma\alpha}g_{\rho\beta}\nabla_\nu\delta g^{\alpha\beta})\Big]. \qquad (2.53)
\end{aligned}$$

We can reduce Eq. (2.53) to fewer and easier to work with terms, by means of tensor symmetries as the ones in Eqs. (2.34), (2.35) and (2.35), which arithmetic manipulation results in

$$\begin{aligned}
2R_{\mu}{}^{\nu\rho\sigma}\delta R^{\mu}{}_{\nu\rho\sigma} &= \delta g^{\alpha\beta}(R^{\eta}{}_{\gamma\rho\sigma}g_{\eta\alpha}g_{\nu\beta}R_{\mu}{}^{\nu\rho\sigma} + R^{\eta}{}_{\nu\rho\sigma}g_{\gamma\alpha}g_{\eta\beta}R_{\mu}{}^{\nu\rho\sigma}) \\
&\quad + \nabla_\rho(-g_{\gamma\alpha}g_{\nu\beta}\nabla_\nu\delta g^{\alpha\beta} + g_{\sigma\beta}g_{\nu\alpha}\nabla_\gamma\delta g^{\alpha\beta})R_{\mu}{}^{\nu\rho\sigma} \\
&\quad + \nabla_\sigma(-g_{\rho\alpha}g_{\nu\beta}\nabla_\gamma\delta g^{\alpha\beta} + g_{\gamma\alpha}g_{\rho\beta}\nabla_\nu\delta g^{\alpha\beta})R_{\mu}{}^{\nu\rho\sigma} \\
&= \delta g^{\alpha\beta}(\cancel{R_{\mu\gamma\rho\sigma}R^{\gamma}{}_{\beta}{}^{\rho\sigma}} + \cancel{R_{\beta\nu\rho\sigma}R_{\alpha}{}^{\nu\rho\sigma}}) + \nabla_\rho(\nabla_\gamma\delta g^{\alpha\beta})R^{\gamma}{}_{\alpha}{}^{\rho}{}_{\beta} \\
&\quad - \nabla_\rho(\nabla_\gamma\delta g^{\alpha\beta})R_{\mu}{}^{\gamma\rho}{}_{\beta} + \nabla_\sigma(\nabla_\gamma\delta g^{\alpha\beta})R_{\alpha}{}^{\gamma}{}_{\beta}{}^{\sigma} - \nabla_\sigma(\nabla_\gamma\delta g^{\alpha\beta})R^{\gamma}{}_{\beta\alpha}{}^{\sigma} \\
&= 4\nabla_\mu(\nabla_\nu\delta g^{\alpha\beta})R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}. \qquad (2.54)
\end{aligned}$$

Regarding the variation of the third and last term of Eq. (2.25) we got to Eq. (2.54). In order to continue and finish our analysis, we must address its context inside the integral that defined the gravitational part of the variation of the action in Eq. (2.4), therefore we shall integrate by parts the only term that resumes Eq. (2.54) with $\sqrt{-g}f'(G)$ that accompanies it inside as the integrand of the action. This procedure has already been used before as the divergence theorem as presented by Eq. (2.23), which proceed as

$$\begin{aligned} 4\sqrt{-g}f'(G)R^{\mu\ \ \nu}_{\ \alpha\ \beta}\nabla_\mu\nabla_\nu\delta g^{\alpha\beta} &= \cancel{\nabla_\mu(4f'(G)R^{\mu\ \ \nu}_{\ \alpha\ \beta}\nabla_\nu\delta g^{\alpha\beta})} - \nabla_\nu\delta g^{\alpha\beta}\nabla_\mu(4f'(G)R^{\mu\ \ \nu}_{\ \alpha\ \beta}) \\ &= \cancel{-\nabla_\mu(\delta g^{\alpha\beta}\nabla_\mu(4\sqrt{-g}f'(G)R^{\mu\ \ \nu}_{\ \alpha\ \beta}))} + \delta g^{\alpha\beta}\nabla_\nu(\nabla_\mu(4\sqrt{-g}f'(G)R^{\mu\ \ \nu}_{\ \alpha\ \beta})) \\ &= 4\sqrt{-g}\delta g^{\alpha\beta}[R^{\mu\ \ \nu}_{\ \alpha\ \beta}(\nabla_\nu\nabla_\mu f'(G)) + f'(G)(\nabla_\nu\nabla_\mu R^{\mu\ \ \nu}_{\ \alpha\ \beta}) \\ &\quad +\nabla_\nu f'(G)\nabla_\mu R^{\mu\ \ \nu}_{\ \alpha\ \beta} + \nabla_\mu f'(G)\nabla_\nu R^{\mu\ \ \nu}_{\ \alpha\ \beta}], \qquad (2.55) \end{aligned}$$

allowing us to substitute Eq. (2.55) back into Eq. (2.54) and the latter into Eq. (2.51),

$$\begin{aligned} \sqrt{-g}f'(G)\delta(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) &= \sqrt{-g}[f'(G)\delta g^{\alpha\beta}(2R_{\alpha\mu\nu\rho}R_\beta^{\ \mu\nu\rho} + \nabla_\nu\nabla_\mu R^{\mu\ \ \nu}_{\ \alpha\ \beta}) \\ &\quad +4\delta g^{\alpha\beta}(R^{\mu\ \ \nu}_{\ \alpha\ \beta}(\nabla_\nu\nabla_\mu f'(G)) + f'(G)(\nabla_\nu\nabla_\mu R^{\mu\ \ \nu}_{\ \alpha\ \beta}) \\ &\quad +\nabla_\nu f'(G)\nabla_\mu R^{\mu\ \ \nu}_{\ \alpha\ \beta} + \nabla_\mu f'(G)\nabla_\nu R^{\mu\ \ \nu}_{\ \alpha\ \beta})] \\ &= \delta g^{\alpha\beta}\sqrt{-g}[f'(G)(2R_{\alpha\mu\nu\rho}R_\beta^{\ \mu\nu\rho} + 4\nabla^\nu(-\nabla_\beta R_{\alpha\nu} + \nabla_\nu R_{\alpha\beta})) \\ &\quad +4(f'(G)(\nabla_\nu\nabla_\mu R^{\mu\ \ \nu}_{\ \alpha\ \beta}) + \nabla_\nu f'(G)\nabla_\mu R^{\mu\ \ \nu}_{\ \alpha\ \beta} \\ &\quad +\nabla_\mu f'(G)\nabla_\nu R^{\mu\ \ \nu}_{\ \alpha\ \beta})], \qquad (2.56) \end{aligned}$$

expression that closes the manipulation carried on with each of the terms that compose Eq. (2.25) regarding the variation of the term G coupled to $\sqrt{-g}f'(G)$ in the gravitational part of the integral of the action in Eq. (2.4).

Assembling the term $\sqrt{-g}f'(G)\delta G$

In Subsec. 2.1.5 we set to express the third term inside the integral of the gravitational part of the action from Eq. (2.4), and last term in deducing the full field equations for this $f(G)$ gravity theory. The term $\sqrt{-g}f'(G)\delta G$, where G is defined by Eq. (2.3), can now be assembled after we have deduced each of the terms that concern the variation δG in Eq. (2.25).

With Eqs. (2.40), (2.50) and (2.56) we can subsequently write that

$$\begin{aligned}
\sqrt{-g}f'(G)\delta G &= \sqrt{-g}\delta g^{\alpha\beta}[f'(G)(2RR_{\alpha\beta} - 2\nabla_\alpha\nabla_\beta R + 2g_{\alpha\beta}\Box R + 2R_{\alpha\mu\nu\rho}R_\beta{}^{\mu\nu\rho} - 4\nabla^\nu\nabla_\beta R_{\alpha\nu} \\
&+4\Box R_{\alpha\beta} - 4\Box R_{\alpha\beta} - 2g_{\alpha\beta}\Box R - 8R_{\mu\alpha\nu\beta}R^{\mu\nu} + 4\nabla_\mu\nabla_\nu R) + 4f'(G)(\nabla_\nu\nabla_\mu R^\mu{}_\alpha{}^\nu{}_\beta) \\
&+4\nabla_\nu f'(G)\nabla_\mu R^\mu{}_\alpha{}^\nu{}_\beta + 4\nabla_\mu f'(G)\nabla_\nu R^\mu{}_\alpha{}^\nu{}_\beta - 4R_{\alpha\beta}\Box f'(G) \\
&-8\nabla^\mu R_{\alpha\beta}\nabla_\mu f'(G) - 4g_{\alpha\beta}R^{\mu\nu}(\nabla_\mu\nabla_\nu f'(G)) - 4g_{\alpha\beta}\nabla^\mu f'(G)\nabla_\mu R \\
&+8R^\mu{}_\alpha(\nabla_\mu\nabla_\beta f'(G)) + 8\nabla_\mu f'(G)\nabla_\beta R^\mu{}_\alpha + 4\nabla_\beta f'(G)\nabla_\alpha R + 4g_{\alpha\beta}\nabla_\mu R\nabla^\mu f'(G) \\
&+2g_{\alpha\beta}R\Box f'(G) - 4\nabla_\beta f'(G)\nabla_\alpha R - 2R(\nabla_\alpha\nabla_\beta f'(G))], \qquad (2.57)
\end{aligned}$$

which, resorting to some rules on the commutation relations between covariant derivatives and the contracted Bianchi identity of Eq. (2.45) that let us simplify some terms as follows

$$\begin{aligned}
-4\nabla_\nu\nabla_\beta R_\alpha{}^\nu &= -4\nabla_\beta\nabla_\nu R^\nu{}_\alpha + 4R^\nu{}_{\mu\beta\nu}R^\mu{}_\alpha - 4R^\mu{}_{\alpha\beta\nu}R^\nu{}_\mu, \\
&= -2\nabla_\beta\nabla_\alpha R - 4R_{\mu\beta}R^\mu{}_\alpha + 4R_{\mu\alpha\nu\beta}R^{\mu\nu} \qquad (2.58) \\
4\nabla_\mu f'(G)\nabla_\nu R^\mu{}_\alpha{}^\nu{}_\beta &= 4\nabla^\mu f'(G)\nabla_\nu R^\nu{}_{\beta\mu\alpha} = -4\nabla^\mu f'(G)\nabla_\alpha R_{\beta\mu} + 4\nabla^\mu f'(G)\nabla_\mu R_{\alpha\beta}, \\
&= -4\nabla^\nu f'(G)\nabla_\beta R_{\alpha\nu} + 4\nabla^\nu f'(G)\nabla_\nu R_{\alpha\beta} \qquad (2.59) \\
4\nabla_\nu f'(G)\nabla_\mu R^\mu{}_\alpha{}^\nu{}_\beta &= 4\nabla^\nu f'(G)\nabla_\mu R^\mu{}_{\alpha\nu\beta} = -4\nabla^\nu f'(G)\nabla_\beta R_{\alpha\nu} + 4\nabla^\nu f'(G)\nabla_\nu R_{\alpha\beta}, \qquad (2.60)
\end{aligned}$$

we are able to express $\sqrt{-g}f'(G)\delta G$ as

$$\begin{aligned}
\sqrt{-g}f'(G)\delta G &= \sqrt{-g}\delta g^{\alpha\beta}[f'(G)(2RR_{\alpha\beta} - 2\nabla_\alpha\nabla_\beta R + 2R_{\alpha\mu\nu\rho}R_\beta{}^{\mu\nu\rho} - 2\nabla_\beta\nabla_\alpha R - 4R_{\mu\beta}R^\mu{}_\alpha \\
&+4R_{\mu\alpha\nu\beta}R^{\mu\nu} - 8R_{\mu\alpha\nu\beta}R^{\mu\nu} + 4\nabla_\alpha\nabla_\beta R) + 4R^\mu{}_\alpha{}^\nu{}_\beta(\nabla_\nu\nabla_\mu f'(G)) - 8\nabla^\mu f'(G)\nabla_\beta R_{\alpha\nu} \\
&+8\nabla^\nu f'(G)\nabla_\nu R_{\alpha\beta} - 4R_{\alpha\beta}\Box f'(G) - 8\nabla^\mu R_{\alpha\beta}\nabla_\mu f'(G) - 4g_{\alpha\beta}R^{\mu\nu}(\nabla_\mu\nabla_\nu f'(G)) \\
&+8R^\mu{}_\alpha(\nabla_\mu\nabla_\beta f'(G)) + 8\nabla^\nu f'(G)\nabla_\beta R_{\nu\alpha} + 2g_{\alpha\beta}R\Box f'(G) - 2R(\nabla_\alpha\nabla_\beta f'(G))], \qquad (2.61)
\end{aligned}$$

by substituting the results derived in Eqs. (2.58), (2.59) and (2.60) into Eq. (2.57).

The final expression we get for the term $\sqrt{-g}f'(G)\delta G$ is then

$$\begin{aligned}
\sqrt{-g}f'(G)\delta G &= \sqrt{-g}\delta g^{\alpha\beta}[f'(G)(2RR_{\alpha\beta} + 2R_{\alpha\mu\nu\rho}R_\beta{}^{\mu\nu\rho} - 4R_{\mu\beta}R^\mu{}_\alpha - 4R_{\mu\alpha\nu\beta}R^{\mu\nu}) \\
&+4R^\mu{}_\alpha{}^\nu{}_\beta(\nabla_\nu\nabla_\mu f'(G)) - 4R_{\alpha\beta}\Box f'(G) - 4g_{\alpha\beta}R^{\mu\nu}(\nabla_\mu\nabla_\nu f'(G)) \\
&+8R^\mu{}_\alpha(\nabla_\mu\nabla_\beta f'(G)) + 2g_{\alpha\beta}R\Box f'(G) - 2R(\nabla_\alpha\nabla_\beta f'(G))]. \qquad (2.62)
\end{aligned}$$

2.2 The full field equations

In review, our objectives for this chapter were to vary the action of Eq. (2.1) with respect to the inverse metric $g^{\mu\nu}$ in order to obtain the full field equations by means of the principle of least action, or in other words, the equations of motion that relate the gravitational field to the matter fields covariantly.

To achieve such a purpose, we addressed each of the terms that compose Eq. (2.4) separately for simplicity and are now able to gather them back together. Accordingly, we replace the results from Eqs. (2.6), (2.13), (2.24) and (2.62) in Eq. (2.4) to retrieve

$$\begin{aligned}\delta S \quad = \quad & \int dx^4 \sqrt{-g}\Big[\frac{1}{2\kappa}\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R + f(G)) + f'(G)(2RR_{\mu\nu} + 2R_{\mu\alpha\beta\chi}R_{\nu}{}^{\alpha\beta\chi} \\ & -4R_{\alpha\nu}R^{\alpha}{}_{\mu} - 4g^{\alpha\chi}g^{\beta\delta}R_{\mu\alpha\nu\beta}R_{\chi\delta}) + 4g^{\alpha\chi}g^{\beta\delta}R_{\mu\chi\nu\delta}(\nabla_{\beta}\nabla_{\alpha}f'(G)) \\ & -4R_{\mu\nu}\Box f'(G) - 4g_{\mu\nu}R^{\alpha\beta}(\nabla_{\alpha}\nabla_{\beta}f'(G)) + 4R^{\alpha}{}_{\mu}(\nabla_{\alpha}\nabla_{\nu}f'(G)) + 4R^{\alpha}{}_{\nu}(\nabla_{\alpha}\nabla_{\mu}f'(G)) \\ & +2g_{\mu\nu}R\Box f'(G) - 2R(\nabla_{\mu}\nabla_{\nu}f'(G))\Big) - \frac{1}{2}T_{\mu\nu}\Big]\delta g^{\mu\nu} \end{aligned} \tag{2.63}$$

after substituting Eq. (2.7) in Eq. (2.6) and acknowledging the symmetry of the term $R^{\alpha}{}_{\mu}(\nabla_{\alpha}\nabla_{\nu}f'(G))$ that allowed us to rewrite it as

$$8R^{\alpha}{}_{\mu}(\nabla_{\alpha}\nabla_{\nu}f'(G)) = 4(R^{\alpha}{}_{\mu}(\nabla_{\alpha}\nabla_{\nu}f'(G)) + R^{\alpha}{}_{\nu}(\nabla_{\alpha}\nabla_{\mu}f'(G))). \tag{2.64}$$

The principle of least action, $\delta S = 0$ sums up to

$$\begin{aligned}\delta S = 0 \quad \Leftrightarrow \quad & 0 = \int dx^4 \left(\frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}}\right)\sqrt{-g}\delta g^{\mu\nu} \\ \Leftrightarrow \quad & 0 = \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}}, \end{aligned} \tag{2.65}$$

where Eq. (2.65) would constitute the equations of motion. For our problem, by comparing Eq. (2.63) with Eq. (2.65) we get the full field equations, or equations of motion,

$$\begin{aligned}\kappa T_{\mu\nu} \quad = \quad & G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(G) + f'(G)[2RR_{\mu\nu} + 2R_{\mu\alpha\beta\chi}R_{\nu}{}^{\alpha\beta\chi} - 4R_{\alpha\nu}R^{\alpha}{}_{\mu} - 4g^{\alpha\chi}g^{\beta\delta}R_{\mu\alpha\nu\beta}R_{\chi\delta}] \\ & -4R_{\mu\nu}\Box f'(G) - 4g_{\mu\nu}R^{\alpha\beta}(\nabla_{\alpha}\nabla_{\beta}f'(G)) + 4R^{\alpha}{}_{\mu}(\nabla_{\alpha}\nabla_{\nu}f'(G)) + 4R^{\alpha}{}_{\nu}(\nabla_{\alpha}\nabla_{\mu}f'(G)) \\ & +2g_{\mu\nu}R\Box f'(G) - 2R(\nabla_{\mu}\nabla_{\nu}f'(G)) + 4g^{\alpha\chi}g^{\beta\delta}R_{\mu\chi\nu\delta}(\nabla_{\beta}\nabla_{\alpha}f'(G)), \end{aligned} \tag{2.66}$$

with $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ the Einstein tensor.

Chapter 3

Order reduced field equations of the $f(G)$ gravity theory

In this Chapter 3 we will study the order reduction technique, its mathematical background and go through some examples of its previous successes in the literature as well as understand why and how we can apply it to our field equations previously assembled in Sec. 2.2 of Chapter 2. Then we will proceed to actually applying it, deriving the lowest order form of each of our variables from Eq. (2.66) to finally construct the order reduced field equations.

3.1 Theoretical background

3.1.1 The mathematical technique

As previously mentioned in Sec. 1.2.3 of Chapter 1, with the aim of obtaining a second-order differential equation that allows a first Friedmann equation of the type of Eq. (1.30), we must apply an order reduction technique to our field equations in Eq. (2.66). This technique should reduce the fourth-order differential equations previously mentioned to second-order ones in a regular and covariant way through considering the second-order system of equations as the real one, and the degrees of freedom added in the higher-order theory as spurious, which is the same as treating the higher-order corrections as a constrained perturbative correction to the underlying, lower-energy effective theory.

Following the method as presented in the example used in Ref. [20], we can take some higher-order derivative generalization of Einstein's gravity theory, in the sense that the Lagrangian of the gravitational part of the action, that will result in the corresponding fourth-order field equations, contains a linear combination of the Ricci scalar, its square, and the square of the Ricci tensor, reads as

$$\mathcal{L} = 2\Lambda + R + \frac{\beta_1}{2}R^2 + \beta_2 R_{\mu\nu}R^{\mu\nu}, \tag{3.1}$$

where all the variables have been presented before in this thesis with the exception of the dimensionless scalar parameters β_1 and β_2, which when $\beta_1 = \beta_2 = 0$ return us the original Einstein's theory plus a cosmological constant, meaning that by simply zeroing the parameters β_1 and β_2 we completely modify the quality of the equations.

In deriving the field equations matching the Lagrangian of Eq. (3.1) we use the principle of least action, varying it with respect to the inverse metric $g^{\mu\nu}$ as we have done before in Chapter 2 for deriving the field equations concerning our problem with the $f(G)$ quantity in the Lagrangian. Resulting from this process for the case of Ref. [20], we have

$$\begin{aligned}\kappa T_{\mu\nu} &= G_{\mu\nu} - \Lambda g_{\mu\nu} + \beta_1 \left[\nabla_\mu \nabla_\nu G - G G_{\mu\nu} - (\Box G - \frac{1}{4}G^2) g_{\mu\nu}\right] \\ &+ \beta_2 \left[\Box G_{\mu\nu} + \nabla_\mu \nabla_\nu G - G G_{\mu\nu} - \left(\Box G + \frac{1}{2}(G_{\alpha\beta}G^{\alpha\beta} - G^2)\right) g_{\mu\nu} + 2R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu} G_{\alpha\beta}\right], \quad (3.2)\end{aligned}$$

where, here, G is the Einsteins tensor's trace, i.e. contraction with the metric as in $G = g^{\mu\nu}G_{\mu\nu}$. As we have mentioned and is clear from Eq. (3.2), by making $\beta_1 = \beta_2 = 0$ the order of Eq. (3.2) drops from four to two, which translates in saying that the equations are singular with respect to these parameters β_1 and β_2. What is the relevance of this statement for our problem? This is equivalent to stating that the set of equations represented in Eq. (3.2) does not depend smoothly on the parameters β_1 and β_2, since the equations change qualitatively when either or both of them vanish. Furthermore, as a consequence there must exist a set of non-generic solutions which constitute the general solution for the underlying set of second-order equations that describe the real theory. This system would be the so-called regular reduction of the initial fourth order one.

The essence of this covariant regular order reduction technique lies in noting that when the energy-momentum tensor is zero, we have for the Einstein's vacuum equations

$$G_{\mu\nu} = \Lambda g_{\mu\nu}, \quad (3.3)$$

which solutions are also solutions for the vacuum field equations of Eq. (3.2), although not the only ones, the other extra solutions may be considered as spurious for the purpose of this reduction technique. After acknowledging this, we are able to understand that constraining an equation such as Eq. (3.2) to select only solutions compatible with Eq. (3.3) is the same as reducing its order, especially through a mechanism that zeroes the parameters β_1 and β_2, i.e. that chooses only the lowest order solutions to function as constraints to our original higher-order theory.

Therefore, what was done here was that lower order solutions such as the ones in Eq. (3.3) (but with a stress-energy tensor different than zero), were taken in consideration as the real solutions for the problem depicted in Eq. (3.2) so that we substituted the Einstein tensor as a function of the stress-energy tensor (first order solution) in Eq. (3.2), lowering its initial derivative order to a second-order differential equation, as we can see in

$$G_{\mu\nu} - \Lambda g_{\mu\nu} + \beta_1 F^{(1,0)}_{\mu\nu} + \beta_2 F^{(0,1)}_{\mu\nu} = \kappa T_{\mu\nu} \quad (3.4)$$

where

$$
\begin{aligned}
F_{\mu\nu}^{(1,0)} &= \kappa(\nabla_\mu\nabla_\nu T - \Box T g_{\mu\nu}) + \kappa^2\left(\frac{1}{4}T^2 g_{\mu\nu} - TT_{\mu\nu}\right) + \Lambda\kappa(Tg_{\mu\nu} - 4T_{\mu\nu}), && (3.5)\\
F_{\mu\nu}^{(0,1)} &= \kappa(\Box T_{\mu\nu} + \nabla_\mu\nabla_\nu T - \Box T g_{\mu\nu}) + 2\kappa R^{\alpha}{}_{\mu}{}^{\beta}{}_{\nu}T_{\alpha\beta}\\
&+\kappa^2\left(\frac{1}{2}T^2 g_{\mu\nu} - TT_{\mu\nu} - \frac{1}{2}T_{\alpha\beta}T^{\alpha\beta}g_{\mu\nu}\right) + \Lambda\kappa(Tg_{\mu\nu} - 2T_{\mu\nu}), && (3.6)
\end{aligned}
$$

where T is the trace of the stress-energy tensor, given by its contraction with the metric: $T = g_{\mu\nu}T^{\mu\nu}$.

In [21] this method is revised and an emphasis to the conditions where we can apply such a mathematical artifice is given. The appeal of higher-order theories as natural corrections to general relativity is usually neglected when some undesirable properties are taken into consideration, usually related to the presence of higher-order derivatives that will result in difficulties for solving the problem such as more degrees of freedom and the lack of a lower-energy bound, meaning that any small perturbation will lead to enormously negative energies, which generates unwanted instability in the solutions. However, we know that if we impose proper constraints to some higher-order theory, which is in its essence just a truncated expansion of a non-local theory, we may avoid those problems. Analogously, we get the same results by treating the same way a theory where the higher-order derivative terms have been added as small corrections - mediated by some small parameter ϵ - to the main theory. To illustrate this scenario, we know that although some higher-derivative theory may behave dramatically different from general relativity, we may be able to perturbatively constrain it so that we get a theory whose effective results are compatible with general relativity's.

The method we have been following in this Sec. 3.1 basically amounts to the imposition of perturbative constraints taken from the truncation of the theory itself. The question to ask ourselves at this point is then, under what conditions is it appropriate to apply such a methodology? Clearly, when we want to interpret our higher-order theory as the lower energy one plus some perturbative corrections, in which case it will make sense to impose perturbative constraints to separate the effects of the higher-order perturbations from the inherent effective field theory. To conclude, employing this order reduction method through perturbative constraints is not an *ad hoc* procedure since we are capable of analysing the consequences in the aftermath. It is actually necessary to resource to these proceedings if we want our higher-order theory to resemble the lower-order one in its behaviour.

3.1.2 Why use it with $f(G)$ gravity and how to implement it?

The modified Gauss-Bonnet $f(G)$ theory possesses, in general, field equations with fourth-order derivatives, which in turn can generate non physical solutions. One way to deal with these solutions is to exclude them whenever they appear. A more interesting way is to devise a method in which these solutions do not appear at all. One such method states that the interesting physical solutions in $f(G)$ theories are the ones that can be found from general relativity in a perturbative way, the other solutions being considered artificial. This is the technique of order reduction [20, 21] which builds solutions perturbatively close to general relativity which are then second order differential equations by construction. This is the approach that we will take as revised in the previous Subsec. 3.1.1.

Historically speaking, higher order theories such as $f(R)$ and $f(G)$ gravities have always been regarded as exact theories, however this does not invalidate the possibility to treat these theories as effective field ones, meaning that we would only consider the solutions perturbatively close to general relativity as physical ones, and discard the rest. These two different approaches will only differ in the way we want to perceive the action: are the extra degrees of freedom fundamental or are they introduced during the derivation of the effective action? In our case, the $f(G)$ action is used as an effective theory to covariantly describe loop quantum cosmology in the limits where general relativity fails.

Since we are looking to get a squared dependence on ρ^2, as explicit in Eq. (1.30), a second-order differential equation is required, and a possible approach is to reduce the order of Eq. (2.66). This is then one of the advantages of adopting such an order reduction technique as the one previously explored.

Briefly, our desire is then to treat $f(G)$ gravity as a theory perturbatively close to general relativity, using the order reduction method that we studied in the previous Subsec. 3.1.1 to select only the physical solutions, already defined as perturbatively close to the lower energy theory (general relativity), instead of discarding the spurious ones after solving the equations. With this purpose, we may parametrize without any loss of generality the function $f(G)$ in the action as

$$f(G) = 2\Lambda + \epsilon\varphi(G), \tag{3.7}$$

where $\varphi(G)$ is some function of the Gauss-Bonnet invariant G (the one we want to find after solving the differential equations), the dimensionless parameter ϵ marks the deviation from general relativity, and Λ is a cosmological constant. The Lagrangian of the gravitational part of the action is then:

$$\mathcal{L} = R + 2\Lambda + \epsilon\varphi(G), \tag{3.8}$$

which is singular with respect to ϵ similarly to Eq. (3.2) with the parameters β_1 and β_2. This parameter also allows us to refer to first order environment when talking about working with ϵ terms, and to lowest order when considering situations with $\epsilon = 0$. With this in mind, our arithmetic challenge in the following sections will be to write each of the terms composing Eq. (2.66) in their lowest order and then substitute them back in ϵ terms considering also the substitution of $f(G)$ and $f'(G)$ with Eq. (3.7) and its respective derivative. To accomplish these objectives a useful equation will be the trace of Eq. (2.66) obtained from their contraction with the inverse metric $g^{\mu\nu}$,

$$\begin{aligned}\kappa T \;=\;& g^{\mu\nu}[G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(G) + f'(G)(2RR_{\mu\nu} + 2R_{\mu\alpha\beta\rho}R_{\nu}{}^{\alpha\beta\rho} - 4R_{\alpha\nu}R^{\alpha}{}_{\mu} - 4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\alpha\nu\beta}R_{\rho\sigma})\\ & -4R_{\mu\nu}\Box f'(G) - 4g_{\mu\nu}R^{\alpha\beta}(\nabla_\alpha\nabla_\beta f'(G)) + 4R^{\alpha}{}_{\mu}(\nabla_\alpha\nabla_\nu f'(G)) + 4R^{\alpha}{}_{\nu}(\nabla_\alpha\nabla_\mu f'(G))\\ & +2g_{\mu\nu}R\Box f'(G) - 2R(\nabla_\mu\nabla_\nu f'(G)) + 4g^{\alpha\rho}g^{\beta\sigma}R_{\mu\rho\nu\sigma}(\nabla_\beta\nabla_\alpha f'(G))]\end{aligned} \tag{3.9}$$

where $T \equiv g^{\mu\nu}T_{\mu\nu}$. Since the Einstein tensor is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, its trace will correspond to $g^{\mu\nu}G_{\mu\nu} = R - (1/2)4R = R - 2R = -R$, and after some mathematical manipulation we obtain for the trace of our equations of motion of Eq. (2.66) the expression

$$
\begin{aligned}
\kappa T &= -R - 2f(G) + f'(G)[2R^2 - 4R_{\mu a}R^{\mu a} + 2R_{\mu abc}R^{\mu abc} - 4R_{ab}R^{ab}] - 2R\Box f'(G) + 8R\Box f'(G) \\
&\quad -4Rf'(G)R + 8R^{\mu a}(\nabla_a \nabla_\mu f'(G)) - 16R^{ab}(\nabla_a \nabla_b f'(G)) + 4g^{ac}g^{bd}R^{\nu}{}_{c\nu d}(\nabla_a \nabla_b f'(G)) \\
\Leftrightarrow \kappa T &= -R - 2f(G) + 2Gf'(G) + 2R\Box f'(G) - 4R^{ab}(\nabla_a \nabla_b f'(G)).
\end{aligned} \tag{3.10}
$$

Closing this subsection, we substitute Eq. (3.7) and its derivative, $f'(G) = \epsilon\varphi'(G)$ in Eqs. (2.66) and (3.10) to obtain our equations of motion and respective trace with the ϵ parameter in evidence. The field equations will then read as

$$
\begin{aligned}
\kappa T_{\mu\nu} &= G_{\mu\nu} - \Lambda g_{\mu\nu} + \epsilon \Big[-\frac{1}{2} g_{\mu\nu}\varphi(G) + \varphi'(G)[2RR_{\mu\nu} + 2R_{\mu\alpha\beta\chi}R_{\nu}{}^{\alpha\beta\chi} - 4R_{\alpha\nu}R^{\alpha}{}_{\mu} - 4R_{\mu}{}^{\chi}{}_{\nu}{}^{\delta}R_{\chi\delta}] \\
&\quad -4R_{\mu\nu}\Box\varphi'(G) - 4g_{\mu\nu}R^{\alpha\beta}(\nabla_\alpha\nabla_\beta\varphi'(G)) + 4R^{\alpha}{}_{\mu}(\nabla_\alpha\nabla_\nu\varphi'(G)) + 4R^{\alpha}{}_{\nu}(\nabla_\alpha\nabla_\mu\varphi'(G)) \\
&\quad +2g_{\mu\nu}R\Box\varphi'(G) - 2R(\nabla_\mu\nabla_\nu\varphi'(G)) + 4g^{\alpha\chi}g^{\beta\delta}R_{\mu\chi\nu\delta}(\nabla_\beta\nabla_\alpha\varphi'(G)) \Big],
\end{aligned} \tag{3.11}
$$

and the trace as

$$
\kappa T = -R - 4\Lambda + \epsilon \left[-2\varphi(G) + 2G\varphi'(G) + 2R\Box\varphi'(G) - 4R^{\alpha\beta}(\nabla_\alpha\nabla_\beta\varphi'(G)) \right]. \tag{3.12}
$$

3.2 Expressing each term at lowest order

3.2.1 Lowest order R or R^T

Due to the form of the field equations, the order reduction amounts to write Eq. (3.11) at order ϵ. Therefore, we have to find the lowest order ($\epsilon = 0$) expressions for R, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ and G so that we may substitute back into Eq. (3.11) where their terms are already explicitly coupled to ϵ.

Firstly we set to find the lowest order scalar curvature R^T, where the superscript T will from now on denote that the quantity to which it appears attached is in its lowest order. With the aim of doing so, we take the trace of Eq. (3.12) which results from having substituted Eq. (3.7) into Eq. (3.9) and set ϵ to zero as

$$
-R - 4\Lambda - 2\overset{0}{\cancel{\epsilon}}\varphi(G) + 2G\overset{0}{\cancel{\epsilon}}\varphi'(G) + 2R\Box(\overset{0}{\cancel{\epsilon}}\varphi'(G)) - 4R^{\mu\nu}(\nabla_\mu\nabla_\nu(\overset{0}{\cancel{\epsilon}}\varphi'(G))) = \kappa T \tag{3.13}
$$

which enables us to write an expression for R, more precisely for R^T since we do it in its lowest order, as we see in

$$
R^T = -4\Lambda - \kappa T. \tag{3.14}
$$

3.2.2 Lowest order $R_{\mu\nu}$ or $R^T_{\mu\nu}$

In a similar manner we take Eq. (3.11), which comes as a result from having substituted Eq. (3.7) into the equations of motion from Eq. (2.66), and set ϵ to zero in order to be able to find an $\epsilon = 0$ order expression for the Ricci tensor $R_{\mu\nu}$, quantity that we will from now on express as $R^T_{\mu\nu}$ to distinguish from the full $R_{\mu\nu}$.

Following our line of thought we may then write

$$
\begin{aligned}
\kappa T_{\mu\nu} \quad &= \quad G_{\mu\nu} - g_{\mu\nu}\Lambda - \frac{1}{2}g_{\mu\nu}\cancelto{0}{\epsilon}\varphi(G) + \cancelto{0}{\epsilon}\varphi'(G)(2RR_{\mu\nu} + 2R_{\mu\alpha\beta\rho}R_{\nu}{}^{\alpha\beta\rho} - 4R_{\alpha\nu}R^{\alpha}{}_{\mu} \\
&\quad -4g^{\alpha\rho}g^{\beta\delta}R_{\mu\alpha\nu\beta}R_{\rho\delta}) - 4R_{\mu\nu}\Box(\cancelto{0}{\epsilon}\varphi'(G)) - 4g_{\mu\nu}R^{\alpha\beta}(\nabla_\alpha\nabla_\beta(\cancelto{0}{\epsilon}\varphi'(G))) \\
&\quad +4R^{\alpha}{}_{\mu}(\nabla_\alpha\nabla_\nu(\cancelto{0}{\epsilon}\varphi'(G))) + 4R^{\alpha}{}_{\nu}(\nabla_\alpha\nabla_\mu(\cancelto{0}{\epsilon}\varphi'(G))) \\
&\quad +2g_{\mu\nu}R\Box(\cancelto{0}{\epsilon}\varphi'(G)) - 2R(\nabla_\mu\nabla_\nu(\cancelto{0}{\epsilon}\varphi'(G))) + 4g^{\alpha\rho}g^{\beta\delta}R_{\mu\rho\nu\delta}(\nabla_\beta\nabla_\alpha(\cancelto{0}{\epsilon}\varphi'(G))) \\
\Leftrightarrow \quad &\kappa T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda
\end{aligned}
\tag{3.15}
$$

which after replacing the scalar curvature R in the first term of the right-hand side of Eq. (3.15) with its lowest order form, Eq. (3.14), will result in the final expression for $R^T_{\mu\nu}$,

$$
R^T_{\mu\nu} = -\frac{\kappa}{2}g_{\mu\nu}T - \Lambda g_{\mu\nu} + \kappa T_{\mu\nu}. \tag{3.16}
$$

3.2.3 Lowest order $R_{\mu\nu\rho\sigma}$ or $R^T_{\mu\nu\rho\sigma}$

For the case of the Riemann tensor, the plot thickens. There are no more equations from the modified Gauss-Bonnet $f(G)$ gravity theory itself to which we may resort in order to set $\epsilon = 0$ in some expression and rewrite it with respect to $R_{\mu\nu\rho\sigma}$, the latter being then the lowest order $R_{\mu\nu\rho\sigma}$ or $R^T_{\mu\nu\rho\sigma}$. One way to overcome this setback is to write the Riemann tensor $R_{\mu\nu\rho\sigma}$ in terms of other quantities for which we already possess their lowest order expression and simply substitute the latter in the $R_{\mu\nu\rho\sigma}$ equation, resulting in a lowest order expression for $R_{\mu\nu\rho\sigma}$, meaning an expression for $R^T_{\mu\nu\rho\sigma}$.

We know that the Riemann tensor, $R_{\mu\nu\rho\sigma}$, can be defined in terms of the Weyl tensor, $C_{\mu\nu\rho\sigma}$, the Ricci tensor, $R_{\mu\nu}$, and the Ricci scalar R as

$$
R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} - \frac{1}{2}(g_{\mu\sigma}R_{\rho\nu} + g_{\nu\rho}R_{\sigma\mu} - g_{\mu\rho}R_{\sigma\nu} - g_{\nu\sigma}R_{\rho\mu}) - \frac{1}{6}(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu})R. \tag{3.17}
$$

Let us now make the simplifying assumption that the spacetime has zero Weyl tensor, $C_{\mu\nu\rho\sigma} = 0$, which is in line with what we will do next, when working with a FLRW line element for which the Weyl tensor vanishes, and substituting Eqs. (3.14) and (3.16) into Eq. (3.17) which turns out to be

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} &= \cancelto{0}{C_{\mu\nu\rho\sigma}} - \frac{1}{2}(g_{\mu\sigma}R_{\rho\nu} + g_{\nu\rho}R_{\sigma\mu} - g_{\mu\rho}R_{\sigma\nu} - g_{\nu\sigma}R_{\rho\mu}) - \frac{1}{6}(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu})R \\
&= -\frac{1}{2}(g_{\mu\sigma}T_{\rho\nu} + g_{\nu\rho}T_{\sigma\mu} - g_{\mu\rho}T_{\sigma\nu} - g_{\nu\sigma}T_{\rho\mu}) + \left(\frac{\kappa T}{4} + \frac{\Lambda}{2}\right)(g_{\mu\sigma}g_{\rho\nu} + g_{\nu\rho}g_{\sigma\mu} \\
&\quad - g_{\mu\rho}g_{\sigma\nu} - g_{\nu\sigma}g_{\rho\mu}) + \left(\frac{\kappa T}{6} + \frac{2\Lambda}{3}\right)(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}). \qquad (3.18)
\end{aligned}
$$

Then we can finally write the lowest order expression for $R_{\mu\nu\rho\sigma}$, or for $R^T_{\mu\nu\rho\sigma}$, which reads as

$$
R^T_{\mu\nu\rho\sigma} = -\frac{\kappa}{2}(g_{\mu\sigma}T_{\rho\nu} + g_{\nu\rho}T_{\sigma\mu} - g_{\mu\rho}T_{\sigma\nu} - g_{\nu\sigma}T_{\rho\mu}) - \frac{1}{3}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\rho\nu})(\Lambda + \kappa T). \qquad (3.19)
$$

3.2.4 Lowest order G or G^T

Similarly to what we have done in Subsec. 3.2.3, our goal in this Subsec. 3.2.4 is to find the lowest order expression for the Gauss-Bonnet invariant G, or G^T, and the simplest way to accomplish that is to write it in terms of $R_{\mu\nu}$ and R and then replace their lowest order expressions as we have obtained in Eqs. (3.14) and (3.16).

We proceed to write G in terms of the Ricci tensor, $R_{\mu\nu}$, and the Ricci scalar, R, by recalling the simplifying, although accurate assumption that the Weyl tensor, $C_{\mu\nu\rho\sigma}$, vanishes for the FLRW line element that we will be working with in the next Chapter. This amounts to

$$
\begin{aligned}
G &= R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
&= R^2 - 4R_{\mu\nu}R^{\mu\nu} + \left[-\frac{1}{2}(g_{\mu\sigma}R_{\rho\nu} + g_{\nu\rho}R_{\sigma\mu} - g_{\mu\rho}R_{\sigma\nu} - g_{\nu\sigma}R_{\rho\mu}) - \frac{1}{6}(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\sigma})\right] \\
&\quad \times \left[-\frac{1}{2}(g^{\mu\sigma}R^{\rho\nu} + g^{\nu\rho}R^{\sigma\mu} - g^{\mu\rho}R^{\sigma\nu} - g^{\nu\sigma}R^{\rho\mu}) - \frac{1}{6}(g^{\mu\rho}g^{\sigma\nu} - g^{\mu\sigma}g^{\rho\nu})\right] \\
&= R^2 - 4R_{\mu\nu}R^{\mu\nu} + \frac{1}{4}(4R_{\mu\nu}R^{\mu\nu} + R^2 - R^{\mu}{}_{\nu}R_{\mu}{}^{\nu} - R_{\mu\nu}R_{\mu\nu} + R^2 - R^{\mu\nu}R_{\mu\nu} - R^{\mu\nu}R_{\mu\nu} \\
&\quad + 4R_{\mu\nu}R^{\mu\nu} - R^{\mu}{}_{\nu}R_{\mu}{}^{\nu} - R_{\mu\nu}R^{\mu\nu} + 4R_{\mu\nu}R^{\mu\nu} + R^2 - R_{\mu\nu}R^{\mu\nu} - R_{\mu\nu}R^{\mu\nu} + R^2 + 4R^{\mu\nu}R_{\mu\nu}) \\
&\quad + \frac{R}{6}(R + R - 4R - 4R - 4R - 4R + R + R) + \frac{R^2}{36}(16 - 4 - 4 + 16) \\
&= R^2 - 4R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu}R^{\mu\nu} + R^2 - 2R^2 + \frac{2}{3}R^2 \\
&= \frac{2}{3}R^2 - 2R_{\mu\nu}R^{\mu\nu}. \qquad (3.20)
\end{aligned}
$$

Let us now substitute Eqs. (3.14) and (3.16) into Eq. (3.20) that gives rise to

$$
\begin{aligned}
G^T &= \frac{2}{3}\left(-4\Lambda-\kappa T\right)^2 - 2\left[\left(\frac{-\kappa T}{2}-\Lambda\right)g_{\mu\nu}+\kappa T_{\mu\nu}\right]\left[\left(-\frac{\kappa T}{2}-\Lambda\right)g^{\mu\nu}+\kappa T^{\mu\nu}\right]\\
&= \frac{2}{3}(16\Lambda^2+8\Lambda\kappa T+\kappa^2T^2)-2\left[4\left(-\frac{\kappa T}{2}-\Lambda\right)^2+2\kappa T\left(-\frac{\kappa T}{2}-\Lambda\right)+\kappa^2T_{\mu\nu}T^{\mu\nu}\right]\\
&= \frac{2}{3}(16\Lambda^2+8\Lambda\kappa T+\kappa^2T^2)-2(\kappa^2T^2+4\kappa T\Lambda+4\Lambda^2-\kappa^2T^2-2\kappa\Lambda T+\kappa^2T_{\mu\nu}T^{\mu\nu})\\
&= \frac{2}{3}(16\Lambda^2+8\Lambda\kappa T+\kappa^2T^2-6\kappa\Lambda T+12\Lambda^2+3\kappa^2T^{\mu\nu}T_{\mu\nu})\\
&= \frac{2}{3}(4\Lambda^2+2\Lambda\kappa T+\kappa^2T^2-6\kappa\Lambda T-3\kappa^2T^{\mu\nu}T_{\mu\nu}), \qquad (3.21)
\end{aligned}
$$

from which we can finally write the lowest order expression for G, or G^T as

$$
G^T = \frac{2}{3}\kappa^2T^2 - 2\kappa^2T_{ab}T^{ab} + \frac{8}{3}\Lambda^2 + \frac{4}{3}\Lambda\kappa T. \qquad (3.22)
$$

3.3 Order reduced field equations of this theory

As we have already seen, the application of order reduction is equivalent to replacing R, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$, and G by R^T, $R^T_{\mu\nu}$, $R^T_{\mu\nu\rho\sigma}$, and G^T in Eq. (2.66). This procedure brings us to the expression

$$
\begin{aligned}
\kappa T_{\mu\nu} &= R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\Lambda g_{\mu\nu}+\epsilon\bigg[-\frac{1}{2}g_{\mu\nu}\varphi^T+\varphi'^T[2R^TR^T_{\mu\nu}+2R^T_{\mu\alpha\beta\chi}g^{\alpha\eta}g^{\beta\zeta}g^{\chi\rho}R^T_{\nu\eta\zeta\rho}\\
&\quad -4R^T_{\alpha\nu}g^{\eta\alpha}R^T_{\eta\mu}-4g^{\chi\rho}g^{\delta\sigma}R^T_{\mu\rho\nu\sigma}R^T_{\chi\delta}]-4R^T_{\mu\nu}\Box\varphi'^T-4g_{\mu\nu}g^{\alpha\eta}g^{\beta\zeta}R^T_{\eta\zeta}(\nabla_\alpha\nabla_\beta\varphi'^T)\\
&\quad +4g^{\alpha\eta}R^T_{\eta\mu}(\nabla_\alpha\nabla_\nu\varphi'^T)+4g^{\eta\alpha}R^T_{\eta\nu}(\nabla_\alpha\nabla_\mu\varphi'^T)+2g_{\mu\nu}R^T\Box\varphi'^T\\
&\quad -2R(\nabla_\mu\nabla_\nu\varphi'^T)+4g^{\alpha\chi}g^{\beta\delta}R^T_{\mu\chi\nu\delta}(\nabla_\beta\nabla_\alpha\varphi'^T)\bigg], \qquad (3.23)
\end{aligned}
$$

where $\varphi^T=\varphi(G^T)$ and $\varphi'^T=\varphi'(G^T)$, with the prime denoting differentiation with respect to the Gauss-Bonnet invariant G^T. Eq. (3.23) is the set of order reduced field equations that we were looking for.

Chapter 4

Bouncing solutions in $f(G)$ gravity

All the work we have done until now - assembling the order reduced equations of motion for the Gauss-Bonnet modified $f(G)$ gravity - will allow us to finally find the first Friedmann equation that fits both the theory and the assumptions made. The ones that we have taken into consideration are: an FLRW metric, a perfect fluid description as well as conservation of energy. In this Chapter 4 we will find the function $f(G)$ which when inserted in the action of Eq. 1.29 returns a bouncing solution. For this, we start by deriving the order reduced form of the modified first Friedmann equation and present it as a second-order solvable differential equation for the function $\varphi(G)$, which has firstly appeared in Eq. (3.7) to parametrize the leading order term in the expansion of $f(G)$, so that we may compare this Friedmann equation with the one in Eq. 1.30 and retrieve the $f(G)$ we are looking for.

4.1 Friedmann-Lemaître-Robertson-Walker metric and other aspects for the first Friedmann equation

4.1.1 Friedmann-Lemaître-Robertson-Walker metric

To carry on with our calculations towards finding a first Friedmann equation from the order reduced field equations in Eq. (3.23), we must choose a line element to set the background on which we will be working with. Since we want to work with an isotropic and homogeneous universe, we assume an FLRW line element,

$$ds^2 = -dt^2 + a^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right], \tag{4.1}$$

where t represents the cosmic time, (r, θ, ϕ) are the spatial coordinates, $a = a(t)$ is the cosmological scale factor, and $k = -1, 0, 1$ yields hyperbolic, flat, and spherical spaces, respectively.

From the line element in Eq. (4.1) we may deduce the FLRW metric which reads as

$$g_{\mu\nu} = \text{diag} \left(-1, \frac{a^2}{1 - kr^2}, a^2 r^2, a^2 r^2 \sin^2 \theta \right), \tag{4.2}$$

and its corresponding inverse metric, that satisfies $g^{\mu\nu}g_{\mu\nu} = 4$, which reads as

$$g^{\mu\nu} = \text{diag}\left(-1, \frac{1-kr^2}{a^2}, \frac{1}{a^2r^2}, \frac{1}{a^2r^2\sin^2\theta}\right). \tag{4.3}$$

4.1.2 Perfect fluid description

We assume a perfect fluid description, so the stress-energy tensor is

$$T^{\mu\nu} = (\rho + p)\, u^\mu u^\nu + p g^{\mu\nu}\,, \tag{4.4}$$

where $\rho = \rho(t)$ and $p = p(t)$ are the energy density and pressure of the fluid, respectively, with

$$p = w\rho\,, \tag{4.5}$$

and where u^μ is the fluid's 4-velocity.

From Eqs. (4.2), (4.3) and (4.4) we are able to write the expressions for the stress-energy tensor and its inverse

$$T_{\mu\nu} = \text{diag}\left(\rho,\, p\frac{a^2}{1-kr^2},\, pa^2r^2,\, pa^2r^2\sin^2\theta\right), \tag{4.6}$$

$$T^{\mu\nu} = \text{diag}\left(\rho,\, p\frac{1-kr^2}{a^2},\, \frac{p}{a^2r^2},\, \frac{p}{a^2r^2\sin^2\theta}\right), \tag{4.7}$$

as well as other useful expressions, such as the trace of the stress-energy tensor T and its contraction with its inverse $T_{\mu\nu}T^{\mu\nu}$, respectively

$$T = g^{\mu\nu}T_{\mu\nu} = -\rho + 3p = \rho(3w-1) \tag{4.8}$$

$$T_{\mu\nu}T^{\mu\nu} = \rho^2 + 3p^2 = \rho^2(3w+1) \tag{4.9}$$

where we also used the equation of state as in Eq. (4.5).

4.1.3 Zero-zero component of the field equations

Before going further to actually computing our modified first Friedmann equation, we have to understand how to derive it. This is done by acknowledging it as the zero-zero component of the full field equations as depicted by Eq. (3.23), meaning that we have to take the equation that corresponds to setting the indices μ and ν to zero, which reads as

$$
\begin{aligned}
\kappa T_{00} &= R_{00} - \frac{1}{2}R + \Lambda + \epsilon\Bigg[\frac{1}{2}\varphi^T + \varphi'^T[2R^T R^T_{00} + 2R^T_{0\alpha\beta\chi}g^{\alpha\eta}g^{\beta\zeta}g^{\chi\rho}R^T_{0\eta\zeta\rho} \\
&\quad -4R^T_{\alpha 0}g^{\eta\alpha}R^T_{\eta 0} - 4g^{\chi\rho}g^{\delta\sigma}R^T_{0\rho 0\sigma}R^T_{\chi\delta}] - 4R^T_{00}\Box\varphi'^T + 4g^{\alpha\eta}g^{\beta\zeta}R^T_{\eta\zeta}(\nabla_\alpha\nabla_\beta\varphi'^T) \\
&\quad +4g^{\alpha\eta}R^T_{\eta 0}(\nabla_\alpha\partial_t\varphi'^T) + 4g^{\eta\alpha}R^T_{\eta 0}(\nabla_\alpha\partial_t\varphi'^T) - 2R^T\Box\varphi'^T \\
&\quad -2R(\partial_t^2\varphi'^T) + 4g^{\alpha\chi}g^{\beta\delta}R^T_{0\chi 0\delta}(\nabla_\beta\nabla_\alpha\varphi'^T)\Bigg],
\end{aligned} \tag{4.10}
$$

where we used directly the following results

$$g_{00} = -1, \tag{4.11}$$

$$\nabla_0\varphi'^T = \partial_t\varphi'^T = \frac{\partial}{\partial t}\varphi'^T, \tag{4.12}$$

$$\nabla_0\partial_t\varphi'^T = \left(\partial_t^2 - \Gamma^\mu_{00}\partial_\mu\right)\varphi'^T = \left(\partial_t^2 - \cancelto{0}{\Gamma^0_{00}\partial_t}\right)\varphi'^T = \partial_t^2\varphi'^T. \tag{4.13}$$

4.2 *Mathematica* and the modified first Friedmann equation

4.2.1 Defining the main tensorial quantities

The position vector, the metric and the inverse metric

Now, we resort to *Mathematica*, a modern technical computing system that will help us with our calculations in this Chapter 4. Furthermore, since we are working with tensor calculus, we shall use the built-in symbol *Table[expr,n]* to build up the vectors, matrices, tensors and other arrays that we may use. This function is responsible for generating a list of *n* copies of *expr*.

Firstly, we defined the geometry of the problem. We began by defining the position vector

$$x = (t, x, \theta, \phi), \tag{4.14}$$

as you may check in Appendix A.1.

The metric and its inverse were defined according to Eqs. (4.2) and (4.3), also evinced in Appendix A.1 where we used the notation g for the metric $g_{\mu\nu}$ and gin for the inverse metric $g^{\mu\nu}$. The code returned as an output for the metric the matrix

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2 a(t)^2 & 0 \\ 0 & 0 & 0 & r^2 a(t)^2\sin^2(\theta) \end{pmatrix}, \tag{4.15}$$

where in the code it is clear that we used $A(t)$ to denote the scale factor a, as you may check with the

Appendices A.1 and A.1 (this was done to avoid confusion since we summed in an index a many times in our code). For the inverse metric the output reads as

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1-kr^2}{a(t)^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2 a(t)^2} & 0 \\ 0 & 0 & 0 & \frac{\csc^2(\theta)}{r^2 a(t)^2} \end{pmatrix}. \tag{4.16}$$

right-hand side of equation Although we here present our results in the form of a matrix, we are actually translating the output produced by *Mathematica* to this more legible form using its in-built function *MatrixForm[list]*. This function is responsible for printing the elements of the list to which it is applied in a regular array, in this case this array is a matrix, further we shall print them as more complicated arrays such as tensors.

The metric connection

After defining the position vector, the metric and its inverse, we were able to construct the metric connection which in this case corresponds to the Christoffel symbol $\Gamma^{\mu}_{\nu\rho}$ as in Eq. (2.28)this is why we named it *Chr* for computational purposes in the code you may consult also in Appendix A.1. The output we got for the metric connection reads as

$$\Gamma^{\mu}_{\nu\rho} = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ \frac{a(t)a'(t)}{1-kr^2} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ r^2 a(t) a'(t) \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ r^2 a(t)\sin^2(\theta) a'(t) \end{pmatrix} \\ \begin{pmatrix} 0 \\ \frac{a'(t)}{a(t)} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{a'(t)}{a(t)} \\ \frac{kr}{1-kr^2} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ -r\left(1-kr^2\right) \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ -r\sin^2(\theta)\left(1-kr^2\right) \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ \frac{a'(t)}{a(t)} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ \frac{1}{r} \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{a'(t)}{a(t)} \\ \frac{1}{r} \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(\theta)(-\cos(\theta)) \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{a'(t)}{a(t)} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{r} \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \\ \cot(\theta) \end{pmatrix} & \begin{pmatrix} \frac{a'(t)}{a(t)} \\ \frac{1}{r} \\ \cot(\theta) \\ 0 \end{pmatrix} \end{pmatrix}, \tag{4.17}$$

where here the prime denotes derivation with respect to time.

The Riemann tensor

After defining the metric connection, we arrive at an expression for the Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$ simply by computing its expression as it is given by Eq. (2.16). In the code, we defined the Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$ as *Riemann* and its output is available at Appendix B.1. We thought it would be useful to also define the Riemann tensor $R_{\mu\nu\rho\sigma}$, with all its indices down, by contracting the variable *Riemann* with the metric *g* (both here read as we defined them in the code), which analytically reads as $R_{\alpha\nu\rho\sigma} = g_{\mu\alpha}R^{\mu}{}_{\nu\rho\sigma}$. We called the resulting variable *riemann* and its output is available in Appendix B.2.

The Ricci tensor

Similarly, the Ricci tensor $R_{\mu\nu}$ is defined by the result of a contraction between the Riemann with all its indices down (in *riemann* in our code) and the inverse metric (*gin* in our code). Which is identical to setting $\mu = \rho$ in the original Riemann tensor $R^{\mu}{}_{\nu\rho\sigma}$ (*Riemann* in our code). We denoted the Ricci tensor as *Ricci* in our code, and its output reads as

$$R_{\mu\nu} = \begin{pmatrix} -\frac{3a''(t)}{a(t)} & 0 & 0 & 0 \\ 0 & \frac{2a'(t)^2+2k+a(t)a''(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2\left(2a'(t)^2+2k+a(t)a''(t)\right) & 0 \\ 0 & 0 & 0 & r^2\sin^2(\theta)\left(2a'(t)^2+2k+a(t)a''(t)\right) \end{pmatrix}. \quad (4.18)$$

The scalar curvature

Again, we define the scalar curvature R from a contraction between the Ricci tensor $R_{\mu\nu}$ (*Ricci* in our code) with the inverse metric $g^{\mu\nu}$ (*gin* in our code). This variable we name *ScalarCurvature* in our notebook, and for its output we obtain

$$R = \frac{6\left(a(t)a''(t)+a'(t)^2+k\right)}{a(t)^2}. \quad (4.19)$$

The Einstein tensor

The Einstein tensor $G_{\mu\nu}$ follows the definition $G_{\mu\nu} = R_{\mu\nu} - 1/2 g_{\mu\nu}R$, which we trivially call *Einstein* in *Mathematica* and its output returns

$$\text{Einstein} = \begin{pmatrix} \frac{3\left(a'(t)^2+k\right)}{a(t)^2} & 0 & 0 & 0 \\ 0 & \frac{a'(t)^2+k+2a(t)a''(t)}{kr^2-1} & 0 & 0 \\ 0 & 0 & -r^2\left(a'(t)^2+k+2a(t)a''(t)\right) & 0 \\ 0 & 0 & 0 & -r^2\sin^2(\theta)\left(a'(t)^2+k+2a(t)a''(t)\right) \end{pmatrix}. \quad (4.20)$$

The Gauss-Bonnet invariant

Another useful definition that we set at this stage is the Gauss-Bonnet invariant G as a function of the other variables that we have already seen, which will then be expressed as $G = R^2 - 4R_{\mu\nu}g^{\alpha\mu}g^{\beta\nu}R_{\alpha\beta} + R_{\mu\nu\rho\sigma}g^{\alpha\mu}g^{\beta\nu}g^{\rho\eta}g^{\sigma\zeta}R_{\alpha\beta\eta\zeta}$, since we have only defined lower indices Riemann and Ricci tensors within *Mathematica* environment. As it is evinced by Appendix A.1, we call this variable *GB* and its output reads as

$$G = \frac{24a''(t)\left(a'(t)^2 + k\right)}{a(t)^3}. \tag{4.21}$$

It is interesting to note before going further that although it is usual to work under the Einstein notation or Einstein summation convention, that implies summation over a set of indexed terms in a formula, this is not directly computed by a tool such as *Mathematica*. Therefore, when defining the tensors, these sums had to be done explicitly in our *Mathematica* notebook as it can be confirmed by checking Appendix A.1 where we have our uncut code concerning these calculations.

Even without further mentioning this again, it is important to notice that we repeat this procedure for every computation involving tensors in every code that we portray in the respective appendices.

The 4-velocity, the stress-energy tensor and its scalar

Considering that we are actually working in order ϵ terms, we need to take into account the expressions for R^T, $R^T_{\mu\nu}$, $R^T_{\mu\nu\rho\sigma}$ and G^T rather than their counterparts R, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ and G. This justifies our choice in the code of Appendix A.2, where we start by defining the 4-velocity vector as

$$u = (1, 0, 0, 0) \tag{4.22}$$

as well as the stress-energy tensor $T_{\mu\nu}$. The latter we call *Tdown* in our *Mathematica* notebook, which by following Eq. (4.6) will return the output

$$T_{\mu\nu} = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & \frac{wa(t)^2\rho(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & r^2wa(t)^2\rho(t) & 0 \\ 0 & 0 & 0 & r^2wa(t)^2\sin^2(\theta)\rho(t) \end{pmatrix}. \tag{4.23}$$

We found it useful, for posterior calculations, to also define in our notebook its inverse $T^{\mu\nu}$ according to Eq. (4.7), which we called *Tup* (in regards of having its indices up) and which output returned

$$T^{\mu\nu} = \begin{pmatrix} \rho(t) & 0 & 0 & 0 \\ 0 & \frac{\left(1-kr^2\right)w\rho(t)}{a(t)^2} & 0 & 0 \\ 0 & 0 & \frac{w\rho(t)}{r^2a(t)^2} & 0 \\ 0 & 0 & 0 & \frac{w\csc^2(\theta)\rho(t)}{r^2a(t)^2} \end{pmatrix}. \tag{4.24}$$

Similarly to what we have done for the scalar curvature R, we contract the stress-energy tensor with the metric to define T, which we called *Tscalar* in the notebook, and with which output reading

$$T = 3w\rho(t) - \rho(t) \tag{4.25}$$

we confirmed what we had analytically found for Eq. (4.8). We were also able to verify the expression that we had analytically found for Eq. (4.9) by doing *Sum[Tdown[[a, b]]*Tup[[a, b]], {a, 4}, {b, 4}]* in *Mathematica*, giving

$$T_{\mu\nu}T^{\mu\nu} = \rho(t)^2 + 3w^2\rho(t)^2. \tag{4.26}$$

The lowest order Ricci scalar or scalar curvature

In agreement with Eq. (3.14), we define the lowest order scalar curvature R^T, which we named *RicciScalarOR* and which returns the output

$$R^T = -4\Lambda - \kappa(3w\rho(t) - \rho(t)). \tag{4.27}$$

The lowest order Ricci tensor

The lowest order Ricci tensor $R^T_{\mu\nu}$ follows the expression of Eq. (3.16), we named it *RicciOR* and the corresponding output reads

$$R^T_{\mu\nu} = \begin{pmatrix} \Lambda + \frac{1}{2}(3w\kappa+\kappa)\rho(t) & 0 & 0 & 0 \\ 0 & \frac{a(t)^2(2\Lambda+(w-1)\kappa\rho(t))}{2kr^2-2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2}r^2a(t)^2(2\Lambda+(w-1)\kappa\rho(t)) & 0 \\ 0 & 0 & 0 & -\frac{1}{2}r^2a(t)^2\sin^2(\theta)(2\Lambda+(w-1)\kappa\rho(t)) \end{pmatrix}. \tag{4.28}$$

We found it useful for future calculations to define the inverse of $R^T_{\mu\nu}$, we named it *RicciORup* and reads as

$$(R^{\mu\nu})^T = \begin{pmatrix} \Lambda + \frac{1}{2}(3w\kappa+\kappa)\rho(t) & 0 & 0 & 0 \\ 0 & \frac{(kr^2-1)(2\Lambda+(w-1)\kappa\rho(t))}{2a(t)^2} & 0 & 0 \\ 0 & 0 & -\frac{2\Lambda+(w-1)\kappa\rho(t)}{2r^2a(t)^2} & 0 \\ 0 & 0 & 0 & -\frac{\csc^2(\theta)(2\Lambda+(w-1)\kappa\rho(t))}{2r^2a(t)^2} \end{pmatrix}. \quad (4.29)$$

The lowest order Riemann tensor

Again, as we had done for the Ricci tensor, we found it useful to define the inverse quantity, $(R^{\mu\nu\rho\sigma})^T$, which we called *RiemannORup* in *Mathematica*. Its output is printed in Appendix B.4.

The lowest order Gauss-Bonnet invariant

Similarly we defined the Gauss-Bonnet invariant G^T with respect to the expression of Eq. (2.3), where we substituted our defined variables in their lowest order: *RicciScalarOR*, *RicciOR*, *RiemannOR* and *RiemannORup*. The output for our lowest order Gauss-Bonnet invariant which we called *GBOR* in our notebook then reads as

$$G^T = \frac{4}{3}(\Lambda - \kappa\rho(t))(2\Lambda + \rho(t)(\kappa + 3\kappa w)). \quad (4.30)$$

Some useful covariant derivatives

It is worth mentioning that we found it useful to define some functions in the *Mathematica* environment respecting the covariant derivative applied to a scalar as we have in the case of the term $\nabla_\mu\partial_\nu\varphi'^T$. This kind of covariant derivative will unfold as

$$\nabla_\mu\partial_\nu\varphi'^T = (\partial_\mu\partial_\nu - \Gamma^\rho_{\mu\nu}\partial_\rho)\varphi'^T. \quad (4.31)$$

In the computational environment we named this function expressed by Eq. (4.31) we built as *CovD*, which we wrote as *CovD[b_, a_, v_] := D[D[v, x[[a]]], x[[b]]] - Sum[Chr[[c, a, b]] D[v, x[[c]]], {c, 4}]*, defining it as a function that would take three arguments: *b* and *a* for the intended indices, for example μ and ν, and *v* for some scalar quantity to which we may apply it, in the case of Eq. (4.31), instead of *v* we would input $\varphi'(G^T)$ in the function.

Another operator we found pertinent to define as a function, is the one expressed by $\Box \equiv \nabla^\mu\nabla_\mu$, which when applied to a scalar as in $\nabla^\mu\partial_\mu\varphi' T$, may be unfolded as

$$\Box\varphi'^T = g^{\mu\nu}\nabla_\mu\partial_\nu\varphi'^T. \quad (4.32)$$

We called this *box* and based on Eq. (4.31) we wrote it as a contraction of the previously defined operator with the inverse metric. It takes only one argument, *v* as the scalar to which it would be applied, since it does not represent a tensor quantity and so needs no definition of the indices.

4.2.2 Vanishing of the Weyl tensor

After delineating the expressions concerning our geometry's variables, we may finally start using them to obtain some results otherwise hard to derive. In this Subsec. 4.2.2 we address the vanishing of the Weyl tensor, previously assumed in Subsec. 3.2.3 of Chapter 3 when we assumed an FLRW metric, which has helped us writing the Riemann tensor in terms of the Ricci tensor and the scalar curvature.

The Weyl tensor is actually a measure of the curvature of the spacetime, more precisely the tidal force that an object would feel when moving along a geodesic. Seeing that the Ricci tensor corresponds to the trace of the Riemann tensor, in order to characterize the Weyl tensor we are left with the traceless part of the Riemann tensor, meaning that this tensor would not only possess the same symmetries as the Riemann in Eq. (2.35) but would also vanish when any couple of its indices are contracted with the metric. For the assumption made back in Chapter 3 we used another property of the Weyl tensor, or conformal tensor, that withal justifies its vanishing: the metric being conformally flat. A metric is conformally flat if there exists some local coordinate system in which the metric is proportional to a constant tensor. The FLRW metric is conformally flat.

To prove the hypothesis that the FLRW metric is indeed conformally flat, we used the platform *Mathematica* where we had already computed the metric $g_{\mu\nu}$, the Riemann tensor $R_{\mu\nu\rho\sigma}$, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R and we simply computed the Weyl tensor through its definition as illustrated by Eq. (3.17). It is important to note that in our *Mathematica* code the Riemann tensor $R_{\mu\nu\rho\sigma}$, which there is defined as *riemann*, is not defined through the Weyl tensor as we have in Eq. (3.17) but rather through a linear combination of metric connections as we may read in Eq. (2.16). This way, we know that our output is true and not a result of defining the Riemann without the contribution of the Weyl in the first place; our Riemann tensor is defined from the choice of the metric that defines the curvature of the spacetime itself.

Finally, we computed the Weyl tensor as it reads in Appendix A.3, calling the Weyl tensor simply *weyl* and defining it as we already explained. Due to the fact that our tensors in *Mathematica* are defined as tables, the result was a table of 4 columns by 4 rows of matrices of 4×4 elements, all of them equal to zero, as expected, proving the vanishing of the Weyl tensor in regard of our choice of an FLRW metric.

4.2.3 Proving conservation of energy

Now, it is important that we verify the conservation of energy in our modified Gauss-Bonnet $f(G)$ gravity theory. This corresponds to guaranteeing that the covariant derivative of the stress-energy tensor is zero, or $\nabla_\mu T^{\mu\nu} = \nabla^\mu T_{\mu\nu}$. Recalling Eq. (2.66), if we want to evaluate the covariant derivative of the stress-energy tensor in the left-hand side of the equation, we might as well calculate the covariant derivative regarding the right-hand side of the equation. In trying to prove

$$
\begin{aligned}
&\nabla^\mu\Bigg[G_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\epsilon\varphi(G)+\epsilon\varphi'(G)[2RR_{\mu\nu}+2R_{\mu\alpha\beta\chi}R_\nu{}^{\alpha\beta\chi}-4R_{\alpha\nu}R^\alpha{}_\mu-4g^{\alpha\chi}g^{\beta\delta}R_{\mu\alpha\nu\beta}R_{\chi\delta}]\\
&-4R_{\mu\nu}\Box\epsilon\varphi'(G)-4g_{\mu\nu}R^{\alpha\beta}(\nabla_\alpha\nabla_\beta\epsilon\varphi'(G))+4R^\alpha{}_\mu(\nabla_\alpha\nabla_\nu\epsilon\varphi'(G))+4R^\alpha{}_\nu(\nabla_\alpha\nabla_\mu\epsilon\varphi'(G))\\
&+2g_{\mu\nu}R\Box\epsilon\varphi'(G)-2R(\nabla_\mu\nabla_\nu\epsilon\varphi'(G))+4g^{\alpha\chi}g^{\beta\delta}R_{\mu\chi\nu\delta}(\nabla_\beta\nabla_\alpha\epsilon\varphi'(G))\Bigg]=0,
\end{aligned}
\tag{4.33}
$$

we start by acknowledging that it is equivalent to proving that the sum of all the covariant derivatives of each of the terms composing Eq. (4.33) is zero.

For this purpose, we create a function in the *Mathematica* environment that computes some covariant derivative ∇^μ of a tensor of type $M_{\mu\nu}$, an auxiliary tool similar to the ones we generated for the Eqs. (4.31) and (4.32). This function would correspond to the analytical expression

$$
\nabla^\mu M_{\mu\nu}=g^{\mu\alpha}\nabla_\alpha M_{\mu\nu}=g^{\mu\alpha}\left(\partial_\alpha M_{\mu\nu}-\Gamma^\beta_{\alpha\mu}M_{\beta\nu}-\Gamma^\beta_{\alpha\nu}M_{\mu\beta}\right). \tag{4.34}
$$

In Appendix A.4 our code evinces that this function, which we there defined as *CovDup2ind[μ_,ν_,V_]*, would take both indices μ and ν as an argument, as well as some tensor of the type $M_{\mu\nu}$ in the place of *V*.

To better implement our long calculation, we decided to assign the tensors $M_{\mu\nu}$ inside the covariant derivatives that together compose Eq. (4.33) to some auxiliary tensor we defined in *Mathematica* as a *Taux* before assembling all the covariant derivatives. As an example we defined an auxiliary tensor which we called *Taux2* to compute the term

$$
Taux2_{\mu\nu}=R_{\mu\eta\zeta\beta}g^{\eta\alpha}g^{\zeta\chi}g^{\beta\rho}R_{\nu\alpha\chi\rho}, \tag{4.35}
$$

we then used μ, ν and *Taux2* as arguments to the function as written in Eq. (4.34) to return the value of the upper index covariant derivative of the tensor of Eq. (4.35). We repeated the procedure for every term, as we can see from the explicit code in Appendix A.4, and found, after some algebraic simplification, that the energy is actually conserved since Eq. (4.33) actually yields a vector with every element equal to zero.

This means that we have proved that $\nabla^\mu T_{\mu\nu}=0$. As a result it will be valuable to note that

$$
\dot\rho=-3H(1+w)\rho\,, \tag{4.36}
$$

equation to which we arrive from choosing $\nu=0$ and by making the necessary substitutions as it reads in

$$
\begin{aligned}
\nabla_\mu T^{\mu 0} &= 0 \Leftrightarrow \partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\alpha} T^{0\alpha} + \Gamma^0_{\mu\alpha} T^{\mu\alpha} = 0 \\
&\Leftrightarrow \partial_t T^{00} + \Gamma^\mu_{\mu 0} T^{00} + \Gamma^0_{\mu\alpha} T^{\mu\alpha} = 0 \\
&\Leftrightarrow \dot{\rho} + \rho \left(\cancelto{0}{\Gamma^0_{00}} + \Gamma^1_{10} + \Gamma^2_{20} + \Gamma^3_{30} \right) + \left(\cancelto{0}{\Gamma^0_{00}} T^{00} + \Gamma^0_{11} T^{11} + \Gamma^0_{22} T^{22} + \Gamma^0_{33} T^{33} \right) = 0 \\
&\Leftrightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3\frac{\dot{a}}{a}p = 0 \\
&\Leftrightarrow \dot{\rho} + 3\frac{\dot{a}}{a}\rho(1+w) = 0. \qquad (4.37)
\end{aligned}
$$

All the substitutions concerning Eq. (4.37) were done according to the symmetry of the stress-energy tensor $T^{\mu\nu}$ that requires $\mu = \nu$ in order to be different from zero (since it is a diagonal tensor as we might recall from Eq. (4.6)), as well as the fact that it only depends on $\rho = \rho(t)$. The values for both the Christoffel and the stress-energy elements were calculated using *Mathematica* as we had already predefined these expressions as explained in Subsec. 4.2.1.

4.2.4 Computing the modified first Friedmann equation

Returning to the zero-zero component of the field equations

After equipping our *Mathematica* notebook with all the tools necessary to compute Eq. (4.10), i.e., after we have defined all the variables (that compose the geometry of the problem) that appear in it by computational means in the previous Subsec. 4.2.1 as well as some useful functions, we are now capable of computing and simplifying the first Friedmann equation in this platform.

Therefore, in Appendix A.5, we focus on writing Eq. (4.10): we use the functions denoting the covariant derivatives of a scalar as explained by Eqs. (4.31) and (4.32), as well as the lowest order variables for the Riemann and the Ricci tensors as well as the scalar curvature, the metric, its inverse and the Einstein tensor. We choose to write the equation as a single line of calculus by passing the term containing the stress-energy tensor to the right-hand side of the Eq. (4.10) so that it becomes easier to manipulate in this computational environment. Now, we may simplify this by multiplying and dividing by whatever factors we find pertinent, without complicating the understanding of the code. In other words, we rewrite Eq. (4.10) as follows

$$
\begin{aligned}
&-\kappa T_{00} + R_{00} + \frac{1}{2}R + \Lambda + \epsilon\bigg[\frac{1}{2}\varphi^T + \varphi'^T[2R^T R^T_{00} + 2R^T_{0\alpha\beta\chi} g^{\alpha\eta} g^{\beta\zeta} g^{\chi\rho} R^T_{0\eta\zeta\rho} - 4R^T_{\alpha 0} g^{\eta\alpha} R^T_{\eta 0} \\
&-4g^{\chi\rho} g^{\delta\sigma} R^T_{0\rho 0\sigma} R^T_{\chi\delta}] - 4R^T_{00}\Box\varphi'^T + 4g^{\alpha\eta} g^{\beta\zeta} R^T_{\eta\zeta}(\nabla_\alpha\nabla_\beta\varphi'^T) + 4g^{\alpha\eta} R^T_{\eta 0}(\nabla_\alpha\partial_t\varphi'^T) \\
&+4g^{\eta\alpha} R^T_{\eta 0}(\nabla_\alpha\partial_t\varphi'^T) - 2R^T\Box\varphi'^T - 2R(\partial_t^2\varphi'^T) + 4g^{\alpha\chi} g^{\beta\delta} R^T_{0\chi 0\delta}(\nabla_\beta\nabla_\alpha\varphi'^T)\bigg], \qquad (4.38)
\end{aligned}
$$

where to be precise, in *Mathematica* instead of the indices going from 0 to 3, they indent from 1 to 4 (g_{00} =*g[[1,1]]* as we have in Appendix A.5).

We then apply the *Mathematica* built-in function *Simplify*, which performs a sequence of algebraic and other transformations on the expression and returns the simplest form it finds. Subsequently, by multiplying the resulting expression by 2, and expanding out the products and positive integer powers in that expression (with *Expand* which rewrites it in the most legible expression possible), we arrive at a zero-zero component for Eq. (3.23) which reads as

$$\frac{6k}{a^2}+6H^2+2\Lambda-2\kappa\rho+\epsilon\varphi^T-\frac{4}{3}\epsilon(\Lambda-\kappa\rho)\left[6H\dot{\varphi}'^T+\varphi'^T\left(2\Lambda+\kappa\rho(1+3w)\right)\right]=0\,, \tag{4.39}$$

where we used the definition for the Hubble function $H=H(t)$

$$H=\frac{\dot{a}}{a}\,, \tag{4.40}$$

with a dot denoting a derivative with respect to time t.

The chain rule

Looking at Eq. (4.39), one of the terms stands out: the one containing $\dot{\varphi}'^T=\dot{\varphi}'(G^T)$. In this case, we have both a time derivative and a differentiation with respect to G applied to the function $\varphi(G)$. For Eq. (4.39) to be trivially solvable for $\varphi(G)$, it would be useful to have all the derivatives with respect to the same variable, more precisely to G.

Working towards this goal, we evince each of the derivatives inside $\dot{\varphi}'^T=\dot{\varphi}'(G^T)$ with the help of the chain rule for differentiation, which reads as

$$\dot{\varphi}'^T=\frac{\partial\varphi' T}{\partial G^T}\frac{\partial G^T}{\partial\rho}\frac{\partial\rho}{\partial t}, \tag{4.41}$$

where we already know that

$$\begin{aligned}\frac{\partial\varphi' T}{\partial G^T} &= \varphi''^T, &(4.42)\\ \frac{\partial G^T}{\partial\rho} &= -\frac{4}{3}\kappa\left(\Lambda+2\rho(\kappa+3\kappa w)-3\Lambda w\right), &(4.43)\\ \frac{\partial\rho}{\partial t} &= \dot{\rho}, &(4.44)\end{aligned}$$

where $\dot{\rho}$ is given by Eq. (4.36). Note that in deriving Eq. (4.43) we simply differentiated Eq. (3.22) with respect to ρ after substituting Eqs. (4.8) and (4.9), that we had previously determined in Subsec. 4.1.2 of this Chapter 4.

Assembling Eqs. (4.42), (4.43) and (4.36) into Eq. (4.41), we finally get

$$\dot{\varphi}'^T=\varphi''^T 4H\kappa\rho(1+w)(\Lambda+2\rho(\kappa+3\kappa w)-3\Lambda w)\,. \tag{4.45}$$

The modified first Friedmann equation

Lastly, when gathering all the elements deduced in order to get to the second-order differential equation that corresponds to the modified first Friedmann equation for the $f(G)$ gravity theory, we just have to replace Eq. (4.45) into Eq. (4.39), giving rise to

$$\begin{aligned} H^2 &= -\frac{1}{6}\bigg(\frac{6k}{a^2}+2\Lambda-2\kappa\rho+\epsilon\varphi^T-\frac{4}{3}\epsilon(\Lambda-\kappa\rho)[6\varphi''^T 4H^2\kappa\rho(1+w)(\Lambda \\ &\quad +2\rho(\kappa+3\kappa w)-3\Lambda w)+\varphi'^T(2\Lambda+\kappa\rho(1+3w))]\bigg). \end{aligned} \tag{4.46}$$

By carefully examining Eq. (4.46), we notice that there is an H^2 dependency coupled to the term containing φ''^T, which is coupled to ϵ. In view of the order reduction technique that we had developed in Chapter 3, we must be able to write this H^2 in its lowest form and then substitute it back into Eq. (4.46). Therefore, we may retrieve the lowest order form of H^2 as we did for other variables such as the Ricci tensor and the scalar curvature back in Chapter 3, by zeroing all the terms coupled to the parameter ϵ in Eq. (4.46) and simplifying it, which then amounts to the expression

$$H^2=-\frac{k}{a^2}-\frac{\Lambda}{3}+\frac{\kappa\rho}{3}. \tag{4.47}$$

Substituting Eq. (4.47) back into Eq. (4.46), we compute at last the modified Friedmann equation for $f(G)$ gravity after a process of order reduction, which reads as

$$\begin{aligned} H^2 &= -\frac{k}{a^2}-\frac{\Lambda}{3}+\frac{1}{3}\kappa\rho+\frac{\epsilon}{18}\bigg[-3\varphi^T+\frac{96\kappa k\rho\varphi''^T(1+w)(\kappa\rho-\Lambda)(\Lambda+2\kappa\rho(1+3w)-3\Lambda w)}{a^2} \\ &\quad +4(\Lambda-\kappa\rho)\bigg(8\kappa\rho\varphi''^T(1+w)(\kappa\rho-\Lambda)(\Lambda+2\kappa\rho(1+3w)-3\Lambda w) \\ &\quad +\varphi'^T(2\Lambda+\kappa\rho(1+3w))\bigg)\bigg]. \end{aligned} \tag{4.48}$$

All of the calculations explained and implemented in this Sec. 4.2.4 were performed and checked under a *Mathematica* environment. As it may be consulted in Appendix A.5, we first wrote a draft expression for the first Friedmann equation which we called *mathematicaresult1* from the simplification of the functional implementation of Eq. (4.48), which we manipulated through a similar process as we had done for the simplification of Eq. (4.38), resulting in what we called *mathematicaresult2*. To *mathematicaresult2* we applied the substitutions $\frac{\dot{a}}{a}->H$ and $\frac{\dot{a}^2}{a^2}->H^2$ turning it into what we termed as *mathematicaresult*. With *mathematicaresult* we were able to set $k=0$ which resulted in *mathematicaresultnok*

$$\begin{aligned} &6H^2+8H\kappa^2\epsilon\rho\dot{\varphi}'^T-8H\Lambda\epsilon\dot{\varphi}'^T+2\Lambda-2\kappa^2\rho+4\kappa^4 w\epsilon\rho^2\varphi'^T \\ &-4\kappa^2\Lambda w\epsilon\rho\varphi'^T+\frac{4}{3}\kappa^4\epsilon\rho^2\varphi'^T+\frac{4}{3}\kappa^2\Lambda\epsilon\rho\varphi'^T-\frac{8}{3}\Lambda^2\epsilon\varphi'^T+\epsilon\varphi^T=0, \end{aligned} \tag{4.49}$$

or $\Lambda = 0$ to give us *mathematicaresultno*Λ giving us

$$\frac{6k}{a^2} + 6H^2 + 8H\kappa^2\epsilon\rho\dot{\varphi}'^T - 2\kappa^2\rho + 4\kappa^4 w\epsilon\rho^2\varphi'^T + \frac{4}{3}\kappa^4\epsilon\rho^2\varphi'^T + \epsilon\varphi^T = 0, \tag{4.50}$$

or to set both $k = \Lambda = 0$ which we denoted by *mathematicaresultnokno*Λ which read as

$$6H^2 + 8H\kappa^2\epsilon\rho\dot{\varphi}'^T - 2\kappa^2\rho + 4\kappa^4 w\epsilon\rho^2\varphi'^T + \frac{4}{3}\kappa^4\epsilon\rho^2\varphi'^T + \epsilon\varphi^T = 0. \tag{4.51}$$

We then proceeded to taking *mathematicaresult* and putting it through some clarifying changes in notation and useful substitutions, finally arriving at what we called *mathematicaresult4*. All the intermediate steps are explicit in the code of Appendix A.5.

After recalling that we were working with a single line of calculus, we now turn it back into a differential equation by retrieving H^2 to a left-hand side of the equation as we clearly write in Appendix A.5, where we label this equation *FirstFriedmann*. Finally we define the lowest order expression for H^2 as *zerothorderH2* and substitute it into its respective place on the right-hand side of *FirstFriedmann* so that the latter becomes *FirstFriedmann1*, corresponding to the second-order differential equation that we want to solve.

The built-in function *DSolve[eqn,u,x]*

When solving *FirstFriedmann1*, we will first put the equation through a series of simplifications and expansions in order to make it easier for us to make the necessary substitutions. Then we will apply the function *DSolve[eqn,u,x]*, which solves a differential equation for the function *u*, with independent variable *x*. This function will not be applied to the first Friedmann equation that we derived, but rather to the comparison between this and the one responsible for a bouncing universe, Eq. (1.30). In the next Chapter 4.3 we will explain further, here we just mention how this *Mathematica* built-in function works in order to better comprehend the code we have in Appendix A.6.

Here, our objective is to just understand that the design of *DSolve* is modular: the algorithms for different classes of problems work independently of one another. Once the differential equation has been classified, the available methods for that class are tried in a specific sequence until a solution is obtained. The code has a hierarchical structure whereby the solution of complex problems is reduced to the solution of relatively simpler problems, for which a greater variety of methods is available.

4.3 Bouncing solutions for the modified first Friedmann equation

4.3.1 Contextualizing the problem

Recapitulating what we set to do in the Sec. 1.2.3 of Chapter 1, concerning the objectives of this thesis: we want to extend general relativity with a function $f(G)$ that provides us a first Friedmann equation which complies with a cyclic cosmology as the one described by Eq. (1.30). In summary, our goal is to find the function $\varphi(G)$ that when substituted into Eq. (4.48), transforms it into a differential equation of the form of Eq. (1.30), which clearly exhibits a bounce when $\rho = \rho_c$. We have then to equate this Eq. (4.48) to Eq. (1.30) and solve the resulting differential equation for $\varphi(G)$.

To comply with the conditions among which the model of Eq. (1.30) was derived, we must first choose the correct values for our constants k, Λ and w to substitute into Eq. (4.48), set in accordance to what was done by the Refs. [12, 13, 14]. More precisely, they quantized and understood a model consisting of a flat, isotropic and homogeneous universe, actually an FLRW spacetime, sourced by a free and massless scalar field in the context of loop quantum cosmology. In other words, they assumed the simplest possible model, i.e., with no cosmological constant, no spatial curvature (spatially flat spacetime), and with the matter being a scalar field (stiff fluid) so that

$$\Lambda = 0\,, \quad k = 0\,, \quad w = 1\,, \tag{4.52}$$

respectively.

With these assumptions it is useful to rewrite the Gauss-Bonnet invariant as a function of ρ, for that we have Eqs. (3.22), (4.4), (4.5), and (4.52) yield

$$\rho^2 = -\frac{3}{16\kappa^2} G^T\,. \tag{4.53}$$

4.3.2 The modified first Friedmann equation of $f(G)$ gravity in this context

When we apply the conditions expressed in Eq. (4.52), as well as substitute all the first order dependencies on ρ using Eq. (4.53), to Eq. (4.48), we are able to show the expression for the modified first Friedmann equation for an $f(G)$ theory in the context of the model we want to study in the framework of loop quantum cosmology, solely in terms of the variable G^T in its ϵ terms. This expression then reads as

$$H^2 = \frac{1}{3}\kappa\rho - \epsilon\left((G^T)^2\varphi''(G^T) - \frac{1}{6}G^T\varphi'(G^T) + \frac{1}{6}\varphi(G^T)\right)\,. \tag{4.54}$$

It is interesting to note, that for a function $\varphi(G) = G$, we retrieve the first Friedmann equation corresponding to the theory of general relativity, as it is expected since G is a topological invariant in four dimensions. This note validates our corrections and assumptions throughout the development of this work.

4.3.3 The source term responsible for the bounce

We now need to be able to rewrite Eq. (4.54) in a way that mimics the form presented by Eq. (1.30). The latter is the equation of motion that evinces a bounce in the framework of loop quantum cosmology and basically reads as

$$H^2 = \frac{1}{3}\kappa\rho + \Psi(\rho), \tag{4.55}$$

where $\Psi(\rho)$ stands for some function of the density ρ, in the case of Eq. (1.30)

$$\Psi_{\text{bouncing}}(\rho) = -\frac{1}{3}\kappa\frac{\rho^2}{\rho_c}. \tag{4.56}$$

The function $\Psi_{\text{bouncing}}(\rho)$ in Eq. (4.56) basically defines the source term that modifies the typical first Friedmann equation for it to return a clear bounce at $\rho = \rho_c$.

For an action with an $f(G)$ term adding to the R responsible for general relativity, such as ours in Eq. (1.29), the first Friedmann equation as depicted by Eq. (4.54) can too be interpreted as an equation of the type of Eq. (4.55) with a source term which reads as

$$\Psi_{f(G)}(G^T) = \epsilon\left(-(G^T)^2\varphi''(G^T) + \frac{1}{6}G^T\varphi'(G^T) - \frac{1}{6}\varphi(G^T)\right). \tag{4.57}$$

Notice that we want to be able to write Eq. (4.56) and (4.57) with the same variables, since our method for finding the function $f(G)$ that retrieves a bouncing universe implicates the comparison between the two. With that in view, we choose to stick to the definition of $\varphi(G^T)$ and write all the equations in terms of G^T. We have then to make some changes in Eq. (4.56), by applying the transformation explicit in Eq. (4.53) that allows us to write every density ρ in terms of the variable G^T. We then get for Eq. (4.56),

$$\Psi_{\text{bouncing}}(G^T) = \frac{G^T}{16\kappa\rho_c}. \tag{4.58}$$

Although we obtained all the results provided in this Subsec. 4.3.3 analytically, we also followed this line of thought in the code of Appendix A.6.

4.3.4 Bouncing cosmology in $f(G)$ theory

Finally, our thesis objective amounts to equating Eq. (4.58) to (4.56), $\Psi_{\text{bouncing}}(G^T) = \Psi_{f(G)}(G^T)$, and solving it for $\varphi(G^T)$. In summary, we need to solve the differential equation

$$\epsilon\left(-(G^T)^2\varphi''(G^T) + \frac{1}{6}G^T\varphi'(G^T) - \frac{1}{6}\varphi(G^T)\right) = \frac{G^T}{16\kappa\rho_c}. \tag{4.59}$$

The differential equation in Eq. (4.59) can be classified as an ordinary differential equation (ODE). Furthermore it is second-order since this is the order of the highest derivative in the equation, it is linear given that the equation is of the first degree in $\varphi(G^T)$ and its derivatives, and the coefficients are a function of its independent variable G^T. Moreover, we are dealing with an inhomogeneous differential equation, meaning that its solution will be the sum of the respective homogeneous equation (if Ψ would be zero) and a particular integral. As previously mentioned, we are dividing our problem in relatively smaller problems which are more easily tackled.

The homogeneous part of our equation corresponds to an Euler equation since it has the form

$$x^2 y''(x) + axy'(x) + by(x) = 0, \tag{4.60}$$

where in our case $x = G^T$, $y(x) = \varphi(G^T)$ and its corresponding derivatives, $a = -\frac{1}{6}$ and $b = \frac{1}{6}$ are constants. The method to solve an Euler equation lies in transforming them into equations with constant derivatives. The general solution of a linear ODE with constant coefficients is then found by using the roots of the characteristic equation for the ODE, which in our case would be

$$-x^2 + \frac{1}{6}x - \frac{1}{6} = 0. \tag{4.61}$$

The solution for Eq. (4.59) reads as

$$\varphi(G) = \bar{c}_1 \sqrt[6]{G} + \bar{c}_2 G - \frac{3G(5\log(G) - 6)}{200\kappa^2 \rho_c \epsilon} \tag{4.62}$$

with $\bar{c}_1$ and $\bar{c}_2$ being two arbitrary constants of integration. In this solution for $\varphi(G)$ we have dropped the superscript T since to ϵ order $f(G)$ and $f(G^T)$ are the same, see Eq. (3.7).

Multiplying $\varphi(G)$ by ϵ we get an equation for $f(G)$ (since we have zeroed the cosmological constant Λ), the quantity we have been after. It is given by

$$f(G) = c_1 \sqrt[6]{G} + c_2 G - \frac{3G(5\ln(G) - 6)}{200\kappa\rho_c}\,, \tag{4.63}$$

where $c_1 = \epsilon\bar{c}_1$ and $c_2 = \epsilon\bar{c}_2$. So the theory has a Lagrangian for the gravity sector given by $\mathcal{L}_{\text{grav}} = R + f(G)$, see Eq. (3.8), which after using Eq. (4.63) gives

$$\mathcal{L}_{\text{grav}} = R + c_1 \sqrt[6]{G} + c_2 G - \frac{3G(5\ln(G) - 6)}{200\kappa\rho_c}. \tag{4.64}$$

The Lagrangian in Eq. (4.64) corresponds to the integrand we were looking for. The resulting covariant action is capable of describing the results achieved in the framework of loop quantum cosmology with the intent to avoid the initial singularity problem, through $f(G)$ theory.

Chapter 5

Conclusions

As we have set ourselves to do in Subsec. 1.2.3, where we stated the objectives for this thesis work, we were able to solve the covariance problem of loop quantum cosmology by having shown that an effective covariant action of the type of an $f(G)$ modification to the Einstein-Hilbert action term, which corresponds to a higher order theory of gravity, describes the dynamics predicted by the quantum theory. This is so, as it leads to the effective Friedmann equation, Eq. (1.30), foreseen in the massless scalar field model of this cosmological framework. Such a conclusion was made possible for a higher order theory such as modified Gauss-Bonnet $f(G)$ gravity theory, given that it was treated with an order reduction method. Using this mathematical technique, where we were able to reduce a fourth order field equation to a second order one, it was possible to retrieve directly only the physical solutions, i.e. the ones perturbatively close to general relativity, instead of having to discard one by one the spurious ones, by constraining a fourth order equation against some boundary conditions. This method implies that the $f(G)$ gravity theory should be seen as an effective field theory, where the extra degrees of freedom correspond to non-physical solutions, therefore dummy solutions, rather than the usual treatment as a fundamental theory.

We were successful in deriving an effective Lagrangian, and so a covariant action, for a modified Gauss-Bonnet $f(G)$ gravity theory which yields a bounce as a cosmological solution for the early-time universe [30]. There has been a growing interest in the understanding of the big bang initial singularity, and cyclic universes with a bounce of the sort we have presented here are viable candidates to finding a solution for this problem, as we have already stated in Chapter 1. Recalling the results mentioned in the state of the art for this thesis, more precisely the ones derived by Sotiriou [18] using metric $f(R)$ gravity theory, it is well noted that our resulting Eq. (5.3) is in contrast with the $f(R)$ theory where it was found that the first order correction is an R^2 term, indeed

$$\mathcal{L}_{\mathrm{grav}f(R)} = R + \frac{1}{18\kappa\rho_c}R^2 \,, \tag{5.1}$$

is compatible with the Lagrangian we found in Eq. (4.64). Furthermore, knowing that in in four dimensions G is a topological invariant, and so can be discarded, we can write our function of G as simply

$$f(G) = c_1 \sqrt[6]{G} - \frac{3}{40\kappa\rho_c}\, G \ln G \,, \tag{5.2}$$

and the Lagrangian for the gravity part of the modified Gauss-Bonnet $f(G)$ gravity action as

$$\mathcal{L}_{\rm grav} = R + c_1 \sqrt[6]{G} - \frac{3}{40\kappa\rho_c}\, G \ln G \,. \tag{5.3}$$

Interesting to note that the Lagrangian we have found in Eq. (5.3) for the $f(G)$ theory, has the zeroth order Einstein-Hilbert term R, plus term with a fractional exponent and a $G \ln G$ term, as the first order correction to general relativity. This result has been previously found through different approaches [27, 31, 32], which strengthens the validity of the order reduction technique and assumptions we used.

In summary, we found that we can obtain a bounce instead of an initial singularity for the early-time cosmology of the universe, in the framework of loop quantum cosmology, through a covariant action of the type

$$S = \frac{1}{2\kappa} \int dx^4 \sqrt{-g} \left(R + c_1 \sqrt[6]{G} - \frac{3}{40\kappa\rho_c}\, G \ln G \right) + S_{\text{matter}}(g_{\mu\nu}, \psi) \,, \tag{5.4}$$

which complies with the conditions imposed for the studied loop quantum cosmology model, where we understand the early universe to be flat, the matter to be stiff and with no cosmological constant.

From Eq. (5.4), we get the corresponding first Friedmann equation, which reads as

$$H^2 = \frac{1}{3}\kappa\rho \left(1 - \frac{\rho}{\rho_c} \right) . \tag{5.5}$$

Eqs. (5.4) and (5.5) are here obtained simply by substitution of the $f(G)$ we found, Eq. (5.2) into the previously derived equations for a general $f(G)$: Eqs.(1.29) and (4.54), respectively for the action and first Friedmann equation. We could do the same for the full field equations: substitute Eq. (5.2) into Eq. (2.66), however, the resulting field equations are visually more complex.

Concerning the growing interest on cyclic cosmologies in general, further experimental evidence [33] is needed to comment on the validity of our choice for this work, although the theoretical motivation that took us in this direction is well posed which is evident from the attention this subject has been attracting in recent years. We may elaborate on an example of parallel work, in Ref. [27], where they develop an effective Gauss-Bonnet extension of loop quantum cosmology, by introducing holonomy corrections in modified $f(G)$ theories of gravity. Their goal is to correct loop quantum cosmology within an $f(G)$ gravity framework in order to comply with various cosmological scenarios, whereas ours is to find a covariant action for loop quantum cosmology through a match with the Friedmann equation predicted by loop quantum cosmology by using a massless scalar field. Through different methods, the resulting functions $f(G)$ of the Gauss-Bonnet invariant G obtained to support bouncing cosmologies are similar in form, both containing logarithmic terms multiplied by G, a power term with positive fractional exponent and a linear term. Such a compatibility between results derived from different approaches provides credibility to the order reduction method we adopted in treating our calculations.

In future work, it would be interesting to follow the same procedure for a more general gravity theory. For example, using a function $f(R^2, R_{\mu\nu}R^{\mu\nu}, R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$ in the gravitational part of the action, or for a more direct comparison with our work, $f(aR^2+bR_{\mu\nu}R^{\mu\nu}+cR_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})$. We know that this more general theory corresponds to $f(G)$ when we set the linear combination parameters to $a=1$, $b=-4$ and $c=1$. This way, by relaxing the constants through all the procedure, we would have a better insight if $f(G)$ was an accurate choice to tackle the problem, in which case there would be a perfect fit between our results and the ones we would get for this more general theory. On the other way, another arrangement could turn out to appear as a more natural choice. Such a test would also provide information on the results obtained in [18], since in this case setting $b=c=0$ in the general theory is equivalent to an $f(R)$ theory of gravity. A similar investigation, could be conducted in order to evaluate our choice concerning the metric: when we used an FLRW line element, which is conformally flat and allowed us to rewrite the Gauss-Bonnet invariant with $a=\frac{2}{3}$, $b=-2$ and $c=0$, we may have lost information. Testing with relaxed coupling parameters might be equivalent to using a more generic metric, without a necessarily vanishing Weyl tensor.

Finally, in regards to the nature of the bounce itself, more specifically, to the moment when according to the evolution for the universe described by Eq. (1.30) a density of $\rho=\rho_c$ is reached and the velocity of expansion comes to zero before turning back, we will only be able to gather more information by means of early time observations. These observations will only be possible through more advanced techniques, probably achieved with further research in the most recently established physics field: gravitational waves astronomy. Until then, we hope to have contributed to the rekindling of this renewed interest around cyclic cosmology and shed some light into alternatives to the current incomplete description of a ultimate initial singularity.

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Appendix A

Mathematica codes

A.1 Tensorial quantities without order reduction

```
x = {t, r, \[Theta], \[CurlyPhi]};
MatrixForm[
 g = {{-1, 0, 0, 0}, {0, (A[t]^2)/(1 - k r^2), 0, 0}, {0, 0,
    r^2*A[t]^2, 0}, {0, 0, 0, (r^2)*A[t]^2 Sin[\[Theta]]^2} }]
MatrixForm[gin = Inverse[g]]

Chr = Table[
   Sum[gin[[a, d]]/
      2*(D[g[[d, b]], x[[c]]] + D[g[[d, c]], x[[b]]] -
       D[g[[b, c]], x[[d]]]), {d, 4}], {a, 4}, {b, 4}, {c, 4}];
Riemann =
  Table[(D[Chr[[\[Mu], \[Nu], \[Beta]]], x[[\[Alpha]]]] -
     D[Chr[[\[Mu], \[Alpha], \[Nu]]], x[[\[Beta]]]] +
     Sum[Chr[[\[Mu], \[CapitalXi], \[Alpha]]]*
        Chr[[\[CapitalXi], \[Nu], \[Beta]]] -
       Chr[[\[Mu], \[CapitalXi], \[Beta]]]*
        Chr[[\[CapitalXi], \[Nu], \[Alpha]]], {\[CapitalXi],
       4}]), {\[Mu], 4}, {\[Nu], 4}, {\[Alpha], 4}, {\[Beta], 4}];
riemann =
  Table[Sum[g[[a, m]]*Riemann[[m, b, c, d]], {m, 4}], {a, 4}, {b,
    4}, {c, 4}, {d, 4}];
Ricci = Simplify[
   Table[Sum[Riemann[[a, \[Mu], a, \[Nu]]], {a, 4}], {\[Mu],
     4}, {\[Nu], 4}]];
ScalarCurvature =
```

```
  Simplify[Sum[gin[[a, b]]*Ricci[[a, b]], {a, 4}, {b, 4}]];
MatrixForm[Einstein = Simplify[Ricci - g*ScalarCurvature/2]];
EinsteinUp =
  Simplify[Table[
    Sum[gin[[a, s]]*gin[[b, j]]*Einstein[[s, j]], {s, 4}, {j, 4}], {a,
      4}, {b, 4}]];
GB = ScalarCurvature^2 -
    4*Sum[gin[[a, c]]*gin[[b, d]]*Ricci[[a, b]]*Ricci[[c, d]], {a,
      4}, {b, 4}, {c, 4}, {d, 4}] +
    Sum[gin[[a, e]]*gin[[b, f]]*gin[[c, j]]*gin[[d, h]]*
      riemann[[a, b, c, d]]*riemann[[e, f, j, h]], {a, 4}, {b, 4}, {c,
      4}, {d, 4}, {e, 4}, {f, 4}, {j, 4}, {h, 4}] // Simplify;
```

A.2 Tensorial quantities with order reduction

```
u={1,0,0,0};
Tup=Table[(\[Rho][t]+w \[Rho][t])u[[a]]u[[b]]
      +w \[Rho][t]gin[[a,b]],{a,4},{b,4}];
Tdown=Simplify[Table[Sum[g[[a,s]]*g[[b,j]]
        *Tup[[s,j]],{s,4},{j,4}],{a,4},{b,4}]];
Tscalar=Sum[g[[a,b]]*Tup[[a,b]],{a,4},{b,4}];

RicciOR=Simplify[Table[(-\[Kappa]/2)g[[a,b]]Tscalar
          -g[[a,b]]\[CapitalLambda]
         +\[Kappa] Tdown[[a,b]],{a,4},{b,4}]];
RicciORup=Simplify[Table[Sum[gin[[a,s]]*gin[[b,j]]
            *RicciOR[[s,j]],{s,4},{j,4}],{a,4},{b,4}]];
RicciScalarOR=-\[Kappa] Tscalar-4\[CapitalLambda];
RiemannOR=Simplify[Table[-1/2(g[[a,d]]RicciOR[[c,b]]
            +g[[b,c]]RicciOR[[d,a]]-g[[a,c]]RicciOR[[d,b]]
             -g[[b,d]]RicciOR[[c,a]])-1/6(g[[a,c]]g[[d,b]]
            -g[[a,d]]g[[c,b]])RicciScalarOR,{a,4},{b,4},{c,4},{d,4}]];
RiemannORup=Simplify[Table[Sum[gin[[a,i]]*gin[[b,j]]
             *gin[[c,l]]*gin[[d,m]]*RiemannOR[[i,j,l,m]],
             {i,4},{j,4},{l,4},{m,4}],{a,4},{b,4},{c,4},{d,4}]];
GBOR=Simplify[RicciScalarOR^2-4*Sum[RicciOR[[a,b]]RicciORup[[a,b]]
             ,{a,4},{b,4}]+Sum[RiemannOR[[a,b,c,d]]
             RiemannORup[[a,b,c,d]],{a,4},{b,4},{c,4},{d,4}]];
```

```
CovD[b_,a_,v_]:=D[D[v,x[[a]]],x[[b]]]-Sum[Chr[[c,a,b]]D[v,x[[c]]],{c,4}]
box[v_]:=Sum[gin[[a,b]]CovD[a,b,v],{a,4},{b,4}]
```

A.3 Vanishing of Weyl

```
Weyl = Table[
   riemann[[a, b, c, d]] +
    1/2 (g[[a, d]]*Ricci[[c, b]] + g[[c, b]]*Ricci[[a, d]] -
       g[[a, c]]*Ricci[[d, b]] - g[[b, d]]*Ricci[[c, a]]) +
    1/6 (g[[a, c]]*g[[d, b]] - g[[a, d]]*g[[b, c]])*
     ScalarCurvature, {a, 4}, {b, 4}, {c, 4}, {d, 4}] // Simplify
```

A.4 Conservation of Energy

```
CovDup2ind[\[Mu]_,\[Nu]_,V_]:=Simplify[Table[Sum[gin[[a,\[Mu]]]
              (D[V[[\[Mu],\[Nu]]],x[[a]]]-Sum[Chr[[c,a,\[Mu]]]V[[c,\[Nu]]]
           +Chr[[c,a,\[Nu]]]V[[\[Mu],c]],{c,4}]),{a,4},{\[Mu],4}],{\[Nu],4}]]

CovDup2ind[\[Mu], \[Nu], Einstein] // Simplify


Taux1 = Simplify[
  Table[ScalarCurvature Ricci[[\[Mu], \[Nu]]], {\[Mu], 4}, {\[Nu],
    4}]]
Taux2 = Simplify[
  Table[Sum[
    riemann[[\[Mu], i, j, z]] gin[[i, m]] gin[[j, n]] gin[[z,
      l]] riemann[[\[Nu], m, n, l]], {i, 4}, {j, 4}, {z, 4}, {m,
     4}, {n, 4}, {l, 4}], {\[Mu], 4}, {\[Nu], 4}]]
Taux3 = Simplify[
  Table[Sum[
    Ricci[[i, \[Nu]]] gin[[i, j]] Ricci[[j, \[Mu]]], {i, 4}, {j,
     4}], {\[Mu], 4}, {\[Nu], 4}]]
Taux4 = Simplify[
  Table[Sum[
    riemann[[\[Mu], i, \[Nu], j]] gin[[i, m]] gin[[j, n]] Ricci[[m,
      n]], {i, 4}, {j, 4}, {m, 4}, {n, 4}], {\[Mu], 4}, {\[Nu], 4}]]
Taux5 = Simplify[
  Table[Ricci[[\[Mu], \[Nu]]] box[
     D[\[Epsilon] \[CurlyPhi][G[t]], G[t]]], {\[Mu], 4}, {\[Nu], 4}]]
Taux6 = Simplify[
```

```
  Table[g[[\[Mu], \[Nu]]] Sum[
     gin[[i, a]] gin[[j, b]] Ricci[[a, b]] CovD[a, b,
       D[\[Epsilon] \[CurlyPhi][G[t]], G[t]]], {i, 4}, {j, 4}, {a,
      4}, {b, 4}], {\[Mu], 4}, {\[Nu], 4}]]
Taux7 = Simplify[
  Table[Sum[
    gin[[a, b]] Ricci[[b, \[Mu]]] CovD[a, \[Nu],
      D[\[Epsilon] \[CurlyPhi][G[t]], G[t]]], {a, 4}, {b, 4}], {\[Mu],
     4}, {\[Nu], 4}]]
Taux8 = Simplify[
  Table[Sum[
    gin[[a, b]] Ricci[[b, \[Nu]]] CovD[a, \[Mu],
      D[\[Epsilon] \[CurlyPhi][G[t]], G[t]]], {a, 4}, {b, 4}], {\[Mu],
     4}, {\[Nu], 4}]]
Taux9 = Simplify[
  Table[g[[\[Mu], \[Nu]]] ScalarCurvature box[
     D[\[Epsilon] \[CurlyPhi][G[t]], G[t]]], {\[Mu], 4}, {\[Nu], 4}]]
Taux10 = Simplify[
  Table[ScalarCurvature CovD[\[Mu], \[Nu],
     D[\[Epsilon] \[CurlyPhi][G[t]], G[t]]], {\[Mu], 4}, {\[Nu], 4}]]
Taux11 = Simplify[
  Table[Sum[
    gin[[a, c]] gin[[b, d]] riemann[[\[Mu], c, \[Nu], d]] CovD[b, a,
      D[\[Epsilon] \[CurlyPhi][G[t]], G[t]]], {c, 4}, {d, 4}, {a,
     4}, {b, 4}], {\[Mu], 4}, {\[Nu], 4}]]

conservationeq =
 CovDup2ind[\[Mu], \[Nu], Einstein] -
   Table[1/2 Sum[
      g[[\[Mu], \[Nu]]] gin[[\[Mu], a]] D[
        2 \[CapitalLambda] + \[Epsilon] \[CurlyPhi][G[t]],
        x[[a]]], {a, 4}, {\[Mu], 4}], {\[Nu], 4}] +
   Table[Sum[
     gin[[\[Mu], a]] D[\[Epsilon] D[\[CurlyPhi][G[t]], G[t]],
       x[[a]]] (2 Taux1[[\[Mu], \[Nu]]] + 2 Taux2[[\[Mu], \[Nu]]] -
        4 Taux3[[\[Mu], \[Nu]]] - 4 Taux4[[\[Mu], \[Nu]]]), {\[Mu],
      4}, {a, 4}], {\[Nu], 4}] + \[Epsilon] D[\[CurlyPhi][G[t]],
     G[t]] (2 CovDup2ind[\[Mu], \[Nu], Taux1] +
      2 CovDup2ind[\[Mu], \[Nu], Taux2] -
```

```
      4 CovDup2ind[\[Mu], \[Nu], Taux3] -
      4 CovDup2ind[\[Mu], \[Nu], Taux4]) -
   4 CovDup2ind[\[Mu], \[Nu], Taux5] -
   4 CovDup2ind[\[Mu], \[Nu], Taux6] +
   4 CovDup2ind[\[Mu], \[Nu], Taux7] +
   4 CovDup2ind[\[Mu], \[Nu], Taux8] +
   2 CovDup2ind[\[Mu], \[Nu], Taux9] -
   2 CovDup2ind[\[Mu], \[Nu], Taux10] +
   4 CovDup2ind[\[Mu], \[Nu], Taux11] // Simplify

conservationeq[[1]] /. {G'[t] -> D[GB, t]} // Simplify
```

A.5 Computing the first Friedmann equation

```
mathematicaresult1 =
 Einstein[[1, 1]] - g[[1, 1]] \[CapitalLambda] -
   1/2 g[[1, 1]] \[Epsilon] \[CurlyPhi][t] + \[Epsilon] \[CurlyPhi]l[
     t] (2 RicciScalarOR RicciOR[[1, 1]] +
      2 Sum[RiemannOR[[1, a, b, c]] gin[[a, d]] gin[[b, e]] gin[[c,
          f]] RiemannOR[[1, d, e, f]], {a, 4}, {b, 4}, {c, 4}, {d,
        4}, {e, 4}, {f, 4}] -
      Sum[4 RicciOR[[a, 1]] gin[[a, b]] RicciOR[[b, 1]], {a, 4}, {b,
        4}] - 4 Sum[
        gin[[a, c]] gin[[b, d]] RiemannOR[[1, a, 1, b]] RicciOR[[c,
          d]], {a, 4}, {b, 4}, {c, 4}, {d, 4}]) -
   2 \[Epsilon] RicciScalarOR CovD[1, 1, \[CurlyPhi]l[t]] -
   2 \[Epsilon] RicciScalarOR box[\[CurlyPhi]l[t]] -
   4 \[Epsilon] RicciOR[[1, 1]] box[\[CurlyPhi]l[t]] +
   8 \[Epsilon] gin[[1, 1]] RicciOR[[1, 1]] CovD[1,
     1, \[CurlyPhi]l[t]] -
   4 \[Epsilon] g[[1, 1]] Sum[
     gin[[a, c]] gin[[b, d]] RicciOR[[c, d]] CovD[a,
       b, \[CurlyPhi]l[t]], {a, 4}, {b, 4}, {c, 4}, {d, 4}] +
   4 \[Epsilon] Sum[
     gin[[a, c]] gin[[b, d]] RiemannOR[[1, c, 1, d]] CovD[a,
       b, \[CurlyPhi]l[t]], {a, 4}, {b, 4}, {c, 4}, {d,
      4}] - \[Kappa] Tdown[[1, 1]] // Simplify

mathematicaresult2 = mathematicaresult1*2 // Expand
```

```
mathematicaresult =
 mathematicaresult2 /. {A'[t]/A[t] -> H, A'[t]^2/A[t]^2 -> H^2}

mathematicaresultnok = mathematicaresult /. {k -> 0}

mathematicaresultno\[CapitalLambda] =
 mathematicaresult /. {\[CapitalLambda] -> 0}

mathematicaresultno\[CapitalLambda]nok =
 mathematicaresultno\[CapitalLambda] /. {k -> 0}

mathematicaresult3 =
 mathematicaresult /. {\[CurlyPhi][t] -> \[CurlyPhi][
     G[t]], \[CurlyPhi]l[t] ->
    D[\[CurlyPhi][G[t]], G[t]], \[CurlyPhi]l'[t] ->
    D[D[\[CurlyPhi][G[t]], G[t]], G[t]] D[GBOR, \[Rho][t]] \[Rho]'[t]}

mathematicaresult4 =
 mathematicaresult3/6 /. {\[Rho]'[t] -> -3 H (1 + w) \[Rho][t]} //
   Simplify

FirstFriedmann = H^2 == Simplify[-mathematicaresult4 + H^2]

zerothorderH2 = FirstFriedmann[[2]] /. {\[Epsilon] -> 0}

FirstFriedmann1 =
 H^2 == (FirstFriedmann[[2]] /. {H^2 -> zerothorderH2})
```

A.6 Solving the first Friedmann equation for $\varphi(G)$

```
AuxEq = FirstFriedmann1[[2]] - zerothorderH2 // Simplify

AuxEq1 = AuxEq // Expand

AuxEq2 = AuxEq1 /. {\[Kappa]^2 \[Rho][
        t]^2 -> (\[Kappa]^2 \[Rho][t]^2/GBOR) G, \[Kappa]^3 \[Rho][
        t]^3 -> (\[Kappa]^2 \[Rho][t]^2/GBOR)^(3/2) G^(3/
         2), \[Kappa]^4 \[Rho][
```

```
        t]^4 -> (\[Kappa]^4 \[Rho][t]^4/GBOR^2) G^2} /. {k -> 0,
   w -> 1, \[CapitalLambda] -> 0}

AuxEq3 = AuxEq2 /. {\[CurlyPhi][G[t]] -> \[CurlyPhi][G], \[CurlyPhi]'[
      G[t]] -> D[\[CurlyPhi][G], G], \[CurlyPhi]''[G[t]] ->
    D[D[\[CurlyPhi][G], G], G]}

\[CapitalPsi] = G/(\[Kappa] \[Rho]c 16)

Solution = DSolve[\[CapitalPsi] == AuxEq3, \[CurlyPhi][G], G]

fG = -((3 G (-6 + 5 Log[G]))/(
      200 \[Epsilon] \[Kappa] \[Rho]c)) \[Epsilon] /. {\[Rho]c ->
      Sqrt[3]/(2 Pi \[Kappa] \[Gamma]^3 lp^2)} // Simplify
```

Appendix B

Mathematica Results

B.1 Output for *Riemann*, i.e. $R^{\mu}{}_{\nu\rho\sigma}$, in *MatrixForm*

$$
\begin{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & \frac{a(t)a''(t)}{1-kr^2} & 0 & 0 \\ -\frac{a(t)a''(t)}{1-kr^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & r^2a(t)a''(t) & 0 \\ 0 & 0 & 0 & 0 \\ -r^2a(t)a''(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & r^2a(t)\sin^2(\theta)a''(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r^2a(t)\sin^2(\theta)a''(t) & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & \frac{a''(t)}{a(t)} & 0 & 0 \\ -\frac{a''(t)}{a(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a'(t)^2r^2+kr^2 & 0 \\ 0 & -a'(t)^2r^2-kr^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & kr^2\sin^2(\theta)+r^2a'(t)^2\sin^2(\theta) \\ 0 & 0 & 0 & 0 \\ 0 & -kr^2\sin^2(\theta)-r^2a'(t)^2\sin^2(\theta) & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & \frac{a''(t)}{a(t)} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a''(t)}{a(t)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{a'(t)^2}{1-kr^2}-\frac{k}{1-kr^2} & 0 \\ 0 & \frac{a'(t)^2}{1-kr^2}+\frac{k}{1-kr^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2a'(t)^2\sin^2(\theta)-\left(1-kr^2\right)\sin^2(\theta)+\sin^2(\theta) \\ 0 & 0 & -r^2a'(t)^2\sin^2(\theta)+\left(1-kr^2\right)\sin^2(\theta)-\sin^2(\theta) & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & \frac{a''(t)}{a(t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{a''(t)}{a(t)} & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{a'(t)^2}{1-kr^2}-\frac{k}{1-kr^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{a'(t)^2}{1-kr^2}+\frac{k}{1-kr^2} & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a'(t)^2r^2-kr^2+\cot^2(\theta)-\csc^2(\theta)+1 \\ 0 & 0 & a'(t)^2r^2+kr^2-\cot^2(\theta)+\csc^2(\theta)-1 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{pmatrix}
\tag{B.1}
$$

B.2 Output for *riemann*, i.e. $R_{\mu\nu\rho\sigma}$, in *MatrixForm*

$$
\begin{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & -\frac{a(t)a''(t)}{1-kr^2} & 0 & 0 \\ \frac{a(t)a''(t)}{1-kr^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & -r^2a(t)a''(t) & 0 \\ 0 & 0 & 0 & 0 \\ r^2a(t)a''(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & -r^2a(t)\sin^2(\theta)a''(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r^2a(t)\sin^2(\theta)a''(t) & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & \frac{a(t)a''(t)}{1-kr^2} & 0 & 0 \\ -\frac{a(t)a''(t)}{1-kr^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{a(t)^2\left(a'(t)^2r^2+kr^2\right)}{1-kr^2} & 0 \\ 0 & \frac{a(t)^2\left(-a'(t)^2r^2-kr^2\right)}{1-kr^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{a(t)^2\left(kr^2\sin^2(\theta)+r^2a'(t)^2\sin^2(\theta)\right)}{1-kr^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{a(t)^2\left(-kr^2\sin^2(\theta)-r^2a'(t)^2\sin^2(\theta)\right)}{1-kr^2} & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & r^2a(t)a''(t) & 0 \\ 0 & 0 & 0 & 0 \\ -r^2a(t)a''(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & r^2a(t)^2\left(-\frac{a'(t)^2}{1-kr^2}-\frac{k}{1-kr^2}\right) & 0 \\ 0 & r^2a(t)^2\left(\frac{a'(t)^2}{1-kr^2}+\frac{k}{1-kr^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2a(t)^2\left(r^2a'(t)^2\sin^2(\theta)-\left(1-kr^2\right)\sin^2(\theta)+\sin^2(\theta)\right) \\ 0 & 0 & r^2a(t)^2\left(-r^2a'(t)^2\sin^2(\theta)+\left(1-kr^2\right)\sin^2(\theta)-\sin^2(\theta)\right) & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & r^2a(t)\sin^2(\theta)a''(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -r^2a(t)\sin^2(\theta)a''(t) & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2a(t)^2\sin^2(\theta)\left(-\frac{a'(t)^2}{1-kr^2}-\frac{k}{1-kr^2}\right) \\ 0 & 0 & 0 & 0 \\ 0 & r^2a(t)^2\sin^2(\theta)\left(\frac{a'(t)^2}{1-kr^2}+\frac{k}{1-kr^2}\right) & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2a(t)^2\sin^2(\theta)\left(-a'(t)^2r^2-kr^2+\cot^2(\theta)-\csc^2(\theta)+1\right) \\ 0 & 0 & r^2a(t)^2\sin^2(\theta)\left(a'(t)^2r^2+kr^2-\cot^2(\theta)+\csc^2(\theta)-1\right) & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{pmatrix}
\tag{B.2}
$$

B.3 Output for *RiemannOR*, i.e. $R^{T}_{\mu\nu\rho\sigma}$, in *MatrixForm*

$$
\begin{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & \frac{a(t)^2(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6-6kr^2} & 0 & 0 \\ \frac{a(t)^2(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6kr^2-6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \frac{1}{6}r^2a(t)^2(2\Lambda+\rho(t)(\kappa+3\kappa w)) & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{6}r^2a(t)^2(2\Lambda+\rho(t)(\kappa+3\kappa w)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & \frac{1}{6}r^2a(t)^2\sin^2(\theta)(2\Lambda+\rho(t)(\kappa+3\kappa w)) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{6}r^2a(t)^2\sin^2(\theta)(2\Lambda+\rho(t)(\kappa+3\kappa w)) & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & \frac{a(t)^2(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6kr^2-6} & 0 & 0 \\ \frac{a(t)^2(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6-6kr^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r^2a(t)^4(\Lambda-\kappa\rho(t))}{3kr^2-3} & 0 \\ 0 & \frac{r^2a(t)^4(\Lambda-\kappa\rho(t))}{3-3kr^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r^2a(t)^4\sin^2(\theta)(\Lambda-\kappa\rho(t))}{3kr^2-3} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{r^2a(t)^4\sin^2(\theta)(\Lambda-\kappa\rho(t))}{3-3kr^2} & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & -\frac{1}{6}r^2a(t)^2(2\Lambda+\rho(t)(\kappa+3\kappa w)) & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{6}r^2a(t)^2(2\Lambda+\rho(t)(\kappa+3\kappa w)) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r^2a(t)^4(\Lambda-\kappa\rho(t))}{3-3kr^2} & 0 \\ 0 & \frac{r^2a(t)^4(\Lambda-\kappa\rho(t))}{3kr^2-3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{3}r^4a(t)^4\sin^2(\theta)(\Lambda-\kappa\rho(t)) \\ 0 & 0 & \frac{1}{3}r^4a(t)^4\sin^2(\theta)(\Lambda-\kappa\rho(t)) & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & -\frac{1}{6}r^2a(t)^2\sin^2(\theta)(2\Lambda+\rho(t)(\kappa+3\kappa w)) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{6}r^2a(t)^2\sin^2(\theta)(2\Lambda+\rho(t)(\kappa+3\kappa w)) & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r^2a(t)^4\sin^2(\theta)(\Lambda-\kappa\rho(t))}{3-3kr^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{r^2a(t)^4\sin^2(\theta)(\Lambda-\kappa\rho(t))}{3kr^2-3} & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3}r^4a(t)^4\sin^2(\theta)(\Lambda-\kappa\rho(t)) \\ 0 & 0 & -\frac{1}{3}r^4a(t)^4\sin^2(\theta)(\Lambda-\kappa\rho(t)) & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{pmatrix}
\tag{B.3}
$$

B.4 Output for *RiemannORup*, i.e. $(R^{\mu\nu\rho\sigma})^T$, in *MatrixForm*

$$
\begin{pmatrix}
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & -\frac{\left(kr^2-1\right)(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6a(t)^2} & 0 & 0 \\ \frac{\left(kr^2-1\right)(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6a(t)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & \frac{2\Lambda+\rho(t)(\kappa+3\kappa w)}{6r^2a(t)^2} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{2\Lambda+\rho(t)(\kappa+3\kappa w)}{6r^2a(t)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & \frac{\csc^2(\theta)(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6r^2a(t)^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\csc^2(\theta)(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6r^2a(t)^2} & 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & \frac{\left(kr^2-1\right)(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6a(t)^2} & 0 & 0 \\ -\frac{\left(kr^2-1\right)(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6a(t)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\left(kr^2-1\right)(\Lambda-\kappa\rho(t))}{3r^2a(t)^4} & 0 \\ 0 & -\frac{\left(kr^2-1\right)(\Lambda-\kappa\rho(t))}{3r^2a(t)^4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\csc^2(\theta)\left(kr^2-1\right)(\Lambda-\kappa\rho(t))}{3r^2a(t)^4} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\csc^2(\theta)\left(kr^2-1\right)(\Lambda-\kappa\rho(t))}{3r^2a(t)^4} & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & -\frac{2\Lambda+\rho(t)(\kappa+3\kappa w)}{6r^2a(t)^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2\Lambda+\rho(t)(\kappa+3\kappa w)}{6r^2a(t)^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\left(kr^2-1\right)(\Lambda-\kappa\rho(t))}{3r^2a(t)^4} & 0 \\ 0 & \frac{\left(kr^2-1\right)(\Lambda-\kappa\rho(t))}{3r^2a(t)^4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\csc^2(\theta)(\Lambda-\kappa\rho(t))}{3r^4a(t)^4} \\ 0 & 0 & \frac{\csc^2(\theta)(\Lambda-\kappa\rho(t))}{3r^4a(t)^4} & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 & -\frac{\csc^2(\theta)(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6r^2a(t)^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\csc^2(\theta)(2\Lambda+\rho(t)(\kappa+3\kappa w))}{6r^2a(t)^2} & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\csc^2(\theta)\left(kr^2-1\right)(\Lambda-\kappa\rho(t))}{3r^2a(t)^4} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{\csc^2(\theta)\left(kr^2-1\right)(\Lambda-\kappa\rho(t))}{3r^2a(t)^4} & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\csc^2(\theta)(\Lambda-\kappa\rho(t))}{3r^4a(t)^4} \\ 0 & 0 & -\frac{\csc^2(\theta)(\Lambda-\kappa\rho(t))}{3r^4a(t)^4} & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{pmatrix}
\tag{B.4}
$$