TÉCNICO LISBOA

Valuation and Financing of Early Stage Technological Companies:

Analysis with a Real Options Structural Model

Daniel Freire da Fonseca

Thesis to obtain the Master of Science Degree in

Industrial Engineering and Management

Supervisor: Prof. João Manuel Marcelino Dias Zambujal de Oliveira

Examination Committee

Chairperson: Prof. Francisco Miguel Garcia Gonçalves de Lima

Members of the Committee: Prof. Maria Margarida Martelo Catalão Lopes de Oliveira Pires Pina

Prof. Maria Isabel Craveiro Pedro

December 2015

Acknowledgments

Solitary though a work like this may be, it would not be, at all, without your help.

A sincere thank you to Professor João Zambujal de Oliveira, who guided me from the absence of a project, through too many ideas and finally to a viable work.

To you, Raquel, I owe nothing short of unending gratitude for your tireless support and contribution.

A special thank you to Vasco Rato for his help in improving my programming skills.

Finally to my parents, who sustain and support me, making possible my academic journey; thankfully completed.

Abstract

This dissertation assesses the impact of a mixed capital structure model, on the valuation of early stage technological companies (ESTCs).

Traditional valuation methods are not suitable for the analysis of ESTCs, not being able to correctly assess the impacts of the high uncertainty and opportunities associated with such ventures. It is due to these same particularities, common to most ESTCs, that these companies face great limitations in the access of debt financing.

The model that is proposed uses the EBIT as the state variable. Modelling the EBIT growth as an arithmetic Brownian motion enables the possibility of negative cash flows. This is essential when dealing with early stage companies, which excludes most of the real options models, developed to study market traded assets, from being viable in assessing the value of ESTCs. The model is developed within a static capital structure framework, which restricts the issuance of debt to a single stage. The proposed model, built on these assumptions, is a flexible base that can be easily implemented and expanded.

A case study based on Fitbit Inc, was conducted, the first application of a structural model to the valuation of an ESTC, in a reality based setting. The results of the model show that the valuation of the company is related with its capital structure and the case study provides a valuation for Fitbit's equity consistent with its market value.

Keywords: Early stage companies; Real options analysis (ROA); Structural model; Arithmetic Brownian motion (ABM); Investment project analysis; Capital structure.

Resumo

A presente dissertação estuda o impacto de uma estrutura de capital mista, na avaliação de empresas tecnológicas em fase inicial de atividade (ETFIAs).

Os métodos tradicionais de avaliação não são os mais adequados à análise das ETFIAs, dado que não permitem estimar corretamente o impacto da elevada incerteza e grandes oportunidades de crescimento associadas a estas empresas. Estas mesmas especificidades, comuns à maioria das ETFIAs, que causam consideráveis restrições no acesso a dívida por parte destas empresas.

O modelo proposto usa o EBIT como variável de estado. Modelar o crescimento do EBIT como um movimento Browniano aritmético permite a existência de valores negativos. Este fator é essencial quando se analisam empresas em fase inicial de atividade e exclui grande parte dos modelos de opções reais, desenvolvidos para estudar ativos negociados nos mercados, de entre as opções viáveis para avaliar ETFIAs. O modelo é desenvolvido assumindo uma estrutura de capital estática, o que restringe a emissão de dívida a uma única fase. O modelo assim definido é uma base flexível, fácil de implementar e expandir.

Foi desenvolvido um caso de estudo baseado na Fitbit Inc. a primeira aplicação de um modelo estrutural à avaliação financeira de uma ETFIA, baseado em dados reais. Os resultados do modelo mostram que a avaliação da empresa está relacionada com a sua estrutura de capital, e a avaliação feita pelo modelo para o capital próprio da Fitbit é consistente com o seu valor de mercado

Palavras-chave: Empresas em fase inicial de atividade; Análise de opções reais (AOR); Modelos estruturais; Movimento Browniano aritmético; Análise de projetos de investimento; Estrutura de capital.

Table of Contents

Figures List

Tables List

Equations List

Acronyms List

ABM – arithmetic Brownian motion

BOPM – binomial option pricing model

BSOPM – Black-Scholes option pricing model

CAPM – capital asset pricing model

CF – cash flow

DCF – discounted cash flow

DER – debt to equity ratio

EBIT – earnings before interest and tax

ESTC – early stage technological company

GAAP – generally accepted accounting principles

GBM – geometric Brownian motion

IP – intellectual property

IPO – initial public offering

NPV – net present value

OI – operating income

ROA – real options analysis

ROV – real options valuation

SDE – stochastic differential equation

WACC – weighted average cost of capital

1. Introduction

1.1. Context and Relevance

Entrepreneurship creates a significant contribution to economic growth in the global economy (Schumpeter, 1934). Globalization shifted the competitive advantage towards knowledge-based economic activity, making it possible for entrepreneurs to re-emerge as a vital factor in modern economies (Audretsch & Thurik, 2001). The fulfillment of this re-emergence depends on the existence of the conditions necessary for new businesses to thrive, the unavailability of credit to early stage companies is a real threat to economic growth. When considering small and medium companies there are significant asymmetries in what concerns accessing finance. Repeatedly, businesses that offer the highest growth possibilities are the ones that face greater difficulties in this endeavor. The cause being that these enterprises have a greater reliance on intangible assets (Martin & Hartley, 2006). Within the last quarter century, intangible assets became the leading asset class, its weight growing from around 32% of the market value of the S&P 500 Index in 1985 to 80% in 2005 as is shown in Figure 1 (Malackowski, 2004). The capital markets and financial system evolved in coherence with this profound transformation in the nature of the assets and structures of companies. The capital markets seek to efficiently allocate capital to promising enterprises. For that connection to occur it’s necessary that the tools required to identify and measure value in those enterprises are available and are deemed credible. The challenge is to create funding mechanisms that can ensure the flow of capital to enterprises that can sustain growth and reward the investors. This cannot happen if willing institutions and individuals, with available resources, cannot identify reliably the true value inherent to new, viable, businesses. Companies, in turn, need to be able to access capital in new ways to finance innovation and expand their businesses (Ellis, 2009).

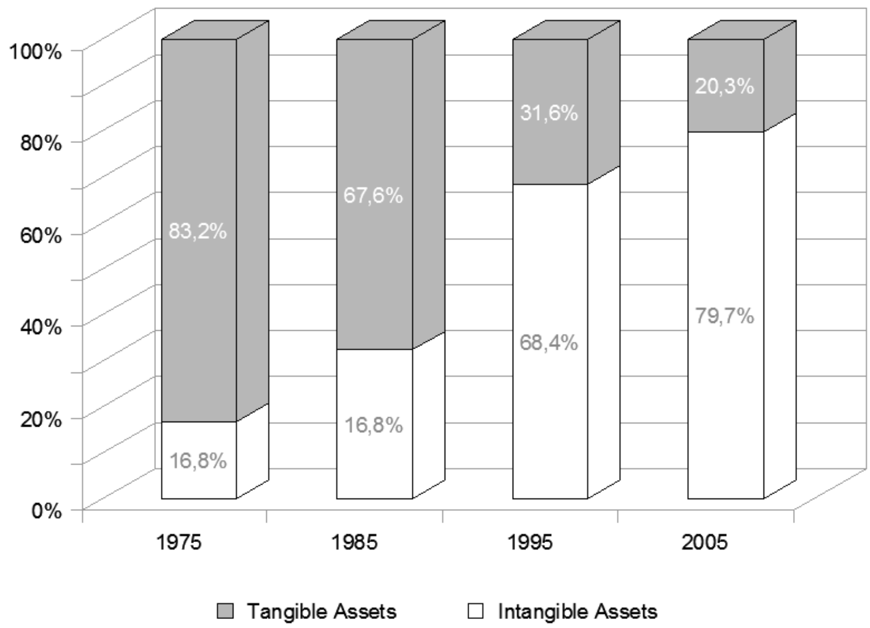

Figure 1 – Components of S&P 500 Market Value (Adapted from: Malakawski, 2004)

The increasing reliance on intangible assets that is particularly relevant in highly technological companies, creates a lag effect between the development of those assets and the incorporation of its value in the valuation of the companies. The benefits generated by those assets are only evident when, directly or not, they result in increased earnings. This has an increased effect in debt financing, due to the lack of flexibility in the valuation methods that debtors use, effectively reducing the offer of financing sources that entrepreneurs can access.

1.2. Objectives and Motivation

The objective of this work is to study the process of valuation of early stage technological companies (ESTCs), in a way that provides some guidance not only in regards to their total value but also considering their capital structure. With this work the author expects to contribute to the literature associated with the study of the valuation of ESTCs. By incorporating debt in the capital structure and valuation process, not only a maximization of the value of the company occurs, but also, more information is created which can be used to justify the access to financing. Instead of developing a method dedicated to the valuation of the IP, the model that will be developed and tested in this work valuates the company with regards to the growth of its cash flows, particularly the earnings before interest and tax (EBIT). The model is grounded on real options analysis (ROA), which enables enough flexibility for the model to be expanded and adapted not only to companies with different characteristics but particularly to include investment options such as expansions, acquisitions, delays, among others, making it useful both for investors and management. Real options analysis also provides adequate properties to the study of ESTCs by the incorporation of the volatility in the model, which reflects the uncertainty inherent to the underlying assets on the results of the valuation. Evidence is provided to support that, for this particular type of company, bank loans are uncommon and difficult to obtain, and that the use of IP assets to back such loans is a viable, and frequently the only, way to obtain approval. Providing a model for the valuation of these companies that demonstrates the advantages of debt can have a positive effect in increasing the offer of financing sources. Investors will be able to benefit from a more comprehensive valuation of investment projects involving ESTCs. For the field of study of Real Options Structural Analysis a case study will be added to the available literature with particular properties that, within the extension of the author’s knowledge upon research, were not yet explored, particularly the application of a structural model to an ESTC.

2. Problem Definition

This work explores the effect of a mixed capital structure, financed by both debt and equity, on the valuation of ESTCs. A brief review of the methods available to valuate properly registered IP, is made to indicate the relevance of intangible assets in enabling both debt and equity financing, the setting that is used in the model. The development and derivation of the model, relying on the existent literature, id made with enough flexibility to allow for the incorporation of further developments and so that it can be adapted to the specifications of each venture. An application of the model is then carried out, with the study of an existing company, Fitbit. The objective of this case study is to validate the use of the model in the context of ESTCs, also illustrating the challenges in valuating early stage firms and suggesting solutions for the unavoidable limitations with a critical analysis of the results.

The model makes use of the real options valuation (ROV) framework, to demonstrate the advantages and possibilities that are enabled by the proper exploration of intangible assets in obtaining debt financing and thus maximizing the value of the company.

In this work ESTCs are defined as enterprises in a stage of development that is ahead of the end of the Start-Up Round and the Second Round, or B Series funding; the company in the case study is evaluated at the moment of the initial public offering (IPO); a more extensive analysis of the development cycle of ventures is presented in the literature review chapter. It is also assumed that ESTCs are high-growth capable companies, with most relevant assets being IP assets. A set of companies defined in such a manner might comprise software, technology, biotechnology or life-sciences companies, among others. These are the base premises on which Fitbit Inc. was chosen for the practical example, being a technological company that relies heavily on the ecosystem created by the integration of the hardware they sell with the digital platform and user interface that accompanies it. Throughout this work, evidence is quoted and presented that deem this set of characteristics to be an accurate representation of highly technological startups as a general group.

3. Literature Review

The literature review for this project encompasses several different topics and it should provide sufficient background to justify the decisions made throughout this work as well as provide assistance in its interpretation.

3.1. Investment Cycle and Early Stage Technological Companies

There are multiple definitions of the life cycle associated with the initial stages of companies. Here follows a description of the early life of a company based on the investment cycle that such companies undergo. This cycle comprehends six separate phases (Bruno and Tyebjee, 1985), companies may undergo financing processes in one, several or all of these steps.

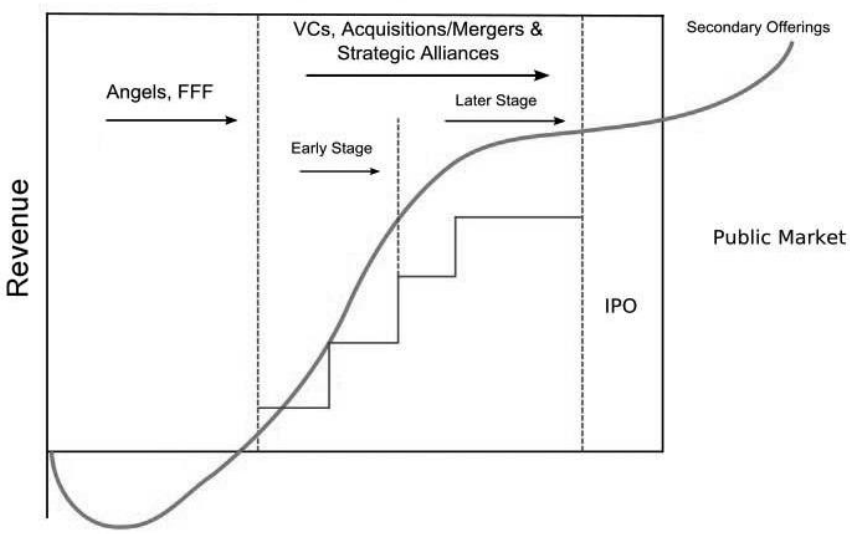

Figure 2 – Investment Cycle (Adapted from Wikipedia Commons, 2009)

- Seed-Money Stage – Usually a small amount of capital, it is normally used to prove a concept or develop a product or prototype.
- Start-Up Funding - Financing for firms in the first years of activity. The application of the funds is frequently related to marketing and product development costs.
- First-Round Financing – Additional capital to sustain the activity of the company. Firms are expected to manufacturing and/or sales at this stage.
- Second-Round Financing - Funds used for working capital for companies that are active and generating revenue but still not making profits.
- Third-Round Financing – Most companies that reach this phase are expected to be breaking-even. This stage is also called mezzanine financing.
- Fourth-Round Financing – Funding round that antecedes the initial public offering. Also known as bridge financing.

Each of these rounds of financing involves one or more entities. It is frequent that funders of the first rounds cannot keep investing in later stages as the monetary amounts and risk levels change considerably along the evolution of the company. For the Seed Stage the most usual sources of financing are the proverbial three F's: Family, Friends and Fools. At this stage the risks are usually too great for institutional funding and the sums involved can the obtained by fundraising from personal relations.

Business Angels on the other hand tend to intervene in the Start-Up stage or possibly in First-Round Financing. They are investment professionals who generally invest their own funds. Angel investors are frequently well-connected, wealthy individuals who can positively influence the results of the enterprise.

The following rounds are usually accomplished by access to venture capital, a professionally managed pool of capital that is invested in equity-related securities of private ventures at different stages of their development (Sahlman, 1990). These high risk investments usually seek to generate return through an eventual realization event, such as an initial public offering (IPO) or a trade sale. The typical venture capital investments are the First and Second Rounds, also called Series A and B respectively. These funds frequently have the network to source new highly qualified team members and to enable relations with flagship customers. Venture Capital investment frequently involves the acquisition of a considerable percentage of the company's equity, this transactions are complex and involve more than just the mere commercialization of property: investors conduct lengthy due diligence, take board seats and secure rights such as liquidation preferences or tag along, drag along rights. It is common for early investors to expand their position in the company if it survives to posterior phases. In 2012, venture capital firms financed 3148 ventures, roughly two thirds of which were follow-up investments into current portfolio companies (Rockthepost, 2014).

Each business and research area has its own particularities when it comes to relevance of IP, funding necessities and sources; without discarding such information, the author still finds it appropriate to extend the scope of this work to what has been defined as the ESTC sector, for the particularities and scenarios that are used and discussed in this study are deemed to be sufficiently comprehensive.

In addition to what has been defined as the characteristics of ESTCs this work adopts the considerations of Domodoran (2009) for young companies:

- No history: being recent ventures, young companies have obviously limited data on previous operations and financials.
- Small or no revenues and operating losses: these companies are still incurring in set-up expenses, some may not even have started to generate revenues. This usually adds up to substantial operating losses.
- Dependent on private equity: they are almost entirely dependent on private financing rather than public markets.
- Many don't survive: many studies prove that the survival rates of young enterprises are very low.
- Multiple claims on equity: the repeated funding rounds expose first investors to the risk of dilution. To protect their positions, equity investors often negotiate terms that include first claims on revenues and liquidation along with control or veto rights.

- Investments are illiquid: since the equity is privately held it represents a less liquid asset than that of publicly traded firms.

3.2. Intangible Assets and Intellectual Property

Intangible assets are knowledge-based assets, form part of a company's intellectual capital, and are sources of firm-level differentiation and competitive advantage (Martin and Hartley, 2006). Intangible assets are not usually included in accounting balance sheets (Citron et al., 2004), either because they do not comply with conventional definitions of assets or because the definition and measuring of their value falls out of standard accounting practices. They comprise proprietary knowledge, skills and relationships that form the basis of the uniqueness of each company and, by nature, some of them cannot easily be acquired in market-based transactions.

The EU Meritum project identifies three distinct categories of intangible assets: 'human capital' which refers to the knowledge and skills of employees; 'relational capital', referring to the supply chain, research and business networks that are available to the firm; and 'structural capital', the organizational competencies of the firm, such as its intellectual property assets.

IP is thusly part of a company's intangible assets. It refers to creations of the mind: inventions, literary and artistic works, symbols, names and images used in commerce. Intellectual property is divided into two categories: Industrial Property includes patents for inventions, trademarks, industrial designs and geographical indications. Copyright covers literary works, films, music, artistic works and architectural design (WIPO, n.d.). The key particularity of intellectual property is that it has been granted specific legal protection and recognition under intellectual property laws.

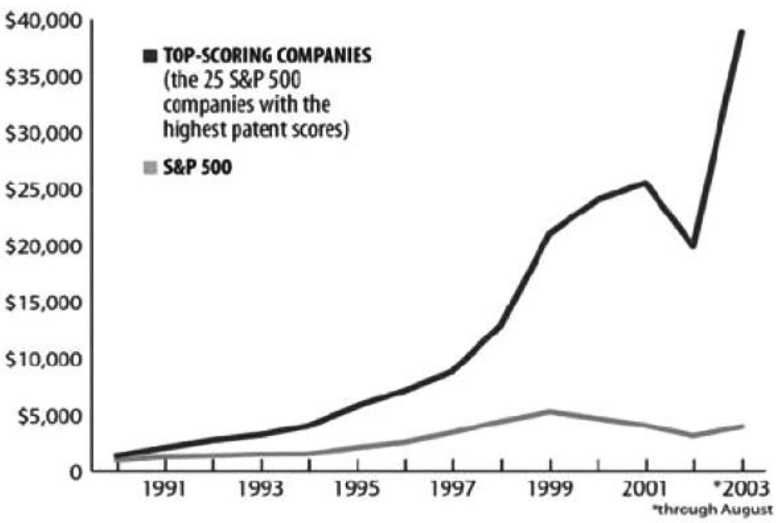

Figure 3 – Performance of companies with the highest patent scores Versus S&P 500 (Adapted from: Malakawski, 2004)

Registering and reporting intangible assets, particularly IP, can provide significant benefits for ESTCs. The European Union's 2006 RICARDIS report encouraged policy initiatives to promote the standardization of the reporting guidelines for research-intensive small and medium enterprises. The report acknowledges the importance of those policies in the effort to increase the access of those companies to financing. A start-up firm's decision to patent is associated with higher annual asset growth of between 8% and 27%. Also, the decision to patent is associated with substantially improved chances to survive the first five years of a start-up's existence (Helmers and Rogers, 2011).

3.3. Sources of Finance

Research has shown that the general constraint for financing resides on the supply-side. The lack of needed capital creates barriers to innovation, condemning young and promising ventures into early abandonment (Blaug and Lekhi, 2009). From the same source were collected several evidences present in the literature that support such claims: in a survey of 171 small and medium enterprises from the United Kingdom, Westhead and Storey (1997) found that for the majority of sample, financial constraints were a cause for limited investment in innovation and constrained development, with increased incidence on companies with intensive R%D. A research by Guidici and Paleari (2000), found that 90% of the surveyed entrepreneurs did not seek debt financing from banks due to a belief that their assets would be undervalued. In another study, which analyses a sample of 130 technological firms, Ullah and Taylor (2007) report that almost 80 per cent of firms are finance constrained. Studies with a scope comprised of companies in the European Union arrived to similar results: Peneder (2008) uses the 2004 European Community Innovation Survey to indicate that small businesses with heavy reliance on intangible assets are the most susceptible to suffer from insufficient funding.

It is also important to note that informational asymmetries also play a part in perturbing the access to finance. Investors and other sources of finance have been found to provide inadequate information to inexperienced entrepreneurs regarding their financial options (Howorth, 2001) with negative consequences on the trust levels between both parties (Berggren et al, 2000).

This reality makes it necessary to explore and discuss the financing alternatives that are available and the impacts that they can have in the performance of a company. As this work considers the case of ESTCs, in which IP assets are of extreme relevance, this review is limited to that scope. Thus the possible sources of finance are analyzed in a context of IP financing, which is intended as the use of these intangible assets for raising capital, be it equity or debt. Loans extended on IP assets are a practice carried out mainly by banks, while venture capitalists and business angels usually provide capital in return for company shares and are referred to also as private equity. Securitization and leaseback consist in using IP assets as security to support the borrower's solvency.

3.3.1. Equity Financing

Equity financing refers to the sale of stake on the ownership of the company, it is intended as a way to obtain financing for business purposes. In the context of ESTCs, which are privately held companies, this usually translates into the six initial funding rounds, led by venture capital firms, business angels or crowdfunding.

3.3.2. IP Focused Debt Financing

Debt financing allows companies to raise capital without diluting the equity of the investors. The capital obtained this way is secured by the assets of the company, in the case of ESTCs this is frequently restricted to IP assets. Debt also has a fiscal advantage over equity. Intangible asset backed loans leverage a portfolio of IP or other intangible assets to secure a loan (Ellis, 2009). Companies with the profile of ESTCs usually face unsurpassable difficulties to obtain debt financing, this is particularly true since the 2008 credit market crisis (Pavlov, 2009). The uncertainty surrounding intangible asset valuation also creates barriers for this kind of deals, as it has an essential role in debt financing, firstly in measuring present and future cash flows for the purpose of servicing the loan repayment, secondly in assessing the value to cover the investment in case of default (Ellis 2009). Although difficult to obtain, loans backed by intangibles have been practiced for centuries. The first trade secrets case in the United States, in 1837, involved debt secured in part by chocolate-making process (Dulken, 2001).

IP Securitizations are a particular form of IP backed debt financing. IP Securitization is a relatively new procedure, with many national level legal particularities. It is becoming more popular and frequent but, at the present, the process is still very complex and of difficult access to small and early stage companies (Tondo-Kramer, 2010). This instrument is used to raise capital as financial security backed by cash flows such as royalties. In this very complex type of financing, the payments stream from IP assets is converted into marketable securities placed with investors. The interest models associated with securitizations can be based on royalties or on an interest over the income, in either case the result for the company is an anticipation of future earnings. The selling companies are achieving an hedge to the risk that is implicit in the uncertainty of future cash flow from the revenue; the estimation of the future viability of the cash flow generating asset is tied to the compensation that the investor demands for bearing the risk. The creation of devices such as IP auctions has decreased the risk to investors, allowing more speculative investments, mainly those based on revenue interest securitizations (Jung and Tamisiea, 2009).

3.3.3. IP Sale and Lease Back

The sale lease-back is a short-term funding strategy that consists in the sale of a portfolio of IP to the investor, and a contract agreement to license the IP to the selling firm, which allows the utilization of the assets the activities of the company, in exchange for a fee. Initially the IP owner transfers the ownership of an IP asset to a leasing company, using the procedures to reinvest in the business. Simultaneously, a lease of the same asset is made to its former owner, under payment of a loan interest. At the end of the leasing contract period, the lessee has the option to buy back the ownership of the asset at a fixed price, by exercising a purchase option (Ellis 2009).

3.4. IP Valuation

Methods for valuation of IP can be divided in two fundamentally distinct groups: quantitative and qualitative methods. The latter group is not strictly speaking for valuation, since those methods don't output a monetary value. They provide, instead, a value guide based on ratings or scores and are thus more correctly identified with evaluation than with valuation**.** Given this explanation, this review will focus on quantitative methods, which attempt to calculate the monetary value of the IP. There are four approaches to achieve this purpose: cost approach, market approach, income approach and real options approach (Mard, 2000; Pavri, 1999, cited in Sudarsanam and Sorwar, 2003).

3.4.1. Cost Approach

Cost based methods are based on the economic principle of substitution, which states that the value of an asset is equal to the cost incurred in the process of creating an equiparable asset, either internally or externally. The cost approach can be carried out by measuring the costs in several ways, namely:

- Historic Cost – the costs to be considered are those that were incurred by the company at the time that the asset was developed.
- Replication Cost – which considers the investment needed to create the same asset at the present moment, taking into account the entire process of research and development.
- Replacement Cost – the cost of replicating the asset, in its function or utility.

The use of cost based approaches is linked to accounting, as it is the only method of valuation that is in accordance with accounting principles. In lack of an alternative an incentive is created by the opportunity to increase awareness towards one's IP in the financial reports of the company.

Apart from its use in accounting, which in itself is an indicator of the inadequacy of the standard financial report tools in what concerns IP, there is little advantage in pursuing methods that assume an equivalence between cost and value. By definition these methods can't take into consideration future benefits that can come from the IP, thus creating an unrealistic representation in which an asset that was costlier to develop is more valuable.

3.4.2. Market Approach

The market approach is straightforward in its definition. The value of the asset is what other actors in the marketplace deemed it to be. There are multiple methods to establish a market approach valuation: The price achieved by bidding in a public auction; the market value achieved by a comparable asset in a previous transaction or the present value of the royalty fees in similar agreements. The pitfall of this approach is not in its definition or in the results that it produces but in the conditions that it requires to function, since there is rarely an active market in which are available public information, price and comparability (AJPark, n.d.).

3.4.3. Income Approach

Income based methods measure the potential future benefits of the IP assets and use them to assess the value of those assets. There are several income based valuation methods: the discounted cash flow model and others more specific to this category of assets, such as the risk adjusted net present value or relief from royalty methods.

a. Discounted Cash Flow model

The DCF model is the widespread of the income based valuation approaches. The model outputs the value of the assets by computing the present value of future cash-flows discounted at a determinate rate, specific to each market and asset. An extensive description of this method of valuation is presented in section 3.5.

b. Relief from Royalty method

This method can be related to the "IP Sale and lease back" explained in section 3.3.3. It involves an assessment of the fee that would have to be payed to license an equivalent IP portfolio from another source. The royalty represents the fee, which would be paid to the licensor if a hypothetical arrangement took place. The method assumes that the value of the IP is defined as the royalty other companies would pay to use it. Estimating this royalty rate is only the first step, a reliable forecast is also required in order to estimate the direct cash flows from the IP asset. As with other income approaches, the royalty rates are then discounted through an appropriated discount rate.

As has been previously stated, income approaches are the most commonly used. Despite this fact, they are also much contested in their ability to produce realistic and fair results. The shortfall of this methods is the unavoidable uncertainty involved in estimating the future cash flows and overall behavior of the variables involved. A further limitation is that the individual sources of risk are not necessarily considered, since the responsibility to incorporate those in the assessment, falls on the analyst and its interpretation of the discount rate. The advantage of this approach is that, once the required data has been obtained it is very simple to assess the value of the assets.

3.4.4. Real Options Approach

While not all intangible assets share real option characteristics, many of them are in essence real options that firms create through their activities, organic investments or acquisitions (Sudarsanam and Marr, 2003). Among these we can consider investments in: human resources, such as education; information technology; research and development and intellectual property. These investments are rarely considered as such, we tend to think of them, and accounting defines them, as expenses. But although they generally don't generate immediate payoffs, many of them come to produce indirect payoffs and some even generate direct inbound cash flows. At least these intangibles create the opportunity for managerial flexibility. When considered as investments it's of no logic leap to understand that they fit under the characteristics that were defined in the previous chapter for ROA. From this conclusion the approaches for ROV explored previously apply to the valuation of IP and intangibles in general.

3.5. Investment Project Valuation

In this section a review of a traditional valuation model is conducted, the Discounted Cash Flow (DCF) model, regarding its application to investment projects. Advanced valuation models, particularly those grounded on real options analysis, are covered in the next section.

3.5.1. Discounted Cash Flow

The DFC model is so embed in the financial world that it has been referred as the heart of most corporate capital-budgeting systems (Luehrman, 1998).

3.5.1.1. Net Present Value

This method is based on the net present value (NPV) of the future cash flows (*CF*) that are expected to be generated by the investment, which are discounted at an appropriate discount rate (*r*).

$$NPV = \sum_{t=0}^{n} \frac{CF_t}{(1+r)^t}$$

Equation 1 – Net Present Value

At this step predictions of the cash flows are required, for the duration of the investment project**.** If the underlying asset for the project is expected to continue to generate cash flows after the end of the project, an estimate for them can be made and discounted – this is called the Terminal Value. The value of the underlying asset can be defined as:

$$Valuation = \sum_{t=0}^{n} \frac{CF_t}{(1+r)^t} + Terminal\ Value$$

Equation 2 – Value of an asset using DCF

3.5.1.2. Weighted Average Cost of Capital

An appropriate discount rate is required to discount the cash flows. Attention is due to the matching of the cash flows and the discount rate that is used, this being especially relevant when valuing companies. An example that arises frequently is the consideration between CFs to the firm and CFs to the equity, with the consequence of considering a discount rate that has implied the cost of capital associated with the entire capital structure or one that only considers equity. The weighted average cost of capital (WACC) is the most widely accepted discount rate (Pratt and Grabowski, 2014).

$$WACC = \frac{E}{V} r_e + \frac{D}{V} r_d (1 - t)$$

Equation 3 – Weighted Average Cost of Capital

E – Equity Value; D- Debt; V – Enterprise Value = E + D; r_e – cost of equity; r_d – unadjusted cost of debt; t – tax rate

3.5.1.3. Cost of Equity

The cost of equity (r_e) is calculated with the use of the capital asset pricing model (CAPM). This model represents the return rate that investors require from the investment, which should be interpreted as the combination of two factors: the compensation of time and risk.

According to the CAPM, the cost of equity can be estimated as:

$$r_e = r_f + (r_m - r_f)\beta$$

Equation 4 – Cost of Equity

r_f – risk free rate; $\mathrm{r_m}$ – market return

The risk free rate (r_f) is the rate of return associated with an investment were the actual return is equal to the expected return, it is a measure of the time value of capital, which implies no variance around the expected return (Domodaran, 2012). According to the same source the rate of a government bond (in the same currency as the investment) provides a good approximation of a true risk free rate. It is also referred that while a long term bond is a good standard there should be some correspondence between the duration of the investment and the maturity of the bond.

The market risk premium ($r_m - r_f$) is the reward required by investors to bear the average risk of investing in the market discounted of the risk free rate. Definition requires it to be greater than zero, increase with the risk aversion of the investors in that market and increase with the average risk in that market (Domodaran, 2012). The standard approach to calculate this premium is to use historical data: define a time period for the estimation; calculate average returns on a stock index during that period (r_m); calculate the difference between that rate and the risk free rate.

The variable beta (β) is the risk that holding the security will add to the investor's portfolio (Rhaiem et al, 2007). It is most frequently derived using linear regression analysis, where the return of the security ($r_{security}$) is the dependent variable and the market return (r_m) is the independent variable. The beta is the slope of the regression line.

$$\beta = \frac{Cov(r_{security}, r_m)}{Var(r_m)}$$

Equation 5 – Beta

This empirical determined input variable is also dependent on the company's historical level of leverage, because higher leverage ratios increase the shareholder's risk. Considering that a company's level of leverage can change considerably during a transaction, the beta might have to be adjusted for this change by unlevering and relevering to the new capital structure (Steiger, 2008).

With non-listed companies there isn't available data to compute the linear regression. For such cases, Damodaran (2012) offers a way to compute an accounting beta: estimating it by regressing the changes in the earnings of the firm against the same variations measured from a market index.

$$\Delta\, Earnings_{company} = a + b * \Delta\, Earnings_{market}$$

Equation 6 – Procedure for Accounting Beta

The lack of accounting data, measured with much less frequency than the prices of a market traded asset, limits the quality of this estimates (Damodaran, 2012). This limitation is increased for early stage companies, where accounting information is restricted to the short life of the firm.

3.5.1.4. Cost of Debt

The cost of debt ($r_d(1-t)$) is the interest rate that the market demands for the debt sold (Steiger, 2008). The author identifies three variables that influence the cost of debt: the base interest rates in the market, which reflect on the risk free rate; the default premium and the firm's effective tax rate. The cost of debt does not correspond to the interest rate that the company pays for the debt that it has inscribed on its books (Damodaran, 2012).

If the firm has currently traded bonds outstanding, the yield to maturity on a long-term bond can be used as the interest rate. Otherwise, if the company is rated by a credible entity that rating can be used to estimate a default spread. For companies that don't fulfill this requirements, two possible approaches are: using the interest rate from a recent long term bank loan or to estimate a synthetic rating to obtain a spread. In its simplest form, the rating can be estimated from the interest coverage ratio:

$$Interest\ Coverage\ Ratio = \frac{EBIT}{Interest\ Expenses}$$

Equation 7 – Interest Coverage Ratio

To calculate the equity and debt weights ($\frac{E}{V}$ and $\frac{D}{V}$) market values should be used. Since debt and equity are traded at market values, and the cost of capital is a measure of the cost of raising the necessary finance to conclude the investment, the WACC is better estimated using market weights (Damodaran, 2012).

3.6. Real Option Analysis

Methods like the DCF model fail to consider the options that are embedded in such actions as a decision to invest, or in making capital structure decisions. The net present value of a project, calculated with the DCF method, cannot incorporate the implied value of the options to delay, expand or abandon a project

(Damodaran, 2012). It also lacks a proper way to deal with uncertainty, and the measurement of the risk, represented by the β coefficient, is of limited application for assets that are not publicly traded. The same source warns of a lack of unanimity among theorists and practitioners for the way in which to use ROA: “some view it as a rhetorical tool that can be used to justify investment, financing and acquisition decisions” but do not recognize the capability to valuate those options with any precision. There are others who argue that quantitative estimation of the value of these options should be used, and built into the decision process.

An option can be defined generally as a right but not an obligation, at or before some specified time, to purchase or sell an underlying asset whose price is subject to some form of random variation. Most obviously though the underlying asset can be a share in a company whose price varies over time as a form of random walk variation (Pitkethly 1997).

First an introduction to the basic concepts of options and option pricing, following Domodaran (Domodaran, 2012). There are two types of options:

3.6.1. Call Options

Call options give the buyer of the option the right to buy the underlying asset at a fixed price, the exercise or strike price. This right can be exercised at any time, until the expiration date of the option. The option is only exercised if, at expiration, the value of the asset is higher than the exercise price. Otherwise it expires and the buyer of the option loses the price paid for it. If the option is exercised the investor buys the stock at the exercise price and the difference between the asset value and the exercise price represents the gross profit of the investment. Subtracting to the gross profit the value paid initially for the option the net profit is achieved.

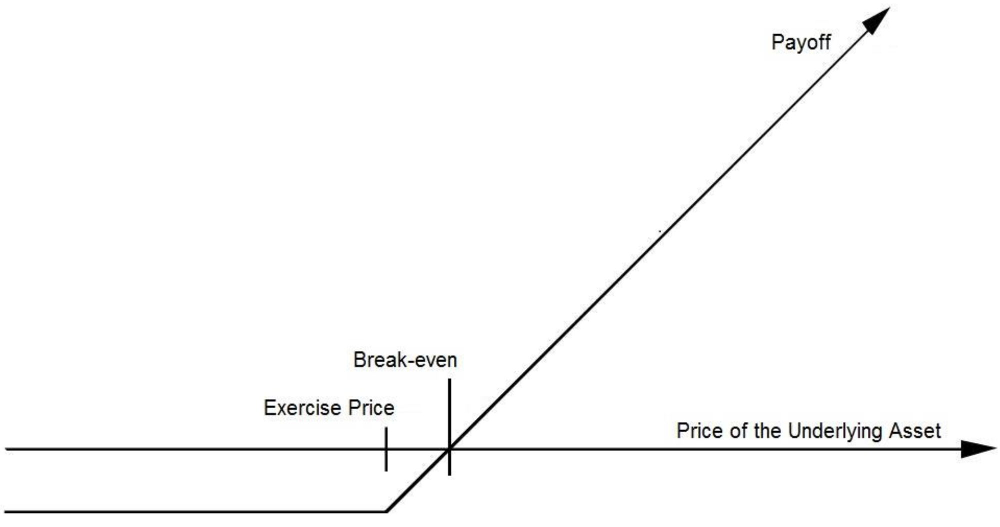

Figure 4 – Payoff Diagram for a Call Option (Adapted from: Damodaran, 2012)

3.6.2. Put Options

Similarly, a put option gives the investor the right to sell the underlying asset at a fixed price. Accordingly, the exercise of the option depends on the price of the underlying asset falling below the strike price. The owner of the option will exercise it and be able to sell the asset for the strike price, earning the difference to the lower market price deducted of the cost of the option.

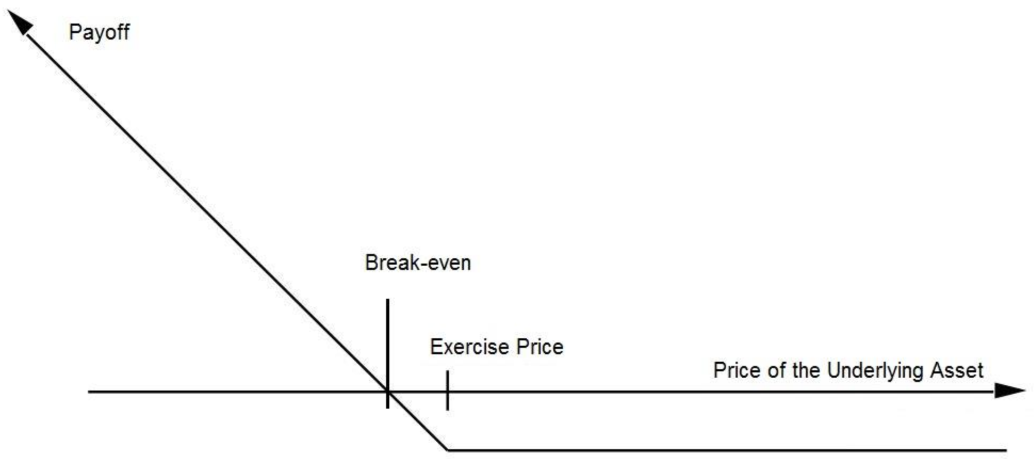

Figure 5 – Payoff Diagram for a Put Option (Adapted from: Damodaran, 2012)

3.6.3. Variables of Option Value

There are a number of variables that influence the value of an option, these variables are related with the underlying asset or the financial markets.

a. Value of the underlying asset – Options derive their value from an underlying asset. An increase in the value of that asset will augment the value of a call option and reduce the value of a put option.
b. Variance in the value of the underlying asset – As the variance is a measure of risk, it usually as an inversely proportional effect on valuation, this does not happen with options. Although counterintuitive, both call and put options increase their value with an increase of the variance in the value of the underlying asset. This derives from the fact that with an option the investor can never lose more than what he paid for it. It logically follows that the potential for large price swings that is translated in the variance creates opportunity for value for the investor.
c. Dividends paid on the underlying asset – this is a derivation of the effect on the value of the underlying asset. As the payment of dividends decreases the value of the underlying asset, being an outbound cash-flow, the value of a call option will also decrease and the value of a put option will increase.
d. Exercise price – one of the key descriptors of an option. With call options the value of the call will decline with the increase of the exercise price. For puts the opposite is true, the value increases directly with the increase of the exercise price.

e. Time to expiration – the effect is similar to what was said about the variance in the value of the underlying asset. Longer times create more opportunity for the price of the underlying asset to change, as the losses are limited to the price paid for the option, this increases the value of options, either calls or puts.
f. Risk free rate – this factor interacts with the value of an option in two different ways. Firstly, as the investor pays up front for the option, there is an opportunity cost which will depend on the market interest rates and time to expiration of the option. Secondly, the risk free rate takes part in the valuation of the option in the computation of the present value of the exercise price. A positive variation of the risk free rate originates an increase in the value of a call option and a decrease on the value of a put option.

3.6.4. American and European Options

Options are classified as American or European according to the possibility to exercise them at times prior to their expiration. American options can be exercised at any moment prior to that date, European options can only be exercised at expiration. The enhanced flexibility that this property confers to the American options makes them more valuable but also increases the difficulty of their valuation. In practical terms this is rarely a limitation as the models for European options can be applied to American options as long as the execution only takes place at expiration. Exercising the option early is frequently undesirable due to the time premium, which is expected to decrease as the expiration nears, and the transaction costs. This means that selling the option is usually preferable to exercising it before the expiration date. A relevant exception is the case where the underlying asset pays large dividends, causing the value of the asset to decrease, with negative impact on call options associated with it.

3.6.5. Option Pricing Models

3.6.5.1. Binomial Model

The Binomial Option Pricing Model (BOPM) follows a simple formulation for the definition of the price of an asset. In this model, at each time period, the price of the asset moves to one of two possible values, this is called a binomial price path.

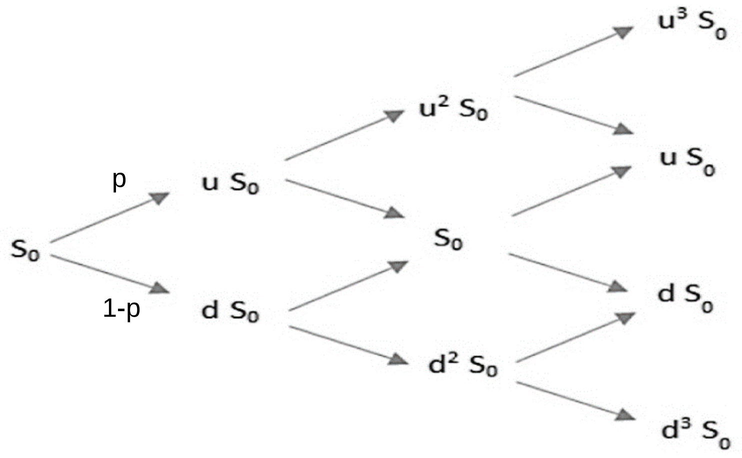

Figure 6 – Binomial Lattice (Adapted from: Brandão, 2005)

The BOPM is a discrete-time model for asset price movements, which follows a time interval (t) between price movements.

The valuation in a binomial process is done iteratively, starting from the last time period and following back in time until the starting point. After each interval an option replicating portfolio is created and evaluated, by the principles of arbitrage the value of the replicating portfolio is equal to the value of the option. The replicating portfolio is a combination of risk-free borrowing or lending and a number of units of the underlying asset, it aims to reproduce the same cash flows as the option being valued. The number of units of the underlying asset is called the option delta (Δ) and is defined, at each time step, as:

$$\Delta = \frac{C_u - C_d}{S_u - S_d}$$

Equation 8 – Option Delta

The binomial option pricing model outputs the composition of the replicating portfolio, which gives us the value of a call option, defined as:

$$C = S\Delta - B$$

Equation 9 – Value of a Call using the BOPM

C – Value of the call; S – Current value of the asset;

Δ – Option delta; B – Borrowing necessary to replicate the option

3.6.5.2. Stochastic Differential Equations

Analytical models for option pricing are based on the assumption that the state variable depends on underlying random processes and is represented by a stochastic differential equation (SDE). Brownian motion is one of the possible random processes that drive SDEs. This continuous time process is frequently called Wiener process, both denominations being interchangeable (Wiersema, 2008).

One SDE, using one Brownian motion as the source of motion is the arithmetic Brownian motion (ABM) or standard Brownian motion. The equation has the form:

$$dx(t) = \mu dt + \sigma dW$$

Equation 10 – Arithmetic Brownian Motion

x – state variable; μ – drift (constant); t – time; σ – standard deviation (constant); W – Wiener process ; d – mathematical differential operator

The coefficient σ is a scaling factor of the random process created by the Brownian motion. The other term, μ, is the drift, it determines the non-random component of the evolution of the variable. In the equation, μ and σ are known constants and $\sigma > 0$. The random variable $x(t)$ is normally distributed, with the distribution parameters $E(x(t)) = x_0 + \mu t$ and $Var(x(t)) = \sigma^2 t$. As this model attributes a normal distribution to the variable $x(t)$, it can take negative values. For this reason it has been discarded in the modelling of stock prices, which are prevented from becoming negative because of the limited liability enjoyed by equity owners. The capability of reaching negative values can be extremely useful when the variable is not the price of a stock, but an underlying financial measure, such as earning.

The SDE that as traditionally been used to model stock prices is the Geometric Brownian motion (GBM). This model is represented by Equation 11, with similarities to that of the ABM (Wiersema, 2008), substituting the variable $x(t)$ by the rate of change of the same variable $\frac{dx(t)}{x(t)}$.

$$dx(t) = \mathrm{x(t)}\mu dt + \mathrm{x(t)}\sigma dW$$

Equation 11 – Geometric Brownian Motion

The parameters μ and σ are constants as in the ABM. Under this model the variable follows a lognormal distribution, which does not allow negative values. The drift and random coefficients, respectively $\mathrm{x(t)}\mu$ and $\mathrm{x(t)}\sigma$, are always proportional to the current value of the variable $\mathrm{x(t)}$, which amounts to exponential growth. The parameters of the lognormal distribution are $E(x(t)) = x_0 e^{\mu t}$ and $Var(x(t)) = {x_0}^2 e^{2\mu t}(e^{t\sigma^2} - 1)$.

3.6.5.3. Black-Scholes Model

The Black Scholes Option Pricing Model (BSOPM) is one of the most outstanding models in financial economics (Sudarsanam and Sorwar, 2003).

Although the model is rarely mentioned as Black-Scholes-Merton model, the contribution of Robert Merton to the field was considered sufficiently relevant by the Swedish Academy to be awarded with the 1997 Nobel Prize in Economics, shared with Scholes.

The BSOPM is a limiting case of the binomial pricing model. As the time interval tends to zero, the limiting distribution tends to one of two forms: a normal distribution, if price changes reduce as the time interval is shortened, in which case the price function becomes continuous; or a Poisson distribution if price jumps are to be allowed. The BSOPM corresponds to the case when the limiting distribution is normal.

The BSOPM applies to European, dividend protected options. Neither early exercise nor dividends are considered in this valuation model.

$$C = SN(d_1) - Ke^{-rt}N(d_2)$$

$$d_1 = \frac{ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

Equation 12 - Black Scholes Option Pricing Model

S – current stock price; K – exercise price; r – risk free rate; σ2 – annualized variance; t – time to expiration.

N (d1) and N (d2) represent the cumulative probability distributions

To value put options we can use the Put-Call parity theorem, which states that:

$$Asset\ Value\ +\ Put\ Value\ =\ Call\ Value\ +\ Present\ Value\ of\ Exercise$$

Equation 13 – Put - Call Parity Theorem

3.6.6. Considerations for Real Options

Real option models are an extension from their origins in financial asset valuation. These models allow us to analyze options on investment projects as if they were options on traded assets (Dzyuma, 2012). Mun (2002) compared in his work the real option valuation method to a road map, full of roads, junctions and turns.

Questions arise from the underlying assumptions of option pricing theory and their validity in the case of real options:

Feasibility of arbitrage – Option pricing models assume that a replicating portfolio can be created using the underlying asset and risk free borrowing or lending. This demands that the underlying asset is traded, when this is not the case arbitrage is not feasible and the values obtained from the models should be interpreted cautiously.

Constancy of variance – Option pricing models consider a constant variance. When long term real options are being considered this is unlikely true. There are modified versions of the pricing models that allow for changing variances as long as the process by which the variance changes can be modeled explicitly (Domodaran, 2012).

Instantaneous exercise - The exercise of most real options takes time, this implies that the true life of a real option is often less than the stated life.

3.6.7. Investment Opportunities as Real Options

Real Options turn a limited commitment into future decision rights (McGrath et al. (2004). The increased flexibility for management decisions and potential future benefits that they enable can, sometimes, lead to the viability of the analyzed project, even if classical NPV analysis yields negative returns.

The real option valuation method lets us evaluate multiple aspects of an investment project that would be neglected with a traditional investment analysis. Some of those options are:

a. Option to abandon

The option to abandon a project only makes sense when we're considering a multi-stage or repeated investment project. In such conditions, if at any moment in time the value of liquidating the project is greater than the remaining value of the project, than it should be liquidated. This analysis is based on the assumption that the liquidation or residual value is constant during the life of the project. The opposite would probably have more realistic application but the use of standard option pricing models wouldn't be adequate. The payoff graph associated with an option to abandon is similar to that of a put, and the option fits that category.

b. Option to delay

It is the most basic condition for the viability of a project that the net present value of its expected cash flows is positive. However, by changes in the underlying conditions or merely on the basis for the estimations, the expected cash flows and the discount rates may change over time, altering the NPV. This means that the viability and the expected return of a project may change by the mere action of a delay. The characteristics of this kind of option are consistent with those of an American call option. It is advisable to take into consideration that the option to delay might create opportunity for other players in a competitive market.

c. Option to expand

Some projects cannot be assessed just by themselves. Consider the case in which the decision to execute a project enables, in a known timeframe, other projects. This is an option to expand a project. Even if the first part of the project has a negative NPV, the valuation of the option to expand can be enough to offset the returns and make the project viable. It is correct to consider this option as a call, at the end of the timeframe to expand, the investor will enter the second phase of the project if the NPV at that time has turned positive. While this rationale may be used to justify initial investments that do not meet the financial requirements in a standard analysis, not all of these investments create valuable options. The value of the option comes from the capability of generating increased returns that is enabled by the strategic action that is associated with it.

d. Other options

Portfolios constituted by a combination of the above can be constructed to simulate such real options as a switch (e.g. the abandonment of a technology for another) or a staged growth, among others (Dzyuma, 2012).

3.6.8. Structural Models

Structural models are based on the assumption that the dynamics of the project variable are described by a stochastic differential equation. From that assumption, it is possible to find the present values of different claims, such as debt, equity and taxes through the solution of a partial differential equation. This pricing framework is based on the work of Black and Scholes (1973) and Merton (1974). If differentiating conditions between debt and equity are incorporated, such as the tax advantage of debt, it is possible to derive the optimal coupon level which maximizes the firm value, thus defining the optimal capital structure. Most structural models, particularly early ones, share some core characteristics: the project variable is the value of the firm’s assets, which is assumed to have dynamics described by a geometric Brownian motion (GBM) and they have a static setting, meaning that debt can be issued only once. The works of Brennan and Schwartz (1978) and Leland (1994) belong to that family of models.

Dynamic capital structure models were first introduced by Kane, Marcus, and McDonald (1984). Later contributors include Fischer et al. (1989) and Goldstein et al. (2001). The first generation of structural models suffers from a particular inconsistency (Ammann and Genser, 2004): the state variable of the models is the unlevered firm value. This implies that both the levered and unlevered value of the firm exist simultaneously, which is not viable in an asset pricing context. Goldstein et al. (2001) proposed a model that uses an income measure that is unaffected by the capital structure decision, the EBIT. This type of model is more consistent with regular practices because debt coupons are financed with earnings, not by issuing new equity. The model divides the present value of the EBIT between several claims.

Ammann and Genser (2004) propose a model which features an alternative assumption for the dynamics of the project variable: instead of the commonly assumed GBM they opt for an arithmetic Brownian motion (ABM). Few other works use ABM as the underlying process, among them are Bank and Lawrenz (2005), and Genser (2006). These models are all in a static setting, although the works of Ammann and Genser (2004) and Genser (2006) do allow for debt with finite maturities.

The use of geometric Brownian motion in all dynamic structural models, so far, is due to the fact that its mathematical properties allow a simplified progression from the static setup to the dynamic one, which is due to a homogeneity property inherent to this particular stochastic process. Since most start-ups do exhibit negative operating income, the assumption of GBM would be fairly limiting from a practical angle (Alexander et al., 2012).

4. Methodology

This chapter covers the definition, justification and development of the model. Using an approach grounded on structural analysis makes it possible to define the optimal capital structure as well as the value of the company and option to invest. By modelling EBIT growth as an ABM, instead of the more frequent assumption of GBM, it becomes possible to incorporate the possibility of negative cash flows, thus making the model more adapted to the reality of ESTCs. The use of debt as well as equity, reaching the optimal capital structure, enables the management of the firm to maximize the value of the company through the tax benefits of debt.

This model is developed within a static capital structure framework. In this setting the optimal leverage is evaluated for a particular EBIT level, without the option of additional issuances of debt in the future. Dynamic optimal leverage has been studied by Fischer et al. (1989), Goldstein et al. (2001), among others. All these works share the underlying assumption that the state variable follows a GBM. To the extent of the author's knowledge, so far, no researcher has published work concerning an ABM based dynamic capital structure model.

A static model can provide not only information on the current capital structure, but also on how far the current leverage is from the optimal one. Used as a reference it provides guidance but maintains management flexibility and adaptability to different scenarios.

4.1. Main assumptions of the valuation model

The choice of the project variable and of the stochastic differential equation that governs its dynamics is a fundamental step of the development of the model. While it can´t be categorically stated that the decisions made in this work represent a solution compatible with all cases, it is possible to present justifications and reasoning to make the case that they provide the better fit for the characteristics inherent to ESTCs. It is also relevant to try to understand why the majority of the literature opts for a different approach, particularly in what concerns the GBM assumption. As was mentioned in section 3.6, the researchers that started the study of structural models, were heavily influenced by the BSOPM and other asset pricing models developed for market traded assets with the market price as state variable, which demanded that the modelling of such variables did not allow negative values, which is warranted by the use of the GBM.

4.1.1. Earnings Before Interest & Tax (EBIT): The project variable

The majority of the real options based models fit into one of two categories, in what concerns the project variables: those that model the value of the firm or of the equity**,** and the ones that model some form of cash flows. The more traditional approach, of modelling firm or equity value, carries some disadvantages that make it unsuitable for this project. One of the difficulties with these models is the definition of the tax benefits, which are treated has inflows of capital, rather than a reduction of the outflow of taxes. Not only is this conceptually wrong, it leads to severe problems in the behavior of such models. Comparative statics analysis shows that this assumption causes the present value of the equity to increase with the tax rate (Goldstein et al, 2001).

The researchers that opted for the use of cash flows as project variables used either EBIT, Free Cash Flow or Revenue. This work uses EBIT, which is the most frequent decision among this class of models. Ammann and Genser (2004) argue that the most important advantage of choosing this project variable is the fact that working with EBIT and DCF based models allows for the cash flows to be split into different claims. Goldstein et al (2001) corroborates: *"The implication is that all contingent claimants (equity, debt, government, etc.) to future EBIT flows are treated in a consistent fashion."* Other strength of this approach is considering that existing claims to future EBIT are mostly insensitive to changes in capital structure. It is standard practice to assume that the EBIT-generating activities of a firm, are independent from capital structure (Modigliani and Miller, 1958; Merton, 1974; Brennan and Schwartz, 1978).

Throughout this work the use of EBIT and Operating Income (OI) will be made without distinguishing between the two. There are differences between the two, but those mainly regard adjustments in special items. The OI is the measure recommended in the generally accepted accounting principles (GAAP). In the context of this academic exercise in company valuation, and more broadly in financial analysis both concepts can be considered equivalent and defined as the revenue subtracted of the operating expenses.

4.1.2. Arithmetic Brownian Motion: Modelling Earnings Before Interest & Tax

Assuming that the market prices of financial derivatives on market-traded assets follow a GBM is a standard assumption (Brandão et al., 2005). A frequent argument favoring GBM is that prices modelled this way do not become negative, a condition that also holds for market traded assets.

The stochastic process that is used to model and simulate the EBIT flows is essential to the model. When dealing with cash flows, which are not market traded assets, they can, and frequently do, become negative. Recognizing this, Copeland and Antikarov (2001), suggest that sometimes ''it may be better to model the value of the underlying as an arithmetic or additive process''.

Here an ABM process is used, in substitution of the most common GBM assumption. Although uncommon this option is not unique and was taken, for purposes similar to those of this work, by Ammann and Genser (2004), Genser (2005) and Bank and Wibmer (2012); as well as for pricing credit spread options (Giacometti and Teocchi, 2005) and valuing real options, Alexander et al (2012).

The most relevant reason not to use GBM in this project is the fact that the implied log-normal distribution of the project variable values, excludes negative ranges (Alexander et al, 2012). This limitation is particularly relevant when one is assessing the cash flows of young firms or trying to evaluate the possible outcomes of a future venture. In the case of ESTCs it's frequently verified that the companies operate with negative EBIT.

4.2. Development of a Structural Model

Having justified the choice of EBIT as the project variable and of ABM as stochastic process, the construction of the model can be carried out. The following process follows analogous steps to those present in the works of Goldstein et al (2001), Ammann and Genser (2004), Genser (2006) and Bank and Wibmer (2012). These and other authors will be referred in particular settings, when necessary to justify decisions or unobvious steps. The structural model computes the present value of the EBIT generated by the company, as estimated by the dynamics of the ABM; then it distributes that total value among the different claimants: in an unlevered setting the division is between the equity owners, the state, which is entitled to taxes, and to closure costs, associated with the abandonment of business if the EBIT falls below a threshold value, the optimal threshold is also evaluated. In a levered setting, besides the already stated claims, there is also a debt claim and one related with the costs of bankruptcy, conditions for which are also estimated in the model. The maximization of the value of the company is achieved because of the tax advantage to debt, this makes it possible to compute an optimal capital structure for the company in the conditions of the valuation model. The value of the company is considered to be the present value of the claims to equity, net of taxes and closure costs, in the unlevered case and to the sum of equity and debt in the levered company.

4.2.1. Value of a claim to cash flows following ABM

The starting point of the model is the stochastic differential equation of the ABM process:

$$dx(t) = \mu dt + \sigma dW$$

Equation 14 – Arithmetic Brownian Motion (repeated)

x – state variable; μ – drift (constant); t – time; σ – standard deviation (constant); W – Wiener process ; d – mathematical differential operator

Discounting the expected cash flows under the risk-neutral measure gives the pricing of the subjacent asset (Goldstein et al., 2001). The expected present value of the EBIT flow is:

$$V_{expected}(x) = E^Q\left[\int_t^{+\infty} xe^{-rt}\,dt\right] = \frac{\mu}{r^2} + \frac{x}{r}$$

Equation 15 – Expected Present Value of the EBIT

E – expected value operator; r – risk free rate

Assuming that no arbitration is possible, defining a risk-neutral measure, Q, guarantees that every claim will receive the expected return for the assumed risk. With $\mu = \mu_m - \sigma\Phi$ being the risk neutral drift, μ_m the measured drift and Φ the risk premium.

The ABM process being an Itô's process, it is possible to affirm, according to Itô's Lemma, that any perpetual claim to $V(x)$ must follow dynamics such as:

$$dV = \left(\frac{dV}{dt} + \frac{dV}{dx}\mu + \frac{1}{2}\frac{d^2V}{dx^2}\sigma^2\right)dt + \frac{dV}{dx}\sigma dW$$ *Equation 16 – Itô's Lemma*[1]

Under the assumption of no arbitrage, with the equivalent martingale measure described above, any claim must earn the risk-free interest rate. The risk-free return on the derivative claim can be defined as:

$$\mathrm{rVdt} = \mathrm{E^Q[dV + xdt]}$$ *Equation 17*

Using Itô's Lemma,

$$\mathrm{rVdt} = \mathrm{E^Q}\left[\left(\frac{\mathrm{dV}}{\mathrm{dt}} + \frac{\mathrm{dV}}{\mathrm{dx}}\mu + \frac{1}{2}\frac{\mathrm{d^2V}}{\mathrm{dx^2}}\sigma^2 + \mathrm{x}\right)\mathrm{dt} + \frac{\mathrm{dV}}{\mathrm{dx}}\sigma\mathrm{dW}\right]$$ *Equation 18*

$$rV = \frac{dV}{dt} + \frac{dV}{dx}\mu + \frac{1}{2}\frac{d^2V}{dx^2}\sigma^2 + x$$ *Equation 19*

Two further alterations must be introduced prior to achieving the differential equation that describes any claim to the EBIT of the firm, and for which a closed form solution can be found. First the term $\frac{dV}{dt}$ is dropped, which equates to the assumption of time independence for the cash flows, which restricts the debt issuance to perpetual debt. Without assuming this, the differential equation would have no closed-form solutions (Leland, 1994). Although perpetual debt is not common for most businesses, it is arguable that debt that is continually renewed can be approximated as perpetual debt. Another frequent setting in the specific case of ESTCs are debt issuance contracts that include provisions to transform that debt into stock options, this also being possible to approximate as perpetual debt.

The other modification needed is to rewrite the intermediary cash flow term, until now x, as a linear function, $x_{current} = mx + k$, which facilitates the task of describing and defining claims with different properties.

The differential equation takes the form:

$$\frac{dV}{dx}\mu + \frac{1}{2}\frac{d^2V}{dx^2}\sigma^2 - rV + mx + k = 0$$ *Equation 20*

[1] The notation in the form $\frac{dX}{dx}$ represents the first partial derivative of the function *X* in order to the variable *x*, the notation $\frac{d^2X}{dx^2}$ is defined accordingly but the exponent [2] indicates a second partial derivative; this notation is used throughout this work.

This being a second order nonhomogeneous differential equation, a solution can be obtained (the complete process is detailed in Annex II):

$$V(x) = \frac{m(\mu_m - \sigma\Phi)}{r^2} + \frac{mx + k}{r} + A_1 e^{\beta_1 x} + A_2 e^{\beta_2 x}$$ *Equation 21*

$$\beta_1 = -\frac{\mu_m - \sigma\Phi}{\sigma^2} - \frac{\sqrt{(\mu_m - \sigma\Phi)^2 + 2r\sigma^2}}{\sigma^2} < 0$$ *Equation 22 – β_1 coefficient*

$$\beta_2 = -\frac{\mu_m - \sigma\Phi}{\sigma^2} + \frac{\sqrt{(\mu_m - \sigma\Phi)^2 + 2r\sigma^2}}{\sigma^2} > 0$$ *Equation 23 – β_2 coefficient*

The equation $V(x)$ is comprised of two distinct parts, a general solution corresponding to the terms starting with A_1 and A_2 and a particular solution comprised of the first two terms. The general solution does not account for intertemporal cash flows, of the form $mx + k$, these are accounted for by the particular solution: the second term affects all the cash flows of this form, the first only those that are affected by the changes of the EBIT level.

At this point is already clear that to maintain economic significance at all times A_2 must me equal to 0, otherwise the value of the claim would increase exponentially with the cash flows (Ammann and Genser, 2004).

4.2.2. Value of unlevered ESTCs

The cash flows generated by an unlevered firm have three potential claimant classes: equity owners, the government (via taxes), and the claimants of abandonment costs. An appropriate consideration of the boundary conditions is enough to obtain solutions for all the remaining unknowns.

To evaluate the present value of all the cash flows generated by the company it is necessary to include all the distributed EBIT. To accomplish this the distributed cash flows must be equal to the EBIT value, or: $mx + k = x$, which is true if $m = 1$ and $k = 0$.The conditions at which the value of the company is zero reveal the threshold at which the model recommends that equity owners abandon their investment, $V(x_a) = 0$. In this boundary it is now possible to solve for A_1:

$$V(x_a) = 0 = \frac{\mu}{r^2} + \frac{x}{r} + A_{1u} e^{\beta_1 x}$$ *Equation 24*

$$A_{1u} = -\frac{\left(\frac{\mu}{r^2} + \frac{x_a}{r}\right)}{e^{\beta_1 x_a}}$$ *Equation 25*

$$V(x) = \frac{\mu}{r^2} + \frac{x}{r} - \left(\frac{\mu}{r^2} + \frac{x_a}{r}\right) e^{\beta_1 (x - x_a)}$$ *Equation 26 – Present Value of the cash flows with abandonment condition*

When the EBIT reaches the boundary value x_a, the costs associated with the abandonment of the investment are incurred and supported by the equity owners. These costs have varied sources, from legal fees to labor related compensations, in the model they take the form of a fixed cost C. At the abandonment threshold x_a the cash flows attributable to closure costs are $\mathrm{CC(x_a) = C}$, as there are no intermediary cash flow distributions, only the general solution is employed:

$$\mathrm{CC(x_a) = C = A_{1C} e^{\beta_1 x_a}}$$ *Equation27*

$$\mathrm{A_{1C} = \frac{C}{e^{\beta_1 x_a}}}$$ *Equation 28*

$$\mathrm{CC(x) = C e^{\beta_1 (x - x_a)}}$$ *Equation 29 – Closure Costs claim*

This claim reaches the value C at the threshold of abandonment and converges exponentially to 0 as the cash flows take higher values.

Tax structures are defined by each government and are highly specific, considering that such considerations are out of the scope of this work the model considers a single tax rate τ, which can be considered for practical purposes as an effective tax rate. The government is entitled to a portion of the total claim over the EBIT, as defined by the tax rate τ, which means that the cash flows distributed as taxes are of the form $\tau(mx + k) = \tau x$, which is true when $x = 1$ and $k = 0$. Under abandonment conditions, $x = x_a$, considering that closure costs are tax deductible, the claim would have to be a tax refund $T(x) = -\tau C$, which leads to:

$$T(x_a) = -\tau \mathrm{C} = \tau\left(\frac{\mu}{r^2} + \frac{x}{r}\right) + A_{1T} e^{\beta_1 x}$$ *Equation 30*

$$A_{1T} = -\frac{\left(\frac{\mu}{r^2} + \frac{x_a}{r} + C\right)}{e^{\beta_1 x_a}}$$

Equation 31

$$T(x) = \tau\left(\frac{\mu}{r^2} + \frac{x}{r}\right) - \tau\left(\frac{\mu}{r^2} + \frac{x_a}{r} + C\right) e^{\beta_1 (x - x_a)}$$

Equation 32 – Tax claim

The equity claim to the unlevered company, $E_u(x)$, is defined as the remaining of the present value of the total claim, $V(x)$, after the deduction of taxes and closure costs.

$$E_u(x) = V(x) - T(x) - CC(x)$$

Equation 33 – Equity claim for the unlevered company

$$E_u(x) = (1 - \tau)\left(\frac{\mu}{r^2} + \frac{x}{r}\right) - (1 - \tau)\left(\frac{\mu}{r^2} + \frac{x_a}{r} + C\right) e^{\beta_1 (x - x_a)}$$

Equation 34 – Equity claim for the unlevered company (explicit form)

The remaining unknown x_a can be found by maximizing the equity value. To achieve this it is necessary to find the conditions at which the partial derivative of the equity value, $\frac{\partial E_u(x)}{\partial x}$, takes the value zero. The "smooth pasting condition" requires that the function and its derivative are null at the abandonment point (Dixit and Pindyck, 1994).

$$\frac{\partial E_u(x)}{\partial x} = 0; \; x = x_a$$

Equation 35 – Smooth pasting condition

$$x_a = \frac{r - C\beta_1 r^2 - \mu\beta_1}{\beta_1 r}$$

Equation 36 – Abandonment threshold

The abandonment threshold, x_a, now defined, completes the model for the unlevered company. As can be seen in Equation 36, this boundary is independent from the EBIT level, x.

4.2.3. Optimal capital structure for ESTCs

Levering the company creates a tax shield, TS, which represents the tax advantage of debt. The total value present in the EBIT created by the company remains constant with the changes in the capital structure. It is only redistributed among the claimants, with the claim due to the government via taxes being reduced. This is in accordance with the pie model of Modigliani and Miller (1958), extended to incorporate taxes and bankruptcy costs.

The optimal capital structure of a company is achieved when the impact of the tax advantage of debt is annulled by the increased cost of debt, which reflects the greater likelihood of financial distress of the indebted firm (Leland, 1994).

This financial distress, in the context of the levered company, can be defined as the condition for bankruptcy. The model assumes that there is a default if the EBIT falls below a threshold x_b, at this level debt holders are entitled to the residual value of the company, after bankruptcy costs, $BC(x_b)$, and taxes are deducted. Bankruptcy is a legal proceeding involving a company that is unable to repay its outstanding debt**.** If the firm does not meet its legal obligations to debtors, when it does not meet a scheduled payment, a default occurs. The existence of a government claim, even in case of default, is based on the assumption that the part of the EBIT-generating structure that survives bankruptcy will be sold: granted solvency for the new ownership, taxes are indeed owed. According to the procedure of Goldstein et al (2001) the costs of financial distress are represented as a fraction, α, of the pre distress asset base attributable to the equity owners, $BC(x_b) = \alpha E_u(x_b)$. That fraction is lost due to bankruptcy in case of default.

$$BC(x_b) = \alpha E_u(x_b) = A_{1BC} e^{\beta_1 x_b}$$ *Equation 37*

$$A_{1BC} = \frac{\alpha E_u(x_b)}{e^{\beta_1 x_b}}$$ *Equation 38*

$$BC(x) = \alpha E_u(x_b) e^{\beta_1 (x - x_b)}$$ *Equation 39 – Bankruptcy Costs claim*

The tax shield, TS, has an upper bound of τc for each period, the incidence of the tax rate on the coupon payment, c. As a consequence it is trivial to find that the tax benefit is null at bankruptcy, given that it implies a default on the coupon payment. The coupon is a fixed out flow such that $mx + k = c$, which demands that $k = c$ and $m = 0$.

$$TS(x_b, c) = 0 = \frac{\tau c}{r} + A_{1TS} e^{\beta_1 x_b}$$ *Equation 40*

$$A_{1TS} = -\frac{\frac{\tau c}{r}}{e^{\beta_1 x_b}}$$ *Equation 41*

$$TS(x,c) = \frac{\tau c}{r} - \frac{\tau c}{r} e^{\beta_1 (x - x_b)}$$ *Equation 42 – Tax shield*

The relation between the total value of the levered firm, $VC(x)$, and the unlevered equity can now be obtained (Leland, 1994):

$$VC(x) = E_u(x) + TS(x) - BC(x)$$ *Equation 43*

Although in comparable conditions the claim of the equity in an unlevered company, $E_u(x)$, is higher than that in the optimally levered company, $E_l(x)$, this still corresponds to maximizing the shareholder wealth: "equity holders receive fair value for the debt claim sold, minus their portion of the restructuring costs" (Goldstein et al, 2001). It is important to keep in mind that the levered and unlevered company do not exist simultaneously, the issuance of debt corresponds to a transference of capital and respective claim from shareholders to debt owners.

$$E_l(x) = VC(x) - D(x)$$ *Equation 44 – Equity claim for the levered company*

Debt value can be set by establishing the boundary condition when EBIT reaches x_b. At this level the claim attributable to debt holders is the remaining fraction of the value after the payment of $BC(x)$. The conditions associated with the coupon level are the same set before, $k = c$ and $m = 0$.

$$D(x_b) = (1 - \alpha) E_u(x_b) = \frac{c}{r} + A_{1D} e^{\beta_1 x_b}$$ *Equation 45*

$$A_{1D} = \frac{(1 - \alpha) E_u(x_b) - \frac{c}{r}}{e^{\beta_1 x_b}}$$ *Equation 46*

$$D(x) = \frac{c}{r} + \left[(1 - \alpha) E_u(x_b) - \frac{c}{r} \right] e^{\beta_1 (x - x_b)}$$ *Equation 47 – Debt claim*

Conditions for the threshold x_b, can be optimized, similarly to what was done for x_a, respecting the smooth-pasting condition (Dixit, 1991):

$$\frac{\partial E_l(x)}{\partial x} = 0; \;\; x = x_b$$ *Equation 48 – Smooth – pasting condition (1)*

Which leads to the result:

$$x_b = \frac{r - \mu\beta_1 + rc\beta_1}{r\beta_1}$$ *Equation 49 – Bankruptcy threshold*

This enables the optimal coupon level to be found, by maximizing firm value, deriving the explicit form of the equation for $VC(x)$ in order to c, and finding the zero of the derivative:

$$\frac{\partial VC(x)}{\partial c} = 0$$ *Equation 50 – Condition for optimal capital structure*

There is no closed solution for this equation, hence a numerical solver is required for the problem. Similar results can be obtained with numerical solvers by maximizing V using the coupon, c, as the variable.

4.2.4. Option to Invest in an ESTC

Until this point, the present analysis concerns the evaluation of a company, and the optimization of its capital structure, given the EBIT value. Underlying this procedure is the option to invest in the firm. The investment, which typically represents the acquisition of an equity stake, has a cost IC. The cost of the investment dictates that it is only logical to invest if EBIT reaches a level x_i. The cash flows associated with the investment option follow a ABM process, with adjusted drift $\mu_I = \mu - \delta$, where δ represents the opportunity costs associated with the wait to invest (Bank and Wibmer, 2012).

$$dx = (\mu - \delta)dt + \sigma dW$$ *Equation 51*

The value of the option to invest, I, follows that of a differential equation, defined similarly to what was presented in subchapter 4.2. Has there are no intermediary cash flows associated with the investment option, an ordinary differential equation is obtained:

$$\frac{dI}{dx}(\mu - \delta) + \frac{1}{2}\frac{d^2I}{dx^2}\sigma^2 - Ir = 0$$ *Equation 52*

A non-trivial solution can be easily found (it has similar form to that of the general solution presented in Annex II):

$$I(x) = A_3 e^{\beta_3 x} + A_4 e^{\beta_4 x}$$ *Equation 53*

$$\beta_3 = -\frac{\mu - \delta}{\sigma^2} - \frac{\sqrt{(\mu - \delta)^2 + 2r\sigma^2}}{\sigma^2} < 0$$

Equation 54

$$\beta_4 = -\frac{\mu - \delta}{\sigma^2} + \frac{\sqrt{(\mu - \delta)^2 + 2r\sigma^2}}{\sigma^2} > 0$$

Equation 55

To solve for x_i three boundary conditions are needed. By interpretation of the results it is possible to set $A_3 = 0$, otherwise the option could increase its value with increasingly negative EBIT values. What is left is an exponential function. To constrain the possible values of the value function, a value-matching condition must be set, defining the option value as equal to the value of the optimally levered firm, net of the tax deductible investment cost:

$$I(x) = VC(x) - (1 - \tau)IC, \ x = x_i$$

Equation 56 – Value-matching condition.

$$A_4 = \frac{VC(x_i) - (1 - \tau)IC}{e^{\beta_4 x_i}}$$

Equation 57

$$I(x) = [VC(x_i) - (1 - \tau)IC]e^{\beta_4 (x - x_i)}$$

Equation 58 – Present value of the option to invest.

The final boundary condition, so that x_i can be optimized, is the smooth-pasting condition:

$$\frac{\partial I(x)}{\partial x} = \frac{\partial VC(x)}{\partial x}; x = x_i$$

Equation 59 – Smooth-pasting condition (2).

An analytical solution could not be found, therefore a numerical solution is required. It is relevant to note that solutions are only valid with $x < x_i$, that is when the current EBIT value is still inferior to the optimal investment value, otherwise the exponential factor in Equation 58 would increase the value of the option above the value of the acquired assets net of the investment cost, which loses practical value. It will be shown in the next chapter that the mentioned validity condition is frequently violated, in which case the option to invest, $I(x)$, should be valued considering $x_i = x$, which transforms equation 64 into equation 62, and nothing can be said about the optimal conditions for the investment. When this happens additional information can be obtained by introducing an extra parameter, λ, representing the minimum stake in the company that the investor should accept in return for the investment cost, IC.

$$I(x) = \lambda VC(x) - (1 - \tau)IC$$

Equation 60

$$\lambda = \frac{I(x) + (1 - \tau)IC}{VC(x)}$$

Equation 61 – Equity stake

The method applied in this subsection can be modified and replicated to reflect different options implied in an investment process, as long as the assumption of dynamics described by an ABM would hold for the real option. As an example, if used by the management of a company it could represent an option to start a new project, in which case $VC(x)$ would represent not the value of the company but of the project, and δ would be measured in the present context of the company.

5. Case Study – Fitbit Inc.

The application of the model developed in the previous chapter to a real company provides insight into the model that is not obtainable otherwise. The choice of Fitbit as the subject of study is a natural one, it is a highly technological company, it went through a very recent and successful initial public offering (IPO) and its success is derived of a unique combination between the fitness trackers that they sell and the complementary software ecosystem, which constitutes a valuable set of IP. The ensuing valuation of the company does not pretend to be a replica of the conditions that an investor faces when deciding on a future investment, it is an academical exercise for which the most complete information available was used. It can be argued that an investor in a pre-IPO moment would have access to undisclosed information, which limits attempts to establish a scenario equivalent to a real case of investment. This is particularly true for the parameters that are related with the simulation of the EBIT, for companies that are not yet obtaining revenue, or that have had revenue for a limited amount of time, the estimates for the simulation would have to be extracted indirectly from other data such as the user base, the market size or more relevantly a valuation of the IP of the company.

5.1. Overview of the company

The term “wearables”, from wearable devices or technology, is transitioning from jargon used only by technology enthusiasts, to a mainstream concept, mentioned daily in all forms of media. The term refers to technological products that are incorporated into clothing or accessories to be worn directly on the body. The number of categories into which wearable devices can be sorted is vast and constantly increasing, including smartwatches, e-textiles, smart glasses, fitness trackers, among others. One of the most successful companies to appear in the wearables market is Fitbit, headquartered in San Francisco, USA. The main products of the company are a range of activity trackers: wireless enabled devices, some used as wristbands others attachable to clothes, which incorporate multiple sensors in order to record information such as the number of steps the user travels daily or the quality and quantity of sleep. On May 2015, the company filed for IPO, to be listed in the New York Stock Exchange, eight years after its funding in 2007. In the years anteceding the IPO, the company was financed by venture capital firms, through four investment rounds, collecting a total of US$66M and reaching a valuation of US$344M at the last round (Fitbit, 2015). The stock price soared from an opening value of US$20 to US$31 in the first hours of trading, indicating a valuation of over US$6 billion (Yahoo, 2015).

Part of what distinguishes Fitbit is their platform, which can be accessed online or with a mobile app. More than keeping record of the activity of each user, the platform performs has a social network keeping users interested in the product, which can be a driver for future sales. The complete ecosystem that sustains the competitive advantage of the company, from the user base to the mobile applications, constitutes a valuable asset that could have been used to access debt capital and reduce the dilution of ownership prior to the IPO, thus retaining more value for founders and early investors.

Despite its success, Fitbit faces relevant competition, from other brands of activity trackers, such as Jawbone or Misfit; to potential substitute products such as Apple Watch and comparable offers from other technological companies. A young company in an extremely competitive market, comprising several of the biggest corporations in the world, carries a significant risk of not being able to sustain its activity in the long run.

5.2. Financial data

One of the difficulties of the valuation of early stage companies, for academic purposes, is that, prior to the IPO, little or no information is available to the general public. As part of the process to become a publicly traded company, Fitbit was required to issue and divulge the S-1 Registration Statement, under the Securities Act of 1993. This document is used as the source of all the financial information related with Fitbit Inc. that is used in the ensuing analysis. The available data reaches back to 2010, including a very comprehensive set of annual financial information.

Table 1 – Fitbit's annual financial data (Fitbit, 2015)

[2]	2010	2011	2012	2013	2014
	Values in million US dollars				
Revenue	5.18	14.454	76.373	271.087	745.433
Operating income	-1.366	-4.313	-3.775	-8.954	157.929
Net income	-1.431	-4.317	-4.216	-51.622	131.777

The same document also provides quarterly data, for the period of June 2013 to March 2015. In order to use the additional information provided by more data points, the measurements for the simulation of EBIT will be based on quarterly values. To obtain the lacking values for the years 2010 to 2012, the annual data for the year will be used, assuming that both the distribution of gross profit and operating expenses follow the same variation pattern has those of the quarters of 2014[3].

Table 2 – Fitbit's quarterly financial data (1) (Fitbit, 2015)

	1Q10	2Q10	3Q10	4Q10	1Q11	2Q11	3Q11	4Q11	1Q12	2Q12	3Q12
	Values in million US dollars										
Gross Profit	0.21	0.27	0.39	0.79	0.66	0.85	1.22	2.50	3.34	4.35	6.23
Operating Expenses	0.44	0.49	0.61	1.48	1.39	1.56	1.93	4.67	4.41	4.96	6.14
Operating Income	-0.23	-0.22	-0.22	-0.69	-0.73	-0.70	-0.70	-2.18	-1.08	-0.61	0.09

---

[2] All the values in this work follow the same notation, using the dot as the decimal mark.

[3] The year 2014 is chosen over 2013 due to the fact that at the end of the year the company had greatly increased expenses due to a product recall.

Table 3 – Fitbit's quarterly financial data (2) (Fitbit, 2015)

	4Q12	1Q13	2Q13	3Q13	4Q13	1Q14	2Q14	3Q14	4Q14	1Q15
	Values in million US dollars									
Gross Profit	12.73	14.99	21.22	41.26	-17.22	44.77	58.39	83.61	170.89	169.21
Operating Expenses	14.90	10.90	14.10	16.78	27.43	28.98	32.56	40.33	97.85	79.27
Operating Income	-2.17	4.09	7.13	24.48	-44.65	15.79	25.83	43.27	73.04	89.94

5.3. EBIT Simulation

The parameters for the EBIT simulation were extracted from the data above. The two parameters that are required are the drift and the volatility, measured by the standard deviation. The average quarterly drift was computed, using an arithmetic mean, at μ_{data} = 4.508 in millions of dollars per quarter; the standard deviation is σ_{data} = 27.93.

The software used was MATLAB, using the default simulation by Euler's method. The minimum number of trials was determined using convergence analysis.

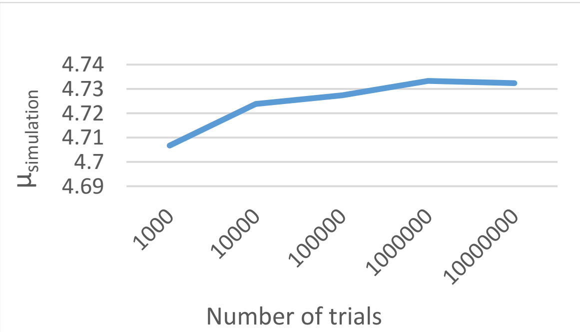

Figure 7 – Drift convergence

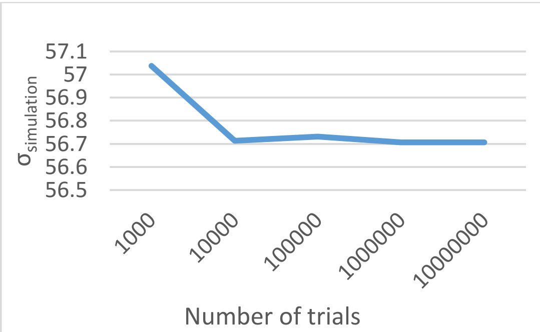

Figure 8 – Standard deviation convergence

Table 4 – Convergence analysis

Number of Trials	$\mu_{simulation}$	Deviation from previous μ	$\sigma_{simulation}$	Deviations from previous σ
10^3	4.7067	-	57.0376	-
10^4	4.7239	0.365%	56.7135	-0.568%
10^5	4.7274	0.074%	56.7308	0.031%
10^6	4.7334	0.127%	56.7069	-0.042%
10^7	4.7324	-0.021%	56.7071	0.0004%

From Table 4 and Figures 7 and 8, it is possible to conclude that with at least 10^4 trials reasonable convergence is reached. Due to the low computational requirements needed to perform the simulation, and with the objective of obtaining better convergence for the drift, a minimum number of trials of 10^6 is recommended. Upon deciding the adequate number of trials, repeated simulations were conducted with the same characteristics to verify the consistency of the results and obtain the estimates. It should be clear from Table 5 that the results of the simulation are sufficiently consistent, with the repeated simulations providing estimates within less than 0.31% from each other.

Table 5 – Consistency analysis

Simulation number	$\mu_{simulation}$	Deviation from average[4]	$\sigma_{simulation}$	Deviation from average
1	4.7334	0.069%	56.7069	0.021%
2	4.7328	0.057%	56.7100	0.026%
3	4.7154	-0.311%	56.6716	-0.042%
4	4.7390	0.188%	56.6963	0.002%
5	4.7300	-0.003%	56.6913	-0.007%
Average	4.7301	-	56.6952	-

From the more exhaustive simulation of 10^7 trials and the consistency test, estimates were chosen as $\mu_{quarterly} = 4.73$ and $\sigma = 56.70$.

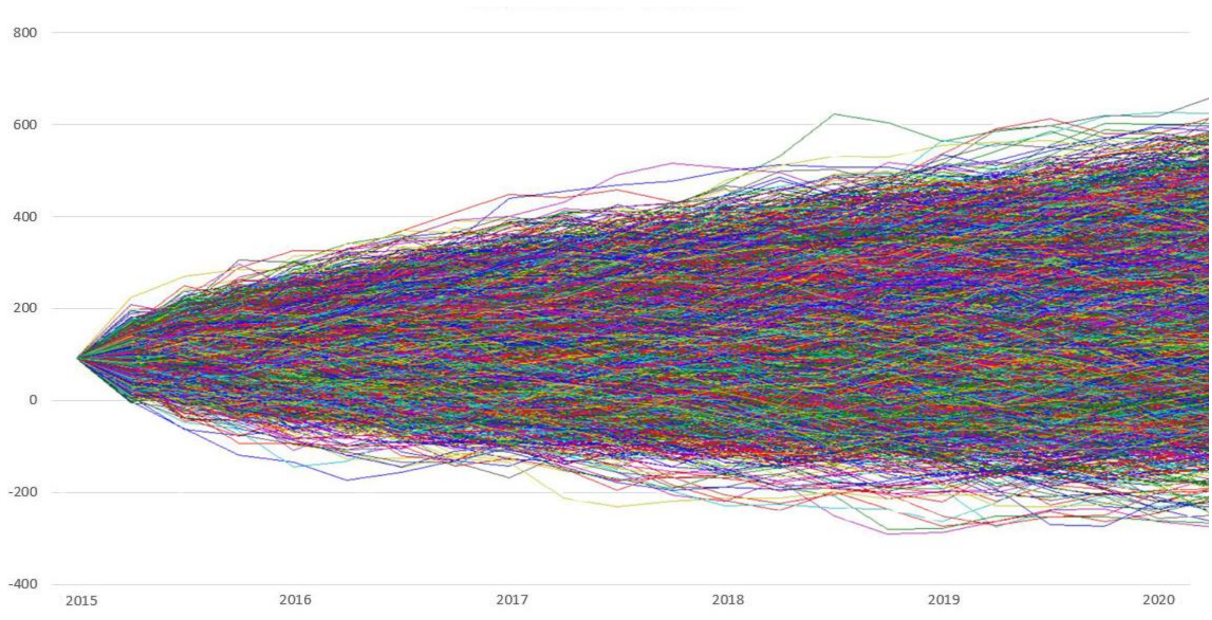

Figure 9 – Fitbit's EBIT simulation using Arithmetic Brownian Motion (in Million US dollars)

[4] Deviations from average calculated according to: $\frac{\mu_{trial} - \mu_{average}}{\mu_{average}}$

In Figure 9 is a representation of all the 10^6 individually simulated trials, offset to start at the EBIT level for the first quarter of 2015, the last available data point, and extending through the first 5 years of simulation. The risk implicit in investing in an early stage company, with high volatility is well implied in the significant amount of simulation trials that reach negative EBIT values. A density plot of the compiled trial data, along with a cumulative probability plot provide a better understanding of the results. The data set that is plotted must be adjusted by removing the starting point, since it is not result of the simulation.

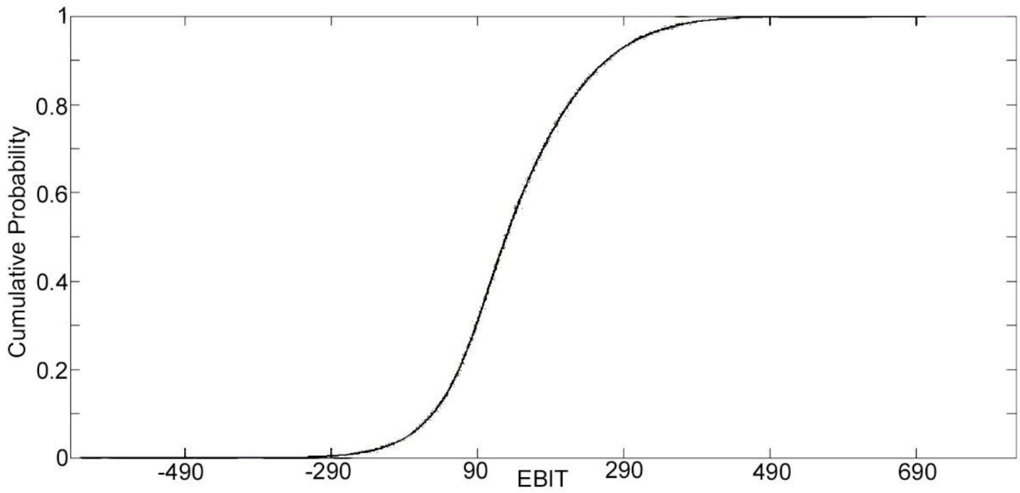

Figure 10 – Cumulative probability plot of the simulated EBIT (in million US dollars)

It's observable in Figure 10 and Figure 11 the extreme range of values that occur within the simulation, with the starting point of $89.94M and a positive quarterly drift there are still 5.5% of negative results. This reflects the uncertainty that is associated with such a young company and which is reflected in the volatility measure.

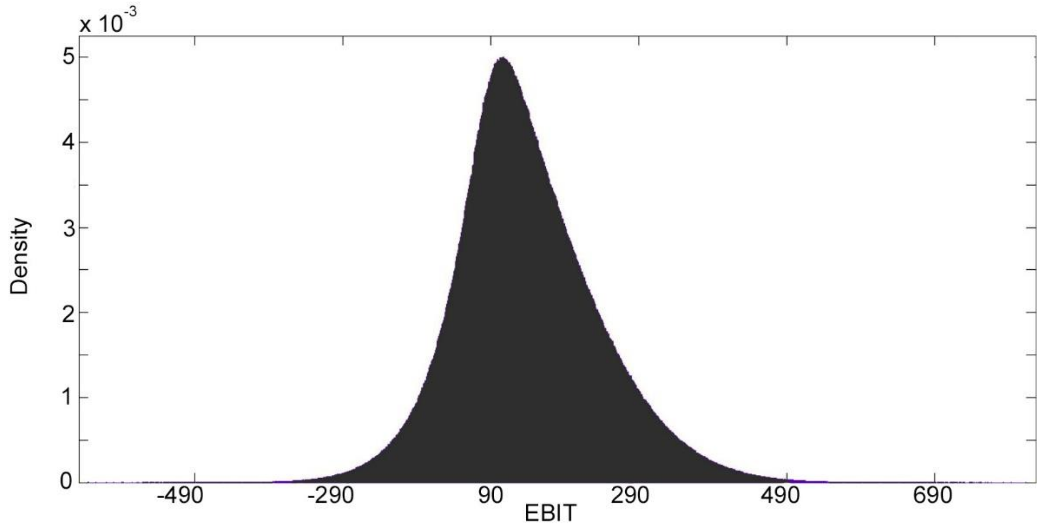

Figure 11 – Probability density plot of the simulated EBIT (in million US dollars)

5.4. Model Parameters

Deciding how to estimate the parameters to apply to the model can have as much impact on the final results as the assumptions for the model itself. Despite this, it is uncommon to find this theme explored in the literature. Using a real company for the case study provides the opportunity to reflect on how to obtain each value and to propose solutions when the procedures are not trivial. The methods that are used in the following steps are suggestions that are meant to provide values extracted from real data; proving the validity of those estimates is outside of the scope of this work, although an analysis of the relevance of each parameter for the results is carried out ahead in section 5.6.

5.4.1. Risk free rate (r)

A risk free asset can be defined as an asset for which the return rate is known with certainty, in other terms: the return rate is equal to the expected return rate (Damodaran, 2012). The same reference goes on to define such an asset as a default risk free, zero coupon bond for a predetermined period of investment. While this is a theoretical definition, a good approximation can be achieved considering long term government bond rates. This approximation will not hold for all cases, but has Fitbit Inc. is an American company, the government can be currently considered default risk free and there are no currency risks because the valuation will be carried out in US dollars.

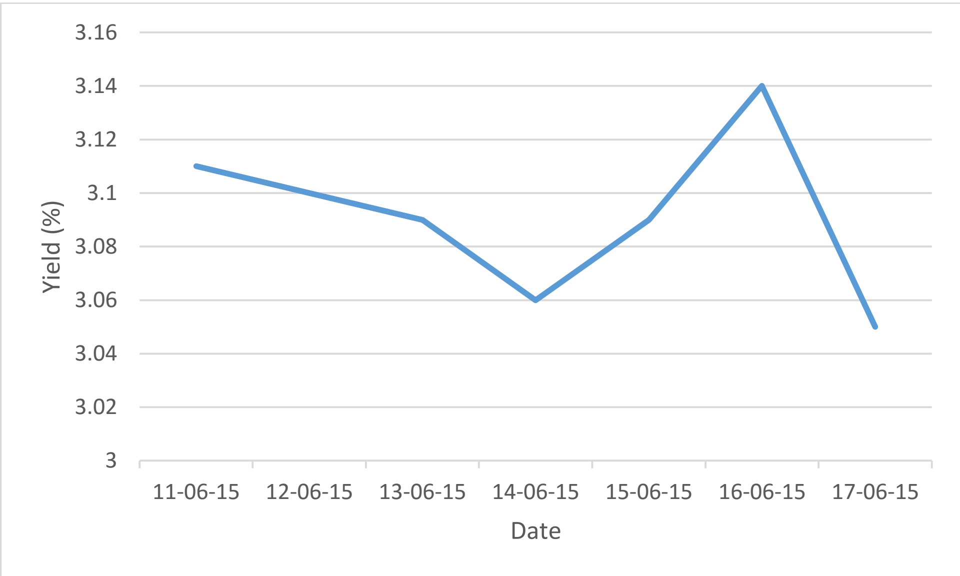

Figure 12 – US 30 year Treasury Bond yields[5] (%) (US Department of the Treasury, 2015)

[5] The method used to obtain the yields is quoted here, as explained by the source (US Department of the Treasury, 2015): "The Treasury's yield curve is derived using a quasi-cubic hermite spline function. Our inputs are the Close of Business (COB) bid yields for the on-the-run securities. Because the on-the-run securities typically trade close to par, those securities are designated as the knot points in the quasi-cubic hermite spline algorithm and the resulting yield curve is considered a par curve. However, Treasury reserves the option to input additional bid yields if there is no on-the-run security available for a given maturity range that we deem necessary for deriving a good fit for the quasi-cubic hermite spline curve. […]"

The value that is used throughout the rest of this analysis is the geometric average of the 30-year Treasury Bond yield rate, obtained from the period of 11-06-15 to 17-06-15, the week before the Fitbits IPO. The yield rate was computed at $r = 3.09\%$, which is the average value of the daily yields for the mentioned week.

5.4.2. EBIT (x)

The starting EBIT is the last known annual value, which corresponds to the year 2014 and is $x_0 = 158$ (Fitbit, 2015), in millions of US dollars.

5.4.3. Drift and standard deviation (μ_m, σ)

Working with the average estimators from the modelling data, after adjusting the quarterly drift to the yearly equivalent, the values used for the model were defined as $\mu_m = 18.92$ in millions of US dollars per year and $\sigma = 56.70$.

5.4.4. Tax rate (τ)

There are several options that can be considered for the appropriate tax rate. Goldstein et al (2001), use an effective tax rate defined as:

$$(1 - \tau_{eff}) = (1 - \tau_c)(1 - \tau_d)$$

Equation 62 – Effective tax rate definition

τ_{eff} *- effective tax rate;* τ_c *- corporate tax rate;*

τ_d *– dividend tax rate*

While this definition is viable, it is based on the assumption of double taxation of dividends, against which there are multiple, country specific covenants. This project follows the more frequent setting of considering a simple tax rate $\tau = \tau_c = 35\%$, the US federal corporate tax rate for 2015 (Deloitte, 2015).

5.4.5. Risk Premium (Φ)

The risk premium can be interpreted in several ways, due to the fact that it is going to be applied to assets that are not publicly traded, the definition that better suits this project is considering the risk premium as the premium requested by the investor to accept the risk inherent to the asset. One of the impracticalities of the model is the fact that the risk premium cannot be thought of as a required rate of return. The reason for this is the expression $\mu = \mu_m - \sigma\Phi$, which requires the product $\sigma\Phi$ to be an absolute drift, in monetary units per time unit. It is of greater practical value to evaluate $\sigma\Phi$ and compute the implied risk premium. According to Festel et al (2013), venture capital investors require a return of 39.5% on their invested capital. It is possible to use Equation 26, setting the remaining parameters to the same values that are being used for the valuation of Fitbit, and arbitrating $\mu = \mu_m$, obtaining a valuation of the same magnitude, and with the use of a numerical solver find the alteration in the drift rate that causes an increase in the value of that claim, according to the model, equal to the required

39,5%. This method obtains a reduction to the drift $\sigma\Phi = 9.52$, which implies $\Phi = 0.168$, and is a measure of the risk premium for companies similar to Fitbit. Risk premium will be inflated, due to the fact that $\mu < \mu_m$, but provides a better estimation than a simple arbitration. The risk neutral drift that results is $\mu = 9.40$.

5.4.6. Investment Cost and Equity Stake (IC, λ)

During their last pre-IPO funding round, in 2013, Fitbit achieved a valuation of US$344M (Fitbit, 2015) As this value dates from two years before the moment that is being considered for this valuation it is gravelly outdated, lacking other suitable reference the value used will be the average value of the last reference valuation for the IPO, which was US$3680M. A total equity stake of λ = 100% will be used, for the cost IC = US$3680M.

5.4.7. Closure and Bankruptcy Costs (C, α)

In the work, *The costs of bankruptcy (*Branch, 2002), the author categorizes and measures the costs caused by financial distress. The author of the paper finds that the net value that remains to be distributed to claimants is equal to 56% of the pre-bankruptcy value of the claims, this amounts to defining our bankruptcy cost $\alpha = 44\%$.

The closure cost C will have almost no effect in the model, for levered companies. This happens because the EBIT level that will trigger firm abandonment will be lower, or more negative, than the bankruptcy threshold. With this in mind, the closure cost can be set for the company in its earlier stages, before any leverage takes place. In such conditions, the closure costs, which equate to staff compensation and legal fees, can be assumed to be related to the general and administrative costs. For the ensuing analysis $C = US\$33.556M$, an arbitrated value equal to the general and administrative costs for 2014, the last year of available data.

5.4.8. Opportunity cost (δ)

Opportunity cost is frequently defined has the return rate that is lost by not using the asset in an alternative project. When investing in ESTCs there is a complementary definition that is relevant but of hard implementation, which is the cost associated with the risk of waiting (e. g. a company that is waiting for investment capital to file for a patent can lose that opportunity by having a competitor file for the patent first). The opportunity cost can be the sum of a number of factors, like the two mentioned above and others that may be particular to each case. One of the factors that is common to most cases of investment in early stage companies is that funding rounds are limited, and conditions for investment change between each round, as the company develops and becomes more or less viable, this constitutes an opportunity cost for potential investors. Taking into account that the model being used considers valuation as a function of EBIT level and drift, it is a reasonable assumption that the opportunity cost has value equal to the drift.

Table 6 – Complete input parameters

Input Parameters[6]									
$\boldsymbol{\mu}$	σ	r	x	C	τ	α	δ	IC	λ
9.40	56.7	3.09%	158	33.556	0.35	0.44	9.40	3680	100%

5.5. Model Results

The results that are discussed below were obtained through an implementation of the model in *Matlab* software.

5.5.1. Unlevered Company

For the unlevered company, an integral claim over the firm's cash flows, would have a present value $V = \$14991\text{M}$, distributed between a tax claim of $T = \$5246\text{M}$ and the equity $E_u = \$9744\text{M}$, with a negligible impact of the closure cost $CC = \$0.28\text{M}$. Shareholders should choose to close the firm only if the EBIT lowered bellow $x_a = \$-427.29\text{M}$.

Table 7 – Model Results – Unlevered Company

Model Results – Unlevered Company				
Values in million US dollars				
$\boldsymbol{V}$	T	E_u	CC	x_a
14991	5246	9744	0.28	-427.29

The value of the unlevered equity, E_u, can be compared to the market value of the equity of Fitbit Inc. In the three months following the IPO of the company, the share price ranged from around US\$30 to US\$52, these values translate into a market value between $\text{US\$6180M}$ and $\text{US\$10700M}$ (Yahoo, 2015). The value resulting from the model, $E_u = US\$9744M$, belongs to the previously stated interval. If one accepts the market price to be an estimate of the fair value of an asset; at least for this case study the estimate of the valuation provided by the model is valid.

5.5.2. Optimally Levered Company

The optimal capital structure is achieved with an annual coupon $c = \text{US\$ }358.81\text{M}$, at that point the present value of the claim over the enterprise value (i.e. net of taxes) would have increased 31.5% to the level $VC = \text{US\$ }12813\text{M}$, distributed between debt $D = \text{US\$ }10233\text{M}$ and equity $E_l = \text{US\$ }2580\text{M}$. This increase in the value of the company is obtained via tax savings of $TS = \text{US\$ }3423\text{M}$, with the risk of financial distress inducing a negative impact of $\text{BC} = \text{US\$ }355\text{M}$ with a bankruptcy threshold of $x_b = \text{US\$ }-67.45\text{M}$.

[6] The units in which the parameters are expressed are: μ, δ – millions of US dollars per year; σ, x, C, IC – Millions of US dollars. The remaining parameters are dimensionless.

Table 8 – Model Results – Optimally Leveraged Company

Model Results – Optimally Leveraged Company							
Values in million US dollars							
$\boldsymbol{VC}$	T	TS	D	E_l	BC	c	x_b
12813	1823	3423	10233	2580	355	358.81	-67.45

In the conditions of this test it is not possible to reach conclusions concerning the optimal conditions for the investment, according to what was explained in section 4.2.4. These conditions are, in any case, favorable to the investor, the option to invest is valued at $I = \text{US\$}10421\text{M}$. The threshold stake in the company, above which the investment is viable is $\lambda = 18.67\%$.

5.6. Sensitivity Analysis and Model Robustness

The analysis that follows explores the impact of each parameter in the model, the explanations that follow each section try to incorporate economical and mathematical interpretations of the results. Each of the parameters that is analyzed in the following sections is inserted into the model with five different values: a reduction of 50%, a reduction of 25%, the original value, an increase of 25% and an increase of 50%, with all remaining parameters unchanged.

5.6.1. EBIT Level

The variation of the EBIT for this analysis is not meant to represent the evolution of the EBIT with the passage of time and changes in market and company conditions. Instead what is evaluated is what different EBIT levels impact in the outputs of the model, for the same market conditions, in the moment of the analysis.

Table 9 – Sensitivity Analysis to x (absolute values)

$\boldsymbol{x}$	$\boldsymbol{VC}$	$\boldsymbol{T}$	$\boldsymbol{D}$	$\boldsymbol{E_l}$	$\boldsymbol{BC}$	$\boldsymbol{c}$	$\boldsymbol{x_b}$	$\boldsymbol{DER}$[7]
79	10542	1603	8289	2255	318	298.45	-127.80	3.68
118.5	11672	1716	9251	2421	337	328.36	-97.89	3.82
158	12813	1823	10233	2580	355	358.81	-67.45	3.97
197.5	13964	1925	11231	2732	371	389.75	-36.50	4.11
237	15124	2022	12245	2879	386	421.14	-5.11	4.25

[7] Debt-to-Equity ratio – this financial ratio will be used in the following sections as a measure of the leverage of the company. It is computed according to: $DER = \frac{D}{E}$.

Table 10 – Sensitivity Analysis to x (percentage values)

x	VC	T	D	E_l	BC	c	x_b	DER
-50%	-17.7%	-12.1%	-19.0%	-12.6%	-10.4%	-16.8%	89.5%	-7.3%
-25%	-8.9%	-5.9%	-9.6%	-6.2%	-5.1%	-8.5%	45.1%	-3.8%
158	12813	1823	10233	2580	355	358.81	-67.45	3.97
+25%	9.0%	5.6%	9.8%	5.9%	4.5%	8.6%	-45.9%	3.5%
+50%	18.0%	10.9%	19.7%	11.6%	8.7%	17.4%	-92.4%	7.1%

As could be expected from Equation 26 the value of all the claims varies directly with the EBIT, this is also in conformity with the economical interpretation of the model, as an increased value for the EBIT represents an increase of the capital to distribute among the claims, all else being equal. Also in conformity with what could be expected from economic interpretation is the increase in the leverage of the company with increased EBIT, as the better performance implied in the increased value would signal the management that a higher debt coupon could be sustained. The lowest future EBIT value that is tolerable before the model recommends a default, is also raised with the increase in the value of the coupon, reflecting the greater stress induced by the debt.

5.6.2. Drift

The drift is the expected average yearly evolution of the EBIT. Different drifts not only mean different present conditions but also different expectations regarding the future performance of the company.

Table 11 – Sensitivity Analysis to μ_m (absolute values)

μ_m	μ	VC	T	D	E_l	BC	c	x_b	DER
9.46	-0.06	5174	1039	3724	1451	211	181.10	-46.02	2.57
14.19	4.67	8514	1487	6440	2073	302	253.90	-61.94	3.11
18.92	9.40	12813	1823	10233	2580	355	358.81	-67.45	3.97
23.65	14.13	17596	1967	14741	2855	351	488.60	-62.99	5.16
28.38	18.86	22572	1975	19629	2943	319	633.14	-53.03	6.67

Table 12 – Sensitivity Analysis to μ_m (percentage values)

μ_m	μ	VC	T	D	E_l	BC	c	x_b	DER
-50%	-100.6%	-59.6%	-43.0%	-63.6%	-43.8%	-40.6%	-49.5%	-31.8%	-35.3%
-25%	-50.3%	-33.6%	-18.4%	-37.1%	-19.7%	-14.9%	-29.2%	-8.2%	-21.7%
18.92	9.40	12813	1823	10233	2580	355	358.81	-67.45	3.97
+25%	50.3%	37.3%	7.9%	44.1%	10.7%	-1.1%	36.2%	-6.6%	30.0%
+50%	100.6%	76.2%	8.3%	91.8%	14.1%	-10.1%	76.5%	-21.4%	68.0%

To better respect *ceteris parabus* conditions, the variated value is the measured drift, as the risk-neutral drift that is used in the model is in itself a computed parameter[8]. The impact that different drifts have in the value of the claims and the leverage follows what could be expected from economic interpretation, the analysis being similar to what was said above for changes in EBIT. An interesting effect is the evolution of the bankruptcy threshold, which increases with either the increase or decrease of the drift. This non-linear comportment translates the interaction between the effects of better expected performance from increased drift and the greater risk that comes with greater leverage.

5.6.3. Volatility

Being a measure of volatility, the standard deviation impacts the risk implied in the investment.

Table 13 – Sensitivity Analysis to σ (absolute values)

σ	VC	T	D	E_l	BC	c	x_b	DER
28.35	13719	1073	12102	1617	166	387.29	45.08	7.48
42.53	13152	1531	10940	2212	278	365.01	-16.01	4.95
56.70	12813	1823	10233	2580	355	358.81	-67.45	3.97
70.88	12671	2018	9842	2829	405	362.96	-112.30	3.48
85.05	12687	2165	9662	3021	439	373.43	-153.08	3.20

Table 14 – Sensitivity Analysis to σ (percentage values)

σ	VC	T	D	E_l	BC	c	x_b	DER
-50%	7.1%	-41.1%	18.3%	-37.3%	-53.2%	7.9%	-166.8%	88.4%
-25%	2.6%	-16.0%	6.9%	-14.3%	-21.7%	1.7%	-76.3%	24.7%
56.70	12813	1823	10233	2580	355	358.81	-67.45	3.97
+25%	-1.1%	10.7%	-3.8%	9.7%	14.1%	1.2%	66.5%	-12.3%
+50%	-1.0%	18.8%	-5.6%	17.1%	23.7%	4.1%	127.0%	-19.4%

The higher risk that is expressed through a higher standard deviation lowers the value of the company and of the debt, as a consequence of lower leverage. The tax claim increases with the volatility, as the lower leverage decreases the positive impact of debt. A less expected result is that, due to the drastic changes in optimal leverage, the equity claim actually raises with the increase volatility, this is a good example of the importance of evaluating the company value instead of restricting the analysis to the equity. The optimal coupon is raised with variations in either direction of the initial value, which creates again an unexpected result: a higher coupon that leads to a lower present value of the debt claim. This effect shows the impact of risk in the valuation of the claims, which is underlined by the increasing values of the bankruptcy costs. The bankruptcy threshold also requires careful evaluation, it is lowered with

[8] $\mu = \mu_m - \sigma\Phi$

increasing risk which appears to be illogical. To better understand this effect it is useful to think of the limit case, where the standard deviation is null, which implies the inexistence of risk. In this theoretical scenario the EBIT of the company would grow year after year by exactly the drift value (as can be perceived from Equation 15) and the bankruptcy threshold would be only infinitesimally lower to the current EBIT, but it would have no meaning whatsoever as the EBIT could never decrease. Given this example is now possible to understand that although the default threshold is lower with decreased standard deviation, the probability that the default ever occurs is actually lower.

5.6.4. Risk free rate

The risk free rate is the parameter that is exogenous to the company that has the greater impact on its valuation, for this reason great caution is needed when evaluating this rate, particularly since its definition allows for different interpretations which can lead to radically different valuations for the same company, performed by different investors or analysts.

Table 15 – Sensitivity Analysis to r (absolute values)

r	VC	T	D	E_l	BC	c	x_b	DER
1.55%	43124	5243	35583	7541	966	600.37	-145.19	4.72
2.32%	20938	2823	16923	4015	539.91	438.00	-96.75	4.21
3.09%	12813	1823	10233	2580	355	358.81	-67.45	3.97
3.86%	8875	1305	7035	1840	256	311.90	-47.50	3.82
4.64%	6615	993	5217	1398	196	280.49	-32.71	3.73

Table 16 – Sensitivity Analysis to r (percentage values)

r	VC	T	D	E_l	BC	c	x_b	DER
-50%	236.6%	187.6%	247.7%	192.3%	172.1%	67.3%	115.3%	18.9%
-25%	63.4%	54.9%	65.4%	55.6%	52.1%	22.1%	43.4%	6.0%
3.09	12813	1823	10233	2580	355	358.81	-67.45	3.97
+25%	-30.7%	-28.4%	-31.3%	-28.7%	-27.9%	-13.1%	-29.6%	-3.8%
+50%	-48.4%	-45.5%	-49.0%	-45.8%	-44.8%	-21.8%	-51.5%	-6.0%

The value of the company and of all the claims varies inversely to the risk free rate. Low values of this rate signify that the agents in the market are not able to obtain decent yields without being exposed to risk. It is implied that in such conditions a greater gap exists between the riskless asset and the rest of the market. If the company is able to maintain its performance in such market conditions it as greater value to the investors as the expected return from the investment represents a greater advantage over the risk bared, which is not a function of the risk free rate. The DER decreases with decreasing risk free rates, moderately when compared to the effect on the company value. The lower optimal leverage also reflects the decreased advantage of bearing risk when the riskless yields increase.

5.6.5. Tax Rate

The government claim that in the form of taxes is a transference of the wealth created by the company into the state. Increased taxes imply that less of the return that was generated by the invested capital will be returned to the investors.

Table 17 – Sensitivity Analysis to τ (absolute values)

τ	VC	T	D	E_l	BC	c	x_b	DER
17.5%	13694	1031	9439	4254	266	316.47	-109.78	2.22
26.25%	13234	1428	9968	3266	329	342.16	-84.09	3.05
35%	12813	1823	10233	2580	355	358.81	-67.45	3.97
43.75%	12414	2222	10362	2053	354	370.66	-55.59	5.05
52.5%	12032	2626	10411	1621	333	379.60	-46.65	6.42

Table 18 – Sensitivity Analysis to τ (percentage values)

τ	VC	T	D	E_l	BC	c	x_b	DER
-50%	6.9%	-43.4%	-7.8%	64.9%	-25.1%	-11.8%	62.8%	-44.1%
-25%	3.3%	-21.7%	-2.6%	26.6%	-7.3%	-4.6%	24.7%	-23.2%
35%	12813	1823	10233	2580	355	358.81	-67.45	3.97
+25%	-3.1%	21.9%	1.3%	-20.4%	-0.3%	3.3%	-17.6%	27.2%
+50%	-6.1%	44.0%	1.7%	-37.2%	-6.2%	5.8%	-30.8%	61.7%

The present value of the EBIT is divided between the company value and the tax claim, the increase in the tax rate only changes the distribution of the claims, and this is observed in the model, as the sum of both claims suffers only negligible changes with the increase of the tax rate. Another effect of the increase in the tax rate is that the optimal leverage increases substantially, due to the fact that debt is exempt from taxes. The bankruptcy threshold behaves as one would expect, increasing to reflect the increased risk from a higher debt coupon level. It is interesting to verify that, according to the model, tax rate increases encourage the companies to raise leverage, operating with greater bankruptcy risk, in order to optimize the return to the investors.

5.6.6. Debt

The level of debt that promotes the optimal value for the company leads to $DER = 3.97$ which is rare, if not inexistent, among real companies. Reasons for this discrepancy were advanced by Goldstein et al (2001): this being a static model, the issuance of debt is considered to be done only once. The consequence of this is that the increase in cost of debt that is experienced by companies in the market when they increase their debt exposure, and consequently their bankruptcy risk, is excluded from the model. That increasing cost of risk offsets the advantages of the tax shield, preventing companies from reaching leverage ratios as high as the ones obtained in static models.

Table 19 – Evolution of the debt to equity ratio with growing EBIT

Debt to Equity Ratio evolution											
EBIT	158	177	196	215	234	253	272	290	309	328	189
DER	3.97	3.57	3.24	2.97	2.73	2.52	2.19	2.19	2.05	1.92	1.81

Although this limitation, as exposed by Goldstein et al (2001), distances the model from the reality, it provides a useful guidance if interpreted correctly. Only one issuance of debt is possible, which produces the optimal capital structure, under the assumption of the model. Evaluating the company at future points in time, assuming EBIT growth consistent with the drift chosen for the scenario, would lead to increasingly smaller debt to equity ratios.

5.6.7. Bankruptcy Stake

The bankruptcy stake is the part of the remaining value that is consumed by costs in case of default.

Table 20 – Sensitivity Analysis to α (absolute values)

α	VC	T	D	E_l	BC	c	x_b	DER
22%	13012	1753	10785	2228	226	379.02	-47.23	4.84
33%	12906	1786	10494	2413	298	368.31	-57.94	4.35
44%	12813	1823	10233	2580	355	358.81	-67.45	3.97
55%	12728	1863	9996	2732	400	350.27	-75.98	3.67
66%	12652	1902	9780	2872	436	342.54	-83.71	3.41

Table 21 – Sensitivity Analysis to α (percentage values)

α	VC	T	D	E_l	BC	c	x_b	DER
-50%	1.6%	-3.8%	5.4%	-13.6%	-36.3%	5.6%	-30.0%	21.9%
-25%	0.7%	-2.0%	2.6%	-6.5%	-16.1%	2.6%	-14.1%	9.6%
44%	12813	1823	10233	2580	355	358.81	-67.45	3.97
+25%	-0.7%	2.2%	-2.3%	5.9%	12.7%	-2.4%	12.6%	-7.6%
+50%	-1.3%	4.3%	-4.4%	11.3%	22.8%	-4.5%	24.1%	-14.1%

The bankruptcy costs increase with the bankruptcy stake, as could be expected. This increase reduces the valuation of the company, reflecting the increased loss sustained by the investors in case of default. Accordingly, the optimization process requires that the debt coupon decreases, lowering the bankruptcy threshold and limiting the negative impact of the bankruptcy costs. The reduction of the coupon lowers the value of the debt claim, resulting in lower leverage.

5.6.8. Investment Costs

The investment cost is the capital that the investor pays for his stake in the company. That value does not affect the value of the company, only impacting the return that the investor obtains from the venture. In this analysis only increases to the IC were considered, the reason for this is that, according to what was explained in section 4.2.4, when the investment threshold is lower than the current EBIT, the computed threshold is not admissible. This means that lower values for the IC would not give any useful information. By studying the impact of higher IC, using pair multiples of the original value, conditions where there was a benefit to delaying the investment were reached.

Table 22 - Sensitivity Analysis to IC (absolute values)

IC	VC	I	x_i	λ
3680	12813	10421[9]	12.16[10]	18.67%
7360	12813	8285[9]	38.96[10]	35.34%
14720	12813	3245[9]	104.66[10]	74.67%
22080	12813	-1368	184.86	-
29440	12813	-3904	267.98	-

Table 23 – Sensitivity Analysis to IC (percentage values)

IC	VC	I	x_i	λ
-50%	0.0%	221.1%[9]	-88.4%[10]	-75.0%
-25%	0.0%	155.3%[9]	-62.8%[10]	-52.7%
14720	12813	3245[9]	104.66[10]	74.67%
+25%	0.0%	-142.2%	76.6%	-
+50%	0.0%	-220.3%	156.0%	-

The increase in the investment costs reduces the value of the investment option. In turn, this results in the need to demand higher equity stakes. When the value of the option becomes negative the threshold to invest raises above the current EBIT level, indicating that the investor should wait until the company shows better results to obtain a return over the fair value of its capital.

[9] These values for the investment option, $I(x)$, correspond to the totality of the equity.
[10] As these values for x_i are lower than the EBIT level, x, they are not valid solutions.

Table 24 – Comparative relevance of the input parameters in the results of the model

	VC	T	D	E_l	BC	c	x_b	DER
Descending order of impact	r	r	r	r	r	μ_m	σ	σ
	μ_m	τ	μ_m	τ	σ	r	r	τ
	x	σ	x	μ_m	α	x	x	μ_m
	τ	μ_m	σ	σ	μ_m	τ	τ	α
	σ	x	τ	α	τ	σ	α	r
	α	α	α	x	x	α	μ_m	x

The results conveyed with Table 24 show the great dependency of almost all the model results on the risk free rate, r. This result is both expected and unavoidable, due to the fact that models in a risk neutral setting require that the cash flows are discounted to the risk free rate. The rate acts as a scaling factor of the value of the claims, and for this reason has low impact on ratios, such as the DER that is a quotient of two claim values. The exponential impact of this rate on the value of the company, is a reason to recommend caution in its estimation: the choice of the value used as input for the model, in section 5.4.1, was taken from a period consistent with the period of the analysis. A cautious investor wanting a long term investment might choose to evaluate the historical behavior of the US 30-year Treasury Bond Yields, but that analysis would be out of scope in an academic valuation. The measured drift, μ_m, is also of great relevance to the results of the model, which suits expectations as it is the measure of the expected growth of the EBIT. In every analysis of ESTCs there will be high uncertainty in the estimation of this parameter, as the short life span that is a defining characteristic of early stage companies means that the data available to derive the estimates will be very limited. The estimation of the drift can be complemented by an analysis to other business specific indicators, such as the size and growth of the market, and the expected performance of the company versus the competition. The bankruptcy threshold and the debt-to-equity ratio are most sensitive to changes in the volatility, measured by the standard deviation, there is some uncertainty in the measurement of this parameter as it is obtained from a simulation based on the available data from Fitbit, a short supply of information, considering the short life of the company. The analysis was not extended to the risk premium parameter due to the fact that its effect on the results would not add significant information, since it only modifies the drift parameter.

6. Conclusions

This work had for its foremost objective to contribute to the literature related with the study of the valuation of ESTCs. By employing a structural model with the assumption that the EBIT of the company can be modelled with an ABM, it was possible to do a valuation of the company and study its optimal capital structure. The application of the model to a case study based on Fitbit Inc lead to a valuation consistent with the market value of the company.

The methodology suited the objectives and object of study. The valuation of these companies requires specific models, suited to deal with their particularities. Real options analysis meets the requirements, particularly the incorporation of the volatility on the valuation to represent the uncertainty associated with the underlying assets, in this case the EBIT of the early stage companies. These models also allow increased flexibility, enabling the incorporation of events such as options to pursue new projects, delay investments or expand existing ones, among many others. Having originated from the pricing of financial options, many of these models share underlying assumptions that make them unsuitable to the conditions faced by ESTCs.

The field of study of structural models, which incorporates the optimization of the capital structure in the valuation process, offers increased possibilities to both management and investors, going beyond the limiting assumption of all equity financing to consider the use of debt in the fulfillment of the capital necessities of the company. This new option is useful for the management, enabling new sources of financing capital, and creates value to investors, maximizing the value of the company through tax savings and avoiding dilution of the equity. Selecting the EBIT as the state variable, creates an economically meaningful representation of the operations, separating the generation of cash flows that is due to the core activities of the company from the claims over those flows.

Early stage companies are highly susceptible to periods of negative earnings, and to not being long lasted. Assuming otherwise is a compromise that skews the reality of the sector and limits the usefulness of the analysis. Basing the dynamics of the state variable on an ABM, although so far uncommon in the literature, is a valid solution to allow negative values for the EBIT. The proposed model, built on these assumptions, is a flexible base that can be easily implemented and expanded.

A careful sensitivity and robustness study lead to a better understanding of the model, revealing its limiting factors and the relevancy of each parameter on the results obtained. A critical assessment of the results of this study created the opportunity to suggest solutions to the limitations found. The results of the valuation are highly sensitive to changes in the risk free rate; there is uncertainty in the long term estimates of this parameter that must be considered. The volatility parameter also has considerable impact in the results, particularly in the optimal leverage proposed by the model. Providing estimates for this parameter is one of the biggest challenges in the application of real options models to ESTCs, due to the lack of available data on which to base the simulations. Working within a static framework for the leverage of the company, leads to unreasonably high leverage ratios. As the value of the coupon remains constant, an assessment of the debt to equity ratio with the expected evolution of the EBIT, shows that the leverage will diminish with time, converging to more reasonable results. An interpretation of the results of this model, that brings it closer to reality, is to treat the optimal coupon as a limit, increasing the debt to reach it as the company evolves, rather than as a starting and fix value.

The application of the model to a reality based setting is a relevant contribution to the available literature, since it is, to the extension of the author knowledge, the first case study for a structural model based on a real ESTC. The results indicate that there is a relation between the value of the company and its capital structure, resulting from the fiscal advantage to debt.

7. Bibliography

AJPark, n.d. *Guide to Royalties and Licence Fees*. Available at: < http://www.ajpark.com/introduction-to-ip/i-want-to-commercialise/guide-to-royalties-and-licence-fees/>.

Alexander, D.R., Mo, M. and Stent, A.F., 2012. Arithmetic Brownian motion and real options. *European Journal of Operational Research*, 219, pp.114–122

Ammann, M. and Genser, M., 2004. *A Testable Credit Risk Framework with Optimal Bankruptcy, Taxes, and a Complex Capital Structure*. Available at: <http://papers.ssrn.com/sol3/papers.cfm?abstract_id=498803>

Audretsch, D.B. and Thurik, R., 2001. What´s New about the New Economy? Sources of Growth in the Managed and Entrepreneurial Economies. *Industrial and Corporate Change*, 10, 1, pp.267-315.

Bank, M. and J. Lawrenz 2005. *Informational Asymmetry Between Managers and Investors in the Optimal Capital Structure Decision*. Available at: <http://papers.ssrn.com/Sol3/papers.cfm?abstract_id=673181>

Bank, M. and Wibmer, K., 2012. *Start-up firm valuation: A real-options approach*. Available at: <http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1928710>

Barton, K. and Lawryshyn, Y., 2010. *Reconciling Real Option Models: An Approach to Incorporate Market and Private Uncertainties*. Available at: <http://www.realoptions.org/papers2010/106.pdf>.

Black, F. and Scholes, M., 1973. The Pricing of Options and Corporate Liabilities*. The Journal of Political Economy*, 81, 3, pp.637–654.

Blaug, R. and Lekhi, R., 2009. Accounting for intangibles: Financial reporting and value creation in the knowledge economy. London: The Work Foundation.

Branch, B., 2002. The costs of bankruptcy. A review. *International Review of Financial Analysis* 11, pp.39–57.

Brandão, L.E., Dyer, J.S. and Hahn, W.J., 2005. Using Binomial Decision Trees to Solve Real-Option Valuation Problems. *Journal Decision Analysis*, 2, 2, pp.69-88.

Brennan, M.J. and Schwartz, E.S., 1978. Corporate Income Taxes, Valuation and the Problem of Optimal Capital Structure. *Journal of Business*, 51, 1, pp.103-114.

Bruno, V. and Tyebjee, T.T., 1985. The Entrepreneur's Search for Capital. *Journal of Business Venturing*, 1, 1, pp.61-74.

Copeland, T.E. and Antikarov, V., 2001. *Real Options for Practitioners*. New York: Texere

Damodaran, A., 2009. *Valuing Young, Start-up and Growth Companies: Estimation Issues and Valuation Challenges*. Available at: <http://people.stern.nyu.edu/adamodar/pdfiles/papers/younggrowth.pdf>

Damodaran, A., 2012. *Investment valuation: Tools and Techniques for Determining the Value of any Asset*. 3rd ed. New Jersey: John Wiley & Sons.

Deloitte, 2015. *Corporate Tax Rates 2015*. Available at: http://www2.deloitte.com/content/dam/Deloitte/global/Documents/Tax/dttl-tax-corporate-tax-rates-2015.pdf

Directorate General for Research of the European Commission, 2006. *Ricardis: Reporting Intellectual Capital to Augment Research, Development and Innovation in SMEs*. Luxembourg: Office for Official Publications of the European Communities.

Dixit, A.K. and Pindyck, R.S., 1994. *Investment under Uncertainty*. New Jersey: Princeton University Press.

Dulken, S., 2001. *Inventing the 19th Century*. New York: New York University Press.

Dzyuma, U., 2012. Real Options Compared To Traditional Company Valuation Methods: Possibilities and Constraints in Their Use. *Financial Internet Quarterly e-Finance*, 8, 2.

Ellis, I., 2009. *Maximizing Intellectual Property and Intangible Assets*. Wachington DC: Athena Alliance.

Festel, G., Wuermseherb, M. and Cattane, G., 2013. Valuation of Early Stage High-tech Start-up Companies. *International journal of business*, 18, 3, pp.217-231.

Fischer, E.O., Heinkel, R. and Zechner, J., 1989. Dynamic Capital Structure Choice: Theory and Tests. *The Journal of Finance*, 44, 1, pp.19–40.

Fitbit, 2015. *Registration* Statement. Available at: <https://www.sec.gov/Archives/edgar/data/1447599/000119312515176980/d875679ds1.htm>

Genser, M., 2006. *Structural Framework for the Pricing of Corporate Securities: Economic and Empirical Issues*. Berlin: Springer Science & Business Media

Ghafele, R., 2010. *Accounting for intellectual property?* Munich Personal RePEc Archive. MPRA Paper No. 37360. Available at: <http://mpra.ub.uni-muenchen.de/37360/>.

Giacometti, R. and Teocchi, M., 2005. On pricing of Credit spread options. *European Journal of Operational Research*, 163, pp.52-64.

Goldstein, R., Ju, N. and Leland, H.E., 2001. An EBIT-Based Model of Dynamic Capital Structure. *The Journal of Business*, 74, 4, pp.483–512.

Helmers, C. and Rogers, M., 2011. *Does patenting help high-tech start-ups?* Available at: <http://www.sciencedirect.com/science/article/pii/S0048733311000680>.

Jaeckel, P., 2002. *Monte Carlo Methods in Finance*. Chichester West Sussex: John Wiley & Sons.

Kane, A., Marcus, A.J. and McDonald, R.L., 1984. How Big is the Tax Advantage to Debt? *The Journal of Finance*, 39, 3, pp.841–853.

Leland, H.E., 1994. Corporate Debt Value, Bond Covenants, and Optimal Capital Structure. *The Journal of Finance*, 49, 4, pp.1213–1252.

Lev, B., 2001. *Intangibles – Management, Measuring and Reporting*. Wachington DC: Brookings Institute.

Luehrman, T.A., 1998. Investment Opportunities as Real Options: Getting Started on the Numbers. *Harvard Business Review*, July-August, pp.51-67.

Malackowski, J.E., 2004. Using Intellectual Property to Grow. *The Beacon*, 3, 4.

Martin, C. and Hartley, J., 2006. *Intangible Assets and SMEs*. London: Association of Chartered Certified Accountants.

McGrath, R.G., Ferrier, W.J. and Mendelow, A.L., 2004. Real Options and Engines of Choice and Heterogeneity. *Academy of Management Review*, 29, 1, pp.86-101.

Merton, R.C., 1974. On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *The Journal of Finance*, 29, 2, pp.449–470.

Modigliani, F. and Miller, M.H., 1958. The Cost of Capital, Corporation Finance and the Theory of Investment. *The American Economic Review*, 48, 3, pp.261–297.

Mun, J., 2002. *Real Option Analysis – Tools and Techniques for Valuing Strategic Investments and Decisions*. New Jersey: John Wiley & Sons.

Pavlov, V., 2009. *Looking Beyond Loans: Where to Find Financing Now*. CNNMoney.

Pitkethly, R., 1997. *The Valuation of Patents : A review of patent valuation methods with consideration of option based methods and the potential for further research*. Cambridge: The Judge Institute of Management Studies.

Pratt, S.P. and Grabowski, R.J., 2014. *Cost of Capital: Applications and Examples*. New Jersey: Jhon Wiley & Sons.

Rhaiem, N., Ben, S. and Mabrouk, A.B., 2007. Estimation of Capital Asset Pricing Model at Different Time Scales. *The International Journal of Applied Economics and Finance*, 1, 2, pp.79-87.

RockThePost, 2014. *Investor Trends 2003-2013*. Available at: <https://onevest.com/cdn-static/files/reports/RockThePost_2013_Investor_Trends_Report.pdf>.

Sahlman, W.A., 1990. The structure and governance of venture-capital organizations. *Journal of Financial Economics*, 27, 2, pp.473-521.

Schumpeter, J.A., 1934. *The Theory of Economic Development: An Inquiry into Profits, Capital, Credit, Interest, and the Business Cycle*. New Brunswick: Transaction Publishers.

Smith, J.E. and Nau, R.F., 1995. Valuing Risky Projects. *Management Science*, 41, 5, pp.795-816.

Steiger, F., 2008. *The Validity of Company Evaluation Using Discounted Cash Flow Methods*. Available at: <http://arxiv.org/ftp/arxiv/papers/1003/1003.4881.pdf>.

Sudarsanam, S., Sorwar, G. and Marr, B., 2003. *Valuation of Intellectual Capital and Real Option Models*. PMA Intellectual Capital Symposium. Cranfield University. Available at: <http://www.realoptions.org/papers2004/SudarsanamIntellCap.pdf>.

Tondo-Kramer, K., 2010. Increasing Access to Startup Financing Through Intellectual Property Securitization, *Journal of Computer & Information Law*, 27, 4, pp.613-646.

US Department of the Treasury, 2015. *Daily Treasury Yield Curve Rates*. Available at: <http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield>

Wiersema, U.F., 2008. *Brownian Motion Calculus*. Chichester West Sussex: John Wiley & Sons.

Wikipedia Commons, 2009. *Startup financing cycle*. Available at: <https://commons.wikimedia.org/wiki/File:Startup_financing_cycle.svg>

World Intellectual Property Organization, n.d. *What is Intellectual Property?* Available at: <http://www.wipo.int/edocs/pubdocs/en/intproperty/450/wipo_pub_450.pdf>.

Yahoo, 2015, *Fitbit Historical Prices*. Available at: <http://finance.yahoo.com/q/hp?s=FIT+Historical+Prices>

8. Annexes

8.1. Annex 1

$$dx(t) = \mu dt + \sigma dW$$

Can be easily integrated into:

$$x(t) = x_0 + \mu t + w_t \sigma$$

With an expected value:

$$E(x) = x_0 + \mu t$$

Thus:

$$E\left(\int_0^\infty x e^{-rt}\,dt\right) = \int_0^\infty e^{-rt}\,E(x)dt = \int_0^\infty e^{-rt}(x + \mu t)dt =$$

$$= \int_0^\infty e^{-rt}(x)dt + \int_0^\infty e^{-rt}(\mu t)dt = \frac{x}{r} + \mu\left[\int_0^\infty e^{-rt} t\,dt\right]$$

Integrating the last term by parts:

$$\int_0^\infty e^{-rt} t\,dt = \left[-\frac{1}{r}e^{-rt}t\right]_0^\infty - \int_0^\infty -\frac{1}{r}e^{-rt}dt = 0 + \frac{1}{r}\int_0^\infty e^{-rt}\,dt = \frac{1}{r^2}$$

Which leads to:

$$E\left(\int_0^\infty e^{-rt}(x)\ dt\right) = \frac{\mu}{r^2} + \frac{x}{r}$$

8.2. Annex 2

This annex provides the detailed process to obtain a solution for the following differential equation:

$$\frac{dV}{dx}\mu + \frac{1}{2}\frac{d^2V}{dx^2}\sigma^2 - rV + mx + k = 0$$

This is a second order nonhomogeneous differential equation, the solution for it is the sum of the general solution for the homogeneous case with the particular solution. The corresponding homogeneous equation takes the form:

$$\frac{dV(x)}{dx}a + \frac{1}{2}\frac{d^2V(x)}{dx^2}b + cV(x) = 0;\ a = \mu, b = \sigma^2 = b, c = -r$$

The solution requires the substitution $V(x) = e^{\beta x}$, with a constant β. The equation requires the derivatives of the substitution: $\frac{d}{dx}\left(e^{\beta x}\right) = \beta e^{\beta x}$ and $\frac{d^2}{dx^2}\left(e^{\beta x}\right) = \beta^2 e^{\beta x}$. With the substitution the equation becomes:

$$a\beta e^{\beta x} + \frac{1}{2}b\beta^2 e^{\beta x} + ce^{\beta x} = 0$$

Or,

$$e^{\beta x}(a\beta + \frac{1}{2}b\beta^2 + c) = 0$$

The zeros, with a finite β, must come from the polynomial under brackets, and those are:

$$\beta_1 = -\frac{a}{b} - \frac{\sqrt{a^2 + 2bc}}{b}$$

$$\beta_2 = -\frac{a}{b} + \frac{\sqrt{a^2 + 2bc}}{b}$$

For arbitrary constants A_1, A_2 the general solution becomes:

$$V(x) = A_1 e^{\beta_1 x} + A_2 e^{\beta_2 x}$$

Now it is necessary to find a particular solution for the nonhomogeneous equation, which will take the simplified form:

$$\frac{dV(x)}{dx}a + \frac{1}{2}\frac{d^2V(x)}{dx^2}b + cV(x) = -k - mx;\ a = \mu, b = \sigma^2 = b, c = -r$$

According to the method of undetermined coefficients, the particular solution is one such as:

$$V_P = p_1 + p_2 x$$

Making the substitution of the particular solution into the equation, taking the required derivatives: $\frac{dV_P}{dx} = p_2$ and $\frac{d^2V_P}{dx^2} = 0$, gives:

$$ap_2 + cp_1 + cp_2 x = -k - mx$$

At the origin ($x = 0$) the equation becomes:

$$ap_2 + cp_1 = -k$$

Substituting this result in the previous equation:

$$cp_2 = -m$$

Which gives the solution for the constants p_1, p_2:

$$p_1 = -\frac{k}{c} + \frac{am}{c^2}; \; p_2 = -\frac{m}{c}$$

And the particular solution becomes:

$$V_P = -\frac{k}{c} + \frac{am}{c^2} - \frac{mx}{c}$$

Which gives, after the substitution for the original constants, the final solution:

$$V(x) = \frac{m\mu}{r^2} + \frac{mx + k}{r} + A_1 e^{\beta_1 x} + A_2 e^{\beta_2 x}$$

$$\beta_1 = -\frac{\mu}{\sigma^2} - \frac{\sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}$$

$$\beta_2 = -\frac{\mu}{\sigma^2} + \frac{\sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}$$