Space Bounded Scatter Machines

(Extended Abstract)

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May 2017

Abstract

This dissertation concerns the computational power of an abstract computation model when bounded in polynomial space. The computation model in study consists of a Turing machine coupled with a physical experiment as oracle (the scatter experiment). This computation model (the scatter machine) is already studied concerning polynomial time restrictions, from where we adapted most of the proof techniques used to this work.

We understood for which functions we are able to build a clock (a specific Turing machine) in polynomial space. There are clocks in polynomial space for functions with at most an exponential growing rate. We also presented a new communication protocol (the generalized protocol) between the analogue and the digital components of the scatter machine which, in this case, allow us to use the proper computational power of the scatter experiment as an oracle.

We introduce a uniform complexity class, which we use as support class for one non-uniform complexity class needed to describe the computational results we obtained. We established the computational results of the scatter machine bounded in polynomial space, for both the sharp wedge and the smooth wedge cases, using in each case both the standard and the generalized communication protocols. We established the results for the three usual precision assumptions (infinite precision, arbitrary precision and fixed precision), and in some cases of the smooth wedge experiment we obtain different results depending on if we use the time schedule (clock to bound the time running of the experiment) or not.

Keywords: Analogue-digital computation; Physical oracle; Non-uniform complexity; Space bounded clocks; Hypercomputation.

1 Introduction

Our aim is to establish the computational power of the scatter machine bounded in polynomial space. The scatter machine is an analogue-digital machine consisting on a Turing machine coupled with a specific physical experiment (see [3]), called scatter experiment (see [6]), it is then an oracle Turing machine with a physical oracle.

A physical experiment as an oracle for a Turing machine arise some interesting issues and we must deal with a few of them, but first, due to the nature of our analogue-digital machine we must introduce the classes we need to describe our results. The scatter machine, as it is an oracle Turing machine, characterizes complexity classes of the non-uniform kind, as for example $PSPACE/poly$ (see [2]).

As the scatter experiment can have a probabilistic behaviour, we need to introduce a uniform complexity class like BPP for polynomial space restrictions. The class of sets decidable by a bounded error probabilistic Turing machine in polynomial space. A bounded error probabilistic Turing machine follows the following decision criterion:

Definition 1. *A bounded error probabilistic Turing machine $\mathcal{M}$ is said to decide a set $A \subseteq \Sigma^*$ if, there is a rational number ϵ with $0 < \epsilon < 1/2$ such that, for every $w \in \Sigma^*$ (a) if $w \in A$, then $\mathcal{M}$ rejects w with probability at most ϵ and (b) if $w \notin A$, then $\mathcal{M}$ accepts w with probability at most ϵ.*

The class of sets which are decidable by a bounded error probabilistic Turing machine clocked in polynomial time is called BPP. Analogously, the class of sets which are decidable by a bounded error probabilistic Turing machine bounded in polynomial space is called $BPPSPACE$.

We intend to use $BPPSPACE$ as support for non-uniform classes, however, simply stating that $A \in BPPSPACE/F$ we may not be stating the more general and intuitive idea about the order of choice between the bounded error and advice. We provide then a different definition analogous to the definition for $BPP//F$ and $BPP//F\star$ (see [3]).

Definition 2. *Let F be a class of advice functions, we denote by $BPPSPACE//F$ the class of sets A for which there exist a probabilistic advice Turing machine $\mathcal{M}$ bounded in polynomial space, a constant ϵ with $0 < \epsilon < 1/2$ and an advice function $f \in F$ such that, for every $w \in \Sigma^*$, (a) if $w \in A$, then $\mathcal{M}$ rejects $\langle w, f(|w|)\rangle$ with probability at most ϵ and (b) if $w \notin A$, then $\mathcal{M}$ accepts $\langle w, f(|w|)\rangle$ with probability at most ϵ.*

Definition 3. *Let F be a class of advice functions, we denote by $BPPSPACE//F\star$ the class of sets A for which there exist a probabilistic advice Turing machine $\mathcal{M}$ bounded in polynomial space, a constant ϵ with $0 < \epsilon < 1/2$ and a prefix advice function $f \in F$ such that, for every $n \in \mathbb{N}$ and $w \in \Sigma^*$ with $|w| \leq n$, (a) if $w \in A$, then $\mathcal{M}$ rejects $\langle w, f(n)\rangle$ with probability at most ϵ and (b) if $w \notin A$, then $\mathcal{M}$ accepts $\langle w, f(n)\rangle$ with probability at most ϵ.*

2 Clocks

With the purpose of bounding the time for the experiment we must have an integrand digital component on the scatter machine. This digital component is a clock which receives as input the query word for the experiment (see [7]). The aim of the clock is to restrict the time of each run of the experiment so that the experiment do not keeps running indefinitely, a clock with this purpose is called time schedule.

As we intend to use analogue-digital machines bounded in polynomial space, we must establish which clocks we are able to construct with this restriction. We proved that the functions which are time constructible in polynomial space have at most an exponential growing rate, and we managed to construct an exponential clock which operates in polynomial space.

Proposition 2.1. *There exists a clock bounded in polynomial space such that, for an input of size n, ticks an exponential amount of transitions on n.*

We depict in Figure 1 a clock that halts after $2n \cdot 2^n + 1$ transitions for an input with size n, operating within space $\mathcal{O}(n)$. We use the input tape (the first tape) as a general work tape.

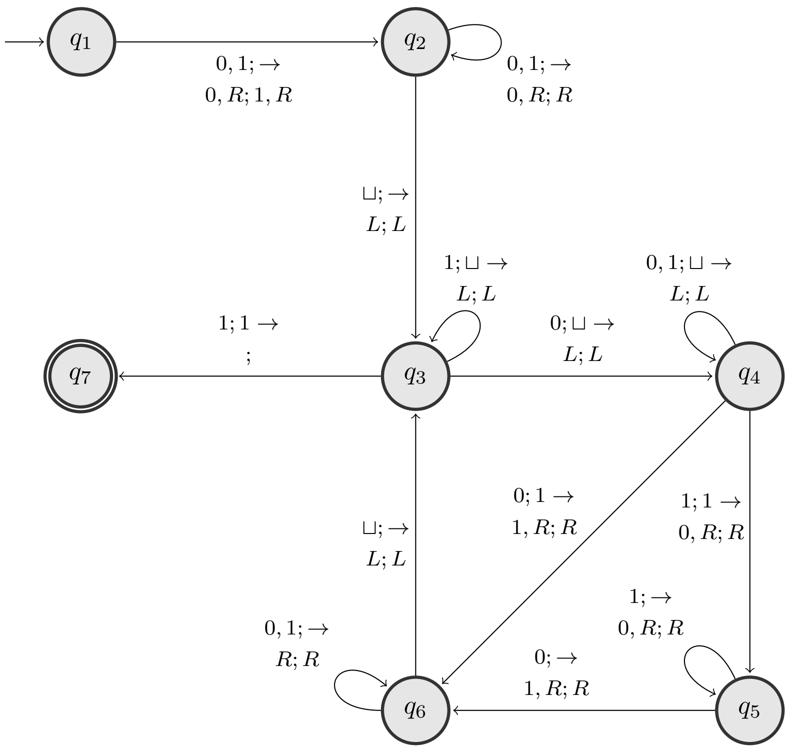

Figure 1: Clock for $f(n) = 2n \cdot 2^n + 1$

We also managed to construct a clock that halts after $2n \cdot 2^{n-(\lfloor \log n \rfloor + 1)} + 1$ transitions for an input with size n, also operating within space $\mathcal{O}(n)$.

Using a common proof technique we are able to prove the following Proposition:

Proposition 2.2. *In polynomial space, any clock ticks at most an exponential amount of transitions.*

3 Scatter experiment

In this work we aim to study a computation model consisting on a physical experiment working as an oracle for a Turing machine, so, an analogue-digital machine. In this section we describe the specific physical experiment we use to our computation model, both of its two versions, and we discuss the physical time for the second version of the experiment, the smooth wedge case.

The two versions of the specific physical experiment we intend to use, the scatter experiment, are the sharp wedge and the smooth wedge cases (see [6] and [5]). The scatter experiment aims to measure the position of the vertex in the wedge by shooting particles from a cannon and checking if they hit the wedge to the right or to the left of the vertex. The sharp scatter experiment is depicted in Figure 2, the scatter experiment with a sharp wedge.

This version of the experiment has a constant time to achieve its conclusions depending on the velocity of the particle v, and the distances d and d' from the cannon to the wedge and from one of the collecting boxes to the other, respectively.

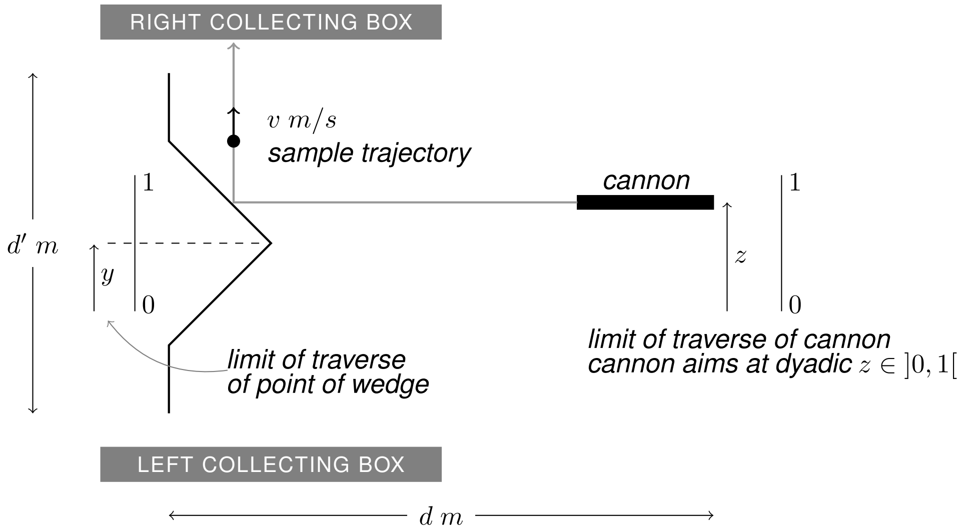

Figure 2: Sharp scatter experiment

The other version of the experiment is the smooth wedge case, where the experiment is almost the same but it has a smooth wedge, also with one "vertex", and needs the collecting boxes to be infinite.

In this version of the experiment we do not have a constant time bound for each run of the experiment as before, the time the experiment takes to achieve its conclusions depends on the difference between the vertex position and the cannon position, respectively y and z. For a curve describing the shape of the wedge being n times continuously differentiable near the vertex position, and with non-zero n^{th} derivative and vanishing derivatives until the $(n-1)^{th}$. With a shot performed at the position z, the time the experiment takes is $t(z)$, where

$$\frac{A}{|y-z|^{n-1}} \leq t(z) \leq \frac{B}{|y-z|^{n-1}}$$

for some real numbers A, $B > 0$, when $|y-z|$ is sufficiently small. (see [5] for details)

4 Scatter machine

In this section we describe how the analogue-digital machine works and we deal with an issue related with the space restriction to the machine, the communication protocol. The scatter machine is a Turing machine coupled with a physical experiment, and we must explain how is the information processed between the Turing machine and the experiment.

The physical experiment, when set to perform a run of the experiment, achieve one of three possible conclusions, the particle is detected in the right collecting box, the particle is detected in the left collecting box or the particle is not detected in any of the boxes. The Turing machine has then three distinguished states in its finite control, the 'right' state, the 'left' state and the 'timeout' state, and the scatter experiment instructs the Turing machine to perform a transition to one of the answer states according to the conclusion of the experiment.

The Turing machine, aiming to set a run for the experiment, must set the parameters for the physical experiment, writing a word in a distinguished tape, the query tape. We assume the physical experiment can be set with three different precision assumptions, usually used on the subject, the infinite precision, the arbitrary precision and the fixed precision (see [8]), and for each one of these assumptions we have a different communication protocol.

The protocols dictate how the experiment behaves after being queried with a specific query word, the word specifies the cannon position, a real number number $z \in]0,1[$, which should be considered as $z = 0.q_1q_2\cdots q_n$, where q_i is th i^{th} digit of q, so, for a query word q the considered cannon position is $z = 0.q$.

Protocol 1. Error-free protocol
Given a word q in the query tape, the experiment sets the cannon position to the real number z (using an infinite precision).

Protocol 2. Error-prone arbitrary precision protocol
Given a word q in the query tape, the experiment sets the cannon position, within a uniform distribution, to a real number in the interval $]z - 2^{|q|}, z + 2^{|q|}[$ (using an arbitrary precision).

Protocol 3. Error-prone finite precision protocol
Given a word q in the query tape, the experiment sets the cannon position, within a uniform distribution, to a real number in the interval $]z - \xi, z + \xi[$, where ξ is a fixed positive real number (using a fixed precision).

These are the three communication protocols usually considered on the subject and already used to the study of polynomial time restrictions on analogue-digital machines (see, for example, [3, 4, 1]). In the case of polynomial space restrictions the query word is bounded by the space restriction to the machine which may restrict the power we can achieve with the scatter experiment. We then considered a different version of the protocol, the space-generalized protocol (or generalized protocol), which consists on do not restrict the space of the query tape, with the provision that this tape must not be used to general scratch computations. This is an useful idea to other subjects, as for example the space bounded reducibility (see [2], §3.8) where the output tape is not space restricted.

We then have an analogue-digital machine in which the transition function can be partitioned in two parts, a part which can read any tape but can only write in the query tape and transit to the 'query' state and another part which cannot read the query tape. The protocols are exactly the same for each precision assumption once the query word is given, the difference is in the length properties of the query word.

We can then introduce the scatter machine for all its different versions. We call sharp scatter machine (ShSM for short) and smooth scatter machine (SmSM for short) to the scatter machine with the sharp wedge and the smooth wedge versions of the experiment, respectively. We call error-free, error-prone arbitrary precision and error-prone finite precision scatter machine to a scatter machine using the infinite, the arbitrary and the fixed precisions, respectively. Finally we call, when not clear by the context, standard scatter machine and generalized scatter machine to a scatter machine with the standard and the generalized communication protocols, respectively.

Note that, for the time restrictions cases the generalized version of the protocol is not useful as we obtain as a corollary of Proposition 4.1 that a scatter machine clocked in polynomial time obtain the same results with both the standard and the generalized protocols.

Proposition 4.1. *A set $A \in \Sigma^*$ is decidable by a generalized scatter machine which is not space restricted if and only if is decidable by a standard scatter machine.*

With different assumptions on the precision to set the experiment we can obtain different kinds of analogue-digital machines, namely the deterministic and the probabilistic kind. The deterministic scatter machine has an acceptance criterion as the deterministic Turing machine and the probabilistic scatter machine follows the bounded error probabilistic decision criterion (see Definition 1).

The probabilistic scatter machines do not need to have a probabilistic digital component as we can use the probabilistic nature of the oracle to simulate the probabilistic nature of a probabilistic Turing machine. This subject is already discussed in [3], for example, for polynomial time restrictions, and we proved that an analogous technique works for the polynomial space restriction. As discussed in [4], for example, we can also simulate the answers from the oracle with a general Turing machine, both in the deterministic and probabilistic cases.

5 Computational power

We have now all we need to study the computational power of the scatter machine bounded in polynomial space. We aim to establish which complexity classes can a scatter machine compute. For both the sharp wedge and the smooth wedge cases, with both the standard protocol and the generalized protocol, and for each of the three precision assumptions.

To the lower bound results we mostly use adaptations of the proof techniques used in [3] and [1]. The idea is to simulate an advice Turing machine with a scatter machine choosing the vertex position which codes the information in the advice function.

To the upper bound results we use adaptations of the proof techniques used in [4] and [1]. The idea is to simulate a scatter machine with a general advice Turing machine, coding the vertex position, or the relevant parameters from the experiment, in the advice function.

Using the standard communication protocol we obtain the following results for the sharp wedge version of the experiment:

Theorem 1. *A set $A \subseteq \Sigma^*$ is decidable by an error-free ShSM in polynomial space if and only if $A \in PSPACE/poly$.*

Theorem 2. *A set $A \subseteq \Sigma^*$ is decidable by an error-prone arbitrary precision ShSM in polynomial space if and only if $A \in BPPSPACE//poly$.*

Theorem 3. *A set $A \subseteq \Sigma^*$ is decidable by an error-prone arbitrary precision ShSM in polynomial space if and only if $A \in BPPSPACE//poly$.*

For the smooth wedge version of the experiment we obtained the following results:

Theorem 4. *A set $A \subseteq \Sigma^*$ is decidable by an error-free SmSM in polynomial space, using an exponential time schedule, if and only if $A \in PSPACE/poly$.*

Theorem 5. *A set $A \subseteq \Sigma^*$ is decidable by an error-prone arbitrary precision SmSM in polynomial space, using an exponential time schedule, if and only if $A \in BPPSPACE//poly$.*

Theorem 6. *A set $A \subseteq \Sigma^*$ is decidable by an error-prone finite precision SmSM in polynomial space, using the time schedule, if and only if $A \in BPPSPACE//poly$.*

This results can then be summarized in the following tables:

	Infinite	Arbitrary	Fixed
Lower Bound	$PSPACE/poly$	$BPPSPACE//poly$	$BPPSPACE//poly$
Upper Bound	$PSPACE/poly$	$BPPSPACE//poly$	$BPPSPACE//poly$

Table 1: Standard communication protocol ShSM

	Infinite	Arbitrary	Fixed
Lower Bound	$PSPACE/poly$	$BPPSPACE//poly$	$BPPSPACE//poly$
Upper Bound	$PSPACE/poly$	$BPPSPACE//poly$	$BPPSPACE//poly$

Table 2: Standard communication protocol SmSM

With the generalized communication protocol we obtain the following results for the sharp wedge version of the experiment:

Theorem 7. *The sets decidable by an error-free ShSM in polynomial space are all the sets $A \in \Sigma^*$.*

Theorem 8. *The sets decidable by an error-prone arbitrary precision ShSM in polynomial space are all the sets $A \in \Sigma^*$.*

Theorem 9. *A set $A \subseteq \Sigma^*$ is decidable by an error-prone finite precision ShSM in polynomial space if and only if $A \in BPPSPACE//poly$.*

For the smooth wedge version of the experiment we have different computational powers according to if we do or do not use the time schedule. Using the time schedule we obtained the following results:

Theorem 10. *A set $A \subseteq \Sigma^*$ is decidable by an error-free SmSM in polynomial space, using an exponential time schedule, if and only if $A \in PSPACE/poly$.*

Theorem 11. *A set $A \subseteq \Sigma^*$ is decidable by an error-prone arbitrary precision SmSM in polynomial space, using an exponential time schedule, if and only if $A \in BPPSPACE//poly$.*

Theorem 12. *A set $A \subseteq \Sigma^*$ is decidable by an error-prone finite precision SmSM in polynomial space, using the time schedule, if and only if $A \in BPPSPACE//poly$.*

Without using the time schedule we obtained the following results:

Theorem 13. *The sets decidable by an error-free SmSM in polynomial space, without using the time schedule, are all the sets $A \in \Sigma^*$.*

Theorem 14. *The sets decidable by an error-prone arbitrary precision SmSM in polynomial space, without using the time schedule, are all the sets $A \in \Sigma^*$.*

With the fixed precision assumption we need to use the time schedule in order to obtain some useful information from the vertex position. We also need to use the time schedule with the arbitrary precision assumption to simulate the probabilistic nature of a probabilistic Turing machine, but we do not need the time schedule to obtain the information from the vertex position to use on the general digital computations.

The results with the generalized protocol can be summarized in the following tables:

	Infinite	**Arbitrary**	**Fixed**
Lower Bound	2^{Σ^*}	2^{Σ^*}	$BPPSPACE//poly$
Upper Bound	2^{Σ^*}	2^{Σ^*}	$BPPSPACE//poly$

Table 3: Generalized communication protocol ShSM

	Infinite	**Arbitrary**	**Fixed**
Lower Bound with time schedule	$PSPACE/poly$	$BPPSPACE//poly$	$BPPSPACE//poly$
Lower Bound without time schedule	2^{Σ^*}	2^{Σ^*}	—
Upper Bound	2^{Σ^*}	2^{Σ^*}	$BPPSPACE//poly$

Table 4: Generalized communication protocol SmSM

6 Conclusion

This work intends to rise the established computational power of the scatter machine. The scatter machine were already studied concerning polynomial time restrictions and we studied it concerning polynomial space restrictions.

We introduced a uniform probabilistic complexity class ($BPPSPACE$). We construct an exponential clock operating in polynomial space and proved that this is the maximum growing rate function which is time constructible in polynomial space. We introduced a communication protocol for the scatter machine aiming to access the whole power of the scatter experiment with space bounded scatter machines. We were able to adapt the proof methods form the time restriction cases for the polynomial space restriction, and to use adaptations to the former simulation techniques in this case.

We then established the computational power of the polynomial space bounded scatter machine. We obtained different results for the smooth wedge version of the experiment with the generalized protocol, according to if we use at most an exponential time or not, equivalently, if we use the time schedule or not.

We suggest for future work, for example, the study of the scatter machine bounded in polynomial space and clocked in quasi-exponential time. Where the notation in the conclusion of [4] can be useful with a new parameter for the space restriction to the machine.

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